Quantum Mechanics under Rotation

Pengfei Zhuang, Tsinghua University

I will discuss Quantum Mechanics under rotation in coordinate space and phase space, and the application to quarkonia.

Introduction

Phase transitions are triggered by external fields.

High temperature and baryon density created in heavy ion collisions:

$T/T_c \sim 2$ *at RHIC and 3-4 at LHC,*

$n_B/n_0 \sim 3$ *at FAIR, NICA and HIAF,*

Strong electromagnetic and rotational fields created in heavy ion collisions:

$|eB|/m_\pi^2 \sim 5$ *at RHIC and* 70 *at LHC,*

$\omega \sim 10^{21}/s$ *at RHIC,*

What we should consider:

1) The external fields $(T, \mu_B, \vec{E}, \vec{B}, \vec{\omega})$ *here are all comparable with the QCD interaction, we must consider their effect on QCD.*

2) When the external fields are extremely strong, Quantum Mechanics works for elementary particles, especially for heavy particles.

Equations of Motion

A system under a rotational field $\vec{\omega}$ can be equivalently regarded as a system at rest in a rotating frame.

See, for instance, Jiang and Liao, PRL117, 192302(2016);
Chen, Fukushima and Huang, PRD93, 104052(2016);
Liu, Gao, Mameda and Huang, PRD99, 085014(2019)

Lagrangian density in the rotating frame:

$$\mathcal{L} = \vec{\bar{\psi}}\left(i\gamma^\mu\partial_\mu - m + \gamma_0\vec{\omega}\cdot\vec{j}\right)\psi$$

$$\vec{j} = \vec{l} + \vec{s}, \qquad \vec{l} = \vec{x}\times\vec{p}$$

Dirac equation:

$$\left(i\gamma^\mu\partial_\mu - m + \gamma_0\vec{\omega}\cdot\hat{\vec{j}}\right)\psi = 0$$

Schrodinger equation:

$$i\frac{\partial}{\partial t}\psi = \hat{H}\psi, \qquad \hat{H} = \frac{\hat{\vec{p}}^2}{2m} - \vec{\omega}\cdot\hat{\vec{j}}$$

Heisenberg equations:

$$\frac{d\hat{\vec{x}}}{dt} = \frac{\hat{\vec{p}}}{m} - \vec{\omega}\times\hat{\vec{x}}, \qquad \frac{d\hat{\vec{p}}}{dt} = -\vec{\omega}\times\hat{\vec{p}}$$

$$m\frac{d^2\hat{\vec{x}}}{dt^2} = -2\vec{\omega}\times\hat{\vec{p}} + m\vec{\omega}\times\left(\vec{\omega}\times\hat{\vec{x}}\right)$$

Coriolis force, Centrifugal force

Effective Gauge Potential

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} - \vec{\omega}\cdot\left(\hat{\vec{x}}\times\hat{\vec{p}} + \vec{s}\right) = \frac{\left(\hat{\vec{p}} - m\vec{\omega}\times\hat{\vec{x}}\right)^2}{2m} - \frac{m}{2}\left(\vec{\omega}\times\hat{\vec{x}}\right)^2 - \vec{\omega}\cdot\vec{s}$$

in comparison with electromagnetic field: $\hat{H} = \frac{\left(\hat{\vec{p}}-q\vec{A}\right)^2}{2m} + qA_0 - \frac{q}{2m}\vec{B}\cdot\vec{s}$

Effective gauge potential:

$$\mathcal{A}_\mu = \left(\mathcal{A}_0, \vec{\mathcal{A}}\right) = \left(-\frac{1}{2}\left(\vec{\omega}\times\hat{\vec{x}}\right)^2, \vec{\omega}\times\hat{\vec{x}}\right)$$

Effective gauge field:

$$\vec{\omega} = \frac{1}{2}\vec{\nabla}\times\vec{\mathcal{A}}$$

Effective charge:

$$m$$

Including Electromagnetic Field

$$\partial_\mu \to \partial_\mu + iqA_\mu \quad \left(\vec{p} \to \vec{p} - q\vec{A}\right)$$

Dirac equation:

$$\left[i\gamma^\mu\left(\partial_\mu + iqA_\mu\right) - m + \gamma_0\vec{\omega}\cdot\left(\vec{x}\times\left(\vec{p} - q\vec{A}\right) + \vec{s}\right)\right]\psi = 0$$

