Blade:

A package for block-triangular form improved Feynman integral decomposition

关鑫

北京大学

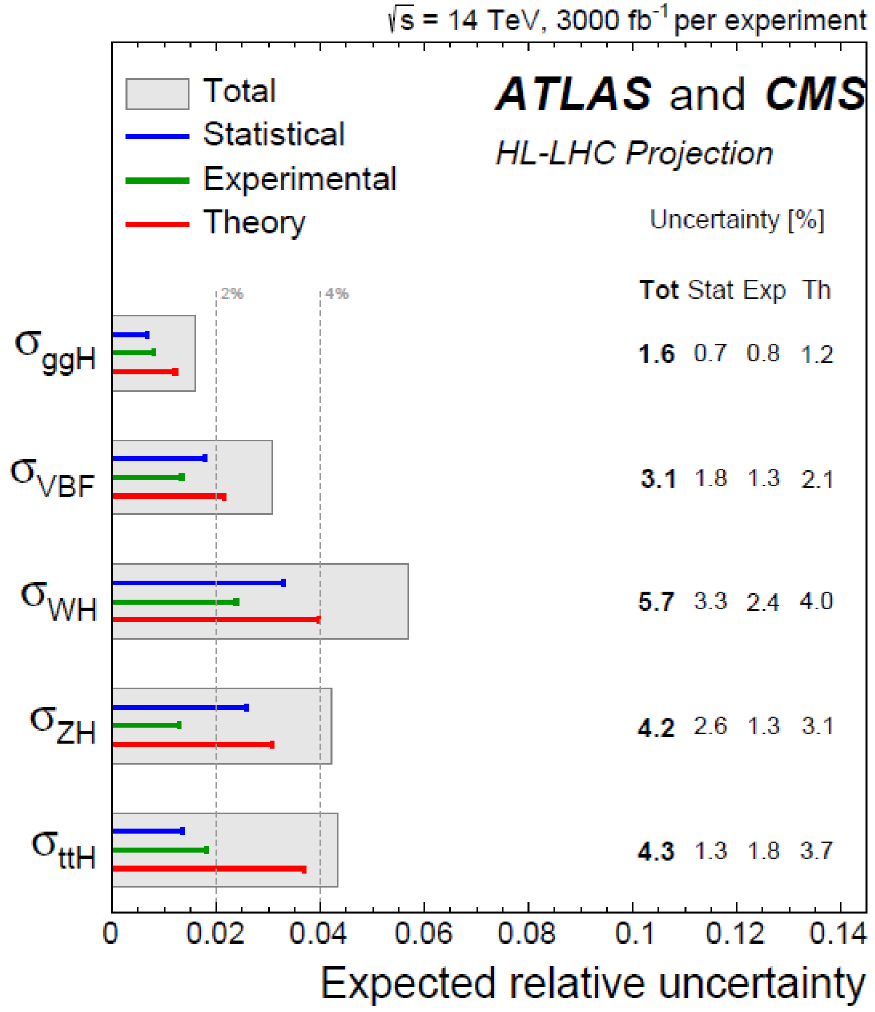
Based on works with
Xiao Liu(刘霄) Yan-Qing Ma(马滟青) and Wen-Hao Wu(武文浩)

第三届量子场论及其应用研讨会

2023/08/16 北京

Collider physics

- **Main way of testing theories**
  - **Success of the Standard Model**
  - **Lack of very clear new physics**
- **Upcoming increased Exp. precision**
  - **Theoretic uncertainty should be reduced further**
  - **High order perturbative calculation**



HL/HE-LHC Workshop, CERN Yellow Rep.Monogr. 7 (2019)

Precision frontier

- **NLO revolution**
- **NNLO revolution**
- **State-of-the art calculations**
  - **Multi-loops**

    $gg \to H$ @ N^3LO_{HTL} C. Anastasiou, C. Duhr, F. Dulat, et al, Phys. Rev. Lett (2015)

    Five loop QCD-β function [P. Baikov, K. Chetyrkin and J. Kuhn, Phys. Rev.Lett(2017), F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, JHEP(2017) T. Luthe, A. Maier, P. Marquard and Y. Schroder, JHEP(2017) K. G. Chetyrkin, G. Falcioni, F. Herzog and J. A. M. Vermaseren, JHEP(2017)]

