Recent lattice QCD studies on multiquark states

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Outline

I. Introduction
II. Heavy flavored multiquark states
III. Charmonium(like) states and their decays
IV. Summary and perspectives

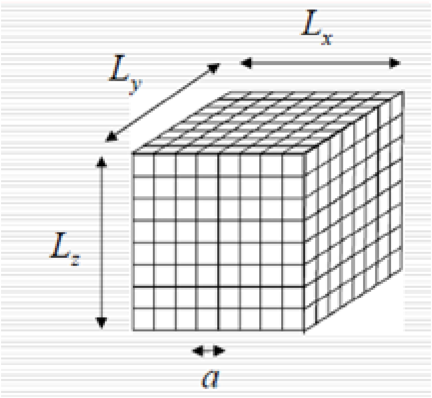
I. Introduction

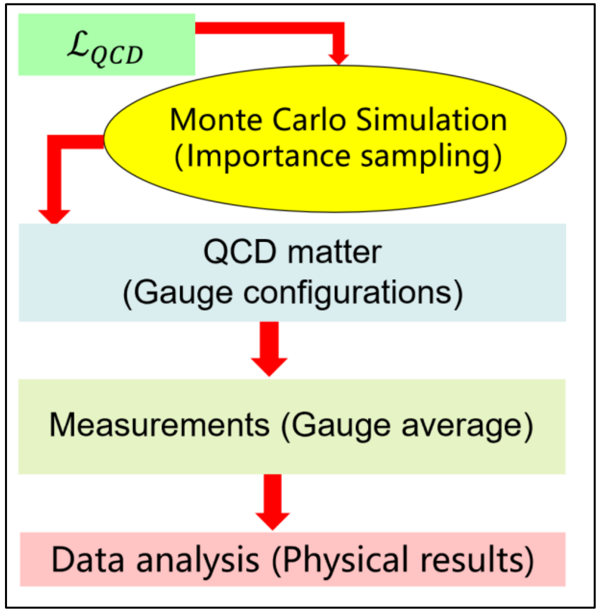
1. Lattice QCD formalism

- **Path integral quantization on finite Euclidean spacetime lattices**

$$Z = \int DAD\psi D\bar{\psi}\, e^{iS[A,\psi,\bar{\psi}]} \to \int DU \det M[U]\; e^{-S_g[U]}$$

$$\langle \hat{\mathcal{O}}[U,\psi,\bar{\psi}]\rangle = \frac{1}{Z}\int DU \;\det M[U]\; e^{-S_g[U]}\;\; \mathcal{O}[U]$$





Green's functions

Field product

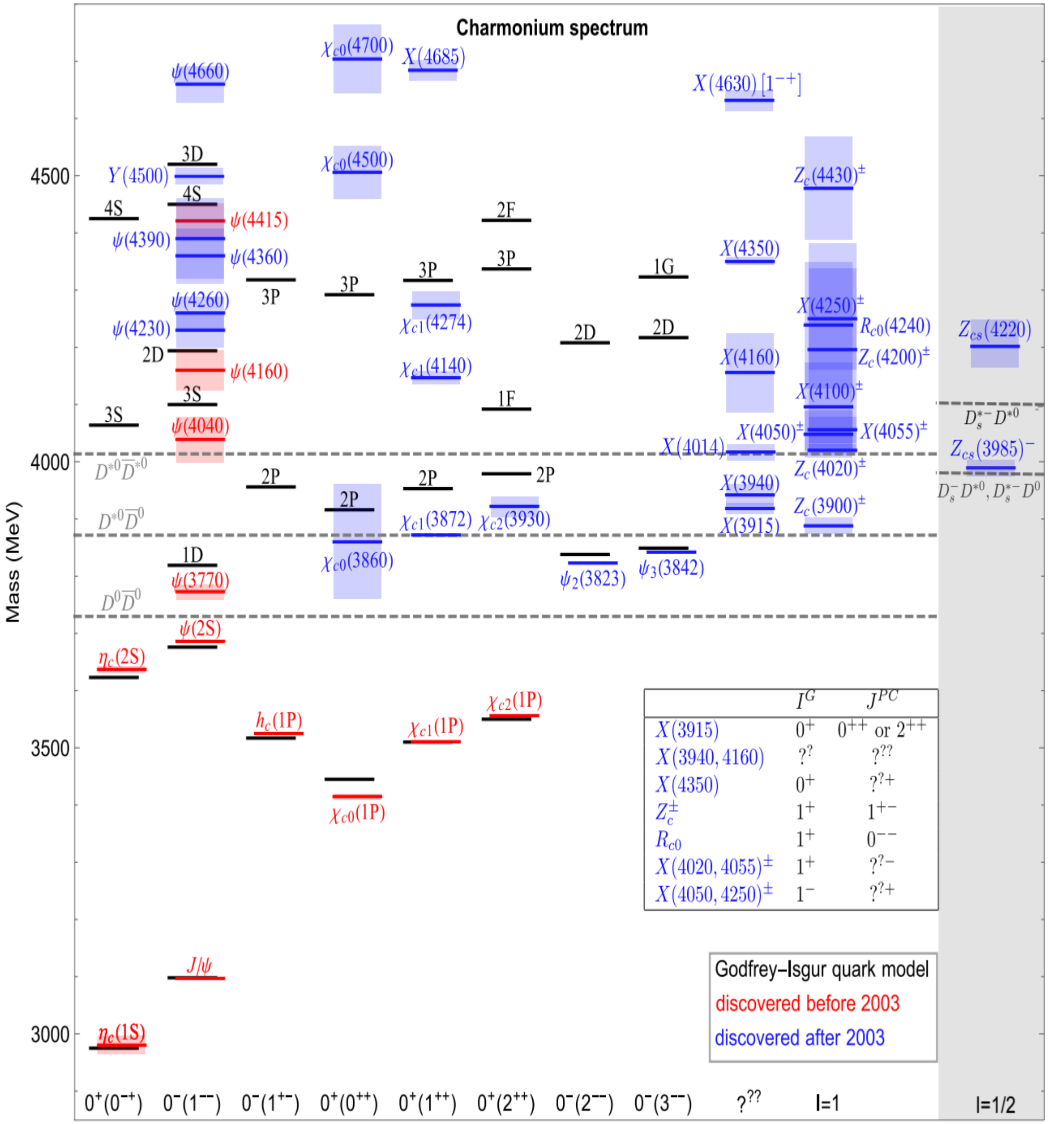
Spacetime discretization

- **Very similar to a statistical physics system**
- **Monte Carlo simulation——importance sampling according to** $\mathcal{P}[U] \propto \det M[U]\; e^{-S_g[U]}$

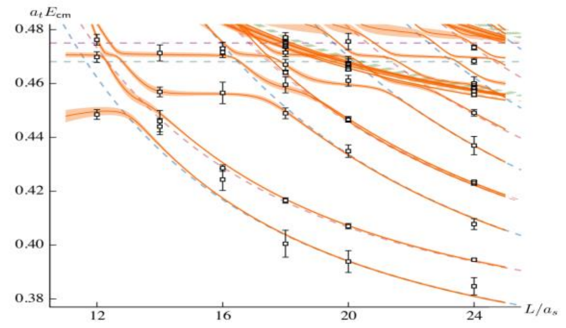
Gauge ensemble: $\{U_i(\text{spacetime}), i = 1, \dots, N\}$ ➡ $\langle \hat{\mathcal{O}}[U,\psi,\bar{\psi}]\rangle = \frac{1}{N}\sum_i \mathcal{O}[U_i] + O\left(\frac{1}{\sqrt{N}}\right)$

2. New hadron states that has heavy quarks

- Ever since the discovery of X(3872), a large number of charmium(-like) structures have been observed by various experiments (BESIII, BaBar, Belle, CDF, D0, ATLAS, CMS and LHCb).
- All of the XYZ states are above or at least in the vicinity of the open-charm thresholds, and are good candidates for hadron molecules.
- Apart from charmium-like states, LHCb observed several P_c states in $J/\psi p$ final states
  $P_c(4312), (4380), P_c(4440), P_c(4457)$
- In 2021, LHCb observed the first doubly charmed structure $T_{cc}^{+}(3875)$.
- More states will be coming.
- Their properties are worthy of a investigation in depth.
- Lattice QCD plays an important role, and are collaborative efforts along with phenomenological studies in this sector.



3. The methodology for studying hadron-hadron scattering in lattice QCD



State-of-art Approach——Lellouch-Lüscher's formalism

(see R. Briceno et al., Rev. Mod. Phys. 90 (2018) 025001 for a review).

$$\det\left[F^{-1}\left(\vec{P},E,L\right)+\mathcal{M}(E)\right]=0$$

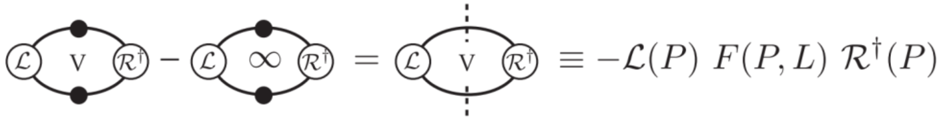
$E_n(L)$: Eigen-energies of lattice Hamiltonian.

- **Interpolation field operator set for a given J^{PC}**
  $\mathcal{O}_i$: $\bar{q}_1\Gamma q_2$ $[\bar{q}_1\Gamma_1 q][\bar{q}\Gamma_2 q_2]$ $[q_1^T\Gamma_1 q][\bar{q}\Gamma_2\bar{q}_2^T]$, ...
- **Correlation function matrix —— Observables**

$$C_{ij}(t) = \left\langle\Omega\middle|\mathcal{O}_i(t)\mathcal{O}_j^\dagger(0)\middle|\Omega\right\rangle = \sum_n \langle\Omega|\mathcal{O}_i|n\rangle\left\langle n\middle|\mathcal{O}_j^\dagger\middle|\Omega\right\rangle e^{-E_n t}$$

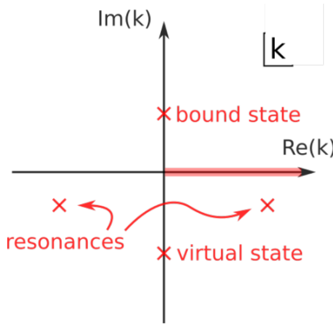
All the energy levels $E_n(L)$ are discretized.

$F\left(\vec{P},E,L\right)$: Mathematically known function matrix in the channel space (the explicit expression omitted



$\mathcal{M}(E)$: Scattering matrix.

- Unitarity requires
  $$\mathcal{M}_{ab}^{-1} = (\mathcal{K}^{-1})_{ab} - i\delta_{ab}\frac{2q_a^*}{E_{cm}}$$
- $\mathcal{K}$ is a real function of s for real energies above kinematic threshold.
- The pole singularities of $\mathcal{M}(s)$ in the complex s-plane correspond to bound states, virtual states, resonances, etc..

