Heavy-quark-pair Production at Lepton Colliders at NNNLO in QCD

Xiang Chen (Peking University)

第三届量子场论及其应用研讨会
August 2023

In collaboration with Xin Guan, Chuan-Qi He, Xiao Liu, Yan-Qing Ma
Based on: [arXiv: 2209.14259v2]

Outline

1. **Introduction: Motivation and Background**
2. Computational Techniques
3. Numerical Results
4. Summary and Outlook

Motivation and Background

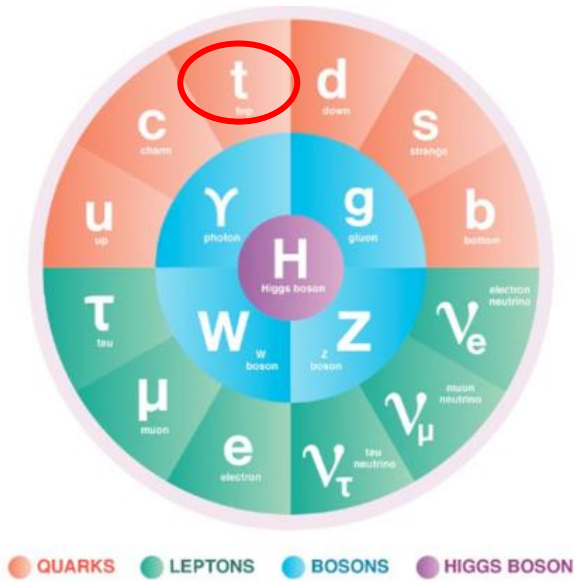
- Standard Model —— High precision/high energy/cosmology ——> New Physics?

- Future lepton colliders:
  - International Linear Collider(ILC)
  - Future Circular Collider(FCC-ee)
  - Circular Electron-Positron Collider(CEPC)

Higher Energy, Cleaner Backgrounds ——> Higher Precision

The International Linear Collider. *(2013)*
The CEPC Study Group. *(2018)*
FCC-ee, A. Abada *et al.* *(2019)*
ILD Concept Group. *(2020)*

- Particles in Standard Model



⟹ New Physics?

Experiment

- Particle: **Top quark**
- Main process: $e^+e^- \to \gamma^*/Z^* \to t\bar{t}$

ILD can determine these form factors separately to accuracies better than 0.3% by measuring the cross section and forward-backward asymmetry for the $t\bar{t}$ pair production with 4 ab^{-1} at 500 GeV [23].

ILD Concept Group. *(2020)*

4. Discussion of systematic uncertainties

In the previous sections measurements of either cross sections or asymmetries have been presented. This section makes an attempt to identify and quantify systematic uncertainties, which may influence the precision measurements.

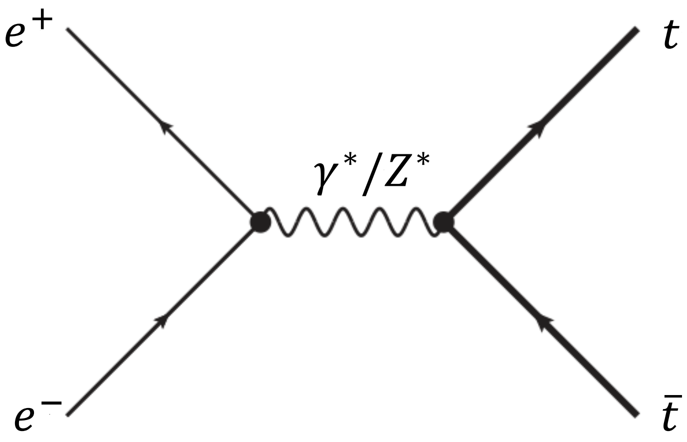
- Luminosity: The luminosity is a critical parameter for cross section measurements only. The luminosity can be controlled to 0.1% [24].

