Some New Progresses of Graphic Rule

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第三届量子场论及其应用研讨会

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Based on the following work:

Xie and Du, to appear; JHEP 12 (2022) 080, Wu, Du,

JHEP12 (2022) 099, Xie, Du, JHEP 01 (2022) 162, Wu, Du

JHEP 04 (2021) 150, Tian, Gong, Xie, Du

JHEP 05 (2020) 008, Du, Hou

JHEP 05 (2019) 012, Hou, Du

JHEP 12 (2017) 038, Du, Feng, Teng

JHEP 09 (2017) 021, Fu, Du, Huang, Feng

Outline

YM and GR

Interactions in nature

$$\begin{array}{ccc} \text{Gravity} & \longleftrightarrow & \text{Strong, Weak, EM} \\ & \downarrow & \\ \text{General relativity} & \longleftrightarrow & \text{Gauge theory} \end{array}$$

Comparing Feynman rules of GR and YM

GR: Infinite number of vertices; Color singlets

YM: Three-point and four-point vertices; Color decomposition formulas

$$M_{\rm YM}^{\rm tree} = \sum_{\boldsymbol{\sigma}\in S_{n-1}} {\rm Tr}(T^{a_1}T^{a_{\sigma(2)}}\dots T^{a_{\sigma(n)}})A(1,\boldsymbol{\sigma})$$

$$M_{\rm YM}^{\rm tree} = \sum_{\boldsymbol{\sigma}\in S_{n-2}} if^{a_1 a_{\sigma(2)} e_1} if^{e_1 a_{\sigma(2)} e_2} \dots if^{e_{n-3} a_{\sigma(n-1)} a_n} A(1,\boldsymbol{\sigma},n)$$

GR as double copy of YM

$$\begin{array}{ccc} \epsilon^{\mu\nu} & \sim & \epsilon^{\mu}\epsilon^{\nu} \\ & \downarrow & \\ \text{GR} & \sim & \text{YM} \times \text{YM} \end{array}$$

GR as double-copy of YM:

- Kawai-Lewellen-Tye (1986)
- Bern-Carrasco-Johansson (2008)
- Cachazo-He-Yuan (2013)

Kawai-Lewellen-Tye double copy

KLT relation:

Closed string tree amplitudes	$\sim$	(Open string tree amplitudes)^2
$\Downarrow$		$\Downarrow$
GR tree amplitudes	$\sim$	(YM tree amplitudes)^2

KLT in field theory:

(Bern, De Freitas,Wong (1999); Bjerrum-Bohr,Damgaard,Feng,Sondergaard (2010))

$$M_n = \sum_{\sigma,\rho} A_n(\rho) S[\rho|\sigma] \widetilde{A}_n(\sigma)$$

M_n :	GR	color-dressed YM
A_n :	YM	ϕ^3 scalar
$\widetilde{A}_n$:	YM	YM

Bern-Carrasco-Johansson double copy

BCJ form of YM amplitudes:

$$M^{\text{YM}} = \sum_{\mathcal{G}} \frac{c_{\mathcal{G}} n_{\mathcal{G}}}{\prod_i D^i_{\mathcal{G}}}$$

$\mathcal{G}$: diagrams with cubic vertices

$D^i_{\mathcal{G}}$: propagators $c_{\mathcal{G}}$: color factors

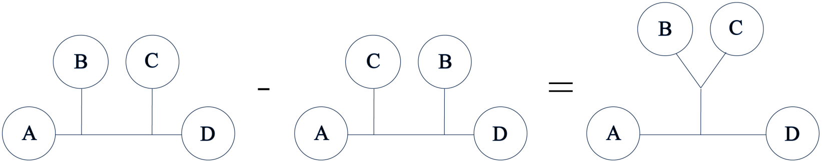
$n_{\mathcal{G}}$: BCJ numerators satisfying

$$c_i = -c_j \quad \Rightarrow \quad n_i = -n_j$$
$$c_i + c_j + c_k = 0 \quad \Rightarrow \quad n_i + n_j + n_k = 0$$

BCJ form of GR amplitudes

$$M^{\text{GR}} = \sum_{\mathcal{G}} \frac{n_{\mathcal{G}} \widetilde{n}_{\mathcal{G}}}{\prod_i D^i_{\mathcal{G}}}$$

Jacobi identity in the BCJ form



Cachazo-He-Yuan double copy

$$\boxed{M=\int d\mu\,\mathcal{I}_L\,\mathcal{I}_R}$$

GR $\quad \mathcal{I}_L=\mathrm{Pf}'(\Psi),\quad \mathcal{I}_R=\mathrm{Pf}'(\Psi)$

Color-ordered YM $\quad \mathcal{I}_L=\frac{1}{z_{12}z_{23}\cdots z_{n1}},\ \mathcal{I}_R=\mathrm{Pf}'(\Psi)$

