QCD under Rotation Workshop

Pion condensate in 3-flavor NJL model with unequal u and d chiral condensate

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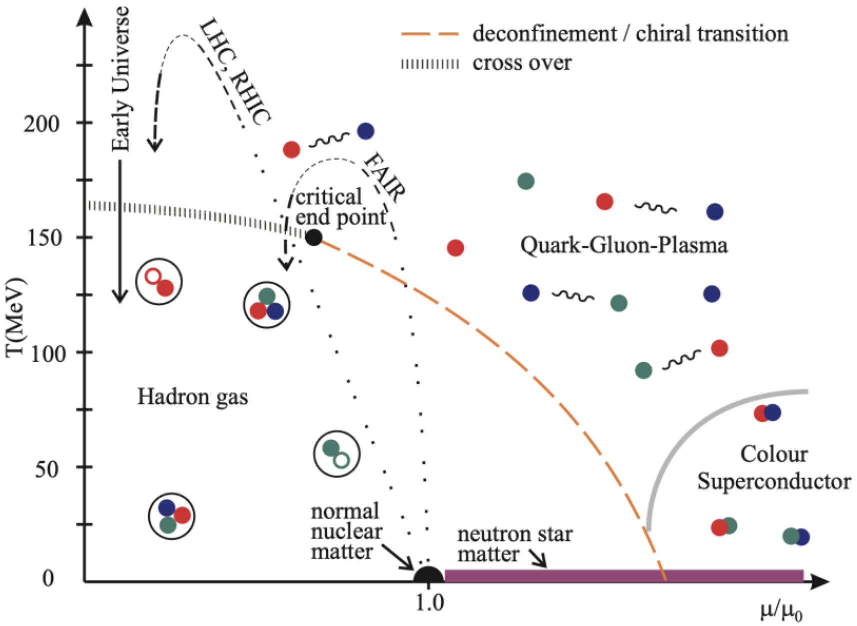
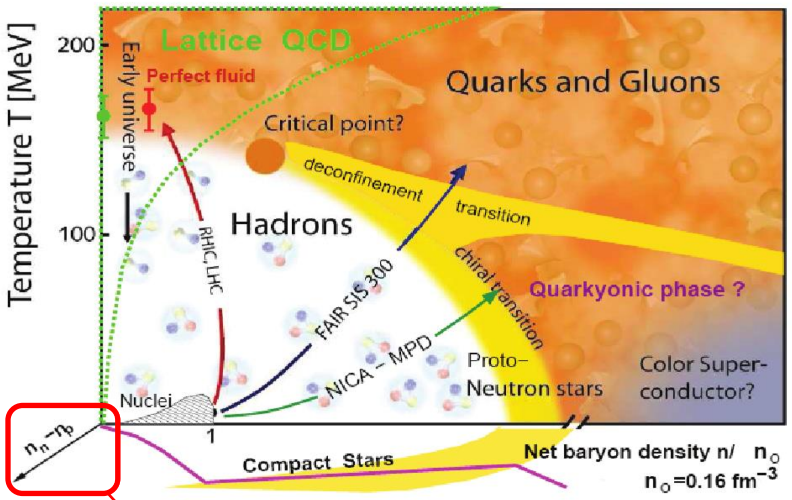
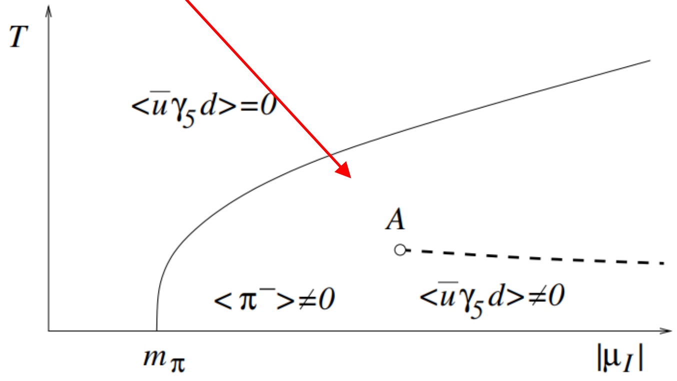
Catalog