M. S. Amjad, *et al. (2015)*

- Experiment precision: A few ‰ to % precision

Theoretical efforts in literature

$e^+e^- \to \gamma^*/Z^* \to t\bar{t}$



- **NNLO(QCD)**: Near-threshold expansion and high energy expansion of cross section B. H. Smith, M. B. Voloshin. *(1994)* K. G. Chetyrkin, A. Kwiatkowski. *(1994)* K. G. Chetyrkin, J. H. Kuhn, M. Steinhauser. *(1996)*
- **NNLO(QCD)**: Cross section and fully differential distributions in the continuum J. Gao, H.-X. Zhu. *(2014)* L. Chen, O. Dekkers, *et al.(2016)*
- **NNNLO(QCD)**: Near-threshold expansion M. Beneke, *et al.* *(2015)*
- **NNNLO(QCD)**: Massive form factors M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *(2022)* M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *(2023)*

- **NNLO(QCD)**: Cross section

$$\sigma_{NNLO} = \sigma_{LO}(1 + \Delta_1 + \Delta_2)$$

$\sqrt{s}$ [GeV]	360	381.3	400	500
Δ_1	0.627	0.352	0.266	0.127
Δ_2	0.281	0.110	0.070	0.020

From L. Chen, O. Dekkers, *et al.(2016)*

- The theoretical uncertainty is about 1%

From J. Gao, H.-X. Zhu. *(2014)*

- Full computation of NNNLO QCD correction is necessary!

Challenges

$$e^+e^- \to \gamma^*/Z^* \to t\bar{t}$$

- Many Feynman diagrams (squared amplitudes)
  - Over 10000+ squared amplitudes
- Many loops
  - VVV: 3 loop integrals + 1 phase-space integral
  - VVR: 2 loop integrals + 2 phase-space integrals
  - VRR: 1 loop integral + 3 phase-space integrals
  - RRR: 0 loop integral + 4 phase-space integrals

→

- Require systematic approach
- Difficult to calculate analytically, calculate numerically

Equipped with the-state-of-art techniques, the high order corrections are available now!!

Outline

Theoretical: The work-flow for high order perturbative computation

➢ Work flow for high order computation:

Generating Integrands ($\mathcal{O}$(few minutes to hours)) → Integrals Reduction ($\mathcal{O}$(hours to days)) → Evaluating master integrals ($\mathcal{O}$(days) typically in frontier)

$$\mathcal{A} = \sum_{i=1}^{\mathcal{O}(10^4)} f_i \times I_i$$

Feynman diagrams

$$I_i = \sum_{j=1}^{\mathcal{O}(10^2)} c_{ij} M_j$$

Integrate-By-Part(IBP) method

$$I_i(p_i \cdot p_j, m_i^2) = \int \frac{d^D l_1 \cdots d^D l_h}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_N^{\nu_N}}$$

Differential equations

Evaluating Feynman integrals is the bottleneck!