Color-ordered BS $\quad \mathcal{I}_L=\frac{1}{z_{12}z_{23}\cdots z_{n1}},\ \mathcal{I}_R=\frac{1}{z_{\sigma_1\sigma_2}z_{\sigma_2\sigma_3}\cdots z_{\sigma_n\sigma_1}}$

$$\left(\mathrm{Pf}'[\Psi]\equiv\frac{perm(ij)}{z_{ij}}\mathrm{Pf}[\Psi^{ij}_{ij}]\right)$$

Cachazo-He-Yuan double copy

Ψ is a $2n \times 2n$ skew-symmetric matrix:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

The $n \times n$ submatrices A, B and C are

	A_{ab}	B_{ab}	C_{ab}
$a \neq b$	$\frac{s_{ab}}{z_{ab}}$	$\frac{2\epsilon_a \cdot \epsilon_b}{z_{ab}}$	$\frac{2\epsilon_a \cdot k_b}{z_{ab}}$
$a = b$	0	0	$-\sum_{c \neq a} \frac{2\epsilon_a \cdot k_c}{z_{ac}}$

Kinematic decomposition

KLT

BCJ $\rightarrow$

CHY

$$M^{\mathrm{GR}} = \sum_{\boldsymbol{\sigma}\in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\mathrm{YM}}(1,\boldsymbol{\sigma},n)$$

$$A^{\mathrm{YM}} = \sum_{\boldsymbol{\sigma}\in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\mathrm{BS}}(1,\ldots,n|1,\boldsymbol{\sigma},n)$$

Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove (2010); Bern and Dennen (2011); Du Feng Fu (2012); Cachazo, He, Yuan (2013)

Expansion coefficients $n_{1,\boldsymbol{\sigma},n} \rightarrow$ BCJ numerator in DDM basis

（Del Duca，Dixon，Maltoni （1999））

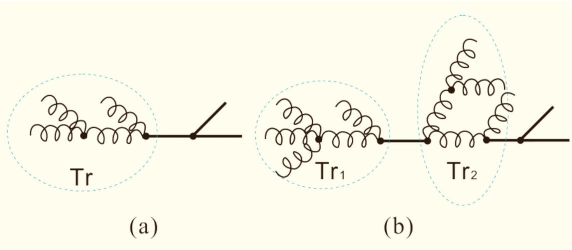
Several Problems

Can we construct local BCJ (polynomial) numerators? $\rightarrow$ On-shell graphic rule

Constructing BCJ numerators in an off-shell way? $\rightarrow$ Off-shell graphic rule

More applications of the graphic rules: inducing new identities, calculations in four dimensions, implications on one-loop

EYM amplitudes: single-trace and multi-trace



EYM=YM+GR+Dilaton+B Fields

Typical Feyman diagrams in EYM

EYM recursive expansion: single-trace case

EYM recursive expansion for single-trace amplitudes

(Stieberger and T. R. Taylor(2016),Nandan, Plefka, Schlotterer, Wen (2016),de la Cruz, Kniss,Weinzierl (2016),Schlotterer (2016),Du, Feng, Fu, Huang (2017); Chiodaroli,Gunaydin,Johansson (2017); Teng, Feng (2017))

$$A(1,2,\ldots,r\|\mathrm{H}) = \sum_{\mathrm{H}\setminus\{h_a\}\to \mathrm{h}|\boldsymbol{h}} \epsilon_{h_a}\cdot F_{\rho_1}\cdots F_{\rho_i}\cdot Y_{\rho_i} \times A(1,\{2,\ldots,r-1\}\shuffle\{\rho_i,\ldots,\rho_1,h_a\},r\|\boldsymbol{h})$$

- $\mathrm{H}\setminus\{h_a\}\to \mathrm{h}|\boldsymbol{h}$: splittings of elements in $\mathrm{H}\setminus\{h_a\}$; Permutations $\boldsymbol{\rho}$ are summed over
- h_a: fiducial graviton; $Y_h = \sum_{j\in\{1,\ldots,r-1\}\text{s.t.}j\prec h} k_j$; $F_i^{\mu\nu} = k_i^\mu\epsilon_i^\nu - k_i^\nu\epsilon_i^\mu$
- $\boldsymbol{A}\shuffle\boldsymbol{B}$: permutations with keeping the relative orders in each set

EYM recursive expansion: multi-trace case

EYM recursive expansion of multi-trace amplitudes (Du, Feng, Teng, (17))

$$\text{Single-trace EYM} \xrightarrow{\text{Gravitons}\to\text{Gluon traces}} \text{Multi-trace EYM}$$

Type-I: Fiducial graviton → Graviton

Type-II: Fiducial graviton → Trace

Expanding GR into EYM

Graviton amplitude can be expanded in terms of single-trace EYM amplitudes (Du, Fu, Feng, Huang 2017)

$$M_n^{\mathrm{GR}} = \sum_{\mathrm{H}\setminus\{h_1,h_2\}\to \mathrm{h}|\boldsymbol{h}} (-1)^{|\mathrm{h}|}\epsilon_{h_1}\cdot F_{\rho_1}\cdots F_{\rho_r}\cdot\epsilon_n A^{\mathrm{EYM}}(h_1,\rho,h_n\|\boldsymbol{h})$$

