Recovering off-shell color-kinematics duality through minimal deformation

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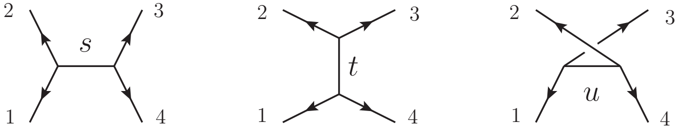
Introduction to Color-Kinematic duality

CK duality

We represent the four-gluon tree amplitude by three trivalent diagrams:



$$A_4^{\text{tree}} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \tag{1}$$

where:

$$c_s = f^{a_1 a_2 a_s} f^{a_s a_3 a_4}, \quad c_t = f^{a_4 a_1 a_t} f^{a_t a_2 a_3}, \quad c_u = f^{a_1 a_3 a_u} f^{a_u a_2 a_4} \tag{2}$$

are color factors and $n_{s,t,u}$ are corresponding numerators.

CK duality

c_s , c_t , c_u satisfy Jacobi relation：

$$c_s = c_t + c_u \tag{3}$$

but we find the numerators also satisfy：

$$n_s = n_t + n_u \tag{4}$$

which is called "dual Jacobi relation". This is the simplest example of CK duality.

More generally, for n-point amplitude(or form factor):

$$A_n^{\text{tree}} = \sum_{i \in \text{ trivalent}} \frac{C_i N_i}{\Pi_a D_{i,a}} \tag{5}$$

we have：

$$C_s = C_t + C_u \quad \Longrightarrow \quad N_s = N_t + N_u \tag{6}$$

Bern,Carrasco,Johansson 0805.3993

CK duality

Bern, Carrasco and Johansson proposed in 0805.3993 that if gauge theory amplitude：

$$A_n^{\text{tree}} = \sum_{i \in \text{ trivalent}} \frac{C_i N_i}{\Pi_a D_{i,a}} \tag{7}$$

satisfy CK duality, then we can obtain gravity amplitude by replacing the C_i by N_i:

$$M_n^{\text{tree}} = \sum_{i \in \text{ trivalent}} \frac{N_i N_i}{\Pi_a D_{i,a}} \tag{8}$$

which is called “double copy” and can be generalized in loop level.（Bern, Carrasco, Johansson 1004.0476）

CK duality

At loop level, CK duality remains a conjecture.

- In N=4 SYM, there are many examples.
- In pure Yang–Mills theory, it is very hard to realize.

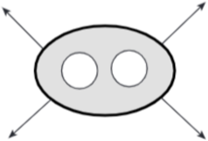
So we will concentrate on pure Yang–Mills theory.

For 4-point amplitude in pYM:

- 1-loop d-dimension. (Z.Bern et.al 1303.6605)
- 2-loop "all-plus". (Z.Bern et.al 1303.6605)
- 2-loop d-dimension with relaxed CK duality. (Z.Bern, S.Davies, J.Nohle ,1510.03448)

CK duality

We also studied the CK duality for d-dimension 4-point 2-loop amplitude:



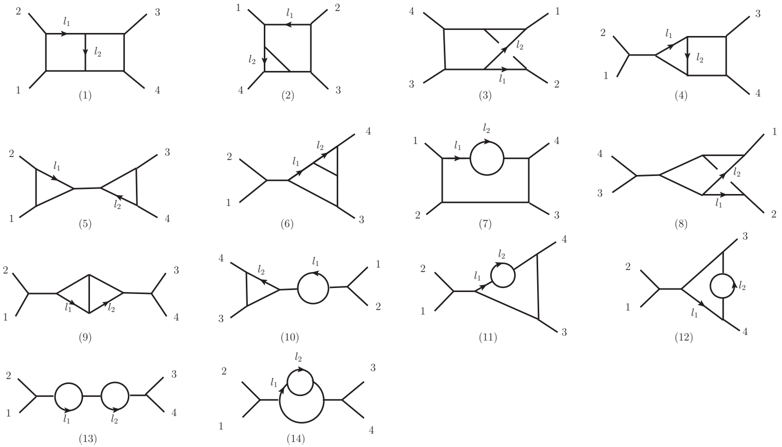
$$: \quad \hat{A}_4^{(2)} = \sum_{S_m} \sum_{\Gamma_i} \int \prod_{j=1}^{2} \frac{d^D l_j}{(2\pi)^D} \frac{1}{S_i} \frac{C_i N_i}{\prod_a D_{i,a}}.$$

Even this has already been studied in 1510.03448, we discovered some new features in our recent study.