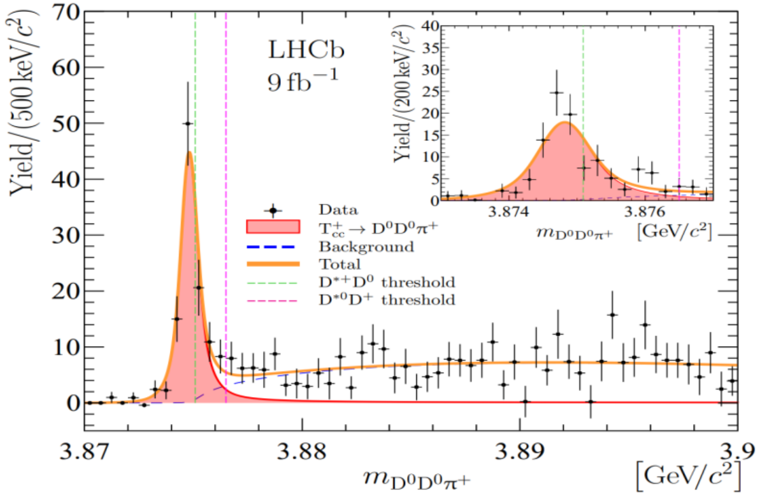


$$k=\frac{1}{2}\sqrt{s-4m^2}$$

II. Heavy flavored multiquark states

1. Lattice studies of $T_{cc}^{+}(3875)$

LHCb discovered $T_{cc}^{+}(3875)$ in 2021 (LHCb, Nature Phys.18, 751 (2022), Nature Comm.13, 3551 (2022))



$$M_{T_{cc}} - (m_{D^0} + m_{D^{*+}}) = -273 \pm 61 \pm 5^{+11}_{-14}\ \mathrm{keV}$$

$$\Gamma_{BW} = 410 \pm 165 \pm 43^{+18}_{-38}\ \mathrm{keV}$$

$$\Gamma_{BW}^{U} = 48 \pm 2^{0}_{-14}\ keV$$

Isospin：Only observed in DD^{*+}, therefore $I = 0$

The minimum quark configuration： $cc\bar{u}\bar{d}$

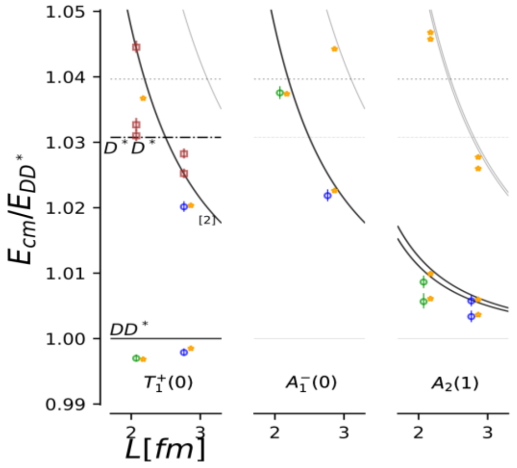
- Spured extensive and intensive phemonenological investigations
- Likely a DD^* hadronic molecule
- A relay race of lattice studies——make the things clearer!

Pole singularity: M. Padmanath and S. Prelovsek, Phys. Rev. Lett. 129 (2022) 032002
Dynamics underlying: S. Chen et al., Phys. Lett. B 833, 137391 (2022)
Interaction potential: Y. Lyu et al., arXiv:2302.04505 (hep-lat)

A. Pole singularity of $DD^*(I=0)$ scattering amplitude from lattice QCD

M. Padmanath and S. Prelovsek, Phys. Rev. Lett. 129 (2022) 032002

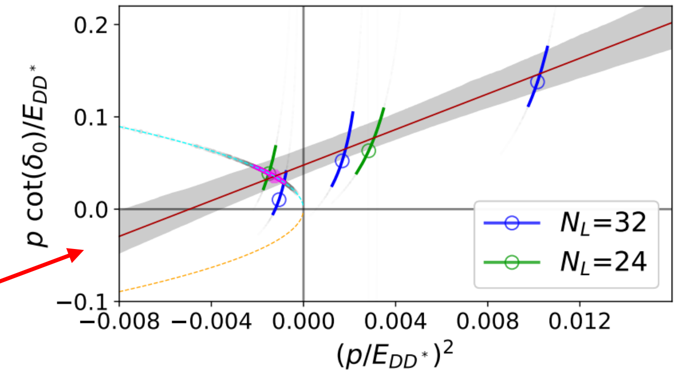
	m_D [MeV]	m_{D^*} [MeV]	M_{av} [MeV]	$a_{l=0}^{(J=1)}$ [fm]	$r_{l=0}^{(J=1)}$ [fm]	$\delta m_{T_{cc}}$ [MeV]	T_{cc}
lat. ($m_\pi \simeq 280$ MeV, $m_c^{(h)}$)	1927(1)	2049(2)	3103(3)	1.04(29)	$0.96(^{+0.18}_{-0.20})$	$-9.9^{+3.6}_{-7.2}$	virtual bound st.
lat. ($m_\pi \simeq 280$ MeV, $m_c^{(l)}$)	1762(1)	1898(2)	2820(3)	0.86(0.22)	$0.92(^{+0.17}_{-0.19})$	$-15.0(^{+4.6}_{-9.3})$	virtual bound st.
exp. [2, 38]	1864.85(5)	2010.26(5)	3068.6(1)	-7.15(51)	[-11.9(16.9),0]	$-0.36(4)$	bound st.



$$e^{2i\delta_l} = 1 + i\,2\rho t_l, \qquad \rho = \frac{2p}{\sqrt{s}}, \qquad \sqrt{s} = E_{cm} = \sqrt{m_D^2 + p^2} + \sqrt{m_{D^*}^2 + p^2}$$

S-wave scattering amplitude: $$t_0 = \frac{\sqrt{s}}{2}\frac{1}{p\cot\delta_0 - ip}$$

Effective range expansion (ERE): $$p\cot\delta_0 = \frac{1}{a_0} + \frac{1}{2}r_0 p^2$$



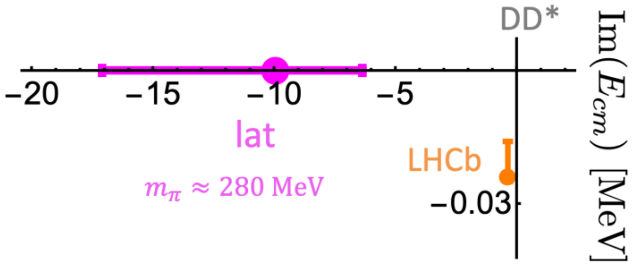
$p = \pm i|p|$
$ip = \mp\sqrt{|p^2|}$,

Pole condition: $p\cot\delta_0 = ip$

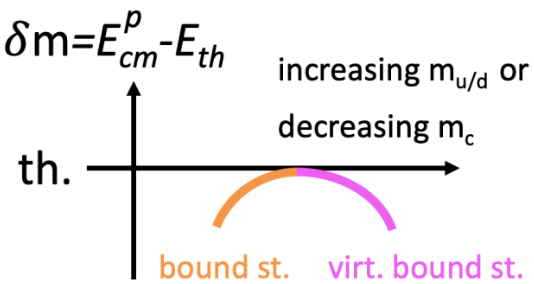
Lüscher's relation:

$$p\cot\delta_0(q^2) = \frac{2}{L\sqrt{\pi}} Z_{00}(1,q^2) = \frac{1}{\pi L}\sum_{\vec{n}\in Z_3}\frac{1}{\vec{n}^2 - q^2}, \qquad q = \frac{Lp}{2\pi}$$

$\delta m_{T_{cc}} = \mathrm{Re}(E_{cm}) - m_{D^0} - m_{D^{*+}}$ [MeV]



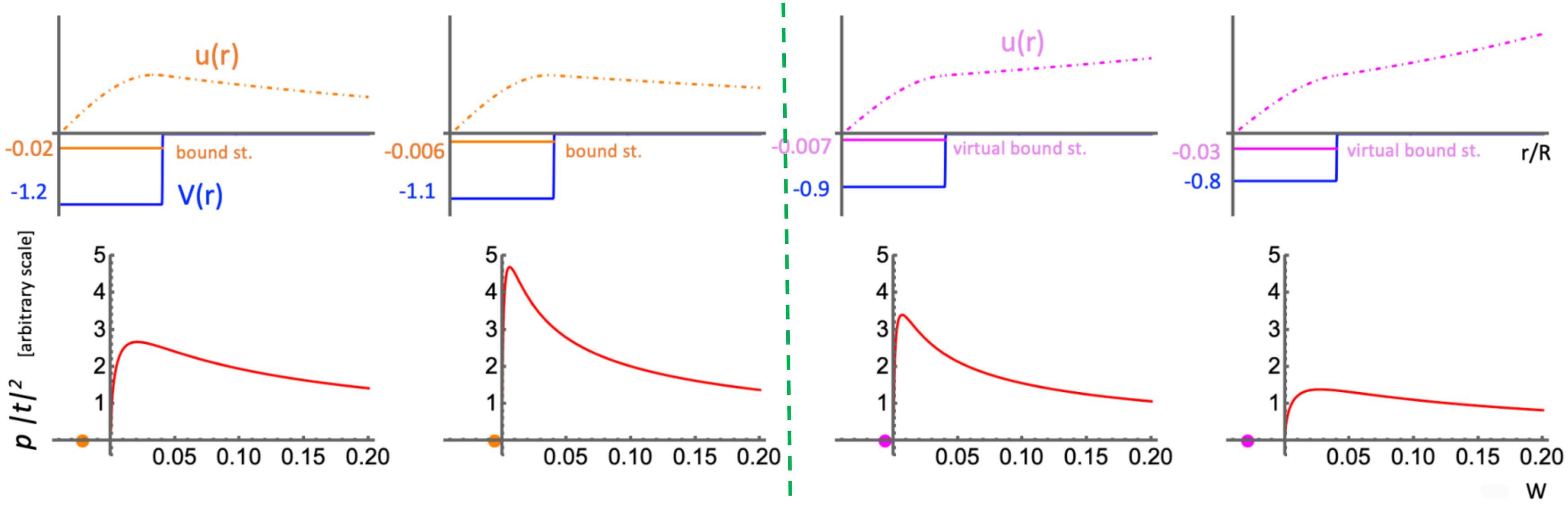
- The quark mass dependence of T_{cc}: when $m_{u/d}$ (m_π) decreases, a virtual state can develop into a bound state.
- This procedure can be illustrated qualitatively as follows:



$$V(r) \propto -\frac{1}{r}e^{-Mr}$$

Bound state: $p = i|p_B| \rightarrow e^{ipr} = e^{-|p_B|r}$

Virtual state: $p = -i|p_B| \rightarrow e^{ipr} = e^{|p_B|r}$



S-wave scattering in a purely attractive potential $V(r)$ (square well potential for instance):

weaker the potential ➡ shallower the bound state ➡ closer the pole to the threshold

even weaker the potential ➡ $i|p_B| \rightarrow -i|p_B|$ ➡ a virtual state

Either bound or virtual, it affects the cross-section and results in an enhancement near the threshold.