    Five loop g-2 S. Volkov, Phys. Rev.D(2019)
  - **Multi-scales**

    $q\,\bar{q} \to \gamma\gamma\gamma$ @ N^2LO_{QCD} Chawdhry, Czakon, Mitov and Poncelet,JHEP(2020)

    $pp \to W\,b\,\bar{b}$ @ N^2LO_{QCD} Hartanto, et al. arxiv: 2205.01687

    $e^+e^- \to H\,Z$ @N^2LO_{EW} A. Freitas, Q. Song, PRL(2023) Xiang Chen, X. G, Chuan-Qi He et al, 2209.14953

Calculation of Scattering Amplitude

➢ General setup

Differential Equation

generate integrand → Integral reduction → Evaluate master integral

$$A = \sum_{i=1}^{\mathcal{O}(10^4)} f_i \times I_i \qquad I_i = \sum_{j=1}^{\mathcal{O}(10^2)} c_{ij} M_j \qquad I_i(p_i \cdot p_j, m_i^2) = \int \dots \int \frac{d^d k_1 \dots d^d k_h}{E_1^{a_1} \dots E_N^{a_N}} \qquad (E_i = q_i^2 - m_i^2 + i0^+)$$

- **AMFlow: any given Feynman integrals can be automatically calculated to high precision as far as the reduction has been achieved**

X. Liu, Y.-Q. Ma, and C.-Y. Wang, Phys. Lett. B (2018),
AMFlow : X. Liu and Y.-Q. Ma, arxiv: 2201.11669

- **Integral reduction is a critical yet formidable task in complicated multiloop processes**

Current status of integral reduction

➢ IBP is taking center stage

- **Integration-by-part identity** Chetyrkin and Tkachov, NPB(1981)

$$\int \prod_{i=1}^{L} \mathrm{d}^D \ell_i \frac{\partial}{\partial \ell_j^{\mu}} [v^{\mu} \mathcal{I}(\vec{v})] = 0$$

➢ Laporta algorithm

Laporta, IJMPA(2000)

- **Gaussian elimination**
- **Widely used, many public codes**

[**Air**, C. Anastasiou and A. Lazopoulos, JHEP(2004),
Reduze, C. Studerus, Comput.Phys.Commun.(2010),
LiteRed, R.NLee., J.Phys.Conf.Ser. 523 (2014)
Fire 6, Smirnov, Comput.Phys.Commun. (2020),
Kira, Usovitsch et al, arXiv:2008.06494,
CARAVEL, Abreu, et al, Comput. Phys. Commun. (2021)]

Current status of integral reduction

➢ Difficulties of IBP method

- Complicated intermediate expression
- Many auxiliary integrals, very sparse
- Resource-consuming due to large scale of linear equations: E.g. Laporta 1910.01248

  Hundreds GB RAM Months of runtime using super computer

  E.g. J. Klappert et al., 2008.06494 E.g. Baikov, Chetyrkin, Kühn, PRL(2017)

➢ Selected developments

- Finite field: avoid intermediate express swell Manteuffel, Schabinger, 1406.4513
- Syzygy equations: trimming IBP system

  Gluza, Kajda, Kosower, 1009.0472
  Larsen, Zhang, et. al., 1511.01071, 1805.01873, 2104.06866
  NeatIBP, Zi-Hao Wu, et al. 2305.08783
- Block-triangular form: minimize IBP system (needs input)

  X.Liu, Y.Q.Ma, 1801.10523, X.G, X.Liu, Y.Q.Ma, 1912.09294
- A better choice of basis: UT basis/ D-factorized