Schrodinger equation:

$$i\frac{\partial}{\partial t}\psi = \hat{H}\psi,$$

$$\hat{H} = \frac{(\hat{\vec{p}} - q\vec{A})^2}{2m} - \vec{\mu}\cdot\vec{B} + qA_0 - \vec{\omega}\cdot\hat{\vec{j}} + q\vec{\omega}\cdot\left(\vec{x}\times\vec{A}\right)$$

$$\vec{\mu} = \frac{q}{m}\vec{s}$$

mixing term

Two-body Systems in $\vec{B}$ Field

For a two-body system with charges $q_1 = -q_2 = q,\ m_1 = m_2 = m,$

$$\hat{H}_B = \frac{\hat{\vec{P}}_{ps}^2}{4m} + \frac{\hat{\vec{p}}^2}{m} + V - \vec{\mu}\cdot\vec{B} + \frac{-2q\left(\hat{\vec{P}}_{ps}\times\vec{B}\right)\cdot\vec{r} + q^2\left(\vec{B}\times\vec{r}\right)^2}{4m}$$

two-body interaction: $V(r, s_1, s_2) = V_c(r) + V_s(r)\vec{s}_1\cdot\vec{s}_2$

magnetic moment: $\vec{\mu} = \frac{q}{m}(\vec{s}_1 - \vec{s}_2)$

pseudo-momentum: $\hat{\vec{P}}_{ps} = \hat{\vec{P}} + \frac{q}{2}\vec{B}\times\vec{r},\quad kinetic\ momentum:\ \hat{\vec{P}}_{kin} = \hat{\vec{P}} - \frac{q}{2}\vec{B}\times\vec{r}$

$$\left[\hat{\vec{P}}, \hat{H}_B\right] \neq 0, \qquad \left[\hat{\vec{P}}_{kin}, \hat{H}_B\right] \neq 0, \qquad \left[\hat{\vec{P}}_{ps}, \hat{H}_B\right] = 0$$

Only the pseudo-momentum $\hat{\vec{P}}_{ps}$ is conserved in a magnetic field!

$$\Psi(\vec{R}, \vec{r}) = e^{i\vec{P}\cdot\vec{R}}\psi(\vec{r}) = e^{i(\hat{\vec{P}}_{ps} - \frac{q}{2}\vec{B}\times\vec{r})\cdot\vec{R}}\psi(\vec{r})$$

$$\left[\frac{\hat{\vec{p}}^2}{m} + V - \vec{\mu}\cdot\vec{B} + \frac{-2q\left(\hat{\vec{P}}_{ps}\times\vec{B}\right)\cdot\vec{r} + q^2\left(\vec{B}\times\vec{r}\right)^2}{4m}\right]\psi(\vec{r}) = \left(E - 2m - \frac{\hat{\vec{P}}_{ps}^2}{4m}\right)\psi(\vec{r})$$

Two-body Systems in $\vec{\omega}$ and $\vec{B}$ Fields

$$\hat{H} = \hat{H}_B + \hat{H}_\omega$$

$$\hat{H}_\omega = -\vec{\omega}\cdot\left(\vec{L}_{ps} + \vec{l} + \vec{s}\right) + \frac{1}{2}q\vec{\omega}\cdot\left(\vec{R}\times\left(\vec{B}\times\vec{r}\right)\right)$$

$$\vec{L}_{ps} = \vec{R}\times\vec{P}_{ps}$$

$mixing\ terms$

There is no more conserved momentum, and there is no way to separate the two-body motion into a center-of-mass motion and a relative motion.

Considering in HIC

$$m\omega \sim m_\pi^2 \ll |eB| \sim (10-70)m_\pi^2$$

we can take $\hat{H}_\omega$ *as a perturbation,*

$$\varepsilon_n = \varepsilon_n^{(0)} + \left\langle\psi_n^{(0)}\middle|\hat{H}'\middle|\psi_n^{(0)}\right\rangle,$$

$$\psi_n = \psi_n^{(0)} + \sum_{m\neq n}\frac{\left\langle\psi_m^{(0)}\middle|\hat{H}'\middle|\psi_n^{(0)}\right\rangle}{\varepsilon_m^{(0)} - \varepsilon_n^{(0)}}\psi_m^{(0)}$$

Quarkonia in External T, $\vec{\omega}$ and $\vec{B}$ Fields

Why heavy quarks ?

