Bootstrapping the form factor with master integral and the maximal transcendentality principle

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Based on the works:
Y. Guo, L. Wang, G. Yang, PRL, 2106.01374
Y. Guo, Q. Jin, L. Wang, G. Yang, JHEP, 2205.12969
Y. Guo, L. Wang, G. Yang, 2209.06816

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High loop and high point physical quantity calculation

Physical quantity in quantum field theory:

- Amplitude: $\mathcal{A}_n = \langle p_1, p_2, \cdots, p_n | p'_1, p'_2, \cdots, p'_m \rangle$
- Form factor: $\mathcal{F}_{\mathcal{O},n} = \langle p_1, \cdots, p_n | \mathcal{O}(q) | \Omega \rangle$, where $q = p_1 + \cdots + p_n$

Traditional Feynman diagram method: the number of Feynman diagrams grows rapidly with the number of externals and loops

Table: Tree-level

gluons	4	5	6	7	8	9	10
Feynman diagram	4	25	220	2485	34300	559405	10525900

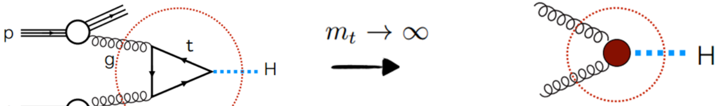
Parke-Taylor formula[1]:

$$\mathcal{A}_n^{(0)}(1^-, 2^-, 3^+, \cdots, n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}. \quad (1)$$

[1] S. J. Parke and T. R. Taylor, Phys. Rev. Lett. 56, 2459 (1986)

Motivation

1 Form factors relate with the effective vertex in an effective theory:



Effective gluon-Higgs vertex:
$$\mathcal{L}_{\text{eff}} = C_0 H \mathrm{Tr}(F_{\mu\nu}^2) + \mathcal{O}(\frac{1}{m_t^2})$$

Figure: The effective process of the gluons produce the Higgs by the top quark loop

2 Form factors in $\mathcal{N}=4$ SYM can be seem as a supersymmetric version of the Higgs scattering.

3 The maximal transcendental parts of form factor in the $\mathcal{N}=4$ SYM and QCD are the same.

4 Helping to develop the more efficient calculation method and to discover the hidden symmetry in the QFT.

Bootstrap method based on master integral

The main idea of bootstrap method: obtaining directly the final result by assuming a general ansatz of result appropriately and utilizing some known physical constraints.

The general form of physical quantum in QFT: arbitrary L-loop n-point result can be expressed linearly by a finite set of master integrals

$$\underbrace{\mathcal{F}_n^{(L)}}_{\text{form factor}} = \underbrace{\mathcal{F}_n^{(0)}}_{\text{tree-level}} \sum_k c_k(s_{ij}, \epsilon) \underbrace{I_{n,k}^{(L)}}_{\text{master integrals}} , \tag{2}$$

$s_{ij} = (p_i + p_j)^2$, $\epsilon = (4 - D)/2$. The coefficients c_k can be solved by the constraints which comes from the physical requirements.

Transcendentality degree

Transcendentality degree: defining transcendentality degreet of ϵ^n is $-n$ and the transcendental function in Feynman integrals is

Table: Definition of transcendentality degree

Function	Rational function	π^n	ζ_n	$\log(x)^n$	$\mathrm{Li}_n(x)$	$\cdots$
Transcendentality degree	0	n	n	n	n	$\cdots$

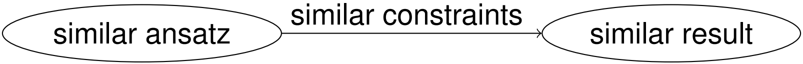
Maximally transcendental part(M.T.): the part of the L-loop result with the transcendentality degree of $2L$, such as

$$\mathcal{I}^{(1)}_{n,\text{M.T.}} = \sum_{i=1}^{n} \Big(-\frac{1}{\epsilon^2} + \frac{\log(s_{i,i+1}/\mu)}{\epsilon} \Big) + \mathcal{O}(\epsilon^0)\,. \tag{3}$$

maximally transcendentality principle

maximally transcendentality principle(MTP)[2]: the maximal transcendental parts of a class of physical quantities in $\mathcal{N}=4$ SYM and QCD are the same, such as the cusp anomalous dimension and the loop corrections of form factors, etc.

The inspiration of bootstrap method:



[2] A. V. Kotikov and L. N. Lipatov, Nucl. Phys. B661 (2003) 19, hep-ph/0208220; A. Kotikov, L. Lipatov, A. Onishchenko and V. Velizhanin, Phys.Lett. B595 (2004) 521, hep-th/0404092.

