

费曼积分在格拉斯曼流形上的 GKZ超几何函数表示

Speaker: 张海斌

Authors: Tai-Fu Feng, Hai-Bin Zhang, Chao-Hsi Chang

第三届量子场论及其应用研讨会

北京计算科学研究中心

2023.8.16



Contents

I. Introduction

II. 3-loop vacuum on compact manifold

Zhang, Feng, GKZ hypergeometric systems of the three-loop vacuum Feynman integrals, *JHEP* 05 (2023) 075 [arXiv: 2303.02795].

III. Feynman integrals on Grassmannian

Feng, Zhang, Chang, Feynman integrals of Grassmannians, *PRD* 106 (2022) 116025 [arXiv: 2206.04224].

V. Summary

I. Introduction

1. Background

- **Higher-order** radiative corrections are more important, with the increasing precision of measurements at the **future colliders**: CLIC, ILC, CEPC, FCC, HL-LHC . . .
- One-loop Feynman integrals are well known analytically in the time-space dimension $D = 4 - 2\varepsilon$.
However, how to **perform analytically multi-loop Feynman integrals** is still a **challenge**.
- **Considering Feynman integrals as the generalized hypergeometric functions, one finds that the D -module of a Feynman diagram is isomorphic to Gel'fand-Kapranov-Zelevinsky (GKZ) D -module.**

I. Introduction

2. Relevant research

- Hypergeometric functions of some Feynman integrals are obtained from **Mellin-Barnes representations**.

Feng, Chang, Chen, Gu, Zhang, *NPB* 927(2018)516 [arXiv:1706.08201]

Feng, Chang, Chen, Zhang, *NPB* 940(2019)130 [arXiv:1809.00295]

Gu, Zhang, *CPC* 43(2019)083102 [arXiv:1811.10429]

Gu, Zhang, Feng, *IJMPA* 35(2020)2050089.

- Using **GKZ hypergeometric system**, we can obtain the fundamental solution systems of Feynman integrals.

Feng, Chang, Chen, Zhang, *NPB* 953(2020)114952, [arXiv:1912.01726]

Feng, Zhang, Chang, *PRD* 106(2022)116025 [arXiv: 2206.04224]

Feng, Zhang, Dong, Zhou, *EPJC* 83(2023)314 [arXiv:2209.15194].

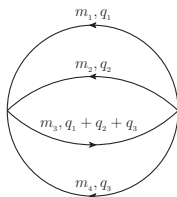
Zhang, Feng, *JHEP* 05(2023)075 [arXiv: 2303.02795].

I. Introduction

3. Generally strategy

- **We can derive GKZ hypergeometric systems of Feynman integrals, basing on Mellin-Barnes representations and Miller's transformation. We can formulate Feynman integrals as hypergeometric functions on general compact manifold or Grassmannian manifold.**
- **Steps:** (1) we write out the **GKZ hypergeometric systems** satisfied by the Feynman integrals on general compact manifold or proper Grassmannian manifold $G_{k,n}$. (2) **fundamental solution systems** are constructed in neighborhoods of regular singularities of the GKZ hypergeometric systems. The combination coefficients can be determined from Feynman integrals with some special kinematic parameters.

II. 3-loop vacuum on compact manifold



- **Feynman integral** of the 3-loop vacuum diagram with 4 propagates is written as

$$U_4 = \left(\Lambda_{\text{RE}}^2\right)^{6-\frac{3D}{2}} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D q_3}{(2\pi)^D} \times \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)((q_1 + q_2 + q_3)^2 - m_3^2)(q_3^2 - m_4^2)}. \quad (2.1)$$

- Zhang, Feng, GKZ hypergeometric systems of the three-loop vacuum Feynman integrals, *JHEP* 05 (2023) 075 [arXiv: 2303.02795].

II. 3-loop vacuum on compact manifold

- Through Mellin-Barnes transformation

$$U_4 = \frac{\left(\Lambda_{\text{RE}}^2\right)^{6-\frac{3D}{2}}}{(2\pi i)^3} \int_{-i\infty}^{+i\infty} ds_1 ds_2 ds_3 \left[\prod_{i=1}^3 (-m_i^2)^{s_i} \Gamma(-s_i) \Gamma(1+s_i) \right] I_q, \quad (2.2)$$

where

$$I_q \equiv \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D q_3}{(2\pi)^D} \frac{1}{(q_1^2)^{1+s_1} (q_2^2)^{1+s_2} ((q_1 + q_2 + q_3)^2)^{1+s_3} (q_3^2 - m_4^2)}. \quad (2.3)$$

II. 3-loop vacuum on compact manifold

- Using **Feynman parametrization** and **Beta function**,

$$B(m, n) = \int_0^1 dx x^{m-1} (1-x)^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad (2.4)$$

one can have

$$I_q = \frac{-i}{(4\pi)^{\frac{3D}{2}}} (-)^{\sum_{i=1}^3 s_i} \left(\frac{1}{m_4^2}\right)^{4 - \frac{3D}{2} + \sum_{i=1}^3 s_i} \left[\prod_{i=1}^3 \Gamma\left(\frac{D}{2} - 1 - s_i\right) \Gamma(1 + s_i)^{-1} \right] \\ \times \Gamma\left(3 - D + \sum_{i=1}^3 s_i\right) \Gamma\left(4 - \frac{3D}{2} + \sum_{i=1}^3 s_i\right). \quad (2.5)$$