1) *Suitable for the study in the frame of Quantum Mechanics,*
2) *Large rotational charge* $m \rightarrow strong\ rotational\ effect,$
3) *Created in the initial stage of HIC where* $\vec{\omega}$ *and* $\vec{B}$ *are strong.*

two-body interaction

$$V(r, s_1, s_2) = V_c(r) + V_s(r)\vec{s}_1 \cdot \vec{s}_2$$

$strong\ interaction\ at\ finitetemperature$

$$V_c(r,T) = -\frac{\alpha}{r}e^{-m_D r} + \frac{\sigma}{m_D}\left[\frac{\Gamma\left(\frac{1}{4}\right)}{2^{\frac{3}{2}}\Gamma\left(\frac{3}{4}\right)} - \frac{\sqrt{m_D r}}{2^{\frac{3}{4}}\Gamma\left(\frac{3}{4}\right)} K_{\frac{1}{4}}\left(m_d^2 r^2\right)\right] - \alpha m_D$$

$$V_c(r,0) = -\frac{\alpha}{r} + \sigma r$$

$screening\ mass\ m_D(T)\ can\ be\ extracted\ from\ lattice\ QCD\ simulations$

$spin\ interaction$

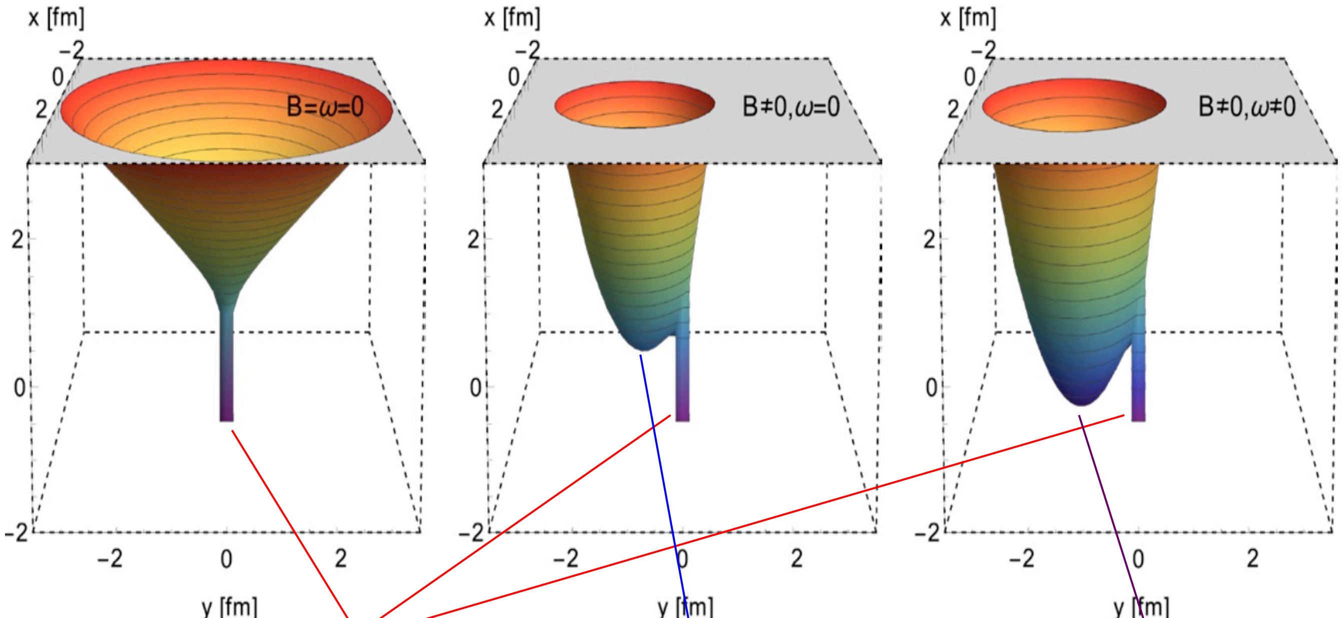
$$V_s(r) = e^{-\beta r}$$

Electromagnetical Bound State

Well-known result: Quarkonia are suppressed by external T field.

What happens when including $\vec{\omega}$ and $\vec{B}$ fields ?