Here

- ★ Sum over splittings of graviton set $\mathrm{H}\setminus\{h_1,h_2\}$ and sum over permutations ρ of elements in h
- ★ $(F_i)^{\mu\nu}\equiv k_i^\mu\epsilon_i^\nu - k_i^\nu\epsilon_i^\mu$, gauge invariance conditions for gravitons (except for h_1 and h_n) are encoded in F_i
- ★ In each term we turn gravitons in $\{h_1,h_n\}\cup \mathrm{h}$ to gluons

Graphic expansions of amplitudes

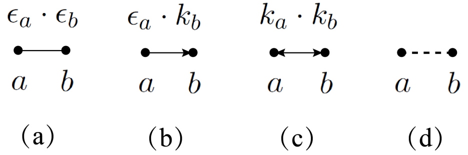
- Expanding amplitudes recursively: GR$\to$EYM $\to$YM

$\Rightarrow$Graphic expansions$\Rightarrow$ local numerators

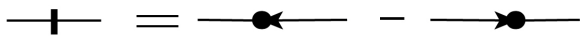
- Properties of graphs $\Rightarrow$ Properties of amplitudes

Lines and chains

Line styles

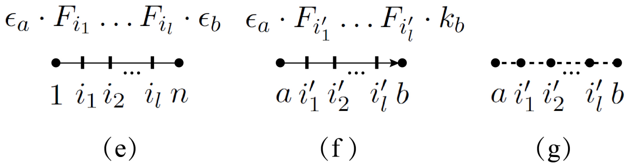


Strength tensors

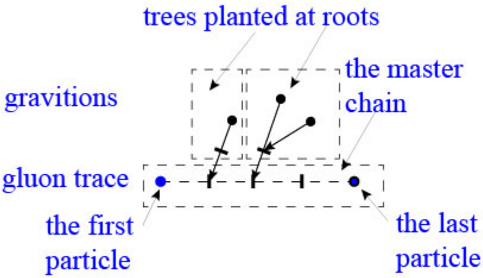


$$F_i^{\mu\nu} = k_i^\mu \epsilon_i^\nu - k_i^\nu \epsilon_i^\mu$$

Chains



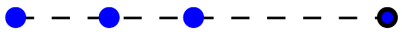
Single-trace EYM



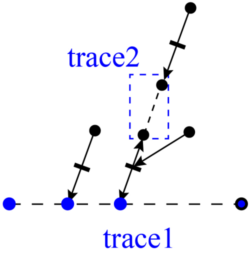
- Reference order: $\mathsf{R} = \{h_{\rho(1)}, \ldots, h_{\rho(s)}\}$
- Root set: $\mathcal{R} = \{1, \ldots, r-1\}$
- Permutations: Shuffling the branches
- $C_{1,\boldsymbol{\sigma},n}$: Summing all graphs contributing $\boldsymbol{\sigma}$

$$A^{\text{EYM}}_{\text{SingleTrace}} = \sum_{\boldsymbol{\sigma}} C_{1,\boldsymbol{\sigma},n} A^{\text{YM}}(1, \boldsymbol{\sigma}, n)$$

Multi-trace EYM

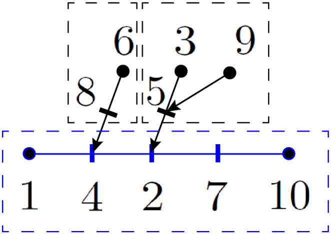
graviton → gluon trace → 

$\boldsymbol{\beta}, a, \boldsymbol{\alpha}, b \qquad (-1)^{N_{\boldsymbol{\beta}}}\ a, \boldsymbol{\alpha} \sqcup\!\sqcup \boldsymbol{\beta}^T, b$



$$A^{\text{EYM}}_{\text{MultiTrace}} = \sum_{\boldsymbol{\sigma}} C_{1,\boldsymbol{\sigma},n} A^{\text{YM}}(1, \boldsymbol{\sigma}, n)$$

YM and GR



$$\boldsymbol{\sigma} \in \{1, 4, \{8, 6\} \shuffle \{2, \{5, \{3\} \shuffle \{9\}\} \shuffle \{7\}\}, 10\}$$

$$M^{\mathrm{GR}} = \sum_{\boldsymbol{\sigma} \in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\mathrm{YM}}(1, \boldsymbol{\sigma}, n) \quad A^{\mathrm{YM}} = \sum_{\boldsymbol{\sigma} \in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\mathrm{BS}}(1, \ldots, n | 1, \boldsymbol{\sigma}, n)$$

Summing over permutations v.s. summing over graphs

$$M = \underbrace{\sum_{\boldsymbol{\sigma}\in S_{n-2}}}_{\text{summing over permutations}} n_{1,\boldsymbol{\sigma},n} A(1,\boldsymbol{\sigma},n)$$

$$\Leftrightarrow M = \underbrace{\sum_{\mathcal{F}}}_{\text{summing over graphs}} C^{\mathcal{F}} \left[\underbrace{\sum_{\boldsymbol{\sigma}^{\mathcal{F}}}}_{\substack{\text{permutations}\\ \text{corresponding to a graph}}} A(1,\boldsymbol{\sigma}^{\mathcal{F}},n) \right]$$