- Phase structure
- Pion condensate
- Gap equation
- Polyakov loop

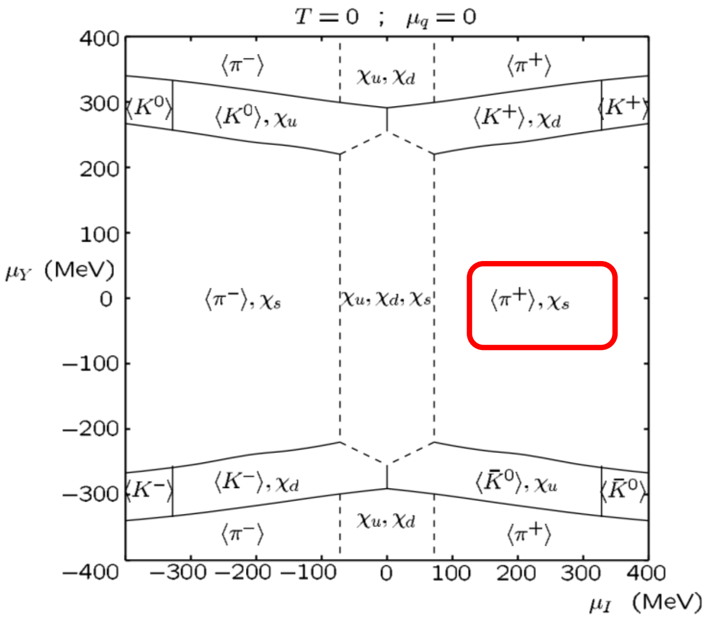
QCD phase structure: pion condensate







D. T. Son and M. A. Stephanov, PRL (2003)



A. Barducci et al., PRD (2005)

$$\pi = \langle\bar{\psi} i\gamma^5\lambda^1\psi\rangle = \langle\bar{u}i\gamma^5 d\rangle + \langle\bar{d}i\gamma^5 u\rangle$$

$$K_u = \langle\bar{\psi} i\gamma^5\lambda^4\psi\rangle = \langle\bar{u}i\gamma^5 s\rangle + \langle\bar{s}i\gamma^5 u\rangle$$

$$K_d = \langle\bar{\psi} i\gamma^5\lambda^6\psi\rangle = \langle\bar{d}i\gamma^5 s\rangle + \langle\bar{s}i\gamma^5 d\rangle$$

$$\mu_B = \frac{3(\mu_u+\mu_d)}{2},$$

$$\mu_I = \mu_u - \mu_d,$$

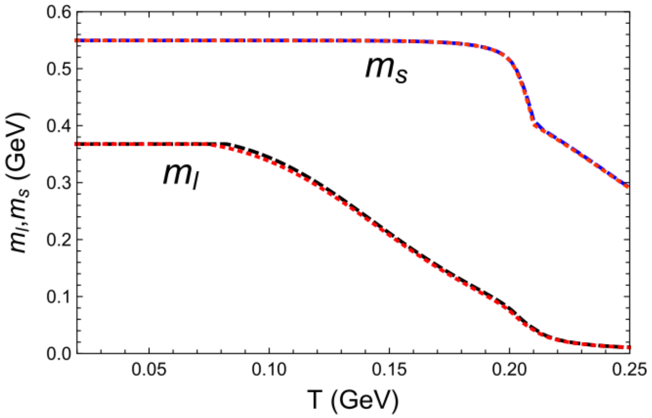
$$\mu_S = \frac{\mu_u+\mu_d}{2} - \mu_s,$$

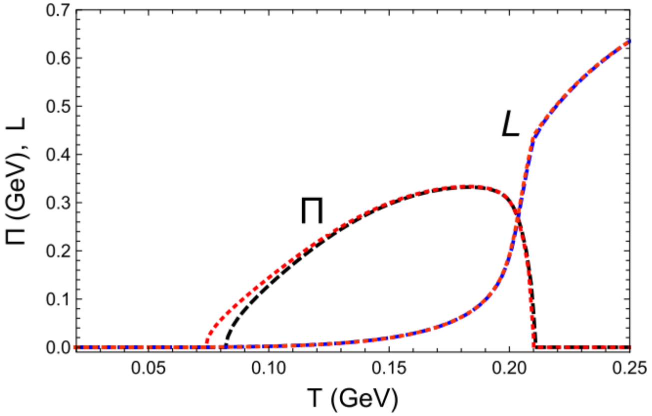
Pion condensate: previous studies

- 3-flavor pNJL model with the approximation of

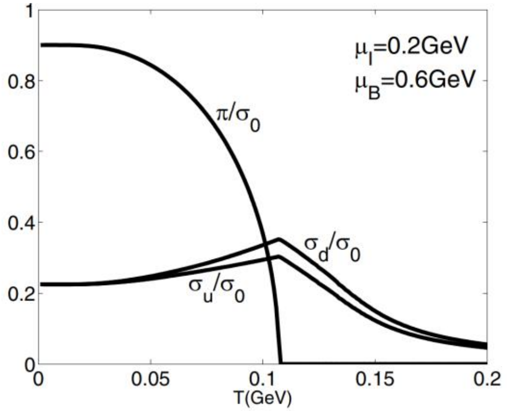
  $m_{0u} \approx m_{0d} = m_{0l}$,

  $\sigma_u \approx \sigma_d = \sigma_l$





G. Cao, L. He, and P. Zhang, PRD (2021)



- 2-flavor NJL model

$$\sigma_u = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{N_c M_q}{E_k} \Big[f_f(E_-^-) + f_f(-E_+^-) - f_f(E_-^+) - f_f(-E_+^+) + \frac{E_k - \mu_I/2}{E_k^-} (f_f(E_-^-) - f_f(-E_+^-)) + \frac{E_k + \mu_I/2}{E_k^+} (f_f(E_-^+) - f_f(-E_+^+)) \Big],$$