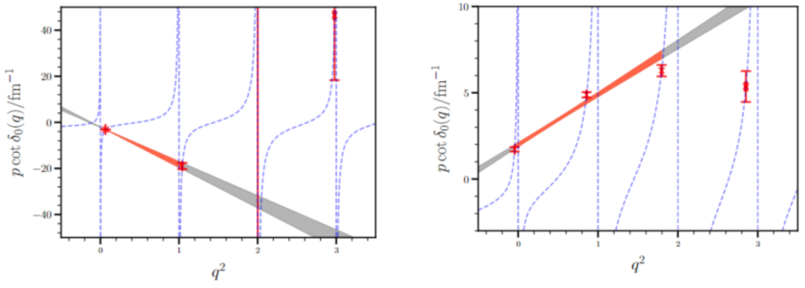
B. Investigation of the isospin-dependent interaction of DD^* scattering

(S. Chen et al., Phys. Lett. B 833, 137391 (2022))

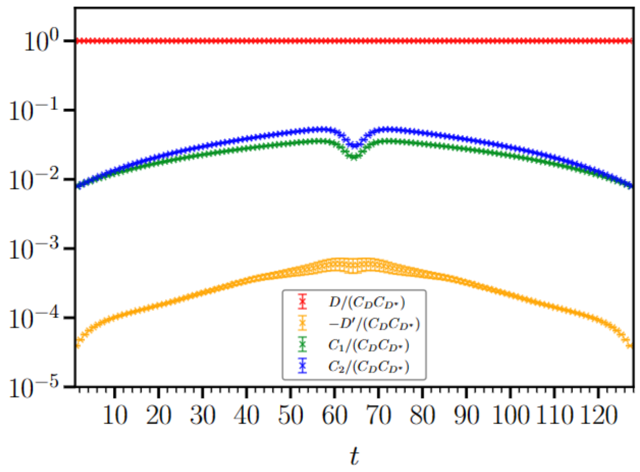
- DD^* energies and scattering momenta can be derived precisely
- Single-channel Lüscher's formula applied
- $I = 1\ DD^*$ is repulsive, $I = 0\ DD^*$ is repulsive (sign of a_0)
- Quark diagrams (after Wick's contraction) contributing to DD^* correlators

$$C^{(I)}_{DD^*}(t) = D + C_1 + (-)^{I+1}(C_2 + D')$$

$$\boldsymbol{p \cot \delta_0(p) = \frac{1}{a_0} + \frac{1}{2} r_0 p^2 + \mathcal{O}(p^4)}$$



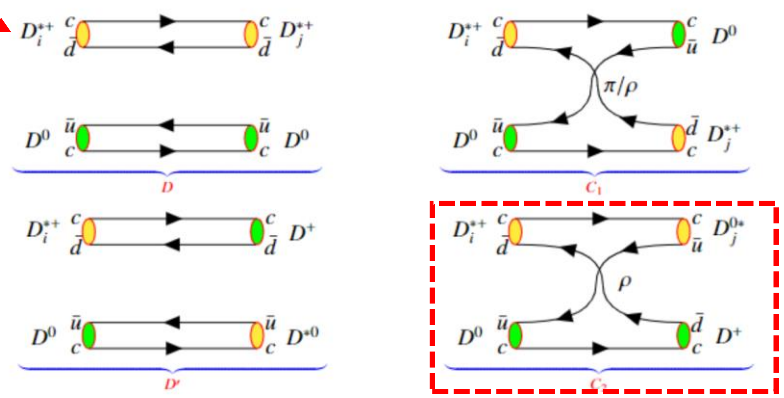
$I = 1$: repulsive $\qquad I = 0$: attractive



$$\Delta E^{(I)}_{DD^*} \approx \epsilon_1 \delta E_1 + (-)^{I+1} \epsilon_2 \delta E_2$$

$$(\epsilon_2 > \epsilon_1 > 0, \qquad \delta E_2 \geq \delta E_1)$$

$$\Rightarrow \begin{cases} \Delta E^{(I=0)}_{DD^*} < 0, \\ \Delta E^{(I=1)}_{DD^*} > 0, \end{cases}$$



Schematic quark diagrams

- Initiatively interprets the underlying physics by analyzing the quark diagrams in lattice QCD calculations

B. Investigation of the isospin-dependent interaction of DD^* scattering

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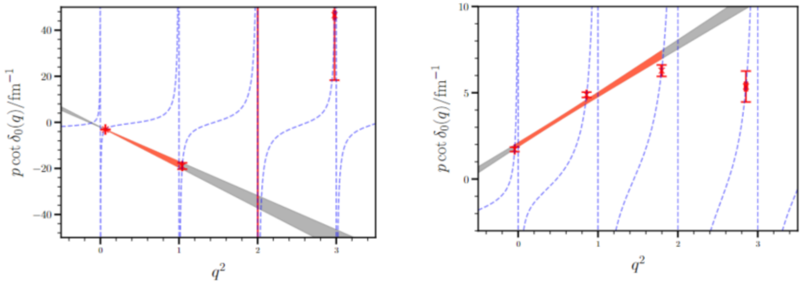
- $I = 1$ DD^* is repulsive, $I = 0$ DD^* is repulsive (sign of a_0)
- Quark diagrams (after Wick's contraction) contributing to DD^* correlators

$$C_{DD^*}^{(I)}(t) = D + C_1 + (-)^{I+1}(C_2 + D')$$

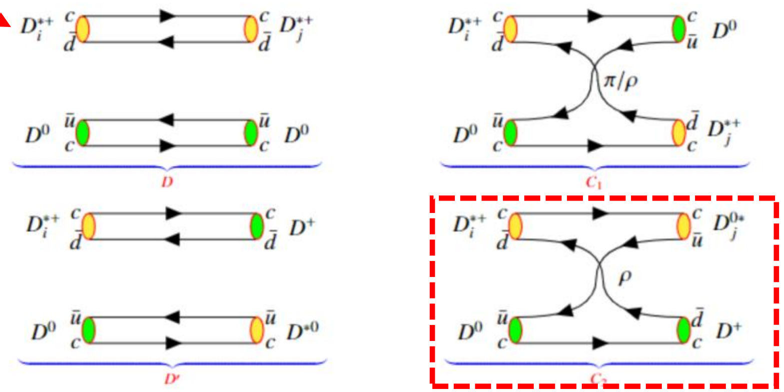
- ✓ D' term is negligible.
- ✓ C_2 term is responsible for the energy difference of $DD^*(I = 1)$ and $DD^*(I = 0)$.
- ✓ C_2 term can be understood as the exchange of charged vector ρ meson, which provides attractive (repulsive) interaction for $I = 0$ $(I = 1)$
- ✓ This is in qualitative agreement with phenomenological studies (Dong et al. CTP73 (2021) 125201, Feijoo et al, PRD104(2021)114015)

- Initiatively interprets the underlying physics by analyzing the quark diagrams in lattice QCD calculations

$$p \cot \delta_0(p) = \frac{1}{a_0} + \frac{1}{2} r_0 p^2 + \mathcal{O}(p^4)$$



$I = 1$: repulsive $I = 0$: attractive



Schematic quark diagrams

C. Hadron-hadron interaction potential——HALQCD approach (Y. Lyu et al., arXiv:2302.04505 (hep-lat))

Nambu-Bethe-Salpeter wave function

- (2+1)-flavor QCD on the 96^4 lattice with $m_\pi = 146.4$ MeV, L=8.1 fm
- Calculate the correlation functions

$$R(\vec{r},t) = e^{(m_{D^*}+m_D)t}\sum_{\vec{x}}\langle 0|D^*(\vec{x}+\vec{r},t)D(\vec{x},t)\bar{J}(0)|0\rangle = \sum_n A_n\psi_n(\vec{r})e^{-\Delta E_n t} + \cdots$$

- The function $R(\vec{r},t)$ satisfies the Shrödinger-type equation

$$\left[\frac{1+3\delta^2}{8\mu}\partial_t^2 - \partial_t - H_0 + \cdots\right]R(\vec{r},t) = \int d\vec{r}'\, U(\vec{r},\vec{r}')R(\vec{r},t), \qquad H_0 = -\frac{\nabla^2}{2\mu}, \qquad \mu = \frac{m_{D^*}m_D}{m_{D^*}+m_D}, \qquad \delta = \frac{m_{D^*}-m_D}{m_{D^*}+m_D}$$

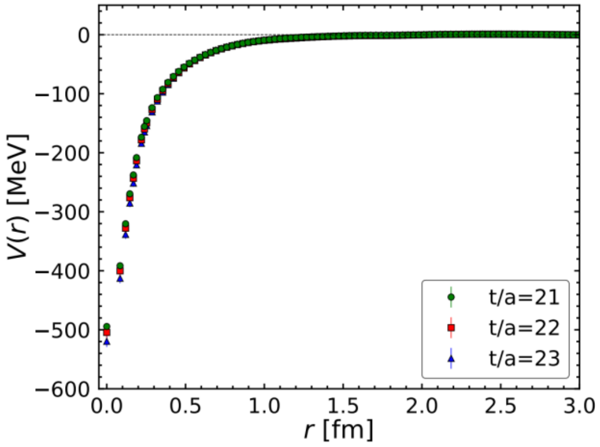
- Takes the leading term of derivative expansion of the non-local $U(\vec{r},\vec{r}')$

$$U(\vec{r},\vec{r}') \approx V(\vec{r})\delta(\vec{r}-\vec{r}'), \qquad V(r) = R^{-1}(\vec{r},t)\left[\frac{1+3\delta^2}{8\mu}\partial_t^2 - \partial_t - H_0 + \cdots\right]R(\vec{r},t)$$

- The DD^* potential in the $(I,J^P) = (0,1^+)$ channel is attractive.
- Short range: attractive diquark-antidiquark $(\bar{u}\bar{d} - cc)$
  Long range: two-pion exchange is favored:

$$V_{fit}^B(r;m_\pi) = \sum_{i=1,2} a_i e^{(-r/b_i)^2} + a_3\left(\frac{1}{r}e^{-m_\pi r}\right)^2 \cdots$$

- Different from phenomenological expectation that ρ-exchange dominates?