Graphic expansions of Berends-Giele currents

Off-shell level: Berends-Giele currents

- A recursive approach to summing Feynman diagrams
- Solution to classical equation of motion
(Lee, Mafra,Schlotterer 2015;E. Bridges, Mafra 2019)
- B-G recursion for YM and BS

B-G currents for YM

Berends-Giele (1987) currents for YM

$$
\begin{aligned}
& J^{\rho}(1,\ldots,n-1) \\
= \; & \frac{1}{s_{1\ldots n-1}}\Bigg[\sum_{1\leq i<n-1} V_3^{\mu\nu\rho} J_{\mu}(1,\ldots,i) J_{\nu}(i+1,\ldots,n-1) \\
& + \sum_{1\leq i<j<n-1} V_4^{\mu\nu\tau\rho} J_{\mu}(1,\ldots,i) J_{\nu}(i+1,\ldots,j) J_{\tau}(j+1,\ldots,n-1)\Bigg]
\end{aligned}
$$

Starting point: $J^{\mu}(a)=\epsilon_a$

B-G currents for BS

B-G currents for BS (Mafra, 2016)

$$
\begin{aligned}
&\phi(1,\dots,n-1\big|\sigma_1,\dots,\sigma_{n-1}) \\
=\ & \frac{1}{s_{1\dots n-1}}\sum_{i=1}^{n-2}\Big[\phi(1,\dots,i\big|\sigma_1,\dots,\sigma_i)\phi(i+1,\dots,n-1\big|\sigma_{i+1},\dots,\sigma_{n-1}) \\
&\quad -\phi(1,\dots,i\big|\sigma_{n-i},\dots,\sigma_{n-1})\phi(i+1,\dots,n-1\big|\sigma_1,\dots,\sigma_{n-i})\Big]
\end{aligned}
$$

Starting point $\phi(a|a)=1$, $\phi(a|b)$ $(a\neq b)$

From on-shell to off-shell

$$
\begin{aligned}
A^{\mathrm{YM}} &= \sum_{\boldsymbol{\sigma}\in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\mathrm{BS}}(1,\ldots,n|1,\boldsymbol{\sigma},n) \\
&\downarrow\ ? \\
J^{\rho} &= \sum_{\boldsymbol{\sigma}\in S_{n-2}} N^{\rho}_{1,\boldsymbol{\sigma},n} \phi^{\mathrm{BS}}(1,\ldots,n|1,\boldsymbol{\sigma},n)
\end{aligned}
$$

In a proper choice of gauge, the second line holds.

A decomposition formula for B-G currents

BG current in Feynman gauge

$$\underbrace{J^{\rho}(1,2,...,n-1)}_{\text{Feynman gauge}} = \underbrace{\widetilde{J}^{\rho}(1,2,...,n-1)}_{\text{BCJ gauge}} + \underbrace{K^{\rho}(1,2,...,n-1) + L^{\rho}(1,2,...,n-1)}_{\text{Vanish in the on-shell limit}}$$

K, L terms

Terms vanishing in the on-shell limit

$$K^{\rho}(1,2,...,n-1) = \frac{1}{s_{12\ldots n-1}}\, k^{\rho}_{1,n-1} \sum_{i=1}^{n-2} \widetilde{J}(1,...,i)\cdot\widetilde{J}(i+1,...,n-1)$$

$$\begin{aligned} L^{\rho}(1,2,...,n-1) \quad = & \sum_{\{a_i,b_i\}\subset\{1,\ldots,n-1\}} (-1)^{l+1} J^{\rho}\Big(S_{1,a_1-1}, K_{(a_1,b_1)}, \\ & S_{b_1+1,a_2-1}, K_{(a_2,b_2)}, ..., K_{(a_l,b_l)}, S_{b_l+1,n-1}\Big) \end{aligned}$$

$$\epsilon\cdot[K(1,2,...,n-1)+L(1,2,...,n-1)]=0$$

$\widetilde{J}^{\rho}$ term

Effective current:

$$\widetilde{J}^{\rho}(1,2,\ldots,n-1) = \sum_{\boldsymbol{\sigma}\in P(2,n-1)} N^{\rho}_{A}(1,\boldsymbol{\sigma})\phi(1,2,\ldots,n-1|1,\boldsymbol{\sigma})$$

- $N^{\rho}_{A}(1,\boldsymbol{\sigma})$: numerators with 1 off-shell line, constructed by graphs
- On-shell limit: $\epsilon_n \cdot N_A(1,\boldsymbol{\sigma}) = n_A(1,\boldsymbol{\sigma},n)$