Generating Integrands

- Generate Feynman amplitudes: **FeynArts** and **QGRAF**

  T. Hahn. *(2001)* P. Nogueira. *(1993)*

- Phase-space integrals

  C. Anastasiou, K. Melnikov. *(1993)*

  reverse unitarity

  Loop integrals

$$\delta(x) \to \frac{1}{2\pi i}\left(\frac{1}{x-i0} - \frac{1}{x+i0}\right)$$

- Numbers of family
  - 62(VVV)+147(VVR)+167(VRR)+99(RRR)=475 families

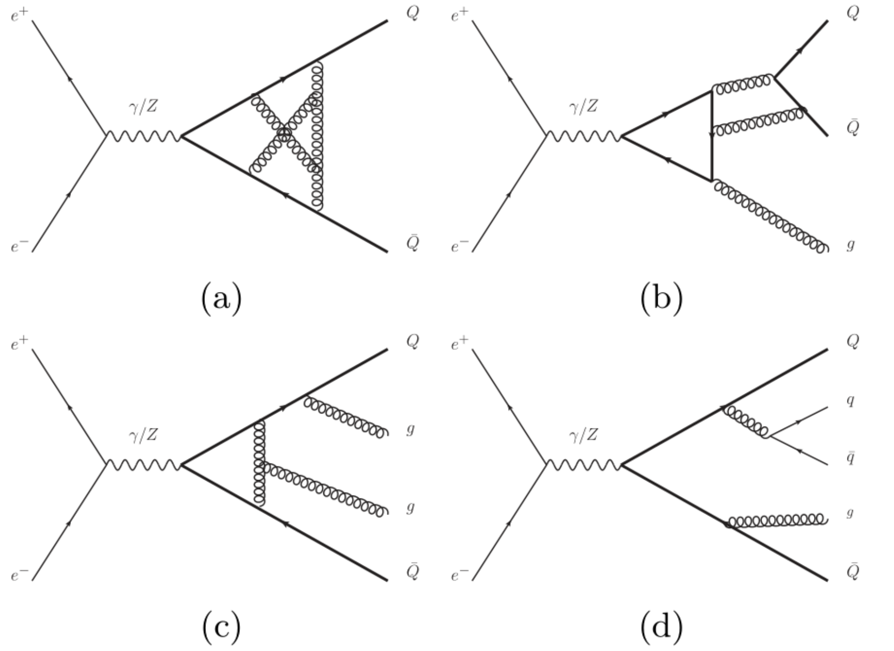
- γ_5 scheme: try to choose a scheme that
  - Keep $\{\gamma_5, \gamma_\mu\} = 0$ in D dimension
  - Avoid much more γ matrix in Trace to handle
  - Avoid additional renormalization(at least in this order)

$\Rightarrow$ KKS scheme: set every axial-vector current **"vertex"** as a reading point and average them

J. G. Korner, D. Kreimer, K. Schilcher. *(1992)*
D. Kreimer. *(1994)*
L. Chen. *(2023)*

Talk by Long Chen

$$e^+e^- \to \gamma^*/Z^* \to t\bar{t}$$



Integrals reduction

- Strategy:
  - Laporta's algorithm S. Laporta. *(2000)*
  - Tools: **`Blade`**, a new package based on the method of block-triangular form, will be released soon.

X. Guan, X. Liu, Y.-Q. Ma. *(2020)*

Talk by Xin Guan

Contribution	Number of families	Computing resources (the most expensive family)
VVV	62	10 hours on 16 cores
VVR	147	5 hours on 16 cores
VRR	167	4 hours on 16 cores
RRR	99	2 hours on 16 cores

Computing resources

  - The computational time of the most expensive family of VVV contributions is about $\mathcal{O}(10\times)$ faster than traditional method
  - Block-triangular form reduction is much more efficient!

Evaluating master integrals

- Method: **numerical differential equations** based on power series expansion

R. N. Lee, V. A. Smirnov. *(2018)*
R. Bonciani. G. Degrassi, P. P. Giardino, R. Grober. *(2019)*
H. Frellesvig, M. Hidding, L. Maestri, F. Moriello, G. Salvatori. *(2020)*
L. Chen, M. Czakon, M. Niggetiedt. *(2021)*
Martijin Hidding. *(2021)*
X. Liu, Y.-Q. Ma. *(2021)*
M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *(2022)*

$$\frac{dI_i(\epsilon,x)}{dx} = \sum_j A_{ij} I_j(\epsilon,x) \qquad I(\epsilon,x) = \sum_{\mu,k,n} c_{\mu,k,n}(\epsilon) x^{\mu(\epsilon)} \log^k x \, x^n$$