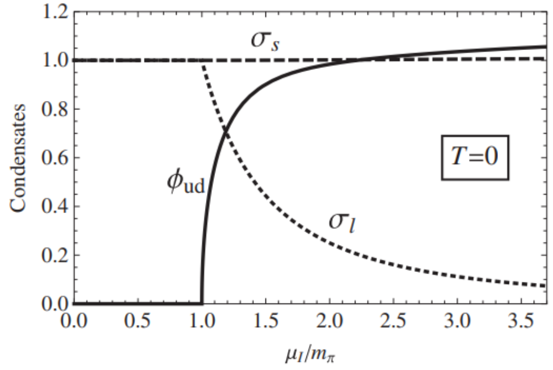
$$\sigma_d = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{N_c M_q}{E_k} \Big[-f_f(E_-^-) - f_f(-E_+^-) + f_f(E_-^+) + f_f(-E_+^+) + \frac{E_k - \mu_I/2}{E_k^-} (f_f(E_-^-) - f_f(-E_+^-)) + \frac{E_k + \mu_I/2}{E_k^+} (f_f(E_-^+) - f_f(-E_+^+)) \Big],$$

$$\pi = -4 N_c G \pi \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[\frac{1}{E_k^-} (f_f(E_-^-) - f_f(-E_+^-)) + \frac{1}{E_k^+} (f_f(E_-^+) - f_f(-E_+^+)) \Big],$$

L. He and P. Zhuang, PRD (2005)

$$M_u = M_d = M_l$$

$$\sigma_u = \sigma_d = \sigma_l, \; m_{0u} = m_{0d} = m_{0l}$$



- 3-flavor NJL model at $\mu_B = 0$

$$\sigma_s = -2 N_c M_s \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1 - 2f(E_\mathbf{k}^s)}{E_\mathbf{k}^s},$$

$$\sigma_l = -N_c M_l \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{E_\mathbf{k}^l} \sum_{\alpha=\pm} \frac{\xi_\mathbf{k}^\alpha}{E_\mathbf{k}^\alpha} (1 - 2f(E_\mathbf{k}^\alpha))$$

$$\phi_{ud} = 2 N_c \Delta \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\alpha=\pm} \frac{1 - 2f(E_\mathbf{k}^\alpha)}{E_\mathbf{k}^\alpha},$$

T. Xia, L. He, and P. Zhuang, PRD (2013)

Pion condensate: the Lagrangian

- The Lagrangian density of the 3-flavor NJL model

$$
\begin{aligned}
\mathcal{L}_{\mathrm{NJL}} &= \mathcal{L}_0 + \mathcal{L}_{\mathrm{S}} + \mathcal{L}_{\mathrm{V}} + \mathcal{L}_{\mathrm{KMT}} + \mathcal{L}_{\mathrm{IS}} + \mathcal{L}_{\mathrm{IV}}, \\
\mathcal{L}_0 &= \bar{\psi}(i\gamma^\mu \partial_\mu - \hat{m})\psi, \\
\mathcal{L}_{\mathrm{S}} &= \frac{G_{\mathrm{S}}}{2} \sum_{a=0}^{8} [(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi} i\gamma^5 \lambda^a \psi)^2],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{V}} &= -\frac{G_{\mathrm{V}}}{2} \sum_{a=0}^{8} [(\bar{\psi}\gamma^\mu\lambda^a\psi)^2 + (\bar{\psi}\gamma^5\gamma^\mu\lambda^a\psi)^2], \\
\mathcal{L}_{\mathrm{KMT}} &= -K[\det\bar{\psi}(1+\gamma^5)\psi + \det\bar{\psi}(1-\gamma^5)\psi], \\
\mathcal{L}_{\mathrm{IS}} &= G_{\mathrm{IS}} \sum_{a=1}^{3} [(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi} i\gamma^5\lambda^a\psi)^2], \\
\mathcal{L}_{\mathrm{IV}} &= -G_{\mathrm{IV}} \sum_{a=1}^{3} [(\bar{\psi}\gamma^\mu\lambda^a\psi)^2 + (\bar{\psi}\gamma^5\gamma^\mu\lambda^a\psi)^2],
\end{aligned}
$$

$$
\begin{aligned}
M_u &= m_u - 2G_{\mathrm{S}}\sigma_u + 2K\sigma_d\sigma_s, \\
M_d &= m_d - 2G_{\mathrm{S}}\sigma_d + 2K\sigma_s\sigma_u, \\
M_s &= m_s - 2G_{\mathrm{S}}\sigma_s + 2K\sigma_u\sigma_d.
\end{aligned}
$$

$$
\sigma_q = -2N_c \int_0^\Lambda \frac{d^3p}{(2\pi)^3} \frac{M_q}{E_q}(1 - f_q - \bar{f}_q),
$$