II. 3-loop vacuum on compact manifold

- Mellin-Barnes representation of the Feynman integral:

$$\begin{aligned}
 U_4 = & \frac{-im_4^4}{(2\pi i)^3 (4\pi)^6} \left(\frac{4\pi \Lambda_{\text{RE}}^2}{m_4^2} \right)^{6 - \frac{3D}{2}} \int_{-i\infty}^{+i\infty} ds_1 ds_2 ds_3 \left[\prod_{i=1}^3 \left(\frac{m_i^2}{m_4^2} \right)^{s_i} \Gamma(-s_i) \right] \\
 & \times \left[\prod_{i=1}^3 \Gamma\left(\frac{D}{2} - 1 - s_i\right) \right] \Gamma\left(3 - D + \sum_{i=1}^3 s_i\right) \Gamma\left(4 - \frac{3D}{2} + \sum_{i=1}^3 s_i\right). \quad (2.6)
 \end{aligned}$$

- It is well known that negative integers and zero are **simple poles** of the function $\Gamma(z)$. As all s_i contours are closed to the right in corresponding complex planes, one finds that the analytic expression of the the three-loop vacuum integral can be written as **the linear combination of generalized hypergeometric functions**.

II. 3-loop vacuum on compact manifold

- Taking the **residue of the pole of $\Gamma(-s_i)$** , ($i = 1, 2, 3$), we can derive one linear independent term:

$$\begin{aligned}
 U_4 \ni & \frac{-im_4^4}{(4\pi)^6} \left(\frac{4\pi\Lambda_{\text{RE}}^2}{m_4^2} \right)^{6-\frac{3D}{2}} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} (-)^{\sum_{i=1}^3 n_i} x_1^{n_1} x_2^{n_2} x_3^{n_3} \\
 & \times \left[\prod_{i=1}^3 \Gamma\left(\frac{D}{2} - 1 - n_i\right) (n_i!)^{-1} \right] \Gamma\left(3 - D + \sum_{i=1}^3 n_i\right) \\
 & \times \Gamma\left(4 - \frac{3D}{2} + \sum_{i=1}^3 n_i\right), \tag{2.7}
 \end{aligned}$$

with $x_i = \frac{m_i^2}{m_4^2}$, ($i = 1, 2, 3$).

II. 3-loop vacuum on compact manifold

$$U_4 \ni \frac{im_4^4}{(4\pi)^6} \left(\frac{4\pi\Lambda_{\text{RE}}^2}{m_4^2} \right)^{6-\frac{3D}{2}} \frac{\pi^3}{\sin^3 \frac{\pi D}{2}} T_4(\mathbf{a}, \mathbf{b} \mid \mathbf{x}), \quad (2.8)$$

with

$$T_4(\mathbf{a}, \mathbf{b} \mid \mathbf{x}) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} A_{n_1 n_2 n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}, \quad (2.9)$$

$$A_{n_1 n_2 n_3} = \frac{\Gamma(a_1 + \sum_{i=1}^3 n_i) \Gamma(a_2 + \sum_{i=1}^3 n_i)}{n_1! n_2! n_3! \Gamma(b_1 + n_1) \Gamma(b_2 + n_2) \Gamma(b_3 + n_3)}. \quad (2.10)$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2, b_3)$ with

$$a_1 = 3 - D, \quad a_2 = 4 - \frac{3D}{2}, \quad b_1 = b_2 = b_3 = 2 - \frac{D}{2}. \quad (2.11)$$

II. 3-loop vacuum on compact manifold

- We can define **auxiliary function**

$$\Phi_4(\mathbf{a}, \mathbf{b} \mid \mathbf{x}, \mathbf{u}, \mathbf{v}) = \mathbf{u}^{\mathbf{a}} \mathbf{v}^{\mathbf{b} - \mathbf{e}_3} T_4(\mathbf{a}, \mathbf{b} \mid \mathbf{x}). \quad (2.12)$$

Through **Miller's transformation**,

$$\begin{aligned} \vartheta_{u_j} \Phi_4(\mathbf{a}, \mathbf{b} \mid \mathbf{x}, \mathbf{u}, \mathbf{v}) &= a_j \Phi_4(\mathbf{a}, \mathbf{b} \mid \mathbf{x}, \mathbf{u}, \mathbf{v}), \\ \vartheta_{v_k} \Phi_4(\mathbf{a}, \mathbf{b} \mid \mathbf{x}, \mathbf{u}, \mathbf{v}) &= (b_k - 1) \Phi_4(\mathbf{a}, \mathbf{b} \mid \mathbf{x}, \mathbf{u}, \mathbf{v}), \end{aligned} \quad (2.13)$$

which naturally induces the notion of GKZ hypergeometric system. Euler operators: $\vartheta_{x_k} = x_k \partial_{x_k}$.

II. 3-loop vacuum on compact manifold

- Through the transformation

$$z_j = \frac{1}{u_j}, \quad z_{2+k} = v_k, \quad z_{5+k} = \frac{x_k}{u_1 u_2 v_k}, \quad (2.14)$$

we have **GKZ hypergeometric system** for the integral

$$\mathbf{A}_4 \cdot \vec{\vartheta}_4 \Phi_4 = \mathbf{B}_4 \Phi_4, \quad (2.15)$$

$$\mathbf{A}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix},$$

$$\vec{\vartheta}_4^T = (\vartheta_{z_1}, \dots, \vartheta_{z_8}),$$

$$\mathbf{B}_4^T = (-a_1, -a_2, b_1 - 1, b_2 - 1, b_3 - 1). \quad (2.16)$$

II. 3-loop vacuum on compact manifold

- Correspondingly the **dual matrix** $\tilde{\mathbf{A}}_4$ of \mathbf{A}_4 is

$$\tilde{\mathbf{A}}_4 = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \quad (2.17)$$

The row vectors of the matrix $\tilde{\mathbf{A}}_4$ induce the **integer sublattice** \mathbf{B} which can be used to construct the formal solutions in hypergeometric series.