- General procedure
  - Set up the differential equations with respect to $x = \frac{4m^2}{s}$ by IBP reduction
  - Obtain boundary condition (**non-trivial**):
    - ✓ We use **AMFlow** to calculate master integrals at $x = \frac{4}{23}$ (arbitrary regular point) X. Liu, Y.-Q. Ma. *(2021)*
    - ✓ Dimensional regulator is set to a small (rational) number $\epsilon = \epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_N \quad N \approx 10$
  - Generalized series expansions around singularities
  - Taylor expansions around regular points
  - Match between two neighboring expansions

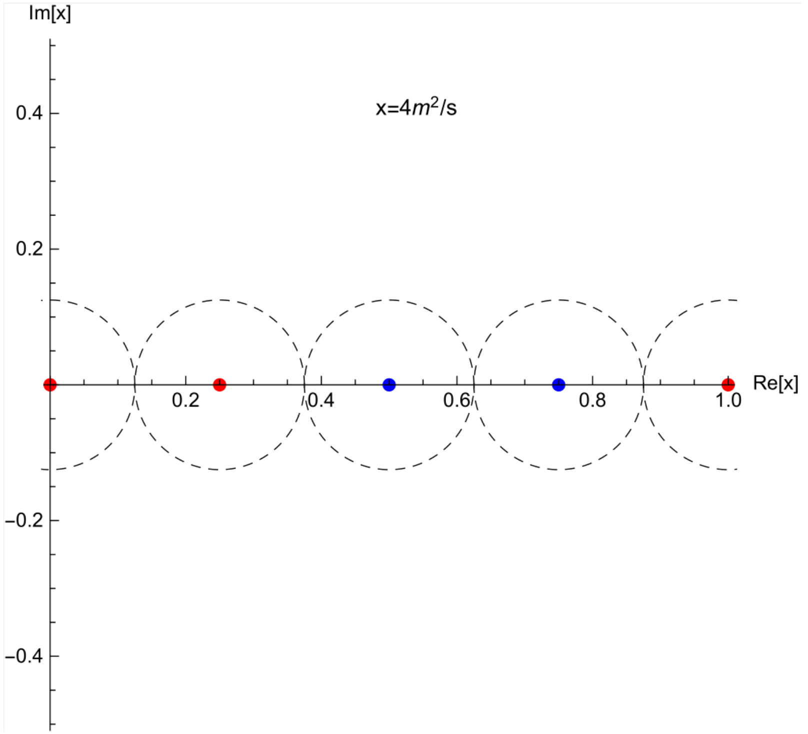
  $\Rightarrow$ Cover physical region

Piecewise function for $e^+e^- \to \gamma^*/Z^* \to t\bar{t}$

- Constructing a piecewise function of $x(=\frac{4m^2}{s})$
  - 5 deeply expanded power series at the following points

$$x \to \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

  - $0, \frac{1}{4}, 1$ are physical singularities, $\frac{1}{2}$ and $\frac{3}{4}$ are regular points
  - The expansion at each x_0 is **valid** in $x \in \left(x_0 - \frac{1}{8}, x_0 + \frac{1}{8}\right)$
  - The whole physical region $x \in (0,1)$ is covered