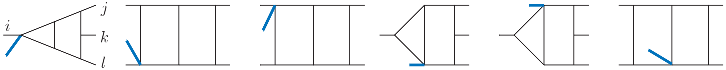


Figure: The maximal topologies have 221 master integrals[3]

ansatz form(inspired by one-loop result):

$$\mathcal{I}^{(2),\text{ansatz}}_{\text{tr}(\phi^3),4}(1^\phi,2^\phi,3^\phi,4^+) = \sum_{k=1}^{221}(a_k B_1 + b_k B_2) I^{(2)}_{4,k}\,, \quad B_1 = \frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle}\,, \quad B_2 = \frac{\langle 14\rangle\langle 23\rangle}{\langle 13\rangle\langle 24\rangle}\,,$$

$$\mathcal{I}^{(2),\text{ansatz}}_{\text{tr}(F^3),4,\text{M.T.}}(1^-,2^-,3^-,4^+) = \sum_{k=1}^{221}\left(a_k B_1 + b_k B_2 + c_k B_1 B_2 + d_k \frac{B_1}{B_2} + e_k \frac{B_2}{B_1}\right) I^{(2)}_{4,k}\,. \tag{4}$$

Physical constraints: (1)IR subtraction; (2)Collinear limit; (3)Spurious pole; (4)Unitarity cut.

[3] S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP 11 (2020) 117, 2005.04195; D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP 01 (2021) 199, 2009.13917

IR subtraction

IR divergence: the IR divergence of higher loop result is determined only by the known lower loop result and the cups anomalous dimension, for two-loop it's

$$\mathcal{I}_n^{(2)} = \underbrace{\frac{A_0}{\epsilon^4} + \frac{A_1}{\epsilon^3} + \frac{A_2}{\epsilon^2} + \frac{A_3}{\epsilon}}_{\text{IR part}} + \underbrace{A_4}_{\text{finite part}} + \mathcal{O}(\epsilon)\,, \qquad \mathcal{I}_n^{(L)} = \mathcal{F}_n^{(L)} / \mathcal{F}_n^{(0)}\,. \quad (5)$$

$\mathcal{N} = 4$ SYM: BDS formula[4]

$$\mathcal{I}_n^{(2)}(\epsilon)\big|_{\text{IR}} = \Big[\frac{1}{2}\big(\mathcal{I}_n^{(1)}(\epsilon)\big)^2 + f^{(2)}(\epsilon)\mathcal{I}_n^{(1)}(2\epsilon)\Big]\Big|_{\text{IR}}\,, \qquad f^{(2)} = -2(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2)\,. \quad (6)$$

QCD: Catani formula[5], its maximal transcendental part is the same as BDS formula.

[4] Z. Bern, L. J. Dixon and V. A. Smirnov, Phys. Rev. D72 (2005) 085001, hep-th/0505205.

[5] S. Catani, Phys. Lett. B427 (1998) 161, hep-ph/9802439.

Collinear limit

The remainder function is defined by the BDS formula:

$$\mathcal{R}_n^{(2)} = \mathcal{I}_n^{(2)}(\epsilon) - \frac{1}{2}\left(\mathcal{I}_n^{(1)}(\epsilon)\right)^2 - f^{(2)}(\epsilon)\mathcal{I}_n^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)\,. \tag{7}$$

Collinear limit: $n+1$-point remainder function $\mathcal{R}_{n+1}^{(L)}$ will degenerate to n-point $\mathcal{R}_n^{(L)}$ with the collinear limit $p_i \parallel p_j$.

$$\mathcal{R}_{n+1}^{(L)} \xrightarrow{p_i||p_{i+1}} \mathcal{R}_n^{(L)}\,. \tag{8}$$

Spurious pole

Spurious pole: when the coefficients contain spurious poles, the form factor result should not diverge at such poles

$$\mathcal{F}_n^{(L)} = \underbrace{\mathcal{F}_n^{(0)}}_{\text{physical poles}} \sum_k \underbrace{c_k(s_{ij}, \epsilon)}_{\text{spurious poles}} I_{n,k}^{(L)} . \tag{9}$$

such as $\mathrm{tr}(\phi^3)$:

$$\mathcal{F}^{(0)}_{\mathrm{tr}(\phi^3),4}(1^\phi, 2^\phi, 3^\phi, 4^+) = \frac{\langle 31\rangle}{\langle 34\rangle\langle 41\rangle}, \quad B_1 = \frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle}, \quad B_2 = \frac{\langle 14\rangle\langle 23\rangle}{\langle 13\rangle\langle 24\rangle}, \tag{10}$$

Physical poles: $s_{i,i+1} \to 0$; spurious poles: $\langle 24\rangle \to 0$.