- Actually the integer sublattice \mathbf{B} indicates that the solutions of the system should satisfy the equations in Eq. (2.15) and the following **hyperbolic equations** simultaneously

$$\frac{\partial^2 \Phi_4}{\partial z_1 \partial z_2} = \frac{\partial^2 \Phi_4}{\partial z_3 \partial z_6}, \quad \frac{\partial^2 \Phi_4}{\partial z_1 \partial z_2} = \frac{\partial^2 \Phi_4}{\partial z_4 \partial z_7}, \quad \frac{\partial^2 \Phi_4}{\partial z_1 \partial z_2} = \frac{\partial^2 \Phi_4}{\partial z_5 \partial z_8}. \quad (2.18)$$

II. 3-loop vacuum on compact manifold

- Defining the **combined variables**

$$y_1 = \frac{z_3 z_6}{z_1 z_2}, \quad y_2 = \frac{z_4 z_7}{z_1 z_2}, \quad y_3 = \frac{z_5 z_8}{z_1 z_2}, \quad (2.19)$$

we write the solutions as

$$\Phi_4(\mathbf{z}) = \left(\prod_{i=1}^8 z_i^{\alpha_i} \right) \varphi_4(y_1, y_2, y_3). \quad (2.20)$$

Here $\vec{\alpha}^T = (\alpha_1, \alpha_2, \dots, \alpha_8)$ denotes a sequence of complex number such that

$$\mathbf{A}_4 \cdot \vec{\alpha} = \mathbf{B}_4, \quad (2.21)$$

namely,

$$\begin{aligned} \alpha_1 + \alpha_6 + \alpha_7 + \alpha_8 &= -a_1, & \alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 &= -a_2, \\ \alpha_3 - \alpha_6 &= b_1 - 1, & \alpha_4 - \alpha_7 &= b_2 - 1, & \alpha_5 - \alpha_8 &= b_3 - 1. \end{aligned} \quad (2.22)$$

II. 3-loop vacuum on compact manifold

- To construct the hypergeometric series solutions of the GKZ hypergeometric system in Eq. (2.15) together with corresponding hyperbolic equations in Eq. (2.18) through triangulation **is equivalent to** choose a set of the linear independent column vectors of the matrix in Eq. (2.17) which spans the dual space.
- We denote the **submatrix** composed of the first, third, and fourth column vectors of the dual matrix of Eq. (2.17) as $\tilde{\mathbf{A}}_{134}$, i.e.

$$\tilde{\mathbf{A}}_{134} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}. \quad (2.23)$$

II. 3-loop vacuum on compact manifold

- Obviously $\det \tilde{\mathbf{A}}_{134} = -1 \neq 0$, and

$$\begin{aligned} \mathbf{B}_{134} &= \tilde{\mathbf{A}}_{134}^{-1} \cdot \tilde{\mathbf{A}}_4 \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}. \end{aligned} \quad (2.24)$$

Taking 3 row vectors of the matrix \mathbf{B}_{134} as the basis of [integer lattice](#), one constructs the [GKZ hypergeometric series solutions](#) in parameter space through choosing the sets of column indices $I_i \subset [1, 8]$ ($i = 1, \dots, 8$) which are consistent with the basis of integer lattice \mathbf{B}_{134} .

II. 3-loop vacuum on compact manifold

- We take the set of column indices $I_1 = [2, 5, 6, 7, 8]$, i.e. the implement $J_1 = [1, 8] \setminus I_1 = [1, 3, 4]$. The choice on the set of indices implies the **exponent numbers** $\alpha_1 = \alpha_3 = \alpha_4 = 0$. Through Eq. (2.22), one can have

$$\begin{aligned} \alpha_2 &= a_1 - a_2, \quad \alpha_5 = b_1 + b_2 + b_3 - a_1 - 3, \\ \alpha_6 &= 1 - b_1, \quad \alpha_7 = 1 - b_2, \quad \alpha_8 = b_1 + b_2 - a_1 - 2. \end{aligned} \quad (2.25)$$

Combined with Eq. (2.11), we can have

$$\alpha_2 = \frac{D}{2} - 1, \quad \alpha_5 = -\frac{D}{2}, \quad \alpha_6 = \frac{D}{2} - 1, \quad \alpha_7 = \frac{D}{2} - 1, \quad \alpha_8 = -1. \quad (2.26)$$

II. 3-loop vacuum on compact manifold

- According to the basis of integer lattice \mathbf{B}_{134} , the corresponding **hypergeometric series solution** with triple independent variables is written as

$$\begin{aligned} \Phi_{[134]}^{(1)}(\alpha, z) &= \prod_{i=1}^8 z_i^{\alpha_i} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} c_{[134]}^{(1)}(\alpha, \mathbf{n}) \left(\frac{z_1 z_2}{z_5 z_8}\right)^{n_1} \left(\frac{z_3 z_6}{z_5 z_8}\right)^{n_2} \left(\frac{z_4 z_7}{z_5 z_8}\right)^{n_3} \\ &= \prod_{i=1}^8 z_i^{\alpha_i} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} c_{[134]}^{(1)}(\alpha, \mathbf{n}) \left(\frac{1}{y_3}\right)^{n_1} \left(\frac{y_1}{y_3}\right)^{n_2} \left(\frac{y_2}{y_3}\right)^{n_3}, \quad (2.27) \end{aligned}$$

with the coefficient is

$$\begin{aligned} c_{[134]}^{(1)}(\alpha, \mathbf{n}) &= \left\{ n_1! n_2! n_3! \Gamma(1 + \alpha_2 + n_1) \Gamma(1 + \alpha_5 - n_1 - n_2 - n_3) \right. \\ &\quad \left. \times \Gamma(1 + \alpha_6 + n_2) \Gamma(1 + \alpha_7 + n_3) \Gamma(1 + \alpha_8 - n_1 - n_2 - n_3) \right\}^{-1}. \end{aligned} \quad (2.28)$$