Heavy Quark Potential at finite $\vec{\omega}$ and $\vec{B}$



Strong interaction potential well

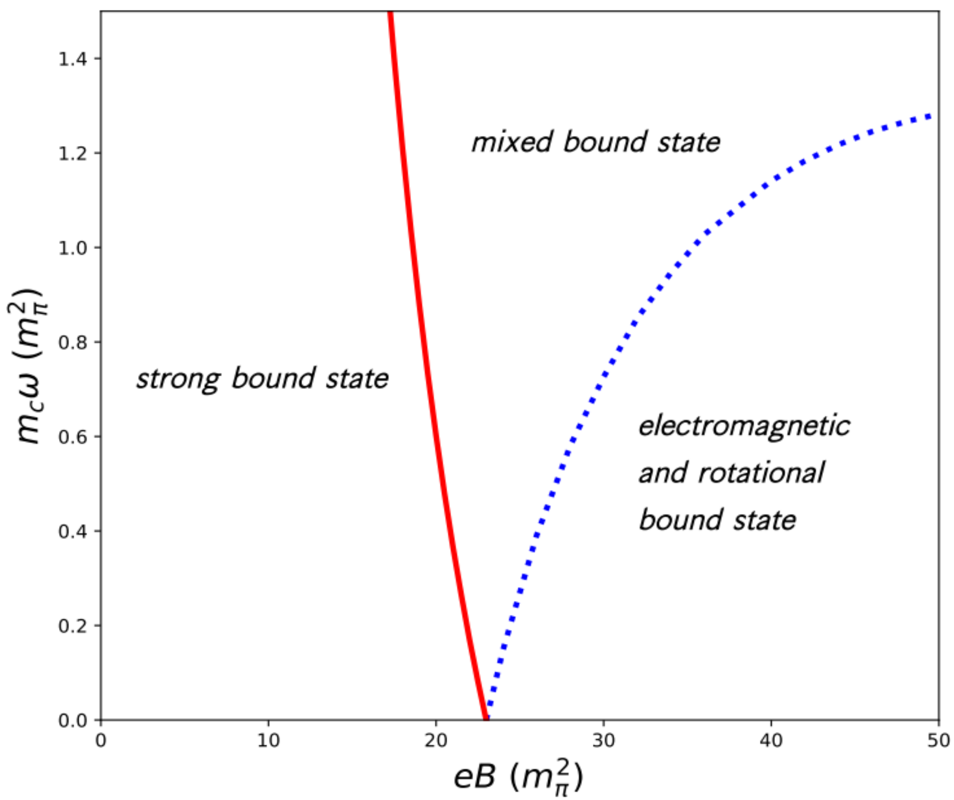
$\vec{B}$ induced potential well

$\vec{B}$ and $\vec{\omega}$ induced potential well

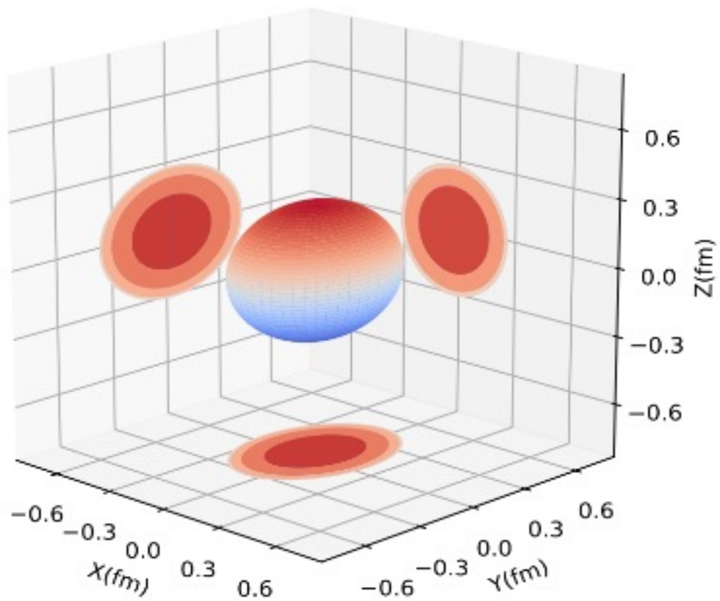
Charmonium Phase Diagram

Phase transition line is determined by the binding energy

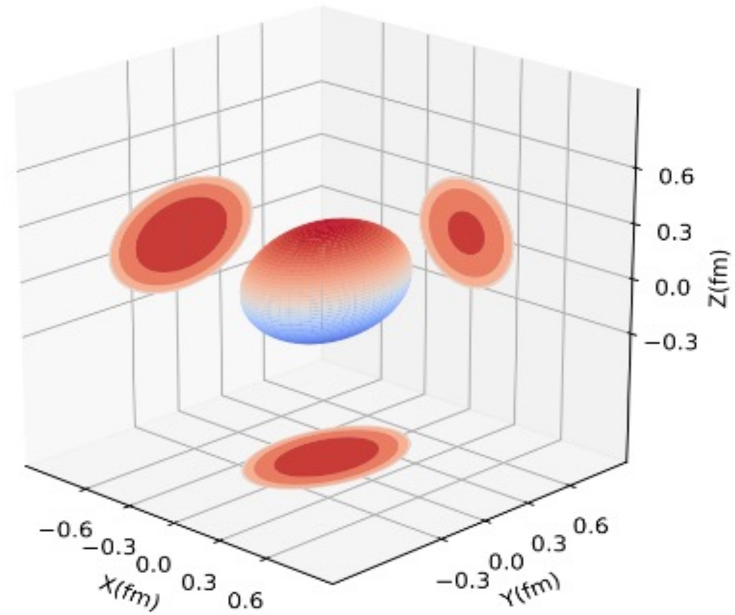
$$\varepsilon(eB, m\omega) = 0$$



Lorentz Force and Coriolis Force Induced Shape Change



$\vec{B}$ induced shape change



$\vec{B}$ and $\vec{\omega}$ induced shape change

Covariant Kinetic Theory

From Quantum Mechanics to Quantum Kinetic Theory

Dirac equation:

$$\left(i\gamma^\mu\partial_\mu - m + \gamma_0\vec{\omega}\cdot\hat{\vec{j}}\right)\psi(x) = 0$$

Covariant Wigner function:

$$W(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{ip\cdot y}\left\langle \psi(x+\frac{y}{2})\bar{\psi}(x-\frac{y}{2})\right\rangle$$

Covariant kinetic equation:

$$\left(\gamma^\mu K_\mu + \frac{\hbar}{2}\gamma^5\gamma^\mu\omega_\mu - m\right)W(x,p) = 0$$

$$K_\mu = \Pi_\mu + \frac{i\hbar}{2}D_\mu, \qquad \omega_\mu = (0,\vec{\omega})$$

$$\Pi_\mu = (p_0+\pi_0,\vec{p}), \qquad \pi_0 = \vec{\omega}\cdot\left(\vec{l}+\frac{\hbar^2}{4}\vec{\nabla}\times\vec{\nabla}_p\right)+\mu_B$$