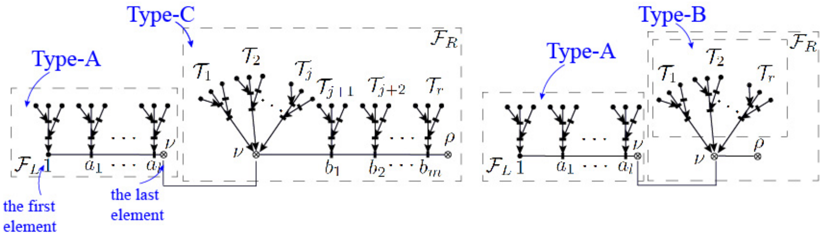
Generalized Strength tensor $\widetilde{F}^{\nu\rho}$

Generalized strength tensor: $\widetilde{F}^{\nu\rho}(A) \equiv 2k_A^{\nu}\widetilde{J}^{\rho}(A) - 2k_A^{\rho}\widetilde{J}^{\nu}(A)$

$$\widetilde{F}^{\nu\rho}_{(1,n-1)} = \sum_{\boldsymbol{\sigma}\in P(1,n-1)} N_C^{\nu\rho}(\boldsymbol{\sigma})\phi(1,...,n-1|\boldsymbol{\sigma}) + \sum_{1\leq i<n-1} 2\left[\widetilde{J}^{\nu}_{(1,i)}\widetilde{J}^{\rho}_{(i+1,n-1)} - \widetilde{J}^{\rho}_{(i+1,n-1)}\widetilde{J}^{\nu}_{(1,i)}\right]$$

- $N_C^{\rho}(1,\boldsymbol{\sigma})$: numerators with 2 off-shell lines, constructed by graphs

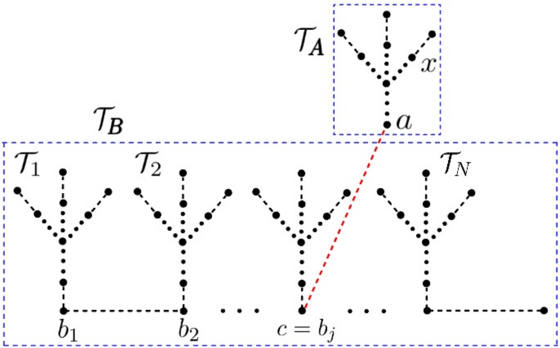
Three types of off-shell numerators



$$\boldsymbol{\sigma} = \{\sigma_2, ..., \sigma_{n-1}\} \in P(2, n-1),\ \boldsymbol{\sigma}_L = \{\sigma_2, ..., \sigma_{i-1}\},\ \boldsymbol{\sigma}_R = \{\sigma_i, ..., \sigma_{n-1}\}$$

$$N_A^{\rho}(1, \boldsymbol{\sigma}) = \left[N_A(1, \boldsymbol{\sigma}_L) \cdot N_C(\boldsymbol{\sigma}_R) - N_A(1, \boldsymbol{\sigma}_L) N_B(\boldsymbol{\sigma}_R) \cdot 2k_{1,i-1}\right]^{\rho}$$

Inducing new relations



Inducing new relations

- Graph-based relations for BG currents in BS: (Du, Wu, 2022)

$$\sum_{a\in\mathcal{T}_A}(-)^{|ax|}\sum_{c\in\mathcal{T}_{\boldsymbol{B}}}\sum_{\boldsymbol{\alpha}\in\mathcal{T}_{\boldsymbol{A}}|_a}\sum_{\boldsymbol{\beta}\in\mathcal{T}_{\boldsymbol{B}}|_b}\left[\sum_{\boldsymbol{\gamma}\in\boldsymbol{\alpha}\sqcup\boldsymbol{\beta}|_{c\prec a}} s_{ac}\,\phi(\boldsymbol{\sigma}\,|\,\boldsymbol{\gamma})\right]$$

$$=\sum_{\boldsymbol{\alpha}\in\mathcal{T}_{\boldsymbol{A}}|_x}\sum_{\boldsymbol{\beta}\in\mathcal{T}_{\boldsymbol{B}}|_b}\left[\phi(\boldsymbol{\sigma}_{1,i}|\boldsymbol{\beta})\phi(\boldsymbol{\sigma}_{i+1,l}|\boldsymbol{\alpha})-\phi(\boldsymbol{\sigma}_{1,l-i}|\boldsymbol{\alpha})\phi(\boldsymbol{\sigma}_{l-i+1,l}|\boldsymbol{\beta})\right]$$

- $\widetilde{J}=\sum N\phi\rightarrow$ Graph-based relations for BG currents YM (Du, Wu, 2022)

- On shell graph-based BCJ relation (Hou, Du, 2018)

Auxiliary lines

Properties of a part of the graphs/amplitude

$\xrightarrow{\text{Auxiliary lines}}$ A full formula with expected property

(Xie, Du, to appear)