- Using the derived potential, the S-wave phase shifts δ_0 is obtained by solving the Schrödinger equation of DD^* system, which is put into the ERE

$$p\cot\delta_0(p) = \frac{1}{a_0} + \frac{1}{2} r_0 p^2 + \mathcal{O}(p^4)$$

- Extrapolate to the physical m_π,

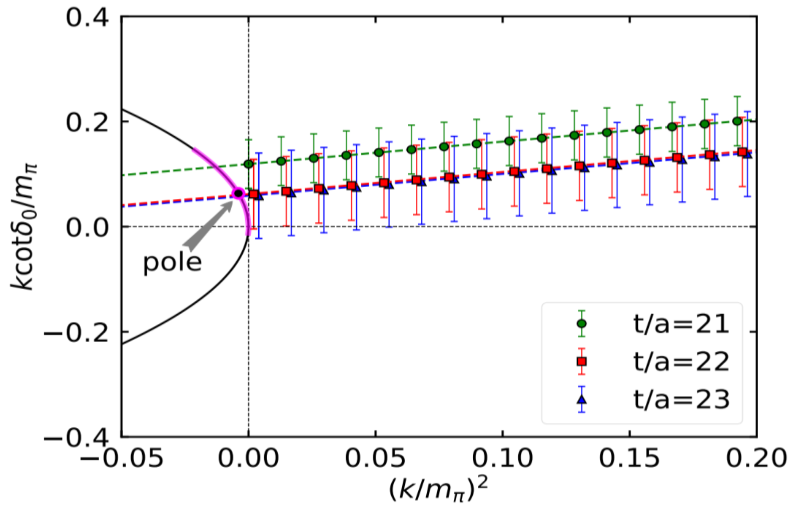
$$V^B_{fit}(r; m_\pi) \rightarrow V^B_{fit}\left(r; m_\pi^{\rm phys}\right)$$

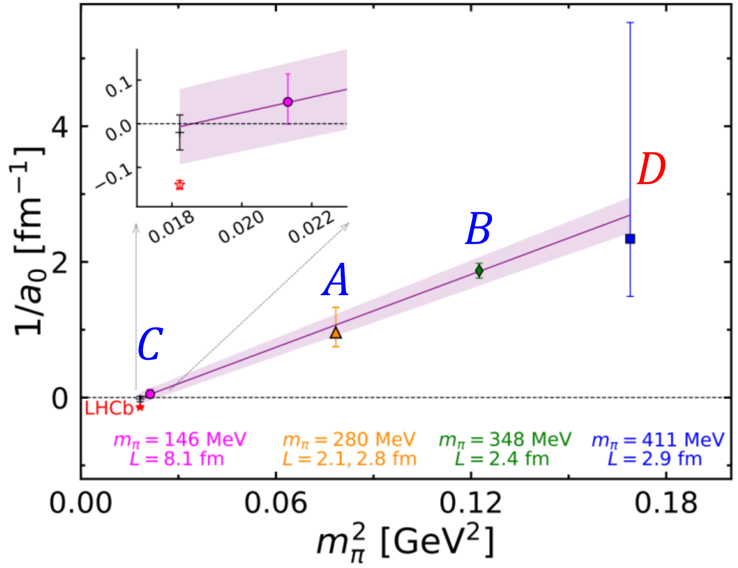
one gets

m_π [MeV]	146.4	135.0
$1/a_0$ [fm^{-1}]	$0.05(5)\,\binom{+4}{-1}$	$-0.02(4)$
$r_{\rm eff}$ [fm]	$1.14(6)\,\binom{+1}{-9}$	$1.14(8)$
$\kappa_{\rm pole}$ [MeV]	$-9(9)\,\binom{+1}{-8}$	$+3(8)$
$E_{\rm pole}$ [keV]	$-45(77)\,\binom{+02}{-99}$	$-10(37)$

consistent with the large negative scattering length a_0 of a bound state ($k = i\kappa_{\rm pole}$).

- This result is consistent with the extrapolated a_0 using the existing lattice results.

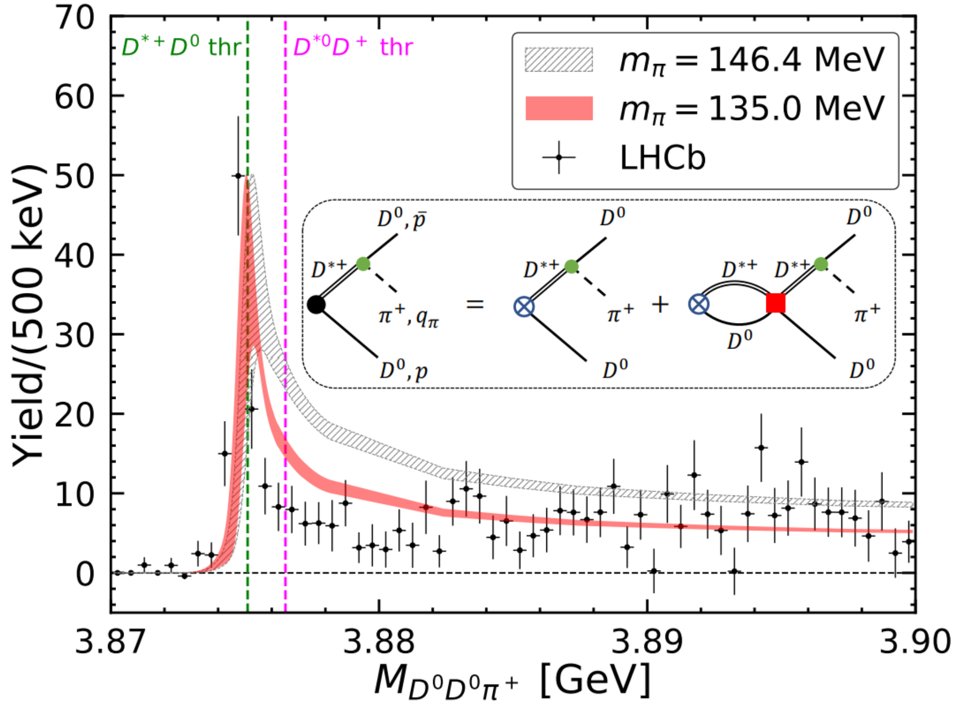




D: Y. Ikeda et al. (HALQCD) Phys. Lett. B 729 (2014) 84-90

- Fit to the $D^0 D^0 \pi^+$ mass spectrum of LHCb experimental data

✓ The gray band: the theoretical obtained by using $V^B_{fit}(r; m_\pi)$ at $m_\pi = 146.4$ MeV
✓ The red band: $D^0 D^0 \pi^+$ mass spectrum obtained by chiral extrapolated $V^B_{fit}(r; m_\pi)$ at $m_\pi = 135.0$ MeV
✓ Consistent with the trend of evolution from a near-threshold virtual state into a loosely bound state.



To summarize,

✓ The existing lattice results of $T^+_{cc}(3875)$ relevant studies are consistent with each other;
✓ These results support the existence of a DD^* bound state in the $I = 0$ channel.
✓ The interaction potential study (C) suggests that the two-pion exchange dominates the long range interaction, while study (B) supports the charged-ρ exchange that provides an attractive interaction for $I = 0$ DD^* system near the threshold, as expected by phenomenological studies.

2. Doubly bottomed counterpart of $T_{cc}^{+}(3875)$

A. BB potential and $\bar{b}\bar{b}ud$ $\left(I(J^P) = 0(0^+)\right)$ tetraquark bound states using lattice QCD

- Static anti-heavy quarks
- The $r_{\bar{b}\bar{b}}$ dependence of the BB system defines the potential.
- The Schrödinger equation is solved to give the binding energy.
- A bound state exists in the $I(J^P) = 0((0,1)^+)$ channel

$$E_B = -90^{+43}_{-36}\ \mathrm{MeV}$$

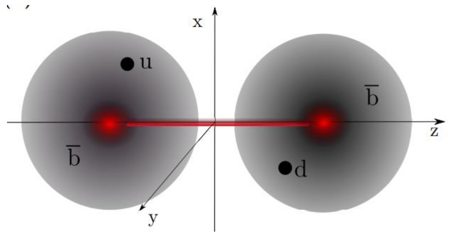
and no binding in the $I(J^P) = 1(1^+)$ channel.

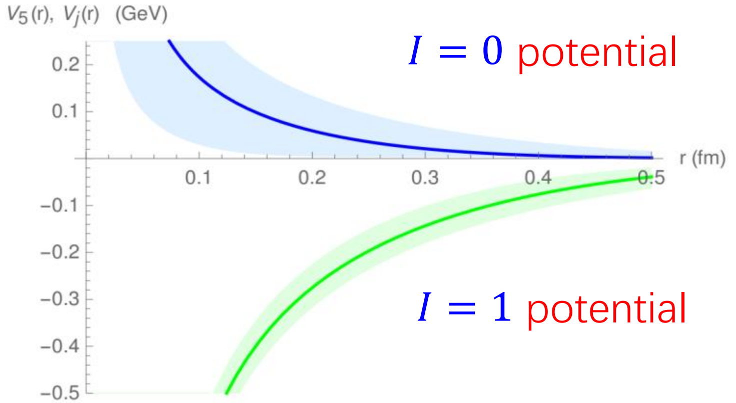
(P. Bicudo et al. Phys. Rev. D 93 (2016) 034507)

- A bound state exists in the $I(J^P) = 0(1^+)$ DD^* and D^*D^* coupled channel

$$E_B = -59^{+30}_{-38}\ \mathrm{MeV}$$

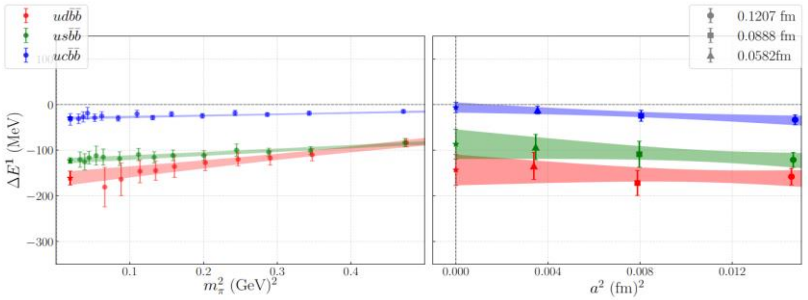
(P. Bicudo et al. Phys. Rev. D 95 (2017) 034502)





B. $\bar{b}\bar{b}ud$ $\left(I(J^P) = 0(0^+)\right)$ tetraquark bound state hinted by negative binding energy

- Chiral extrapolation
- Continuum extrapolation



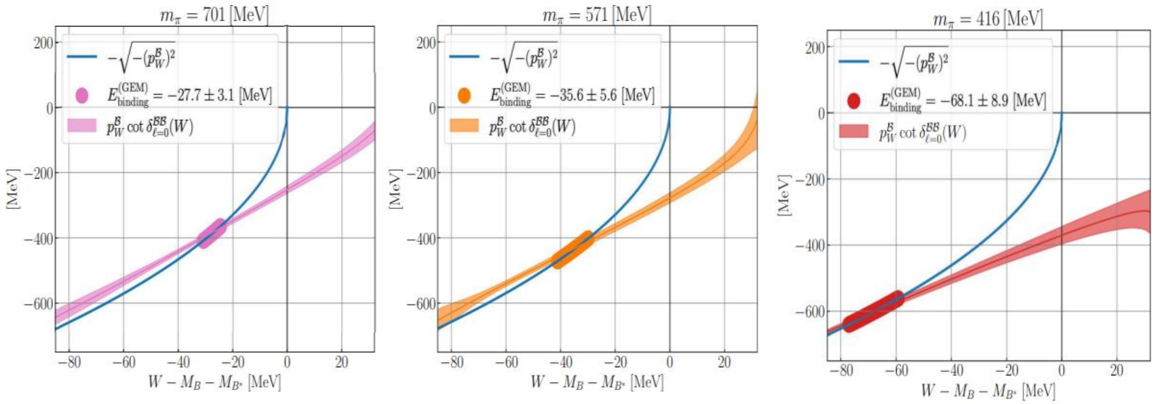
P. Junnarkar et al.,
Phys. Rev. D 99, 034507 (2019)