$$D_\mu = \left(d_t,\vec{\nabla}\right), \qquad d_t = \partial_t - \vec{\omega}\cdot\left(\vec{x}\times\vec{\nabla}+\vec{p}\times\vec{\nabla}_p\right)$$

Spin Decomposition

16 spin components:

$$W(x,p) = \frac{1}{4}\left[F(x,p) + i\gamma_5 P(x,p) + \gamma^\mu V_\mu(x,p) + \gamma^\mu \gamma_5 A_\mu(x,p) + \frac{1}{2}\sigma^{\mu\nu} S_{\mu\nu}(x,p)\right]$$

16 transport equations + 16 constraint equations:

$$2\Pi^\mu V_\mu + \hbar\omega^\mu A_\mu = 2mF,$$
$$\hbar D^\mu A_\mu = 2mP,$$
$$4\Pi_\mu F - 2\hbar D^\nu S_{\nu\mu} - \hbar\epsilon_{\mu\nu\alpha\beta}\omega^\nu S^{\alpha\beta} = 4mV_\mu,$$
$$-\hbar D_\mu P + \epsilon_{\mu\nu\alpha\beta}\Pi^\nu S^{\alpha\beta} - \hbar\omega_\mu F = 2mA_\mu,$$
$$\hbar(D_\mu V_\nu - D_\nu V_\mu) + 2\epsilon_{\mu\nu\alpha\beta}\Pi^\alpha A^\beta + \hbar\epsilon_{\mu\nu\alpha\beta}\omega^\alpha V^\beta = 2mS_{\mu\nu}$$
$$\hbar D^\mu V_\mu = 0,$$
$$2\Pi^\mu A_\mu + \hbar\omega^\mu V_\mu = 0,$$
$$\hbar D_\mu F + 2\Pi^\nu S_{\nu\mu} - \hbar\omega_\mu P = 0,$$
$$4\Pi_\mu P + \hbar\epsilon_{\mu\nu\alpha\beta}D^\nu S^{\alpha\beta} + 2\hbar\omega^\nu S_{\mu\nu} = 0,$$
$$2(\Pi_\mu V_\nu - \Pi_\nu V_\mu) - \hbar\epsilon_{\mu\nu\alpha\beta}D^\alpha A^\beta + \hbar(\omega_\mu A_\nu - \omega_\nu A_\mu) = 0.$$