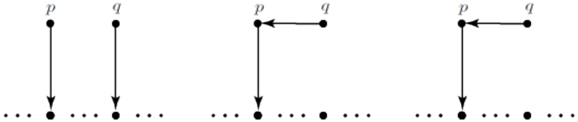
A simple example: Amplitude with two gravitons p and q
A symmetric form of the MHV sector (the part with no $\epsilon \cdot \epsilon$)

$$\begin{aligned} I_{\text{MHV}}(1, ..., r||p, q) &= \sum_{\shuffle} (\epsilon_p \cdot X_p(\shuffle)) (\epsilon_q \cdot X_q(\shuffle)) \\ &\quad \times A(1, \{2, \cdots, r-1\} \shuffle \{p\} \shuffle \{q\}, r) \end{aligned}$$

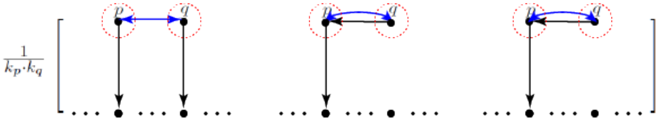
$X_p^\mu(\shuffle)$: sum of momenta on the left of p

Auxiliary lines

Graphs for I_{MHV}:



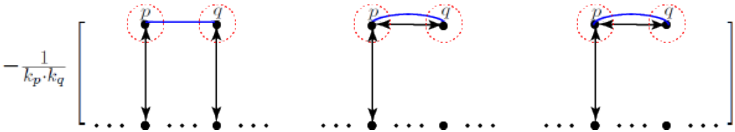
Adding auxiliary lines to graphs



Auxiliary lines

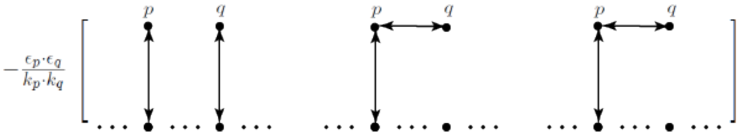
Gauge invariance condition $\epsilon_p^\mu \to k_p^\mu,\ \epsilon_q^\mu \to k_q^\mu \ \Rightarrow\ k^\mu\epsilon^\nu \to -k^\nu\epsilon^\mu$

$\Rightarrow$ Graphs for I_{NMHV}: A symmetric expression for I_{NMHV}



Auxiliary lines

$\Rightarrow$



$\Rightarrow$

$$I_{\text{NMHV}}(1, ..., r||p, q) = -\sum_{\sqcup\!\sqcup} \frac{\epsilon_p \cdot \epsilon_q}{k_p \cdot k_q}(k_p \cdot X_p(\sqcup\!\sqcup))(k_q \cdot X_q(\sqcup\!\sqcup))$$
$$\times A(1, \{2, \cdots, r-1\} \sqcup\!\sqcup \{p\} \sqcup\!\sqcup \{q\}, r)$$

A new expansion formula at tree level

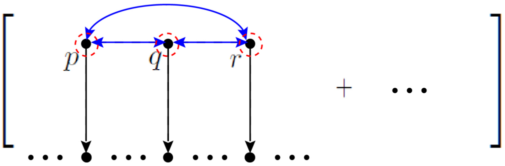
$\Rightarrow$ A new symmetric expression for single-trace EYM

$$A(1,...,r||p,q) = \sum_{\shuffle}\left[\epsilon_p^{\mu}\epsilon_q^{\nu} - \frac{\epsilon_p\cdot\epsilon_q}{k_p\cdot k_q}(k_p)_{\mu}(k_q)_{\nu}\right] X_p^{\mu}(\shuffle)X_q^{\nu}(\shuffle)$$
$$\times A(1,\{2,\cdots,r-1\}\shuffle\{p\}\shuffle\{q\},r)$$

Three-graviton case

Amplitude with three gravitons

$$\frac{1}{(k_p\cdot k_q)(k_q\cdot k_r)(k_r\cdot k_p)}$$



Three-graviton case

A symmetric formula for amplitude with three gravitons

$$
\begin{aligned}
& A(1, ..., n||p, q, r) \\
= \; & \sum_{\shuffle} \Bigg[\epsilon_p^{\mu}\epsilon_q^{\nu}\epsilon_r^{\rho} - \frac{\epsilon_p \cdot \epsilon_q}{k_p \cdot k_q}(k_p)_\mu (k_q)_\nu (\epsilon_r)_\rho - \frac{\epsilon_p \cdot \epsilon_r}{k_p \cdot k_r}(k_p)_\mu (\epsilon_q)_\nu (k_r)_\rho \\
& - \frac{\epsilon_q \cdot \epsilon_r}{k_q \cdot k_r}(\epsilon_p)_\mu (k_q)_\nu (k_r)_\rho \Bigg] X_p^{\mu}(\shuffle) X_q^{\nu}(\shuffle) X_r^{\rho}(\shuffle) \\
& \times A(1, \{2, \cdots, n-1\} \shuffle \{p\} \shuffle \{q\} \shuffle \{r\}, n)
\end{aligned}
$$

One more example

- We cannot find such a formula where all terms have the form $X_p^{\mu} X_q^{\nu} X_r^{\rho} X_s^{\gamma}$ contracted with ϵ^{μ} or k^{μ} for amplitudes with four gravitons.