II. 3-loop vacuum on compact manifold

- And then, through Eq. (2.26), the corresponding **hypergeometric series solution** can be written as

$$\Phi_{[134]}^{(1)}(\alpha, z) = y_1^{\frac{D}{2}-1} y_2^{\frac{D}{2}-1} y_3^{-1} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} c_{[134]}^{(1)}(\alpha, \mathbf{n}) \left(\frac{1}{y_3}\right)^{n_1} \left(\frac{y_1}{y_3}\right)^{n_2} \left(\frac{y_2}{y_3}\right)^{n_3}, \quad (2.29)$$

with the coefficient is

$$c_{[134]}^{(1)}(\alpha, \mathbf{n}) = \frac{\Gamma(\frac{D}{2} + n_1 + n_2 + n_3) \Gamma(1 + n_1 + n_2 + n_3)}{n_1! n_2! n_3! \Gamma(\frac{D}{2} + n_1) \Gamma(\frac{D}{2} + n_2) \Gamma(\frac{D}{2} + n_3)}. \quad (2.30)$$

Here, the **convergent region** is

$$\Xi_{[134]} = \{(y_1, y_2, y_3) \mid 1 < |y_3|, |y_1| < |y_3|, |y_2| < |y_3|\}, \quad (2.31)$$

which shows that $\Phi_{[134]}^{(1)}(\alpha, z)$ is in neighborhood of regular singularity ∞ .

II. 3-loop vacuum on compact manifold

- In a similar way, we can obtain other seven hypergeometric solutions which are consistent with the basis of integer lattice \mathbf{B}_{134} , and the convergent region is also $\Xi_{[134]}$.
- The above **eight** hypergeometric series solutions $\Phi_{[134]}^{(i)}(\alpha, z)$ whose convergent region is $\Xi_{[134]}$ can constitute a **fundamental solution system**.
- Multiplying one of the row vectors of the matrix \mathbf{B}_{134} by -1, the induced integer matrix can also be chosen as a basis of the integer lattice space of certain hypergeometric series.

II. 3-loop vacuum on compact manifold

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\begin{aligned} \mathbf{B}_{\bar{1}34} &= \text{diag}(-1, 1, 1) \cdot \mathbf{B}_{134} \\ &= \begin{pmatrix} -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}, \end{aligned} \quad (2.32)$$

one obtains eight hypergeometric series solutions

$\Phi_{[\bar{1}34]}^{(i)}(\alpha, z)$ ($i = 1, \dots, 8$) similarly. The convergent region is

$$\Xi_{[\bar{1}34]} = \{(y_1, y_2, y_3) \mid |y_1| < 1, |y_2| < 1, |y_3| < 1\}, \quad (2.33)$$

which shows that $\Phi_{[\bar{1}34]}^{(i)}(\alpha, z)$ are in neighborhood of regular singularity 0 and can constitute a fundamental solution system.

II. 3-loop vacuum on compact manifold

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\begin{aligned} \mathbf{B}_{1\bar{3}4} &= \text{diag}(1, -1, 1) \cdot \mathbf{B}_{134} \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}, \end{aligned} \quad (2.34)$$

one obtains eight hypergeometric series solutions

$\Phi_{[1\bar{3}4]}^{(i)}(\alpha, z)$ ($i = 1, \dots, 8$) similarly. The convergent region is

$$\Xi_{[1\bar{3}4]} = \{(y_1, y_2, y_3) \mid 1 < |y_1|, |y_2| < |y_1|, |y_3| < |y_1|\}, \quad (2.35)$$

which shows that $\Phi_{[1\bar{3}4]}^{(i)}(\alpha, z)$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

II. 3-loop vacuum on compact manifold

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\begin{aligned} \mathbf{B}_{13\bar{4}} &= \text{diag}(1, 1, -1) \cdot \mathbf{B}_{134} \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}, \end{aligned} \quad (2.36)$$

one obtains eight hypergeometric series solutions

$\Phi_{[13\bar{4}]}^{(i)}(\alpha, z)$ ($i = 1, \dots, 8$) similarly. The convergent region is

$$\Xi_{[13\bar{4}]} = \{(y_1, y_2, y_3) \mid 1 < |y_2|, |y_1| < |y_2|, |y_3| < |y_2|\}, \quad (2.37)$$

which shows that $\Phi_{[13\bar{4}]}^{(i)}(\alpha, z)$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

III. 3-loop vacuum on compact manifold

3-loop vacuum with 4 propagates

- i9-8th, 32GB: GKZ: ~ 1.5 s, FIESTA: ~ 160 s

```
SumLim = 15;
ParameterSub = {De  $\rightarrow$  4 - 2  $\times$  0.001,  $\epsilon \rightarrow$  0.001,  $a_1 \rightarrow$  1,
   $a_2 \rightarrow$  1,  $a_3 \rightarrow$  1,  $a_4 \rightarrow$  1,  $m_4 \rightarrow$  0.01,  $m_1 \rightarrow$  0.02,  $m_2 \rightarrow$  10,  $m_3 \rightarrow$  0.04 };
NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

Numerical result = -7.5628×10^8

Time Taken 1.55064 seconds

```
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
```

FIESTA Value = -7.56285×10^8

Time Taken 160.037 seconds

- i9-13th, 64GB: GKZ: ~ 1 s, FIESTA: ~ 50 s