  S. Abreu, et al., PRL (2019)
  Usovitsch, 2002.08173
  A. V. Smirnov, V. A. Smirnov , 2002.08042

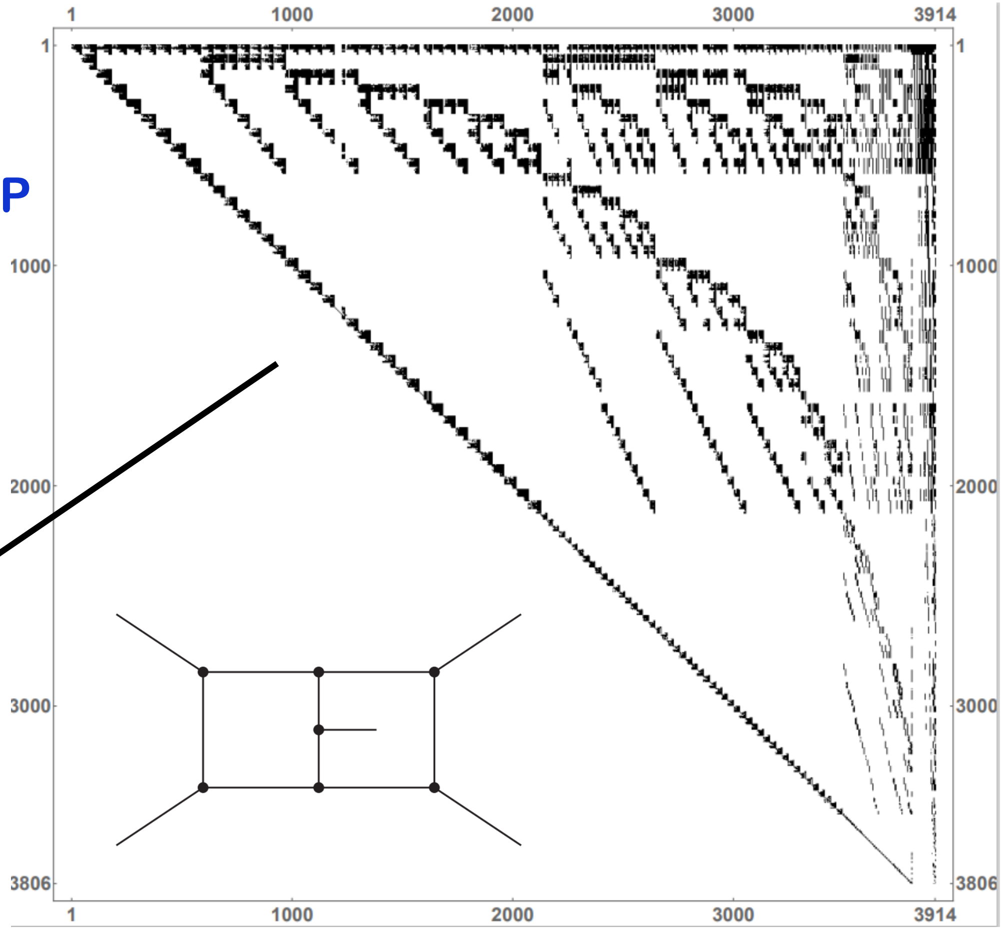
Block-triangular form

➢ Improved linear system

X. Liu, YQM, PRD(2019)
X. Guan, X. Liu, YQM, CPC(2020)

- Simple relations among given Feynman integrals
- Several orders of magnitude equations less than IBP
- Nice block-triangular structure
- Ideal for numerical sampling
  (finite field / floating number)

$$
\begin{aligned}
Q_{11}\, I_1 + Q_{12}\, I_2 + Q_{13}\, I_3 + Q_{14}\, I_4 + \ldots + Q_{1N}\, I_N &= 0 \\
Q_{21}\, I_1 + Q_{22}\, I_2 + Q_{23}\, I_3 + Q_{24}\, I_4 + \ldots + Q_{2N}\, I_N &= 0 \\
Q_{33}\, I_3 + Q_{34}\, I_4 + \ldots + Q_{3N}\, I_N &= 0 \\
Q_{43}\, I_3 + Q_{44}\, I_4 + \ldots + Q_{4N}\, I_N &= 0 \\
\ldots
\end{aligned}
$$



Block-triangular form of double-pentagon topology

Search algorithm

- **Decomposition of $Q_i(\vec{s},\epsilon)$** $\sum Q_i(\vec{s},\epsilon)\, I_i(\vec{s},\epsilon) = 0$

$$Q_i(\vec{s},\epsilon) = \sum_{\mu_0=0}^{\epsilon_{max}} \sum_{\mu} \tilde{Q}_i^{\mu_0\mu_1\ldots\mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r}$$

- $\tilde{Q}_i^{\mu_0\mu_1\ldots\mu_r}$ are unknowns
- $\mu_1 + \cdots + \mu_r = d_i$