Unitarity cut: let the propagator be on-shell

$$\frac{i}{p^2 + i\epsilon} \to 2\pi\delta_+(p^2)\,. \tag{11}$$

Obtaining the coefficients of the master integrals by the unitarity cuts

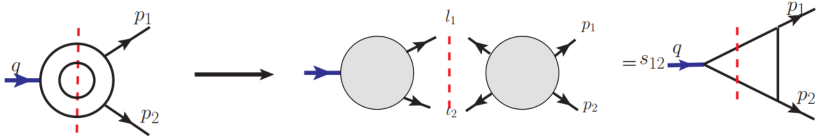


Figure: The unitarity cuts of the one loop

The process of solving the constraints

Table: The process of the parameters to be solved

	Physical constraints	$\mathrm{tr}(\phi^3)$	$\mathrm{tr}(F^3)$
	Symmetry	221	560
symbol	IR subtraction	82	207
symbol	Collinear limit	38	119
symbol	Spurious poles	22	53
numeric	IR subtraction	17	24
numeric	Collinear limit	10	20
numeric	Unitarity cuts	0	0

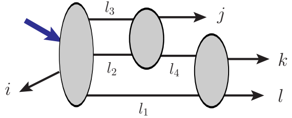


Figure: D-dimensional unitarity cuts

Bootstrapping the form factor with master integrals and the maximal transcendentality principle

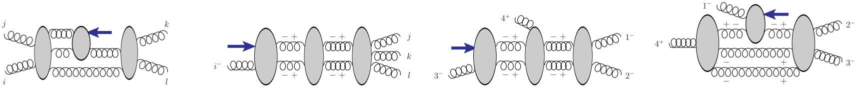
The freedom degree of the maximal transcendental part

Table: The process of the parameters to be solved

	Physical constraints	$\mathrm{tr}(F^3)$
	Symmetry	560
symbol	IR subtraction	207
symbol	Collinear limit	119
symbol	Spurious poles	53
numeric	IR subtraction	24
numeric	Collinear limit	20
	Unitarity cuts	0

The anastz and the physical constraints between QCD and $\mathcal{N} = 4$ SYM only differ in the unitarity cut step: the difference is determined by the 20 parameters? (The answer will be no.)

The unitarity cuts provided the same result



(a) (b) (c) (d)

Figure: The unitarity cuts contain only gluon contribution

(a)-(d) can constrain the 18 of the remaining 20 parameters: the maximal transcendental part of the result in $\mathcal{N}=4$ SYM and QCD can determined only by 2 parameters

$$\underbrace{\tilde{\Delta}^{(2)}_{\text{M.T.}}}_{\text{difference}} = \frac{B_1}{B_2}\left(c_1\tilde{G}^{(2)}_1 + c_2\tilde{G}^{(2)}_2\right) + (p_1 \leftrightarrow p_3)\,. \tag{12}$$

Maximal transcendentality principle in this case

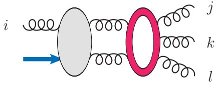


Figure: The unitarity cuts determined the potential difference.

Only the fermion contributes to maximal transcendental part.

QCD (fundamental) $\xrightarrow{n_f \to 4N_c,\ N_c \to \infty}$ $\mathcal{N} = 4$(adjoint)

$$\mathcal{F}^{(2),\text{QCD}}_{4,\text{tr}(F^3),\text{M.T.}}\Big|_{n_f \to 4N_c, N_c \to \infty} = \mathcal{F}^{(2),\mathcal{N}=4}_{4,\text{tr}(F^3),\text{M.T.}} \,. \tag{13}$$

Discussion

Main results: We first bootstrapping a series two-loop four-point form factor result, (1)$\mathrm{tr}(\phi^3)$ in planar $\mathcal{N}=4$ SYM[6], (2)the maximal transcendental part of $\mathrm{tr}(F^3)$ in $\mathcal{N}=4$ SYM, pure-YM and QCD[7] and (3)$\mathrm{tr}(\phi^2)$ with lightlike limit in planar $\mathcal{N}=4$ SYM[8].

Outlook:

1. Bootstrapping the more complex case, such as non-planar part, lower transcendentality part and QCD result.
2. Comprehending and deciphering the maximal transcendentality principle via the bootstrap method.
3. Exploring other possible physical constraint, such as Regge limit, FFOPE and $\bar{Q}$-equation, etc.

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[6] Y. Guo, L. Wang, G. Yang, Phys. Rev. Lett. 127, 151602, arXiv:2106.01374.

[7] Y. Guo, Q. Jin, L. Wang, G. Yang, JHEP 09 (2022) 161, arXiv:2205.12969.

[8] Y. Guo, L. Wang, G. Yang, arXiv:2209.06816.

Thanks!