```
SumLim = 15;
ParameterSub = {De  $\rightarrow$  4 - 2  $\times$  0.001,  $\epsilon \rightarrow$  0.001,  $a_1 \rightarrow$  1,
   $a_2 \rightarrow$  1,  $a_3 \rightarrow$  1,  $a_4 \rightarrow$  1,  $m_4 \rightarrow$  0.01,  $m_1 \rightarrow$  0.02,  $m_2 \rightarrow$  10,  $m_3 \rightarrow$  0.04 };
NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

Numerical result = -7.5628×10^8

Time Taken 1.16336 seconds

```
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
```

FIESTA Value = -7.56285×10^8

Time Taken 53.2344 seconds

III. 3-loop vacuum on compact manifold

3-loop vacuum with 5 propagates

- i9-8th, 32GB: GKZ: ~ 3 min, FIESTA: ~ 30 min

```
SumLim = 15;
ParameterSub = {De -> 4 - 2 * 0.001, e -> 0.001, a1 -> 1, a2 -> 1,
  a3 -> 7 / 8, a4 -> 3 / 4, a5 -> 1, m1 -> 0.1, m2 -> 5, m3 -> 0.3, m4 -> 0.3, m5 -> 100};
NumericalSum[SeriesSolution, ParameterSub, SumLim, RunInParallel -> True];

In[43]:=
Numerical result = 1.42136 * 10^7
In[43]:=
Time Taken 192.386 seconds

FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = 1.42136 * 10^7
Time Taken 1847.93 seconds
```

- i9-13th, 64GB: GKZ: ~ 2 min, FIESTA: ~ 10 min

```
SumLim = 15;
ParameterSub = {De -> 4 - 2 * 0.001, e -> 0.001, a1 -> 1, a2 -> 1, a3 -> 7 / 8,
  a4 -> 3 / 4, a5 -> 1, m1 -> 0.1, m2 -> 5, m3 -> 0.3, m4 -> 0.3, m5 -> 100};
NumericalSum[SeriesSolution, ParameterSub, SumLim, RunInParallel -> True];

In[46]:=
Numerical result = 1.42136 * 10^7
In[46]:=
Time Taken 120.792 seconds

FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = 1.42136 * 10^7
Time Taken 623.497 seconds
```

III. Feynman integrals on Grassmannian

1. 1-loop self-energy: $m_1^2 = m_2^2 = 0$

- Adopting **Feynman parametric representation**, we get the integral of zero virtual masses as

$$\begin{aligned}
 A_{\text{1SE}}(p^2, 0, 0) &= \left(\Lambda_{\text{RE}}^2\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q+p)^2} \\
 &= \frac{i\Gamma(2 - \frac{D}{2}) \left(\Lambda_{\text{RE}}^2\right)^{2-D/2}}{(4\pi)^{D/2}} \int_0^1 dt_1 t_1^{D/2-2} (p^2 t_1 - p^2)^{D/2-2} \\
 &= \frac{i\Gamma(2 - \frac{D}{2}) \left(\Lambda_{\text{RE}}^2\right)^{2-D/2}}{(4\pi)^{D/2}} \int_0^1 \omega_2(t) t_1^{D/2-2} t_2^{-D} (p^2 t_1 + p^2 t_2)^{D/2-2}, \quad (3.1)
 \end{aligned}$$

with the **homogeneous coordinate** $t_2 = -1$, the volume element of projective line $\omega_2(t) = t_2 dt_1 - t_1 dt_2$.

- Feng, Zhang, Chang, Feynman integrals of Grassmannians, *PRD* 106 (2022) 116025 [arXiv: 2206.04224].**

III. Feynman integrals on Grassmannian

1. 1-loop self-energy: $m_1^2 = m_2^2 = 0$

- The integral

$$A_{1SE}(p^2, 0, 0) \propto \int_0^1 \omega_2(t) t_1^{D/2-2} t_2^{2-D} (p^2 t_1 + p^2 t_2)^{D/2-2}, \quad (3.2)$$

can be embedded in the subvariety of the [Grassmannian](#) $G_{2,3}$, with [splitting local coordinates](#) as

$$A^{1SE} = \begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & p^2 \end{pmatrix}. \quad (3.3)$$

- first row: integration variable t_1 , second row: t_2 ,
 first column: power function $t_1^{D/2-2}$, second column: t_2^{2-D} ,
 third column: $(z_{1,3} t_1 + z_{2,3} t_2)^{D/2-2} = (t_1 p^2 + t_2 p^2)^{D/2-2}$.

III. Feynman integrals on Grassmannian

1. 1-loop self-energy: $m_1^2 = m_2^2 = 0$

Splitting local coordinates: $A^{1SE} = \begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & p^2 \end{pmatrix}$.

$$A_{1SE}(p^2, 0, 0) \propto \int_0^1 \omega_2(t) t_1^{D/2-2} t_2^{2-D} (p^2 t_1 + p^2 t_2)^{D/2-2},$$

satisfies the following **GKZ-system**

$$\begin{aligned} \left\{ \vartheta_{1,1} + \vartheta_{1,3} \right\} A_{1SE} &= -A_{1SE}, & \left\{ \vartheta_{2,2} + \vartheta_{2,3} \right\} A_{1SE} &= -A_{1SE}, \\ \vartheta_{1,1} A_{1SE} &= \left(\frac{D}{2} - 2 \right) A_{1SE}, & \vartheta_{2,2} A_{1SE} &= (2 - D) A_{1SE}, \\ \left\{ \vartheta_{1,3} + \vartheta_{2,3} \right\} A_{1SE} &= \left(\frac{D}{2} - 2 \right) A_{1SE}, \end{aligned} \quad (3.4)$$

where the Euler operator $\vartheta_{i,j} = z_{i,j} \partial / \partial z_{i,j}$.