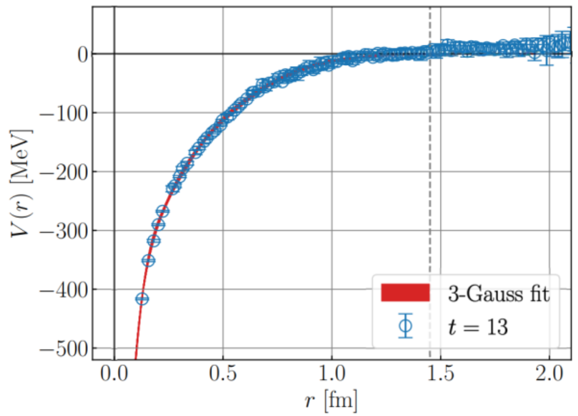
2. Doubly bottomed counterpart of $T_{cc}^{+}(3875)$

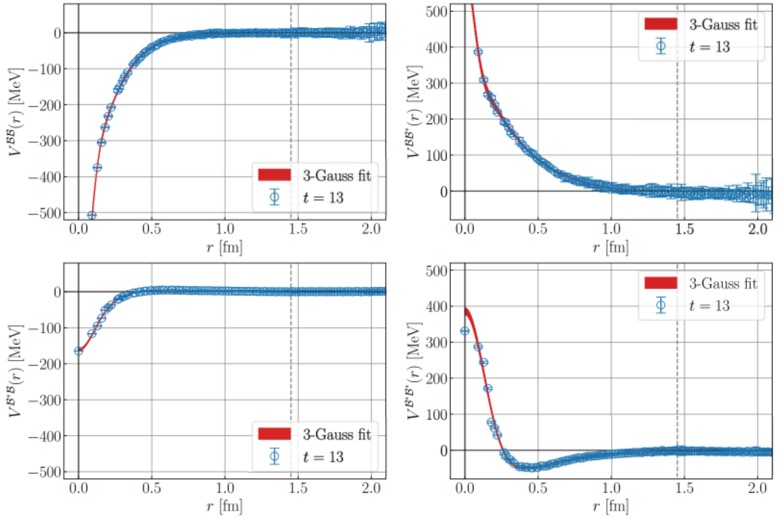
C. $BB^* - B^*B^*$ coupled channel potential and $\bar{b}\bar{b}ud$ $\left(I(J^P) = 0(1^+)\right)$ — HALQCD formalism

(S. Aoki and T. Aoki, PoS LATTICE2022, 049 (2023))

- Calculate the NBS wave function to derive the potential of $BB^* - B^*B^*$ coupled channel potential.
- Solve the Lippmann-Schwinger equation to get the scattering phase of the BB^* single channel.







- The linear chiral extrapolation of the binding energy in m_π^2 gives

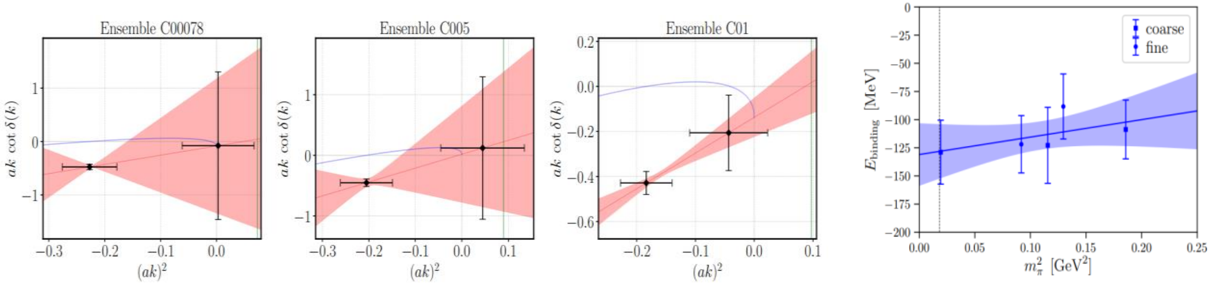
$$E_B^{\rm single} = -154.8 \pm 17.2 \text{ MeV},$$
$$E_B^{\rm couple} = -83.0 \pm 10.2 \text{ MeV}$$

- Corroborate the previous lattice results.

2. Doubly bottomed counterpart of $T_{cc}^{+}(3875)$

D. $\bar{b}\bar{b}qq'$ (1^{+}) systems explored in the Lellouch-Lüscher formalism

- For the $\bar{b}\bar{b}ud$ $(0(1^{+}))$ system, phase shifts $\delta_0(k)$ are calculated at five m_π values.



All the cases give negative E_B, which are extrapolated to the value at the physical m_π:

$$E_B = -128 \pm 24 \pm 10 \text{ MeV}$$

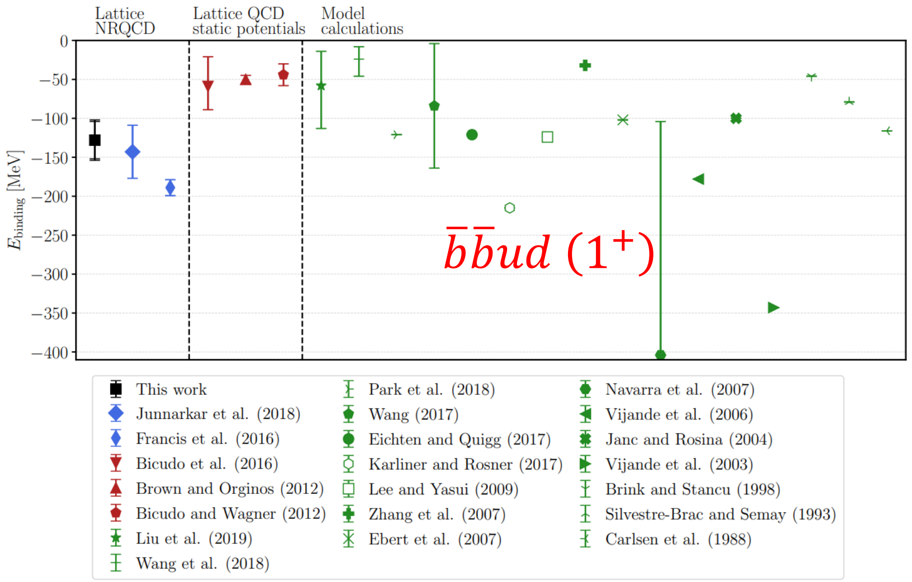
(L. Leskovec et al. Phys. Rev. D 100 (2019) 014503)

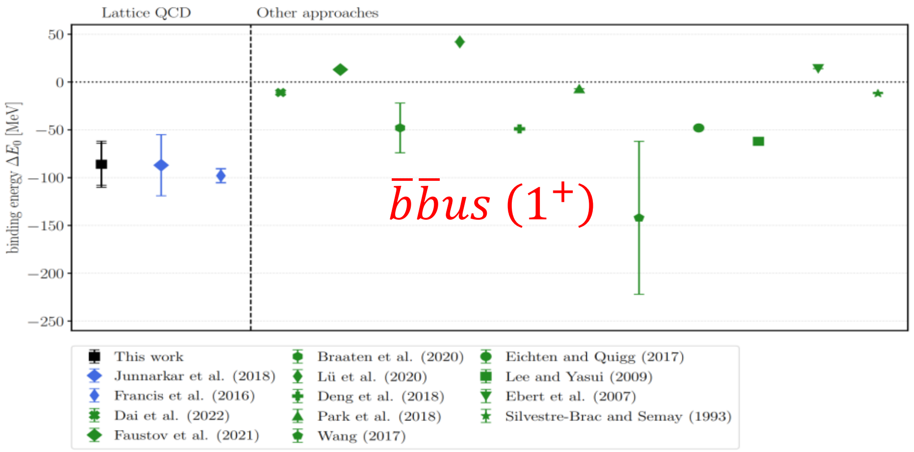
- Clear evidence for a $\bar{b}\bar{b}us$ (1^{+}) tetraquark:

$$E_B = -86 \pm 22 \pm 10 \text{ MeV}$$

but strong discrepancies, even on the qualitive level, between non-lattice results.

(S. Meinel et al. Phys. Rev. D 106 (2022) 034507)



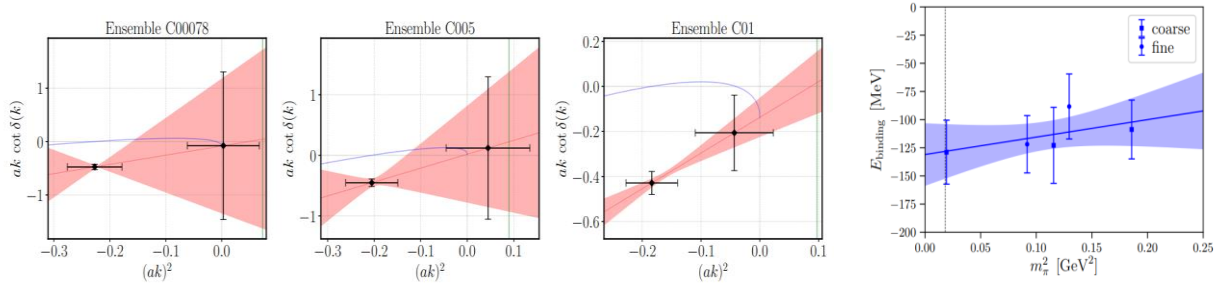


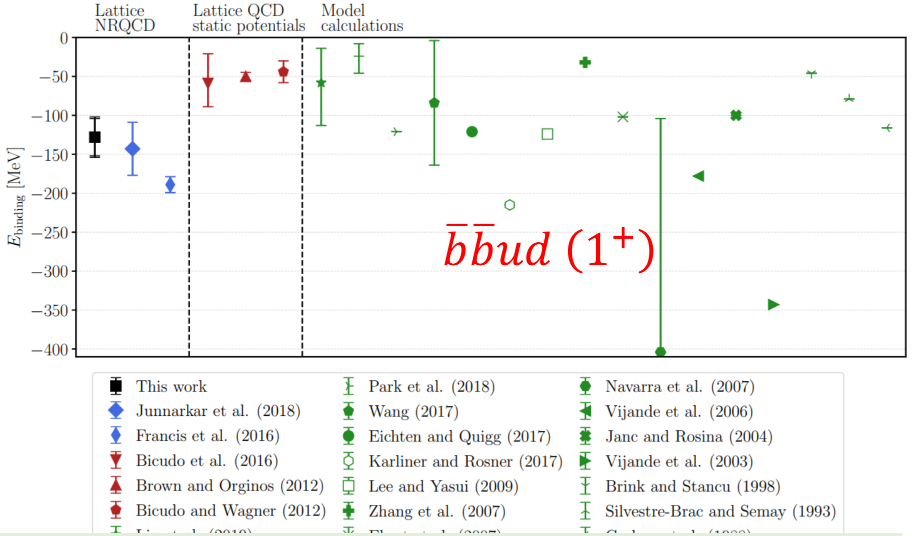
Blue: LQCD, Green: Pheno.

2. Doubly bottomed counterpart of $T_{cc}^{+}(3875)$

D. $\bar{b}\bar{b}qq'$ (1^{+}) systems explored in the Lellouche-Luescher formalism

- For the $\bar{b}\bar{b}ud$ $(0(1^{+}))$ system, phase shifts $\delta_0(k)$ are calculated at five m_π values.





All the cases give negative E_B which are extrapolated t…

To summarize:

✓ All the existing lattice QCD studies indicate the existence of $T_{bb}(0(1^{+}))$

✓ However, the predicted banding energy E_B varies in the range $(-40)-(-130)$ MeV .