- **Input from numerical IBP** $I_i(\vec{s},\epsilon) = \sum_{j=1}^{n} C_{ij}(\vec{s},\epsilon) M_j(\vec{s},\epsilon)$

$$\Rightarrow \sum_{\mu_0,\mu} \sum_{j=1}^{n} \tilde{Q}_i^{\mu_0\ldots\mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r} C_{ij}(\vec{s},\epsilon) M_j(\vec{s},\epsilon) = 0$$

- **Linear equations:** $\sum_{\mu_0,\mu} \tilde{Q}_i^{\mu_0\ldots\mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r} C_{ij}(\vec{s},\epsilon) = 0$

- With enough constraints $\Rightarrow \tilde{Q}_i^{\mu_0\ldots\mu_r}$
- With finite field technique, equations can be efficiently solved
- Relations among $G \equiv \{I_1, I_2, \ldots I_N\}$ can be determined

Reduction

- **With $G = G_1 \cup G_2$, satisfy**
  - G_1 is more complicated than G_2
  - G_1 can be reduced to G_2

$$Q_{11}\, I_1 + Q_{12}\, I_2 + Q_{13}\, I_3 + Q_{14}\, I_4 + \ldots + Q_{1N}\, I_N = 0$$
$$Q_{21}\, I_1 + Q_{22}\, I_2 + Q_{23}\, I_3 + Q_{24}\, I_4 + \ldots + Q_{2N}\, I_N = 0$$
$$Q_{33}\, I_3 + Q_{34}\, I_4 + \ldots + Q_{3N}\, I_N = 0$$
$$Q_{43}\, I_3 + Q_{44}\, I_4 + \ldots + Q_{4N}\, I_N = 0$$
$$\ldots$$

- **Algorithm** *Search for efficient relations*
  1. Set degree bound
  2. Search relations among G
  3. If obtained relations are enough to determine G_1 by G_2, stop;
     else, increase degree bound and go to step 2

- **Conditions for G_1 and G_2**
  1. Relations among G_1 and G_2 are not too complicated: easy to find
  2. $\#G_1$ is not too large: numerically diagonalize relations easily

Adaptive search strategy

Too many unknowns?

- **Semi-analytic**
  - Keep a subset of variables analytic ⇒ easy to search
  - The integral set is the same ⇒ still very efficient
  - More than one block-triangular form is needed

$$Q_i(\vec{z}) = \sum_{\mu} \tilde{Q}_i^{\mu_1 \dots \mu_r} z_1^{\mu_1} \dots z_r^{\mu_r}$$

$$Q_i(z_{1,0}, \dots z_{r-1,0}, z_r) = \sum_{\mu_r} \tilde{Q}_i^{\mu_0} z_r^{\mu_r}$$

$$Q_i(z_{1,0}, \dots z_{r-2,0}, z_{r-1}, z_r) = \sum_{\mu_{r-1},\mu_r} \tilde{Q}_i^{\mu_{r-1}\mu_r} z_{r-1}^{\mu_{r-1}} z_r^{\mu_r}$$

… …

- **Adaptive search**
  1. $n = 1$
  2. Search n-variable block-triangular form within time limit T
  3. If search succeed, $n++$ and go to step 2, otherwise go to step 4
  4. Perform reduction by solving the most efficient linear system(i-variable)
- **Exploit full potential of block-triangular form**

Implementation: Blade

➢ Framework

Integral extension
- Easy to search
- Delicate scheme
- User-defined scheme
- …

Determine master integrals (maximal-cut)

→ **Generate IBP**
- Automatic seeding
- Trim seed integrals
- …

→ **Adaptive search strategy**
- Delicate scheme
- Flexible polynomial ansatz
- …

→ **Reconstruction**

FiniteFlow, Peraro, JHEP (2019)

➢ All in one function

- BLReduce
- BLDifferentialEquation

Compute derivatives (BLDifferentialEquation → BLReduce)

Usage of Blade

X. G, Xiao Liu, Yan-Qing Ma, Wen-Hao Wu, 2308.xxxx

➢ Download

Link: https://gitlab.com/multiloop-pku/blade

B

Blade

Project ID: 42197761

75 Commits · 1 Branch · 0 Tags · 2.4 MB Project Storage

Block-triangular form improved Feynman integral decomposition .

```
chmod +x auto_install
./auto_install
```