Exponent matrix: $\begin{pmatrix} \frac{D}{2} - 2 & 0 & 1 - \frac{D}{2} \\ 0 & 2 - D & D - 3 \end{pmatrix}$. (3.5)

III. Feynman integrals on Grassmannian

1. 1-loop self-energy: $m_1^2 = m_2^2 = 0$

- Exponent matrix:

$$\begin{pmatrix} \frac{D}{2} - 2 & 0 & 1 - \frac{D}{2} \\ 0 & 2 - D & D - 3 \end{pmatrix}.$$

- $G_{2,3}$ splitting local coordinates:

$$A^{1SE} = \begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & p^2 \end{pmatrix}.$$

- One obtains solution of the GKZ-system

$$A_{1SE}(p^2, 0, 0) = C_{1SE}^{(0)}(p^2)^{1-D/2} (p^2)^{D-3} = C_{1SE}^{(0)}(p^2)^{D/2-2}. \quad (3.6)$$

III. Feynman integrals on Grassmannian

2. 1-loop self-energy: $m_1^2 = 0, m_2^2 \neq 0$

- Adopting Feynman parametric representation

$$\begin{aligned}
 A_{1SE}(p^2, 0, m_2^2) &= \left(\Lambda_{\text{RE}}^2\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2((q+p)^2 - m_2^2)} \\
 &= \frac{i\Gamma(2 - \frac{D}{2}) \left(\Lambda_{\text{RE}}^2\right)^{2-D/2}}{(-)^{2-D/2} (4\pi)^{D/2}} \int_0^1 dt_1 dt_3 \frac{t_1^{D/2-2} \delta(t_1 + t_3 - 1)}{(t_3 p^2 - m_2^2)^{2-D/2}} \\
 &= \frac{i\Gamma(2 - \frac{D}{2}) \left(\Lambda_{\text{RE}}^2\right)^{2-D/2}}{(-)^{2-D/2} (4\pi)^{D/2}} \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}} \quad (3.7)
 \end{aligned}$$

with homogeneous coordinate $t_2 = -1$, volume element of projective space $\omega_3(t) = t_1 dt_2 dt_3 - t_2 dt_1 dt_3 + t_3 dt_1 dt_2$.

III. Feynman integrals on Grassmannian

2. 1-loop self-energy: $m_1^2 = 0, m_2^2 \neq 0$

- The integral

$$A_{1SE}(p^2, 0, m_2^2) \propto \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}},$$

can be embedded in the subvariety of the **Grassmannian** $G_{3,5}$, with **splitting local coordinates** as

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix}. \quad (3.8)$$

- first row: t_1 , second row: t_2 , third row: t_3 ,
 first column: $t_1^{D/2-2}$, second column: t_2^{2-D} , third column:
 $t_3^0 = 1$, fourth column represents the function $\delta(t_1 + t_2 + t_3)$,
 fifth column: $(z_{1,5} t_1 + z_{2,5} t_2 + z_{3,5} t_3)^{D/2-2} = (t_2 m_2^2 + t_3 p^2)^{D/2-2}$.

III. Feynman integrals on Grassmannian

2. 1-loop self-energy: $m_1^2 = 0, m_2^2 \neq 0$

$$A_{1SE}(p^2, 0, m_2^2) \propto \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}},$$

satisfies the following **GKZ-system**

$$\left\{ \vartheta_{1,1} + \vartheta_{1,4} \right\} A_{1SE} = -A_{1SE}, \quad \left\{ \vartheta_{2,2} + \vartheta_{2,4} + \vartheta_{2,5} \right\} A_{1SE} = -A_{1SE}, \quad (3.9)$$

$$\left\{ \vartheta_{3,3} + \vartheta_{3,4} + \vartheta_{3,5} \right\} A_{1SE} = -A_{1SE}, \quad \vartheta_{1,1} A_{1SE} = \left(\frac{D}{2} - 2 \right) A_{1SE}, \quad \vartheta_{2,2} A_{1SE} = (2 - D) A_{1SE},$$

$$\vartheta_{3,3} A_{1SE} = 0, \quad \left\{ \vartheta_{1,4} + \vartheta_{2,4} + \vartheta_{3,4} \right\} A_{1SE} = -A_{1SE}, \quad \left\{ \vartheta_{2,5} + \vartheta_{3,5} \right\} A_{1SE} = \left(\frac{D}{2} - 2 \right) A_{1SE}.$$

Exponent matrix:

$$\begin{pmatrix} \frac{D}{2} - 2 & 0 & 0 & 1 - \frac{D}{2} & 0 \\ 0 & 2 - D & 0 & \alpha_{2,4} & \alpha_{2,5} \\ 0 & 0 & 0 & \alpha_{3,4} & \alpha_{3,5} \end{pmatrix}, \quad (3.10)$$

$$\alpha_{2,4} + \alpha_{2,5} = D - 3, \quad \alpha_{3,4} + \alpha_{3,5} = -1, \quad \alpha_{2,4} + \alpha_{3,4} = \frac{D}{2} - 2, \quad \alpha_{2,5} + \alpha_{3,5} = \frac{D}{2} - 2.$$