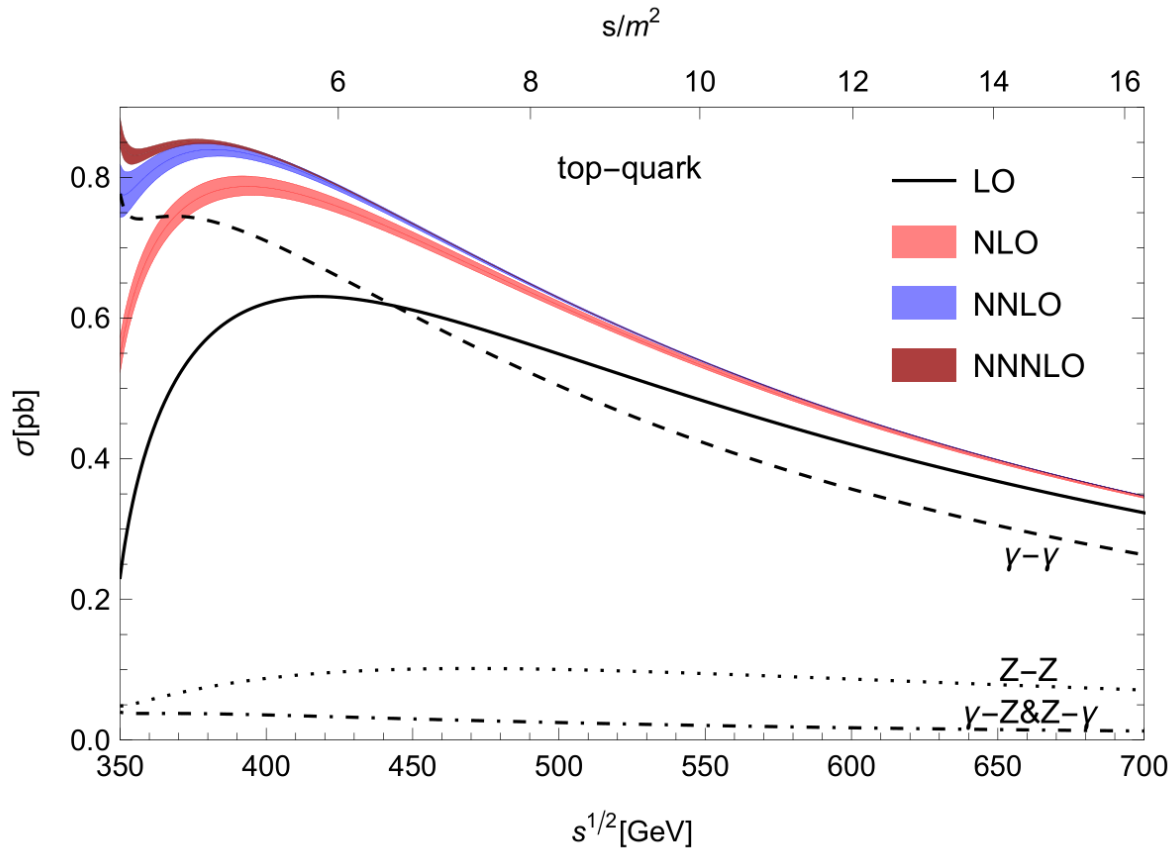


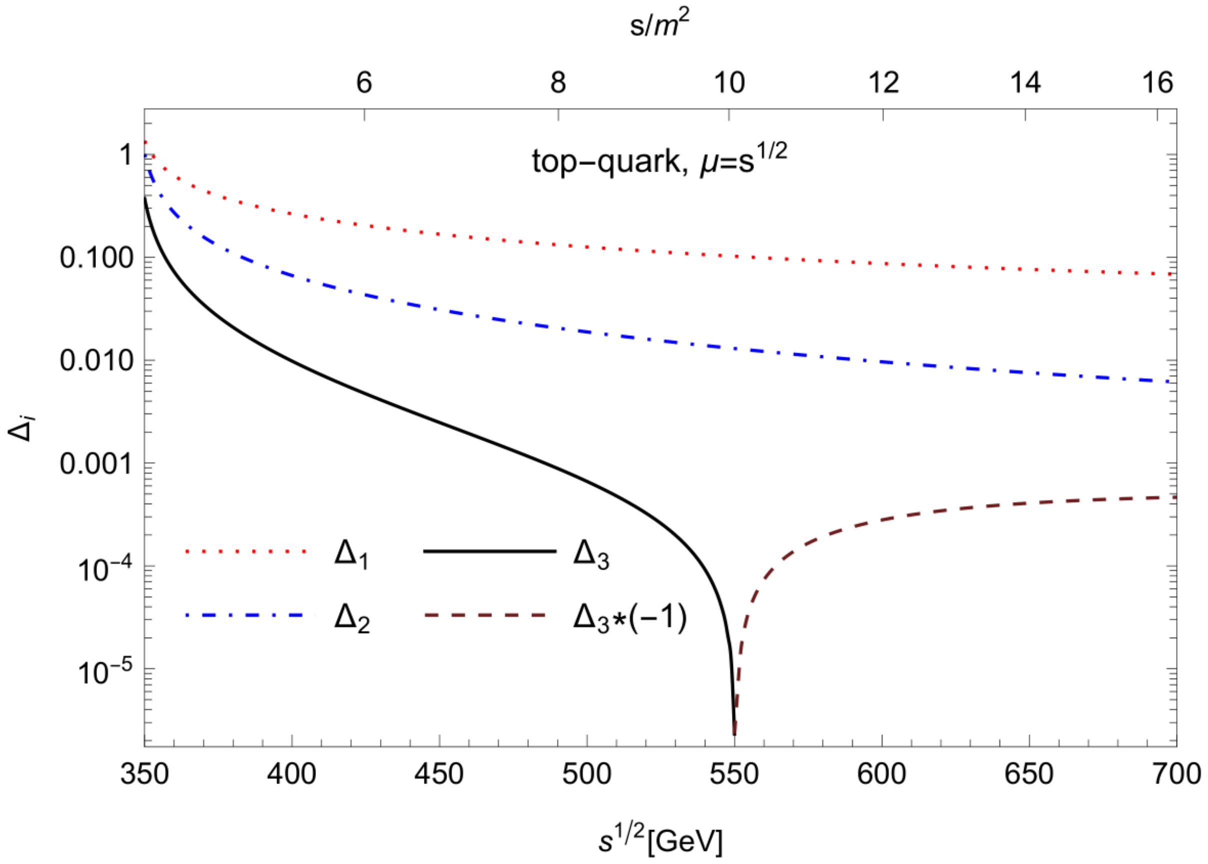
- cross section $\sigma(\epsilon_i, x)$ → Fit ϵ_i → cross section $\sigma(\epsilon, x)$ → $\epsilon \to 0$ due to UV, IR poles have been canceled → cross section $\sigma(x)$

Outline

Numerical Results for $e^+e^- \to \gamma^*/Z^* \to t\bar{t}$ cross section

- Input values: $m_t = 173.34GeV$, $\alpha = 1/132.2$, $\alpha_s^{n_f=5}(m_Z = 91.1876GeV) = 0.1181$
- $\sigma_{NNNLO} = \sigma_{LO}(1 + \Delta_1 + \Delta_2 + \Delta_3)$