➢ Examples

- 1_automatic - introduction to automatic reduction of Feynman loop integrals;
- 1_preferred_masters - introduction to automatic reduction with user-defined master integrals;
- 1_userdefined_target - introduction to automatic reduction of user defined target integrals;

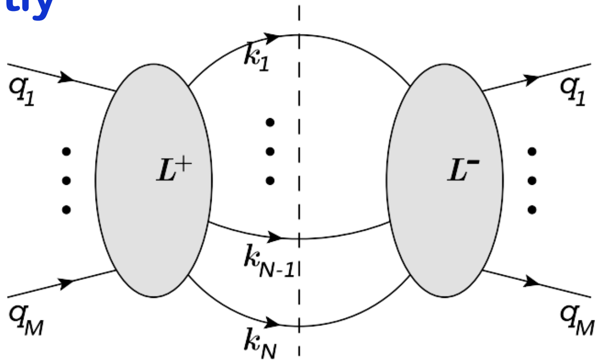
...

New features of Blade

X. G, Xiao Liu, Yan-Qing Ma, Wen-Hao Wu, 2308.xxxx

- Usually improve the efficiency of IBP reduction significantly

- Detect symmetries of cut diagram correctly



- Generalized integrand

E.g. $G_{\vec{\nu}}(\vec{x}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{e^{Z^{\vec{x}}}}{\prod_{i=1}^{N} \mathcal{D}_i^{\nu_i}}$

⟹ Recurrence relations for any given integral family

X. G, Xiang Li and Yan-Qing Ma, 2306.02927

- Complex mode

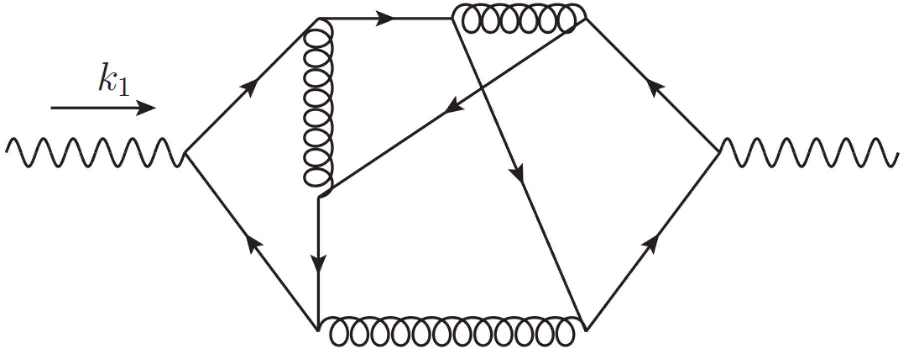
$m^2 \to 1 + \mathrm{i}$

- …

Example 1： 4-loop 2-point one massive internal line

➢ $e^+ e^- \to \gamma^*/Z^* \to t\,\bar{t}$ @ N^3LO_{QCD}

Xiang Chen, X. G, Chuan-Qi He, Xiao Liu, and Yan-Qing Ma. 2209.14259



- **Two variables:** $\epsilon\,, m_t^2.\,(k_1^2 \to 1)$

369 MIs

➢ **Significant improvement of IBP**

Mode	s_{max}	t_{IBP}	t_{BL}	Probes for search	Probes for reconstruction	CPU·h
Blade	3	118s	0.93s	672	28007	115
	4	439s	1.1s	512	48328	280
Kira2.2	3	354s	-	-	31197	3100
	4	2400s	-	-	?	?