CK duality for d-dimension 4-point 2-loop amplitude

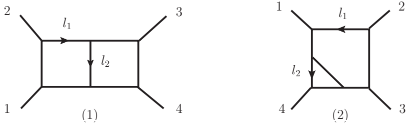
Trivalent diagrams

All the trivalent diagrams for 4-point 2-loop amplitude are:

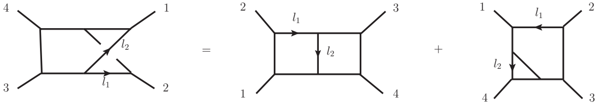


We exclude topologies with scaleless integral.

Master topologies:



All other numerators can be deduced by them, for example:



⋮

We denote the master numerators as n_1 and n_2.

Make ansatz

We only need to make ansatz for n_1 and n_2:

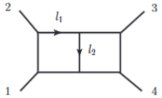
$$n_i = \sum_k a_{ik} M_k, \qquad i = 1, 2 \tag{9}$$

- a_{ik} : undetermined parameters.
- M_k : monomials made of local function of $\varepsilon_i \cdot \varepsilon_j$, $\varepsilon_i \cdot p_j$, $p_i \cdot p_j$.

We will introduce 20020 parameters in total.

Symmetry and global CK

- Demand the numerators share the same symmetry property as its topology:



$$\begin{cases} n_1 = n_1[p_2, p_1, p_4, p_3, p_3 + p_4 - l_1, -l_2] \\ n_1 = n_1[p_4, p_1, p_4, p_3, l_2 - l_1, l_2] \end{cases}$$

- Demand all the CK relations are consistent with each other.

The parameters reduce to 1382 after above constraints are satisfied. We call n_i satisfy "global CK relations".

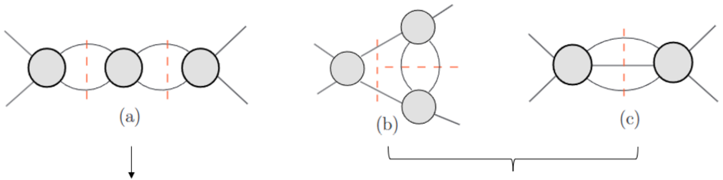
Then we need to apply unitarity cuts.

Unitarity cut

Crucial point:

The global CK integrand n_i can not pass all unitarity cuts.

The spanning set of cuts for 4-point 2-loop amplitude:



Can not satisfy

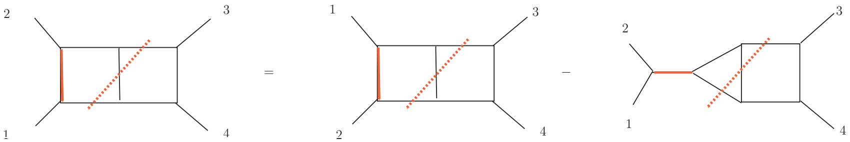
Can satisfy

Overcome difficulty

Basic aspects for dealing with this difficulty:

- Enlarge ansatz.
- Consider helicity amplitude.
- Release symmetry or CK constraints.

In 1510.03448, authors proposed that we can release the CK identities to hold only on unitarity cuts:



without loosing the double-copy property.

Old Strategy

To achieve this goal, they made ansatz for each topology:

$$n_i = \sum_k a_{ik} M_k, \qquad i = 1, 2, ..., 14 \tag{10}$$

- 120904 parameters in total, 28204 remained after symmetry constraints satisfied.
- Require CK identities satisfied while taking unitarity cuts.

Successfully constructed the 4-point 2-loop amplitude integrand with relaxed CK constraints.

In the final result, there still exist 6322 free parameters.

Disadvantages

Disadvantages

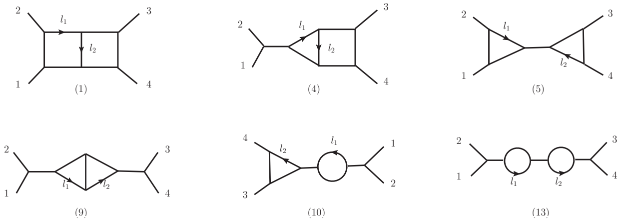
- Introduce much more parameters. $(1382 \to 28204)$
- Very hard to be generalized.
- Hard to understand the breaking of CK duality.