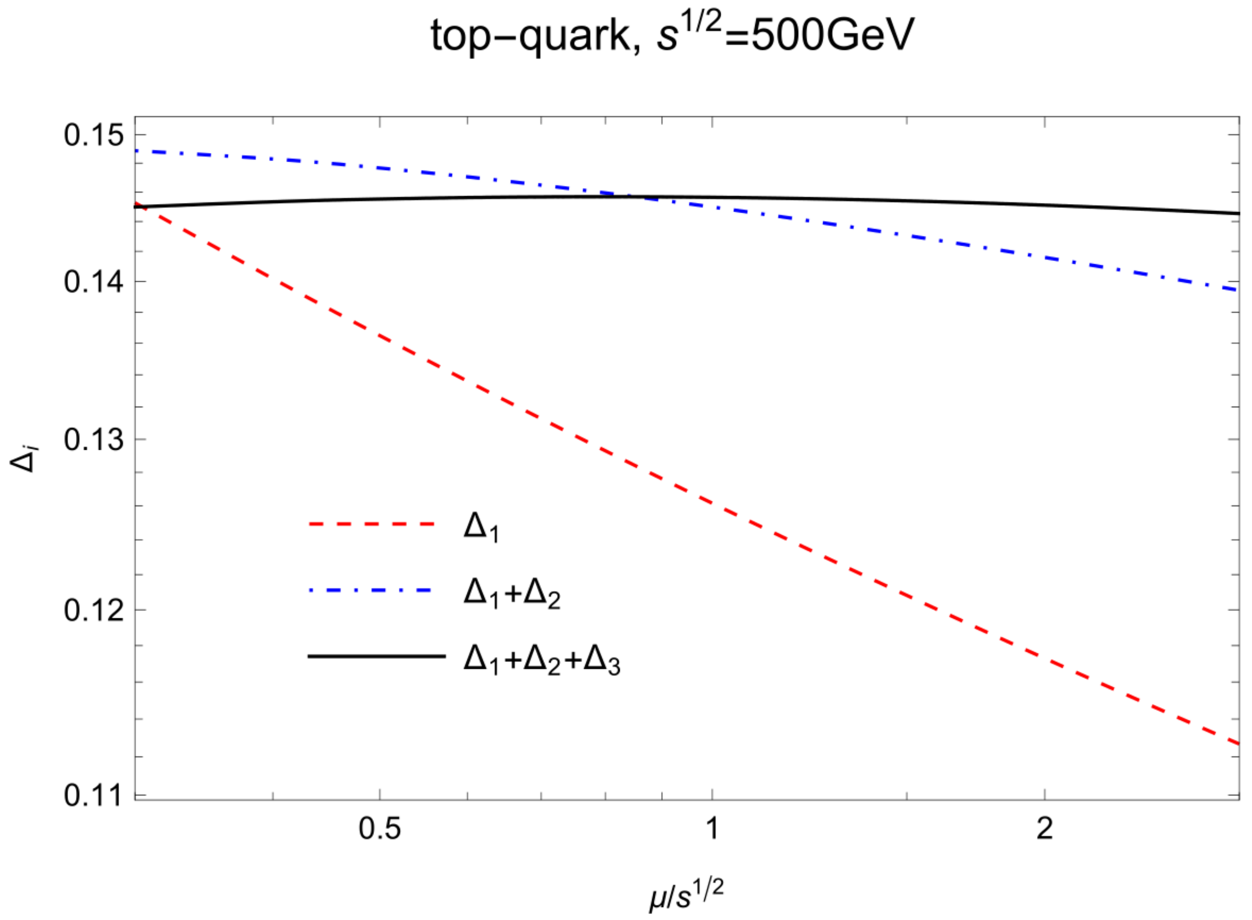




The middle lines correspond to $\mu = \sqrt{s}$, the upper and lower lines correspond to $\mu = 2\sqrt{s}$ and $\mu = \sqrt{s}/2$

Numerical Results for $e^+e^- \to \gamma^*/Z^* \to t\bar{t}$ cross section

- Input values: $m_t = 173.34GeV$, $\alpha = 1/132.2$, $\alpha_s^{n_f=5}(m_Z = 91.1876GeV) = 0.1181$
- $\sigma_{NNNLO} = \sigma_{LO}(1 + \Delta_1 + \Delta_2 + \Delta_3)$

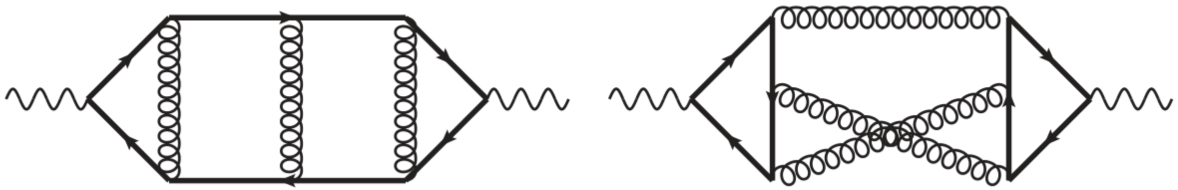


The scale dependence has been reduced from 0.94% at **NNLO** to 0.11% at **NNNLO** for a collider energy of $\mathbf{500GeV}$, which meets the precision (1‰ to 1%) requested by future lepton colliders!