* s_{max} denotes the maximal rank of Feynman integrals

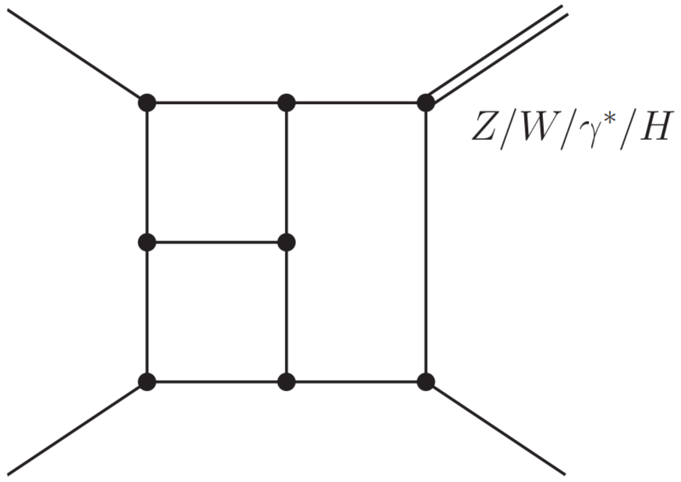
* Kira2.2 is highly competitive with other reduction packages on the market

* Blade adopt default options， Kira2.2 is called with run_firefly: ture, --bunch_size=2

Example 2： 3-loop 4-point one massive external line

- $pp \to V + j \ @N^3LO, pp \to H + j \ @N^3LO_{HTL}$
  - **Three variables：** $\epsilon, m^2, t \, . (s \to 1)$
  - **Leading color approximation**

  T. Gehrmann, P. Jakubčík, C. C. Mella et al. 2307.15405



121 MIs

- **Comparison**

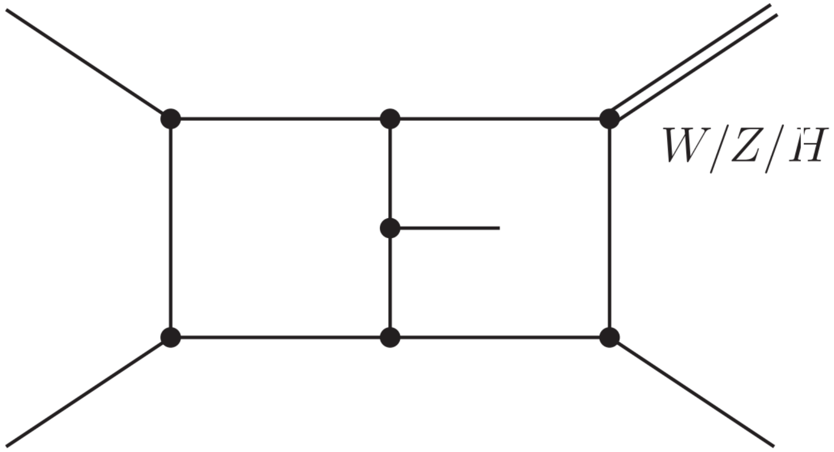
s_{max}	CPU· h_{Blade}	CPU · $h_{Kira2,2}$	Kira2.2/Blade
2	35	130	3.7
3	60	750	12.5
4	180	5400	30

- **Blade is about one order of magnitude faster than Kira in this problem**

Example 3： 2-loop 5-point one massive external line

- $pp \to Wjj,\ pp \to Hb\bar{b}, \ldots @N^2LO_{QCD}$
  - Six variables: $\epsilon, m^2, s_{23}, s_{34}, s_{45}, s_{51}, (s_{12} \to 1)$
  - Leading color approximation

S. Abreu, F. Febres Cordero, H. Ita et al. JHEP(2022)
H. B. Hartanto, R. Poncelet, A. Popescu et al. PRD(2022)
S. Badger, H. B. Hartanto, J. Kryś et al. JHEP(2021)
S. Badger, H. B. Hartanto, J. Kryś et al. JHEP(2022)



142 MIs

- Nonplanar double pentagon
  - One of the most complicated topology
  - Trick: $\epsilon = 10^{-3}$

Mode	s_{max}	t_{IBP}	t_{BL}	Probes for search	Time for search	Probes for reconstruction
Blade	5	12.4s	0.16s	4000	6.5d	?

- The block-triangular form is ideal for numeric evaluation

* 1000 probes are sufficient to fit the template relations in other prime fields.
* 8h are sufficient to fit the template relations in other primes fields.

Summary and outlook

➢ Summary

- We present Blade, the first public implementation for the block-triangular form improved Feynman integral decomposition
- The Blade package offers several notable improvements, which can be applied to more general cases
- Examples: the method of block-triangular form is powerful

➢ Outlook

- To tackle cutting-edge problems
- To optimize Blade, e.g. syzygy equations, finite-field reconstructor, parallelization

Thank you!