Equal-time Kinetic Theory

To compare with experimental measurement, we should go from Covariant (8-dimensional) to Equal-time (7-dimensional) Kinetic Theory

$$W(x,p) \to W(x,\vec{p})$$

Equal-time Wigner functions:

$$W_n(x,\vec{p}) = \int dp_0 p_0^n W(x,p)\gamma_0$$

16 spin components:

$$W_0(x,\vec{p}) = \frac{1}{4}[f_0(x,\vec{p}) + \gamma_5 f_1(x,\vec{p}) - i\gamma_0\gamma_5 f_2(x,\vec{p}) + \gamma_0 f_3(x,\vec{p}) + \gamma_5\gamma_0\vec{\gamma}\cdot\vec{g}_0(x,\vec{p}) + \gamma_0\vec{\gamma}\cdot\vec{g}_1(x,\vec{p}) - i\vec{\gamma}\cdot\vec{g}_2(x,\vec{p}) - \gamma_5\vec{\gamma}\cdot\vec{g}_3(x,\vec{p})]$$

16 transport equations + 16 constraint equations:

$$\hbar(d_t f_0 + \nabla\cdot\mathbf{g}_1) = 0,$$
$$\hbar(d_t f_1 + \nabla\cdot\mathbf{g}_0) = -2m f_2,$$
$$\hbar d_t f_2 + 2\boldsymbol{p}\cdot\mathbf{g}_3 = 2m f_1,$$
$$\hbar d_t f_3 - 2\boldsymbol{p}\cdot\mathbf{g}_2 = 0,$$
$$\hbar(d_t\mathbf{g}_0 + \nabla f_1) - 2\boldsymbol{p}\times\mathbf{g}_1 + \hbar\boldsymbol{\omega}\times\mathbf{g}_0 = 0,$$
$$\hbar(d_t\mathbf{g}_1 + \nabla f_0) - 2\boldsymbol{p}\times\mathbf{g}_0 + \hbar\boldsymbol{\omega}\times\mathbf{g}_1 = -2m\mathbf{g}_2,$$
$$\hbar(d_t\mathbf{g}_2 + \nabla\times\mathbf{g}_3) + 2\boldsymbol{p} f_3 + \hbar\boldsymbol{\omega}\times\mathbf{g}_2 = 2m\mathbf{g}_1,$$
$$\hbar(d_t\mathbf{g}_3 - \nabla\times\mathbf{g}_2) - 2\boldsymbol{p} f_2 + \hbar\boldsymbol{\omega}\times\mathbf{g}_3 = 0,$$

$$2\int \mathrm{d}p_0 p_0 F = \hbar\nabla\cdot\mathbf{g}_2 - 2\pi_0 f_3 + 2m f_0 - \hbar\boldsymbol{\omega}\cdot\mathbf{g}_3,$$
$$2\int \mathrm{d}p_0 p_0 P = -\hbar\nabla\cdot\mathbf{g}_3 - 2\pi_0 f_2 - \hbar\boldsymbol{\omega}\cdot\mathbf{g}_2,$$
$$2\int \mathrm{d}p_0 p_0 V_0 = 2\boldsymbol{p}\cdot\mathbf{g}_1 - 2\pi_0 f_0 + 2m f_3 - \hbar\boldsymbol{\omega}\cdot\mathbf{g}_0,$$
$$2\int \mathrm{d}p_0 p_0 A_0 = -2\boldsymbol{p}\cdot\mathbf{g}_0 + 2\pi_0 f_1 + \hbar\boldsymbol{\omega}\cdot\mathbf{g}_1,$$
$$2\int \mathrm{d}p_0 p_0 \mathbf{V} = \hbar\nabla\times\mathbf{g}_0 - 2\boldsymbol{p} f_0 + 2\pi_0\mathbf{g}_1 - \hbar\boldsymbol{\omega} f_1,$$
$$2\int \mathrm{d}p_0 p_0 \mathbf{A} = -\hbar\nabla\times\mathbf{g}_1 - 2\boldsymbol{p} f_1 + 2\pi_0\mathbf{g}_0 + \hbar\boldsymbol{\omega} f_0 - 2m\mathbf{g}_3,$$
$$2\int \mathrm{d}p_0 p_0 S^{0i}\mathbf{e}_i = \hbar\nabla f_3 - 2\boldsymbol{p}\times\mathbf{g}_3 + 2\pi_0\mathbf{g}_2 + \hbar\boldsymbol{\omega} f_2,$$
$$\int \mathrm{d}p_0 p_0 \epsilon^{ijk} S_{jk}\mathbf{e}_i = \hbar\nabla f_2 - 2\pi_0\mathbf{g}_3 - 2\boldsymbol{p}\times\mathbf{g}_2 - \hbar\boldsymbol{\omega} f_3 + 2m\mathbf{g}_0 .$$