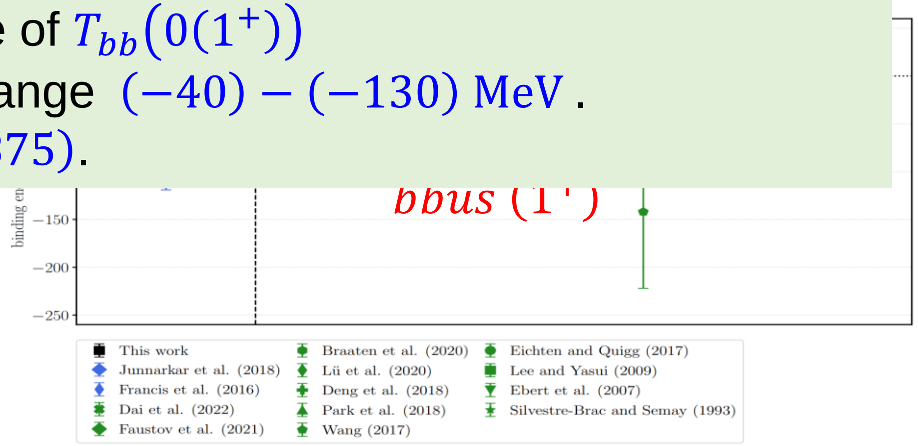
✓ The absolute value $|E_B|$ is quite larger than that of $T_{cc}^{+}(3875)$.

- C…

$$E_B = -86 \pm 22 \pm 10 \text{ MeV}$$

but strong discrepancies, even on the qualitive level, between non-lattice results.

(S. Meinel et al. Phys. Rev. D 106 (2022) 034507)

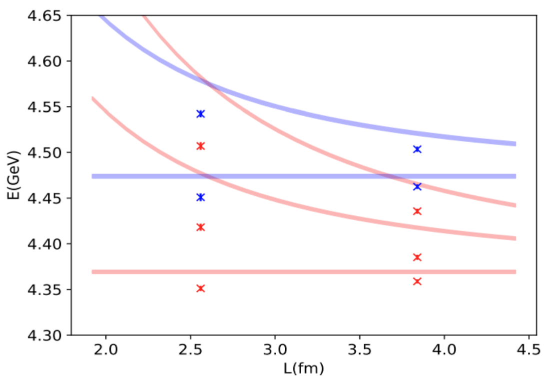


Blue: LQCD, Green: Pheno.

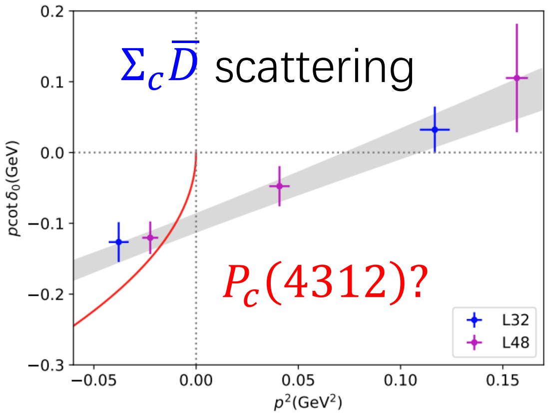
3. P_c states and $\Sigma_c D(D^*)$ scatterings (H. Xing et al., arXiv:2210.08555)

- LHCb observed several P_c states in $J/\psi p$ final state $P_c(4312),\ P_c(4380),\ P_c(4440),\ P_c(4457)$ which must have the minimal quark configuration $uudc\bar{c}$.
- The $J^P = \frac{1}{2}^-$ $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ scatterings are investigated via the Leuscher's method:

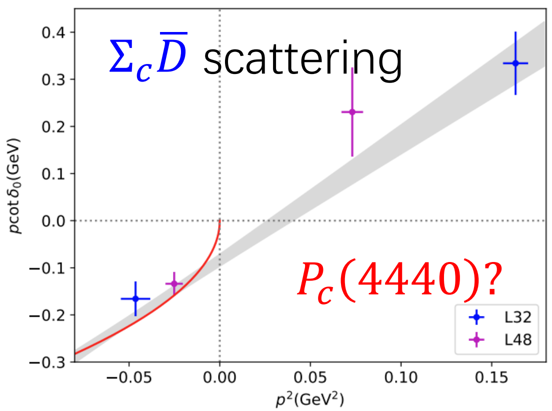
$$p\cot\delta_0(p(E)) = \frac{2}{L\sqrt{\pi}}\mathcal{Z}_{00}(1;q^2(E)) \qquad\qquad p\cot\delta_0(p) = \frac{1}{a_0} + \frac{1}{2}rp^2 \quad (ERE)$$



- ✓ Points: $E_n(L)$ $\Sigma_c\bar{D}$ (red) and $\Sigma_c\bar{D}^*$(blue)
- ✓ Curves: $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ free energies.



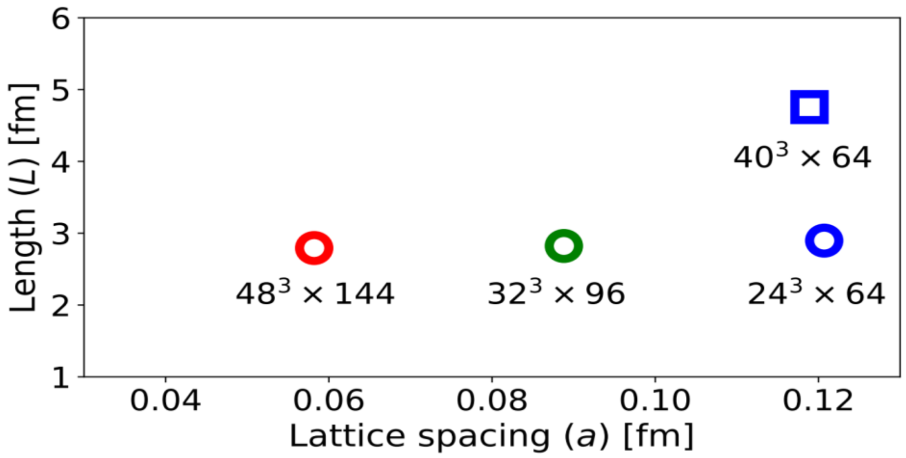
$$\begin{aligned} a_0(\Sigma_c\bar{D}) &= -2.0(3)(5)\mathrm{fm}, \\ r_0(\Sigma_c\bar{D}) &= 0.46(6)(17)\mathrm{fm}, \\ E_B(\Sigma_c\bar{D}) &= 6(2)(2)\mathrm{MeV}, \end{aligned}$$

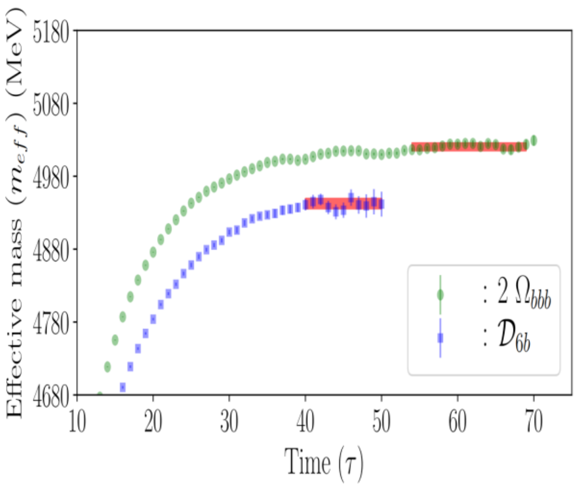


$$\begin{aligned} a_0(\Sigma_c\bar{D}^*) &= -2.3(5)(1)\mathrm{fm}, \\ r_0(\Sigma_c\bar{D}^*) &= 1.01(8)(10)\mathrm{fm}, \\ E_B(\Sigma_c\bar{D}^*) &= 7(3)(1)\mathrm{MeV}, \end{aligned}$$

- Comment: The $J/\psi\, p - \Sigma_c D^{(*)}$ coupled channel effects have not been considered. They can be important, since P_c states are observed in the $J/\psi\, p$ invariant mass spectrum.

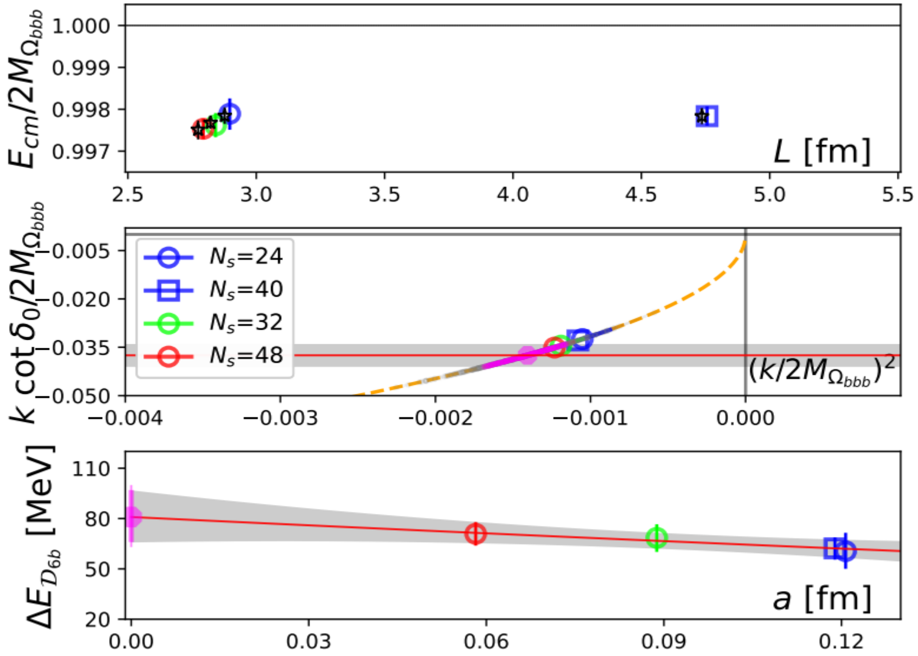
3. Dibaryon $\Omega_{bbb}\Omega_{bbb}$ from lattice QCD (N. Mathur et al., Phys. Rev. Lett. 130 (2023) 111901)





Ensemble	ΔE	Ensemble	ΔE
$24^3 \times 64$	−61(11)	$40^3 \times 64$	−62(7)
$32^3 \times 96$	−68(9)	$48^3 \times 144$	−71(7)

TABLE I. Energy difference $\Delta E = M_{\mathcal{D}_{6b}} - 2M_{\Omega_{bbb}}$ in MeV on different ensembles.