III. Feynman integrals on Grassmannian

2. 1-loop self-energy: $m_1^2 = 0, m_2^2 \neq 0$

- **Dual space** of the GKZ-system: 3×5 matrix $(0_{3 \times 3} | E_3^{(1)})$ with

$$E_3^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (3.11)$$

- **Integer lattice** $(0_{3 \times 3} | nE_3^{(1)})$ ($n \geq 0$) is compatible with two choices of the exponents. Through

$$\alpha_{2,4} + \alpha_{2,5} = D - 3, \quad \alpha_{3,4} + \alpha_{3,5} = -1, \quad \alpha_{2,4} + \alpha_{3,4} = \frac{D}{2} - 2, \quad \alpha_{2,5} + \alpha_{3,5} = \frac{D}{2} - 2,$$

the first choice is written as

$$\alpha_{2,4} = 0, \quad \alpha_{2,5} = D - 3, \quad \alpha_{3,4} = \frac{D}{2} - 2, \quad \alpha_{3,5} = 1 - \frac{D}{2}. \quad (3.12)$$

III. Feynman integrals on Grassmannian

2. 1-loop self-energy: $m_1^2 = 0, m_2^2 \neq 0$

Splitting local coordinates: $A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix}.$

Integer lattice $(\mathbb{0}_{3 \times 3} | nE_3^{(1)})$: $nE_3^{(1)} = \begin{pmatrix} 0 & 0 \\ n & -n \\ -n & n \end{pmatrix}.$

Exponent matrix: $\begin{pmatrix} \frac{D}{2} - 2 & 0 & 0 & 1 - \frac{D}{2} & 0 \\ 0 & 2 - D & 0 & 0 & D - 3 \\ 0 & 0 & 0 & \frac{D}{2} - 2 & 1 - \frac{D}{2} \end{pmatrix}.$

Hypergeometric function:

$$\begin{aligned} \psi_{\{1,2,3\}}^{(1)} &\sim (m_2^2)^{\alpha_{2,5}} (p^2)^{\alpha_{3,5}} \sum_{n=0}^{\infty} \frac{\Gamma(-\alpha_{2,5} + n) \Gamma(-\alpha_{3,4} + n)}{\Gamma(1 + \alpha_{2,4} + n) \Gamma(1 + \alpha_{3,5} + n)} (m_2^2)^{-n} (p^2)^n \\ &\sim (m_2^2)^{D-3} (p^2)^{1-D/2} \sum_{n=0}^{\infty} \frac{\Gamma(3 - D + n)}{n!} \left(\frac{p^2}{m_2^2}\right)^n. \end{aligned} \quad (3.13)$$

III. Feynman integrals on Grassmannian

2. 1-loop self-energy: $m_1^2 = 0, m_2^2 \neq 0$

- For integer lattice $(0_{3 \times 3} | nE_3^{(1)})$ ($n \geq 0$), second choice is written as

$$\alpha_{3,5} = 0, \alpha_{2,4} = \frac{D}{2} - 1, \alpha_{2,5} = \frac{D}{2} - 2, \alpha_{3,4} = -1. \quad (3.14)$$

- Adopting integer lattice and the corresponding exponents matrices, we obtain hypergeometric function as

$$\begin{aligned} & \psi_{\{1,2,3\}}^{(2)}(p^2, 0, m_2^2) \\ & \sim (m_2^2)^{\alpha_{2,5}} (p^2)^{\alpha_{3,5}} \sum_{n=0}^{\infty} \frac{\Gamma(-\alpha_{2,5} + n) \Gamma(-\alpha_{3,4} + n)}{\Gamma(1 + \alpha_{2,4} + n) \Gamma(1 + \alpha_{3,5} + n)} (m_2^2)^{-n} (p^2)^n \\ & \sim (m_2^2)^{D/2-2} \sum_{n=0}^{\infty} \frac{\Gamma(2 - \frac{D}{2} + n)}{\Gamma(\frac{D}{2} + n)} \left(\frac{p^2}{m_2^2}\right)^n. \end{aligned} \quad (3.15)$$

III. Feynman integrals on Grassmannian

2. 1-loop self-energy: $m_1^2 = 0, m_2^2 \neq 0$

- For integer lattice $(0_{3 \times 3} \mid -nE_3^{(1)})$ ($n \geq 0$), two possibilities:

$$\alpha_{2,5} = 0, \alpha_{2,4} = D - 3, \alpha_{3,4} = 1 - \frac{D}{2}, \alpha_{3,5} = \frac{D}{2} - 2; \quad (3.16)$$

$$\alpha_{3,4} = 0, \alpha_{2,4} = \frac{D}{2} - 2, \alpha_{2,5} = \frac{D}{2} - 1, \alpha_{3,5} = -1. \quad (3.17)$$

- Adopting **integer lattice and exponents matrices**, we obtain two linear independent hypergeometric functions as

$$\psi_{\{1,2,3\}}^{(3)}(p^2, 0, m_2^2) \sim (p^2)^{D/2-2} \sum_{n=0}^{\infty} \frac{\Gamma(3-D+n)}{n!} \left(\frac{m_2^2}{p^2}\right)^n; \quad (3.18)$$

$$\psi_{\{1,2,3\}}^{(4)}(p^2, 0, m_2^2) \sim \frac{(m_2^2)^{D/2-1}}{p^2} \sum_{n=0}^{\infty} \frac{\Gamma(2-\frac{D}{2}+n)}{\Gamma(\frac{D}{2}+n)} \left(\frac{m_2^2}{p^2}\right)^n. \quad (3.19)$$

III. Feynman integrals on Grassmannian

2. 1-loop self-energy: $m_1^2 = 0, m_2^2 \neq 0$

- $\det(A_{\{1,2,3\}}^{(1)}) = 1$, $G_{3,5}$ splitting local coordinates

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix}. \quad (3.20)$$

Convergent regions: $|1/x| \leq 1, |x| \leq 1$ ($x = m_2^2/p^2$). Neighborhoods: $x = \infty, 0$.