Semi-classical ($\hbar$) Expansion

At order $\hbar^0$, only f_0 (number distribution) and $\vec{g}_0$ (spin distribution) are independent components.

16 transport equations are reduced to

$$\left[\partial_t + \left(\pm\frac{\boldsymbol{p}}{\epsilon_p} + \boldsymbol{x}\times\boldsymbol{\omega}\right)\cdot\nabla - (\boldsymbol{\omega}\times\boldsymbol{p})\cdot\nabla_p\right] f_0^{(0)\pm} = 0,$$

$$\left[\partial_t + \left(\pm\frac{\boldsymbol{p}}{\epsilon_p} + \boldsymbol{x}\times\boldsymbol{\omega}\right)\cdot\nabla - (\boldsymbol{\omega}\times\boldsymbol{p})\cdot\nabla_p\right] \mathbf{g}_0^{(0)\pm} = -\boldsymbol{\omega}\times\mathbf{g}_0^{(0)\pm}.$$

Coriolis force *spin-rotation coupling*

16 constraint equations are reduced to

$$on - shell\ energy \quad E_p^{\pm} = \pm\varepsilon_p - (\vec{\omega}\cdot\vec{l} + \mu_B), \qquad \varepsilon_p = \sqrt{m^2+p^2}$$

$$f_1^{(0)\pm} = \pm\frac{1}{\epsilon_p}\boldsymbol{p}\cdot\mathbf{g}_0^{(0)\pm},$$

$$f_2^{(0)\pm} = 0,$$

$$f_3^{(0)\pm} = \pm\frac{m}{\epsilon_p}f_0^{(0)\pm},$$

$$\mathbf{g}_1^{(0)\pm} = \pm\frac{\boldsymbol{p}}{\epsilon_p}f_0^{(0)\pm},$$

$$\mathbf{g}_2^{(0)\pm} = \frac{1}{m}\boldsymbol{p}\times\mathbf{g}_0^{(0)\pm},$$

$$\mathbf{g}_3^{(0)\pm} = \pm\frac{1}{m\epsilon_p}\left[\epsilon_p^2\mathbf{g}_0^{(0)\pm} - \boldsymbol{p}(\boldsymbol{p}\cdot\mathbf{g}_0^{(0)\pm})\right].$$

One can systematically calculate the higher order contributions.

Quark Spin Matrix

The particle spin of an statistical ensemble is determined by the spin density matrix.

For the ensemble of quarks, the density operator in coordinate space

$$\hat{\rho}_q = \sum_{s,x} f(s,x)|s,x\rangle\langle s,x|$$

f: *quark distribution determined by Schrodinger equation*

and in phase space

$$\hat{\rho}_q = \sum_{s,xy,p} f(s,x,p)e^{-ipy}|s,x_+\rangle\langle s,x_-|, \qquad x_\pm = x \pm y/2$$