ILD Concept Group. *(2020)*

Verification for $e^+e^- \to \gamma^*/Z^* \to t\bar{t}$ (I)

- We calculate the cross section in another totally different way
  - Optical theorem: imaginary part of forward scattering amplitude $\Longrightarrow$ cross section
  - Same computational strategy
  - Sample Feynman diagrams:



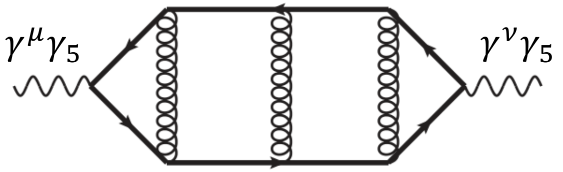
  - Using block-triangular relations is expected to be 2 orders of magnitude faster than plain IBP system
  - Contributions from **massless final states** without any top quark must be subtracted
  - Contributions from **four-heavy-quark production** should be subtracted when we consider the region above its threshold, i.e. $x < 1/4$

Verification for $e^+e^- \to \gamma^*/Z^* \to t\bar{t}$ (II)

- We calculate all the contribution below NNNLO and the **axial-axial contribution for singlet case** at NNNLO in another γ_5 scheme (Larin scheme)
  - **Non-singlet**: the external current couples directly to the fermion line of the final-state quarks
  - **Singlet**: the external current does not couple to the fermion line of the final-state quarks
- Classify contribution, the current could be **vector(V)** or **axial-vector(A)** :
  - **V-V**: checked using optical theorem
  - **V-A**: equals to **0** due to

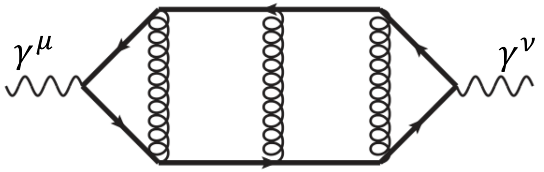
$$Tr(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \to \epsilon^{\mu\nu\rho\sigma} \text{ and } \epsilon^{\mu\nu\rho\sigma} g_{\rho\sigma} = 0,\ \epsilon^{\mu\nu\rho\sigma} p_\rho p_\sigma = 0$$

  - **A-A**:
    - Non-singlet:

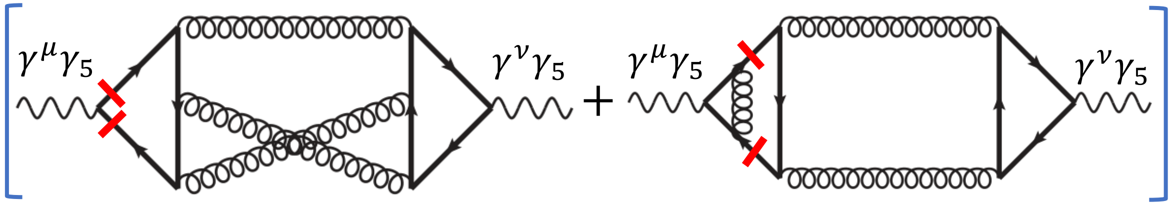


$\{\gamma_5, \gamma_\mu\} = 0$

$\gamma_5^2 = 1$



    - **Singlet**:



KKS Scheme(avoid additional renormalization)
‖ checked
Larin scheme(need additional renormalization)

Red symbols ❙ represent the reading points

Outline

Summary and Outlook

- **Block-triangular relations** and **auxiliary mass flow** are very powerful tools for high order calculations.
- Our result for $e^+e^- \to \gamma^*/Z^* \to t\bar{t}$ NNNLO cross section significantly **reduce theoretical uncertainty**, which **meets the precision** requested by future lepton colliders.
- Our strategy is **applicable** for many other processes.