- $\det(A_{\{1,2,5\}}^{(1)}) = p^2$, $G_{3,5}$ splitting local coordinates

$$\left(A_{\{1,2,5\}}^{(1)}\right)^{-1} \cdot A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -\frac{m_2^2}{p^2} & 1 - \frac{m_2^2}{p^2} & 0 \\ 0 & 0 & \frac{1}{p^2} & \frac{1}{p^2} & 1 \end{pmatrix}. \quad (3.21)$$

Convergent regions: $|1/(1 - 1/x)| \leq 1, |1 - 1/x| \leq 1$. Neighborhoods: $x = 0, 1$.

- $\det(A_{\{1,3,5\}}^{(1)}) = -m_2^2, \det(A_{\{2,3,4\}}^{(1)}) = 1, \det(A_{\{2,4,5\}}^{(1)}) = -p^2,$
 $\det(A_{\{3,4,5\}}^{(1)}) = m_2^2.$
- 24 fundamental solutions

III. Feynman integrals on Grassmannian

3. 1-loop self-energy: $m_1^2 \neq 0, m_2^2 \neq 0$

- Adopting Feynman parametric representation

$$\begin{aligned}
 A_{1SE}(p^2, m_1^2, m_2^2) &= \left(\Lambda_{\text{RE}}^2\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m_1^2)((q+p)^2 - m_2^2)} \\
 &\propto \int_0^1 dt_1 dt_2 \delta(t_1 + t_2 - 1) (t_1 t_2 p^2 - t_1 m_1^2 - t_2 m_2^2)^{D/2-2} \\
 &\propto \int \omega_3(t) \delta(t_1 + t_2 + t_3) t_3^{2-D} (t_1 t_2 p^2 + t_1 t_3 m_1^2 + t_2 t_3 m_2^2)^{D/2-2}. \quad (3.22)
 \end{aligned}$$

- The integral can be embedded in the subvariety of the Grassmannian $G_{3,6}$, with splitting local coordinates as

$$A^{(1S)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & p^2 \\ 0 & 1 & 0 & 1 & 1 & m_1^2 \\ 0 & 0 & 1 & 1 & 1 & m_2^2 \end{pmatrix}. \quad (3.23)$$

- 72 fundamental solutions

III. Feynman integrals on Grassmannian

4. 1-loop triangle: $m_1^2 = m_2^2 = m_3^2 = 0$

- Adopting Feynman parametric representation

$$\begin{aligned}
 C(p_1^2, p_2^2, p_3^2) &= \left(\Lambda_{\text{RE}}^2\right)^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q-p_1)^2(q+p_2)^2} \\
 &\propto \int_0^1 dt_1 dt_2 dt_3 \frac{\delta(t_1 + t_2 + t_3 - 1)}{(t_1 t_2 p_3^2 + t_2 t_3 p_1^2 + t_1 t_3 p_2^2)^{3-D/2}} \\
 &\propto \int_0^1 \omega_4(t) t_4^{3-D} \frac{\delta(t_1 + t_2 + t_3 + t_4)}{(t_1 t_2 p_3^2 + t_2 t_3 p_1^2 + t_1 t_3 p_2^2)^{3-D/2}}. \quad (3.24)
 \end{aligned}$$

- Grassmannian $G_{4,7}$, with splitting local coordinates as

$$A^{(1T)} = \begin{pmatrix} 1 & 0 & 0 & 0 & p_2^2 & 1 & 1 \\ 0 & 1 & 0 & 0 & p_3^2 & 1 & 1 \\ 0 & 0 & 1 & 0 & p_1^2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}. \quad (3.25)$$

- 192 fundamental solutions

III. Feynman integrals on Grassmannian

5. 2-loop vacuum: $m_1^2 \neq 0, m_2^2 \neq 0, m_3^2 \neq 0$

- Adopting Feynman parametric representation

$$\begin{aligned}
 A_{2\text{vac}}(m_1^2, m_2^2, m_3^2) &\propto \int_0^\infty \prod_{i=1}^3 dt_i \frac{(t_1 t_2 t_3)^{1-D/2} \delta(t_1 t_2 t_3 - t_1 t_2 - t_1 t_3 - t_2 t_3)}{(t_1 m_1^2 + t_2 m_2^2 + t_3 m_3^2)^{3-D}} \\
 &\propto \int \omega_4(t) \frac{(t_1 t_2 t_3)^{1-D/2} t_4^{D/2-1} \delta(t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4)}{(t_1 m_1^2 + t_2 m_2^2 + t_3 m_3^2)^{3-D}}. \quad (3.26)
 \end{aligned}$$

- Grassmannian $G_{4,8}$, with splitting local coordinates as

$$A^{(2V)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & m_1^2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & m_2^2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & m_3^2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}. \quad (3.27)$$

- 48 fundamental solutions

IV. Summary

- Using **Mellin-Barnes representation and Miller's transformation**, we derive **GKZ hypergeometric systems** of Feynman integrals. In the neighborhoods of **origin 0 including infinity ∞** , we can obtain **analytical hypergeometric series solutions** through GKZ-systems.
- **Feynman integrals** also can be taken as functions on the subvarieties of **Grassmannians through homogenizing the parametric representation**. The GKZ-systems can be obtained in splitting local coordinates. **Fundamental solution systems** in neighborhoods of **all regular singularities**, using **matrices of integer lattice and exponent matrices**.
- **Multi-loop diagrams**: the sizes of Grassmannians are too large to construct the fundamental systems through **Feynman parametric representations**. To efficiently derive the solution, we can embed the integrals into the subvarieties of Grassmannians using **α -parametric representation**.



河北大学

河北省量子场论精细计算与应用重点实验室

河北省计算物理基础学科研究中心

河北大学物理科学与技术学院 张海斌

THANKS!