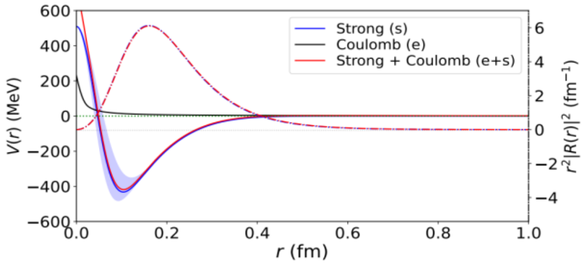
Continuum extrapolation

$$k\cot\delta_0 = -\frac{1}{a_0^{(0)}} + \frac{a}{a_0^{(1)}}$$

$$a_0^{(0)} = 0.18^{+0.02}_{-0.02}\ \text{fm}, \qquad a_0^{(1)} = -0.18^{+0.18}_{-0.11}\ \text{fm}^2$$

$$\Delta E_{\mathcal{D}_{6b}} = -81^{+14}_{-16}\ \text{MeV}$$

$V_s(r)$:Multi-Gaussian attractive potential



Radial distribution

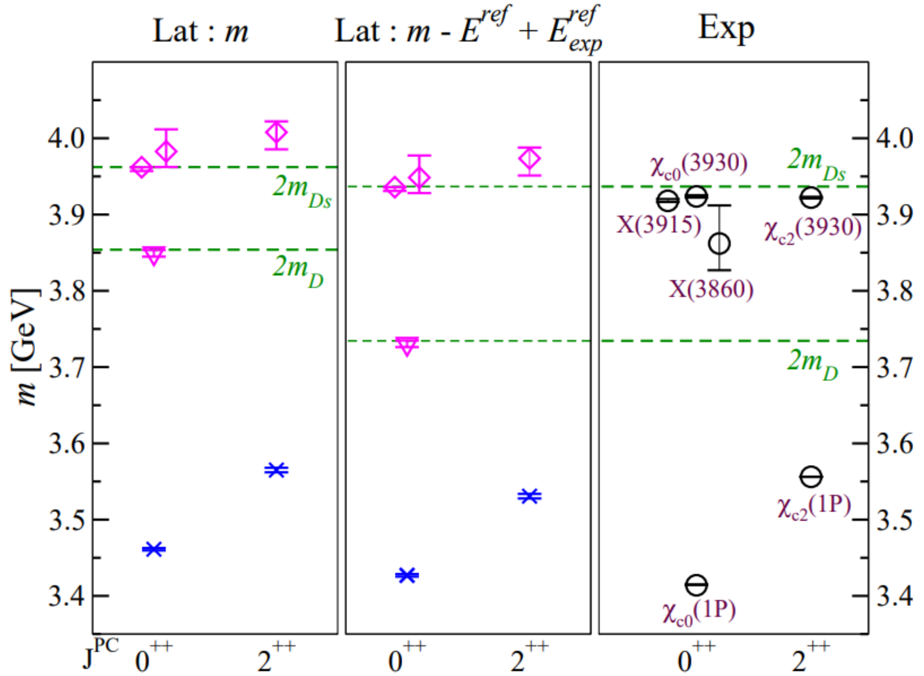
III. Charmonium(like) states and their decays

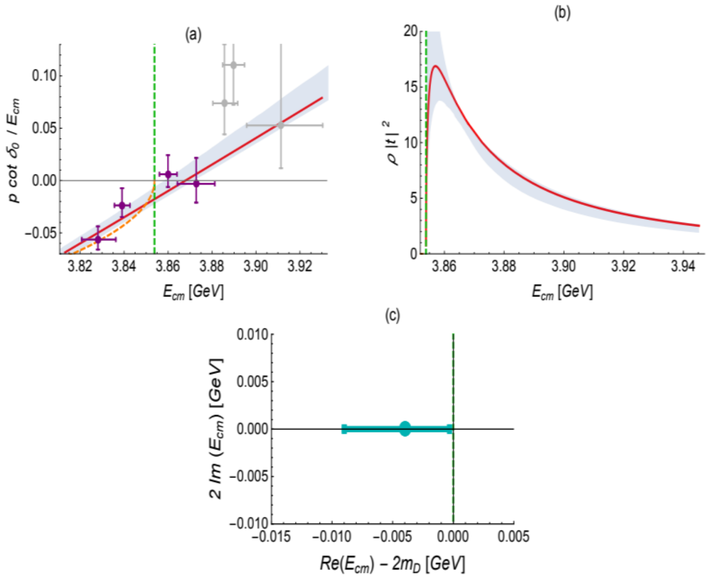
1. $J^{PC} = (0,2)^{++}$ charmoniumlike resonances in coupled $D\bar{D}$ and $D_s\bar{D}_s$ scattering

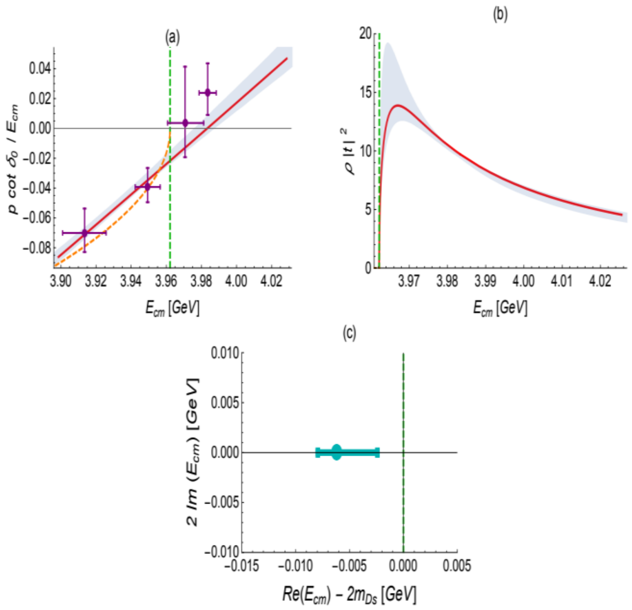
(S. Prelovsek et al., JHEP 06 (2021) 035)

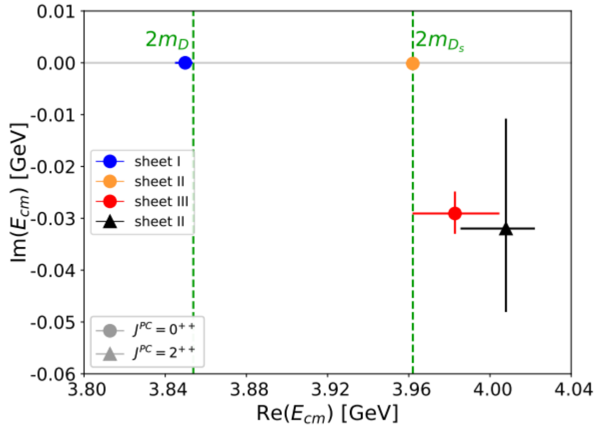
- Relevant to $X(3860)$, $X(3930)$ and $X(3915)$, which are near $D\bar{D}$ and $D_s\bar{D}_s$ thresholds.
- The operator set includes $\bar{c}c$ operators and ($D\bar{D}$, $D_s\bar{D}_s$) operators with different relative momenta.
- Lellouch-Luescher formalism is implemented.

✓ A 0^{++} shallow bound state ($E_B \sim -4$ MeV) is observed right below the $D\bar{D}$ threshold.
✓ A narrow resonance appears just below the $D_s\bar{D}_s$ threshold, which may have connections with $\chi_{c0}(3930)$ and $X(3915)$
✓ Consistent with the trend of evolution from a near-threshold virtual state into a loosely bound state.
✓ The single channel analysis of $L = 2$ $D\bar{D}$ scattering find a 2^{++} resonance, whose properties are consistent with $\chi_{c2}(3930)$.









A 0^{++} shallow bound state in s-wave $D\overline{D}$ scattering

A 0^{++} shallow bound state in s-wave $D_s\overline{D}_s$ scattering

2. Decays of charmoniumlike 1^{-+} hybrid η_{c1} (C. Shi et al., arXiv: 2306.12884 (hep-lat))

- There exist candidates for light 1^{-+} hybrids, such as $\pi_1(1600)$ and $\eta_1(1855)$.
- The charmonium like counterpart η_{c1} of η_1 is expected. Lattice QCD predicts $m_{\eta_{c1}} \sim 4.2 - 4.4$ GeV.
- Two body decay modes of η_{c1}: $D_1\bar{D},\ D^*\bar{D}, D^*\bar{D}^*, \chi_{c1}\eta(\eta'), \eta_c\eta(\eta'), J/\psi\omega(\phi)$
- The first lattice QCD calculation of the partial widths of these decays is presented.

Lattice methodology (C. McNeile & C. Michael, Phys. Lett. B 556 (2003) 177)

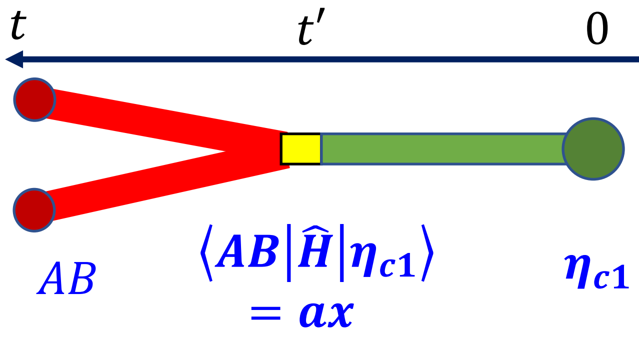
For the two-body decay $\eta_{c1} \to AB$, in the space spanned by $|\eta_{c1}\rangle$ and $|AB\rangle$ $(m_{\eta_{c1}} > E_{AB})$

$$|\eta_{c1}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |AB\rangle = \begin{pmatrix}0\\1\end{pmatrix} \quad \hat{H} = \begin{pmatrix} m_{\eta_{c1}} & x \\ x & E_{AB}\end{pmatrix}$$

$$\Rightarrow \quad \hat{T}(a) = e^{-a\hat{H}} = e^{-a\bar{E}} \begin{pmatrix} e^{-a\Delta/2} & ax \\ ax & e^{a\Delta/2}\end{pmatrix}$$

$$\bar{E} = \frac{m_{\eta_{c1}} + E_{AB}}{2}, \qquad \Delta = m_{\eta_{c1}} - E_{AB}$$

The transition takes place at any t' between 0 and t:



$$\langle \Omega | \mathcal{O}_{AB} | \eta_{c1} \rangle \approx 0 \qquad \langle \Omega | \mathcal{O}_{\eta_{c1}} | AB \rangle \approx 0$$

$$\frac{\mathcal{C}_{\eta_{c1},AB}(t)}{\sqrt{\mathcal{C}_{\eta_{c1}}(t)\mathcal{C}_A(t)\mathcal{C}_B(t)}} \to -ax\,t\left(1 + \frac{1}{24}(a\Delta\, t)^2\right)$$

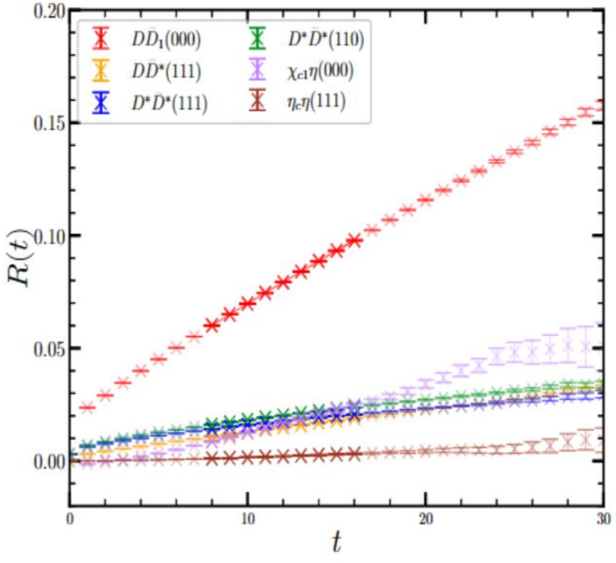
Amplitudes for $\eta_{c1} \to AB$ from the Lagrangian