- In the case with five gravitons, the expansion formula where terms have the form $X_p^{\mu} X_q^{\nu} X_r^{\rho} X_s^{\gamma} X_t^{\beta}$ contracted with ϵ^{μ} or k^{μ} conflicts to gauge invariance.

Application at one-loop

EYM → YM ⇔ YMS → BS

One loop CHY （Geyer, L. Mason，15； He, Yuan 15, Cachazo, He,Yuan,15； EYM and YMS see Porkert,Schlotterer,22； Zhou 22.）:

$$M^{\text{1-loop}}_{n,\text{B}\otimes\text{C}} = \int \frac{\mathrm{d}^D l}{l^2} \lim_{k_\pm \to \pm l} \int \mathrm{d}\mu^{\text{tree}}_{n+2} I^{\text{1-loop}}_{\text{B}} I^{\text{1-loop}}_{\text{C}}$$

Scattering equations at one-loop level:

$$\frac{l \cdot k_i}{\sigma_i} + \sum_{\substack{j=1 \\ j\neq i}} \frac{k_i \cdot k_j}{\sigma_{ij}} = 0$$

Application at one-loop

One-loop CHY integrands for YMS:

$$I_{\mathrm{C}}^{\text{1-loop}} \to \mathrm{PT}(\sigma_1,...,\sigma_n) \equiv \sum_{i=1}^{n} \mathrm{PT}^{\mathrm{tree}}(+,\sigma_i,\sigma_{i+1},...,\sigma_{i-1},-)$$

$$I_{\mathrm{B}}^{\text{1-loop}} \to \sum_{\boldsymbol{\rho}\in\text{perms of gluons}} \sum_{\shuffle} C(+,\boldsymbol{\rho}\shuffle\{1,...,r\},-) \times \mathrm{PT}^{\mathrm{tree}}(+,\boldsymbol{\rho}\shuffle\{1,...,r\},-) + \mathrm{cyclic}(1...r)$$

- $1...r$ scalar trace
- The coefficients generated by graphic rule
- Multi-trace cases are obtained from single trace cases

Application at one-loop

One loop CHY $\Rightarrow$ Linear loop propagators

$\Downarrow$ Graphic rule (Xie and Du, to appear)

Feynman diagrams $\Rightarrow$ Quadratic loop propagators

Approach-1: From symmetric expansion formula

Tensorial PT factors (Feng, He, Zhang, Zhang,22)

$$\mathrm{PT}^{\mu_1\mu_2\ldots\mu_r}(1,2,\ldots,n)=\sum_{i=1}^{n}\mathrm{PT}^{\mathrm{tree}}(+,i,i+1,\ldots,i-1,-)\prod_{j=1}^{r}(l^{\mu_j}-k^{\mu_j}_{1,2,\ldots,i-1})$$

The tensorial PT factor can produce quardratic propagators

$$\frac{1}{l^2}\int d\mu^{\mathrm{tree}}_{n+2}\mathrm{PT}(1,\rho(2,\ldots,n))\mathrm{PT}^{\mu_1\mu_2\ldots\mu_r}(1,\sigma(2,\ldots,n))$$

$$=\sum_{\substack{(A_1,A_2,\ldots,A_m)=(1,\rho)\\(\widetilde{A}_1,\widetilde{A}_2,\ldots,\widetilde{A}_m)=(1,\sigma)\\\{A_j=\widetilde{A}_j\},2\le m\le n}}(l+k_{\widetilde{A}})^{\mu_1}\ldots(l+k_{\widetilde{A}})^{\mu_r}\mathrm{gon}(A_1,A_2,\ldots,A_m)\prod_{i=1}^{m}\phi_{A_i|\widetilde{A}_i}$$

where $\mathrm{gon}(A_1,A_2,\ldots,A_m)=1/(l^2 l^2_{A_1}l^2_{A_{12}}\ldots l^2_{A_{12\ldots m-1}})$

Approach-1: From symmetric expansion formula

$$
\begin{aligned}
& X_p^{\mu}(\shuffle)X_q^{\nu}(\shuffle)X_r^{\rho}(\shuffle) \\
& \quad \times \mathrm{PT}^{\mathrm{tree}}(+,\{1,\cdots,n\}\shuffle\{p\}\shuffle\{q\}\shuffle\{r\},-)+\mathrm{cyclic}(1,...,r) \\
= \quad & (l+X'_p)^{\mu}(\shuffle)(l+X'_q)^{\nu}(\shuffle)(l+X'_r)^{\rho}(\shuffle) \\
& \quad \times \mathrm{PT}^{\mathrm{tree}}(+,\{1,\cdots,n\}\shuffle\{p\}\shuffle\{q\}\shuffle\{r\},-)+\mathrm{cyclic}(1,...,r) \\
\to \quad & \{l^{\mu}l^{\nu}l^{\rho}+[l^{\mu}l^{\nu}X_r'^{\rho}+...]+[l^{\mu}X_q'^{\nu}X_r'^{\rho}+...] \\
& \qquad +X_p'^{\mu}X_q'^{\nu}X_r'^{\rho}+...\}\mathrm{PT}^{\mathrm{tree}}+\mathrm{cyclic}(1,...,r) \\
\sim \quad & \mathrm{PT}^{\mu\nu\rho} \qquad \mathrm{PT}^{\mu\nu} \qquad \mathrm{PT}^{\mu} \qquad \mathrm{PT}
\end{aligned}
$$