We develop a new strategy which can greatly refine above disadvantages.

New strategy

We will base on the global CK integrand n_i.

Since only cut (a) can not be satisfied, we will concentrate on topologies that will contribute to cut (a):



Add some "simple deformations" to corresponding n_i:

$$N_i = n_i + \Delta_i \tag{11}$$

other numerators remain unchanged, and we wish N_i can pass all unitarity cuts and obey CK identities under cuts.

New strategy

Features about Δ_i:

1. Δ_i should vanish in cut (b) and (c):

$$\Delta_i|_{cut(b),(c)} = 0 \tag{12}$$

2. Δ_i also satisfy CK relations on unitarity cuts:

$$(N_s - N_t - N_u)|_{cut} = 0 \quad \Longrightarrow \quad (\Delta_s - \Delta_t - \Delta_u)|_{cut} = 0 \tag{13}$$

These features will significantly simplify Δ_i. CK relations between Δ_i allows us to only make ansatz for Δ_1:

$$\Delta_1 \xrightarrow{\text{CK}} \Delta_4 = \Delta_1 - \Delta_1[p_3, p_4, p_2, p_1, l_1 - l_2 + p_1 + p_2, -l_2] \quad \cdots$$

restricted in cut (a) and not global

New strategy

Now we introduce how to determine Δ_i.

Divide n_i into 3 parts:

$$n_i = n_i^{(1)} + n_i^{(2)} + n_i^{(3)} \tag{14}$$

- $n_i^{(1)}$: $(\varepsilon_i \cdot \varepsilon_j)(\varepsilon_k \cdot \varepsilon_l)[S]^3$
- $n_i^{(2)}$: $(\varepsilon_i \cdot \varepsilon_j)(\varepsilon_k \cdot p)(\varepsilon_l \cdot p)[S]^2$
- $n_i^{(3)}$: $(\varepsilon_1 \cdot p_i)(\varepsilon_2 \cdot p_j)(\varepsilon_3 \cdot p_k)(\varepsilon_4 \cdot p_l)[S]$

$[S]$: set of all mandelstam variables.

Correspondingly:

$$\Delta_i = \Delta_i^{(1)} + \Delta_i^{(2)} + \Delta_i^{(3)} \tag{15}$$

We first focus on $\Delta_i^{(1)}$.

Determine $\Delta_1^{(1)}$

For $n_i^{(1)}$, only terms proportional to $(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4)$ can not pass cut (a). So $\Delta_1^{(1)}$ should satisfy:

1. Proportional to $(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4)$.
2. Proportional to $l_2^2 \quad \longleftrightarrow \quad$ Vanish in cut (b), (c).

So we propose $\Delta_1^{(1)}$ to be:

$$\Delta_1^{(1)} = (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) l_2^2 (\sum_k c_k^{(1)} A_k), \tag{16}$$

where $A_k \in [S]^2$.

29 parameters in $\Delta_1^{(1)}$ with symmetry satisfied.

Determine $\Delta_1^{(1)}$

Other deformations $\Delta_i^{(1)}$ are deduced by $\Delta_1^{(1)}$ through CK relations.

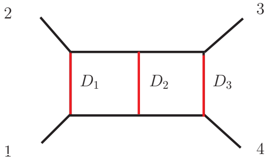
$N_i^{(1)}$ now can indeed pass cut (a).

$c_k^{(1)}$ in $\Delta_1^{(1)}$ mix with a_{ik} in $n_i^{(1)}$ and are not uniquely fixed.