IE	$N_s^3 \times N_t$	β	a_t^{-1}(GeV)	ξ	m_π(MeV)	N_V	$N_{\rm cfg}$
L16	$16^3 \times 128$	2.0	6.894(51)	~ 5.3	~ 350	70	708
L24	$24^3 \times 192$	2.0	6.894(51)	~ 5.3	~ 350	160	171

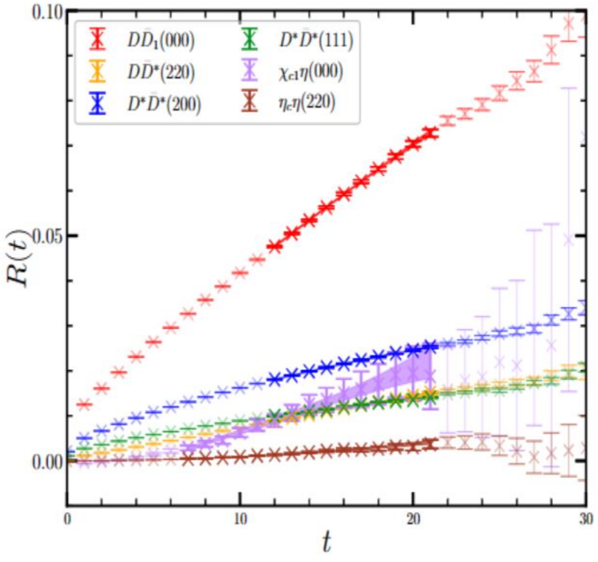
$$
\begin{aligned}
x^{\lambda'\lambda}_{AP} &= g_{AP} m_{\eta_{c1}} \vec{\epsilon}_\lambda(\vec{0}) \cdot \vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}), \\
x^{\lambda}_{PP} &= 2 g_{PP} \vec{\epsilon}_\lambda(\vec{0}) \cdot \vec{k}, \\
x^{\lambda'\lambda}_{D^*\bar{D}} &= g_{D^*\bar{D}} \vec{\epsilon}_\lambda(\vec{0}) \cdot (\vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}) \times \vec{k}), \\
x^{\lambda'\lambda''\lambda}_{D^*\bar{D}^*} &= 2 g_{D^*\bar{D}^*} \vec{\epsilon}_\lambda(\vec{0}) \cdot \left(\vec{k} \times \left[\vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}) \times \vec{\epsilon}^{\,*}_{\lambda''}(-\vec{k})\right]\right)
\end{aligned}
$$

$$
\frac{\mathcal{C}_{\eta_{c1},AB}(t)}{\sqrt{\mathcal{C}_{\eta_{c1}}(t)\mathcal{C}_A(t)\mathcal{C}_B(t)}} \to -(ax)\, t \left(1 + \frac{1}{24}(a\Delta\, t)^2\right)
$$

Efffective couplings g_{AB} are derived as follows:

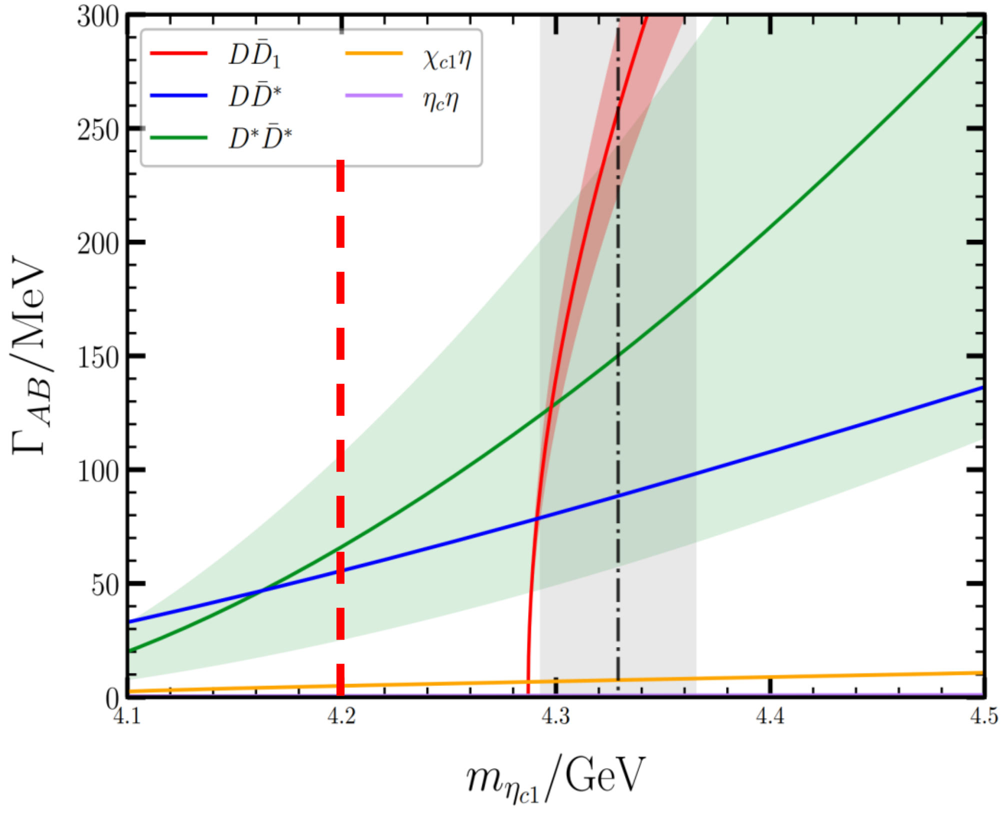


L16



L24

Mode (AB)	$\hat{k}$(IE)	r_1 $(\times 10^{-3})$	g_{AB}	g_{AB} (ave.)	Γ_{AB} (MeV)
$D_1\bar{D}$	(0, 0, 0)(L16)	4.95(5)	4.27(5)	4.6(6)	258(133)
	(0, 0, 0)(L24)	3.10(26)	4.92(41)		
$D^*\bar{D}$	(1, 1, 1)(L16)	1.11(3)	8.35(21)	8.3(7)	88(18)
	(2, 2, 0)(L24)	0.78(7)	8.34(74)		
$D^*\bar{D}^*$	(1, 1, 1)(L16)	1.00(3)	3.44(12)	4.6(1.8)	150(118)
	(1, 1, 0)(L16)	1.15(4)	3.79(12)		
	(2, 0, 0)(L24)	1.05(9)	5.06(42)		
	(1, 1, 1)(L24)	0.67(7)	6.31(58)		
$\chi_{c1}\eta_{(2)}$	(0, 0, 0)(L16)	2.04(26)	1.31(2)	1.35(45)	–
	(0, 0, 0)(L24)	1.18(38)	1.39(45)		
$\eta_c\eta_{(2)}$	(1, 1, 1)(L16)	0.20(6)	0.62(18)	0.55(22)	–
	(2, 2, 0)(L24)	0.10(3)	0.47(12)		



The $m_{\eta_{c1}}$-dependence of partial decay widths

$$\left|D^*\bar{D}^*\right\rangle^{(I=0)}_{(C=+)} = \frac{1}{\sqrt{2}}\left(\left|D^{*+}D^{*-}\right\rangle + \left|D^{0*}\bar{D}^{0*}\right\rangle\right)^{(S=1)}_{(L=1)}$$

$L + S = \text{even}$

- For $m_{\eta_{c1}} = 4329(36)$ MeV, we have

$$\Gamma_{D_1\bar{D}} = 258(133)\ \mathrm{MeV}$$
$$\Gamma_{D^*\bar{D}^*} = 150(118)\ \mathrm{MeV}$$
$$\Gamma_{D^*\bar{D}^*} = 88(18)\ \mathrm{MeV}$$

$$\Gamma_{\chi_{c1}\eta} = \sin^2\theta \cdot 44(29)\ \mathrm{MeV}$$
$$\Gamma_{\eta_c\eta'} = \cos^2\theta \cdot 0.93(77)\ \mathrm{MeV}$$

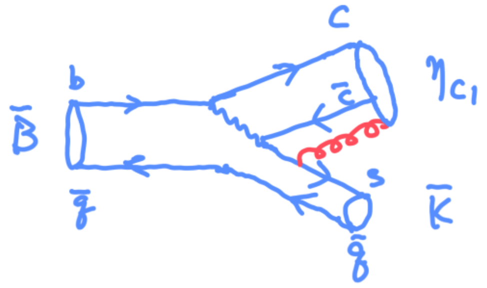
- Given the mass above, η_{c1} seems too wide to be identified easily in experiments.

- However, $\Gamma_{\eta_{c1}}$ is very sensitive to $m_{\eta_{c1}}$.
- If $m_{\eta_{c1}} \sim 4.2$ GeV, then $\Gamma_{\eta_{c1}} \sim 100$ MeV. The dominant decay channels are $D^*\bar{D}$ and $D^*\bar{D}^*$.
- Especially for $D^*\bar{D}^*$, the measurement of the polarization of D^* and $\bar{D}^*$ will help distinguish a 1^{-+} states from 1^{--} states.

- It is suggested that LHCb, BelleII and BESIII to search for η_{c1} in $D^*\bar{D}$ and $D^*\bar{D}^*$ systems !

- η_{c1} production on e^+e^- collider $\quad e^+e^- \to \psi(nS) \to \gamma\eta_{c1} \quad (\psi(4415)\ etc.)$

- η_{c1} production in B meson decays (LHCb and Belle II)



$$B \to \bar{K}X, \qquad X = X(3872), Z_c(4430), Z_c(3900), \qquad etc.$$

- η_{c1} decay modes

Flux-tube model selection rules:
1) Modes of two S-wave mesons are suppressed, SP-modes are favored.
2) Modes of two identical mesons are prohibited.

$$\langle AB|H_I|H\rangle \propto \int d^3\vec{r}\ (\phi_H(\vec{r})\cdots)\int_0^1 d\xi\cos(\xi\pi)\ \phi_A(\xi\vec{r})\phi_B\big((1-\xi)\vec{r}\big)$$

(P. Page et al., Phys. Rev. D 59 (1999) 034016)

But these rules for η_{c1} decys are not supported by the lattice calculation.

V. Summary

- Lattice QCD makes a rapid progress in the study of heavy flavor spectroscopy.
- Multiquark states are hot topics of lattice QCD studies.
- The existing lattice QCD results relevant to $T_{cc}^{+}(3875)$ are consistent with each other and support the existence of a shallow $DD^{*}(I=0)$ bound state.
- Similar studies are extended to the beauty counterpart T_{bb} of T_{cc}, and suggest the existence of a (deeply) bound $I(J^P)=0(1^+)\ BB^*$ state.
- A deeply bound dibaryon $\Omega_{bbb}\Omega_{bbb}$ is predicted.
- There are also developments in the study of charmoniumlike resonance.
- The decay properties of charmoniumlike hybrid η_{c1} are predicted by lattice QCD.
- More interesting works is underway.

Thank you for your attention!