Approach-1: From symmetric expansion formula

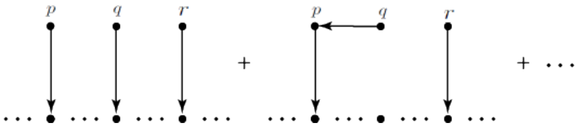
Tree-level symmetric expansion for $r+2$-scalar, 3-gluon amplitude implies

$$
\begin{aligned}
& I_{\mathrm{B}}^{\text{1-loop}}(1,\ldots,n||p,q,r;l) \\
\rightarrow \quad & \sum_{\shuffle}\left[\epsilon_p^{\mu}\epsilon_q^{\nu}\epsilon_r^{\rho}-\frac{\epsilon_p\cdot\epsilon_q}{k_p\cdot k_q}(k_p)_{\mu}(k_q)_{\nu}(\epsilon_r)_{\rho}-\frac{\epsilon_p\cdot\epsilon_r}{k_p\cdot k_r}(k_p)_{\mu}(\epsilon_q)_{\nu}(k_r)_{\rho}\right. \\
& \left.-\frac{\epsilon_q\cdot\epsilon_r}{k_q\cdot k_r}(\epsilon_p)_{\mu}(k_q)_{\nu}(k_r)_{\rho}\right]X_p^{\mu}(\shuffle)X_q^{\nu}(\shuffle)X_r^{\rho}(\shuffle) \\
& \qquad\times\mathrm{PT}^{\text{tree}}(+,\{1,\cdots,n\}\shuffle\{p\}\shuffle\{q\}\shuffle\{r\},-)+\text{cyclic}(1,\ldots,r)
\end{aligned}
$$

$\Rightarrow$ Quardratic propagators

Approach-2: From the graphic expansion with reference order

graphs for terms with $(\epsilon \cdot \epsilon)^0$

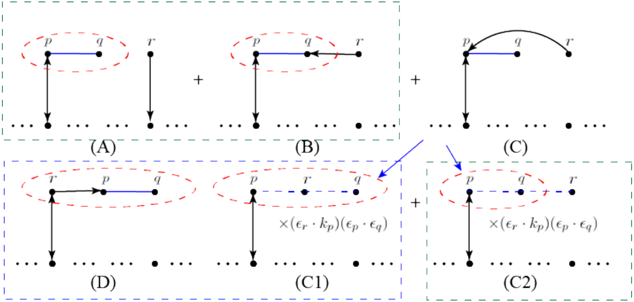


$$(\epsilon_p \cdot X_p)(\epsilon_q \cdot X_q)(\epsilon_r \cdot X_r)\mathrm{PT}^{\mathrm{tree}}(+, \{1, \cdots, n\} \sqcup \{p\} \sqcup \{q\} \sqcup \{r\}, -)$$
$$+\mathrm{cyclic}(1, ..., n)$$

$\Rightarrow$ Tensorial PT factors$\Rightarrow$ Quardratic propagators

Approach-2: From the graphic expansion with reference order

graphs for terms with $(\epsilon \cdot \epsilon)^1$



(A)+(B)+(C1)

$\Downarrow$

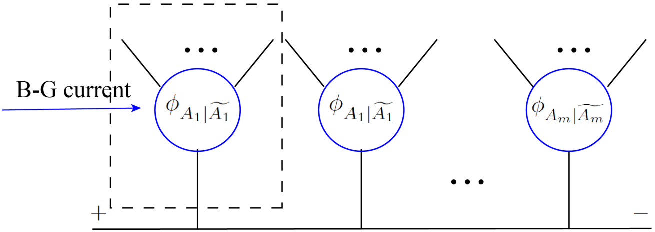
Double-trace one gluon with $q \prec r$, $\mathrm{Tr}_2 = \{p, q\}$

(D)+(C2)

$\Downarrow$

Double-trace pure scalar $\mathrm{Tr}_2 = \{r, p, q\}$

Approach-2: From the graphic expansion with reference order



$$\frac{1}{l^2 s_{A_1,l} \cdots s_{A_1 A_2,l} \cdots s_{A_1 A_2 \ldots A_m,l}}$$

Approach-2: From the graphic expansion with reference order

- If Tr_2 is involved in a full current $\Rightarrow$ Quadratic propagators

- If Tr_2 is divided into several parts $\Rightarrow$ Quadratic propagators or cancel out (by the help of graph-based BCJ relation for BG currents)

- This approach seems to be straightforwardly generalized to amplitudes with an arbitrary number of traces and gluons, and also YM and gravity

Summary and further discussions

Summary

- Graphic expansion for on-shell amplitudes
- Graphic expansion for B-G currents
- Auxiliary lines, new expansion formula
- Applications at one-loop level

End

谢 谢!