Surprisingly, we find there exist a very simple special solution for $\Delta_1^{(1)}$:

$$\Delta_1^{(1)} = (d-2)^2(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4)D_1 D_2 D_3 \tag{17}$$

where D_1, D_2 and D_3 are:



Determine $\Delta_1^{(2)}$

Very Similar to $\Delta_1^{(1)}$, we make ansatz for $\Delta_1^{(2)}$:

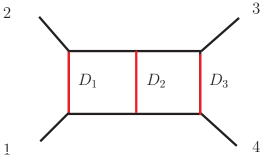
$$\Delta_1^{(2)} = ((\varepsilon_1 \cdot \varepsilon_2)(\sum_{k_1} c_{k_1}^{(2)} B_{1,k_1}) + (\varepsilon_3 \cdot \varepsilon_4)(\sum_{k_2} c_{k_2}^{(2)} B_{2,k_2}))l_2^2 \tag{18}$$

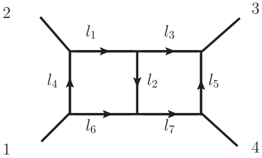
B_{1,k_1}: $(\varepsilon_3 \cdot p)(\varepsilon_4 \cdot p)[S]$, B_{2,k_2}: $(\varepsilon_1 \cdot p)(\varepsilon_2 \cdot p)[S]$.

102 parameters in $\Delta_1^{(2)}$ with symmetry satisfied.

Again, we find a very simple special solution for $\Delta_1^{(2)}$ after $N_i^{(2)}$ pass all cuts:

$$\Delta_1^{(2)} = -4(d-2)^2((\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot l_5)(\varepsilon_4 \cdot l_5)D_1 D_2 + (\varepsilon_3 \cdot \varepsilon_4)(\varepsilon_1 \cdot l_4)(\varepsilon_2 \cdot l_4)D_2 D_3) \tag{19}$$





Determine $\Delta_1^{(3)}$

Ansatz for $\Delta_1^{(3)}$ is:

$$\Delta_1^{(3)} = (\sum_k c_k^{(3)} C_k) l_2^2 \ , \tag{20}$$

C_k: $(\varepsilon_1 \cdot p_i)(\varepsilon_2 \cdot p_j)(\varepsilon_3 \cdot p_k)(\varepsilon_4 \cdot p_l)$. 76 parameters with symmetry satisfied.

As we expect, $N_i^{(3)}$ can pass all unitarity cuts now.

The simplest form we find for $\Delta_1^{(3)}$ contains 6 terms.

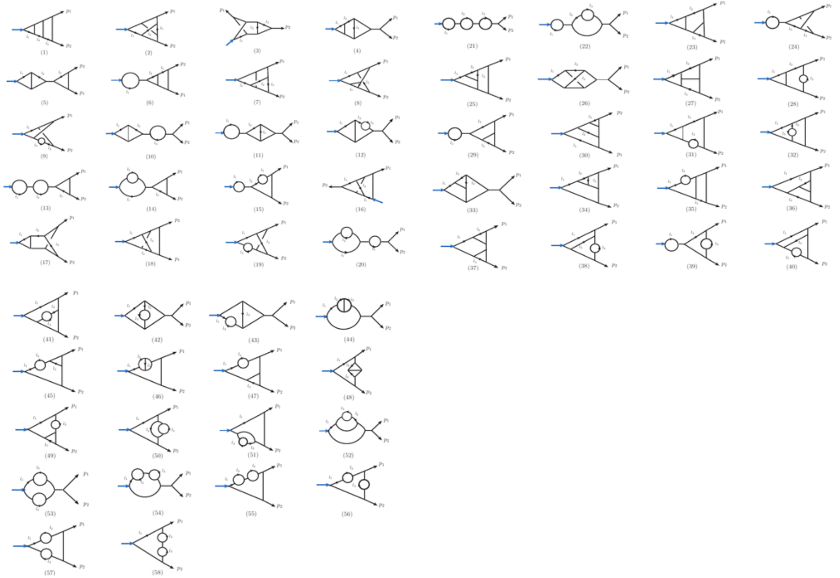
Advantages

Now we successfully construct N_i, which satisfy relaxed CK relations and can pass all unitarity cuts.

Advantages

1. Much less parameters.
2. Easy to be generalized. (3-loop Sudakov form factor)
3. Reveal the breaking of CK duality more precisely.

3-loop Sudakov form factor



Summary and Outlook

Summary and Outlook

We study the CK duality of d-dimension 4-point 2-loop amplitude:

- Develop a new strategy to construct integrand with relaxed CK duality.
- The relaxed CK integrand N_i is closest to global CK integrand.
- The strategy can be generalized in 3-loop Sudakov form factor.

Outlook:

- General rule for deformations Δ_i?
- What about relaxing symmetry constraints?
- Towards higher-loop or higher-point amplitudes?

Thank you!