$f = f_0 \pm f_1$: *quark distribution determined by kinetic equation*

Density matrix elements

$$\langle s',x'|\hat{\rho}_q|s'',x''\rangle = \sum_{s,xy,p} f(s,x,p)e^{-ipy}\langle s',x'_+|s,x_+\rangle\langle s,x_-|s'',x''_-\rangle$$

$$= \sum_{p} f(s',x',p)\,\delta_{s's''}\delta_{x'x''}$$

Quarkonium Spin Matrix

$q\bar{q}$ state: $|s_1, s_2, x_1, x_2\rangle$

For the ensemble of $q\bar{q}$, the density operator in coordinate space

$$\hat{\rho}_{q\bar{q}} = \sum_{s,x} f_q(s_1, x_1) f_{\bar{q}}(s_2, x_2) |s_1, s_2, x_1, x_2\rangle\langle s_1, s_2, x_1, x_2|$$

and in phase space

$$\hat{\rho}_{q\bar{q}} = \sum_{\substack{s_1, x_1, y_1, p_1 \\ s_2, x_2, y_2, p_2}} f_q(s_1, x_1, p_1) f_{\bar{q}}(s_2, x_2, p_2)\, e^{-ip_1 y_1} e^{-ip_2 y_2} |s_1, s_2, x_1^+, x_2^+\rangle\langle s_1, s_2, x_1^-, x_2^-|$$

$$x_i^{\pm} = x_i \pm y_i/2$$

Spin density matrix in meson space (diagonal elements):

$$\rho_{q\bar{q}}(s, s_z, X, x) = \langle s, s_z, X_+, x_+ | \hat{\rho}_{q\bar{q}} | s, s_z, X_-, x_- \rangle$$

$$\rho_{q\bar{q}}(s, s_z, X, P) = \sum_{x,p} f_q\left(s_1, X + \frac{x}{2}, \frac{P}{2} + p\right) f_{\bar{q}}\left(s_2, X - \frac{x}{2}, \frac{P}{2} - p\right) W(s, s_z, X, x, P, p)$$

Quark Wigner function — Meson Wigner function

$$W(s, s_z, X, x, P, p) = \sum_{y,Y} e^{-iPY} e^{-ipy} \langle s, s_z, X_+, x_+ | s_1, s_2, x_1^+, x_2^+ \rangle \langle s_1, s_2, x_1^-, x_2^- | s, s_z, X_-, x_- \rangle$$

$$\langle s_1, s_2, x_1^-, x_2^- | s, s_z, X_-, x_- \rangle = \Psi(s, s_z, X_-, x_-) \psi_q^*(s_1, x_1^-) \psi_{\bar{q}}^*(s_2, x_2^-)$$

J/ψ Spin Alignment

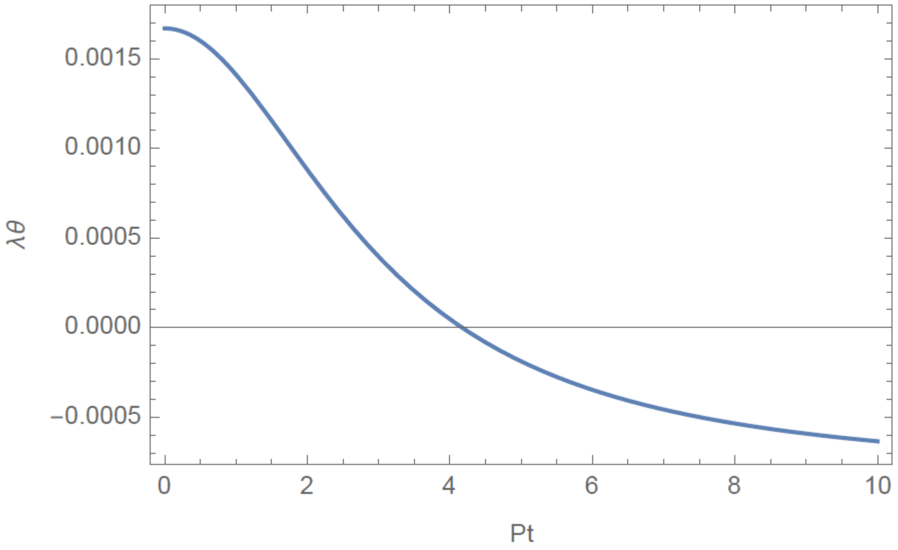
Taking the following approximations:

1) *the center-of-mass motion as a plane wave and the relative motion as a Gauss-like wave packet,*

2) *thermalized J/ψs with the medium.*

The angular parameter of J/ψ distribution

$$\lambda_\theta = \frac{1-3\rho_{00}}{1+\rho_{00}}$$



Summary

1) We derived the general two-body Schrodinger equation under rotation.

2) We derived the general equal-time kinetic equations for fermions under rotation.

3) We applied the general theories to quarkonia under rotation.

I thank Dr. Shile Chen very much, the talk is mainly based on her PhD thesis.