Rigorous results in 3-flavor NJL model??

$$\Rightarrow \quad M_u \neq M_d,\ \sigma_u \neq \sigma_d$$

- 2-flavor NJL model

$$
\sigma_u = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{N_c M_q}{E_k} \Big[f_f(E^-_-) + f_f(-E^-_+) - f_f(E^+_-) - f_f(-E^+_+) + \frac{E_k - \mu_I/2}{E_k^-}(f_f(E^-_-) - f_f(-E^-_+)) + \frac{E_k + \mu_I/2}{E_k^+}(f_f(E^+_-) - f_f(-E^+_+)) \Big],
$$

$$
\sigma_d = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{N_c M_q}{E_k} \Big[-f_f(E^-_-) - f_f(-E^-_+) + f_f(E^+_-) + f_f(-E^+_+) + \frac{E_k - \mu_I/2}{E_k^-}(f_f(E^-_-) - f_f(-E^-_+)) + \frac{E_k + \mu_I/2}{E_k^+}(f_f(E^+_-) - f_f(-E^+_+)) \Big],
$$

$$
\pi = -4N_c G\pi \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[\frac{1}{E_k^-}(f_f(E^-_-) - f_f(-E^-_+)) + \frac{1}{E_k^+}(f_f(E^+_-) - f_f(-E^+_+)) \Big],
$$

L. He and P. Zhuang, PRD (2005)

- 3-flavor NJL model at $\mu_B = 0$

$$
\sigma_s = -2N_c M_s \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1 - 2f(E^s_{\mathbf{k}})}{E^s_{\mathbf{k}}},
$$

$$
\sigma_l = -N_c M_l \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{E^l_{\mathbf{k}}} \sum_{\alpha=\pm} \frac{\xi^\alpha_{\mathbf{k}}}{E^\alpha_{\mathbf{k}}}(1 - 2f(E^\alpha_{\mathbf{k}}))
$$

$$
\phi_{ud} = 2N_c\Delta \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\alpha=\pm} \frac{1 - 2f(E^\alpha_{\mathbf{k}})}{E^\alpha_{\mathbf{k}}},
$$

T. Xia, L. He, and P. Zhuang, PRD (2013)

Pion condensate: quark propagator

$$\begin{aligned}\mathcal{L}_{\rm MF} &= \bar\psi(i\gamma^\mu\partial_\mu+\hat\mu\gamma^0+\tilde\mu_{\rm V}\gamma^0+\tilde\mu_{\rm IV}\gamma^0-\hat m+\Sigma_{\rm S}+\Sigma_{\rm IS}+\Sigma_{\rm K})\psi\\ &\quad -\mathcal{V}_{\rm S}-\mathcal{V}_{\rm IS}-\mathcal{V}_{\rm K}-\mathcal{V}_{\rm V}-\mathcal{V}_{\rm IV}\\ &= \bar\psi(i\gamma^\mu\partial_\mu+\hat\mu\gamma^0-\hat M)\psi-\mathcal{V}\\ &= \bar\psi S^{-1}\psi-\mathcal{V},\end{aligned}$$

$$S^{-1}(p)=\begin{pmatrix}\gamma^\mu p_\mu+\tilde\mu_u\gamma^0-M_u & \Delta_\pi & \Delta_{K_u}\\ \Delta_\pi & \gamma^\mu p_\mu+\tilde\mu_d\gamma^0-M_d & \Delta_{K_d}\\ \Delta_{K_u} & \Delta_{K_d} & \gamma^\mu p_\mu+\tilde\mu_s\gamma^0-M_s\end{pmatrix}$$

$$\begin{aligned}\Delta_\pi &= i\left(G_{\rm S}+2G_{\rm IS}-K\sigma_s\right)\pi\gamma^5+\frac{K}{2}K_uK_d,\\ \Delta_{K_u} &= i\left(G_{\rm S}-K\sigma_d\right)K_u\gamma^5+\frac{K}{2}\pi K_d,\\ \Delta_{K_d} &= i\left(G_{\rm S}-K\sigma_u\right)K_d\gamma^5+\frac{K}{2}\pi K_u.\end{aligned}$$

$$\begin{aligned}\pi &= \langle\bar\psi i\gamma^5\lambda^1\psi\rangle=\langle\bar u i\gamma^5 d\rangle+\langle\bar d i\gamma^5 u\rangle\\ K_u &= \langle\bar\psi i\gamma^5\lambda^4\psi\rangle=\langle\bar u i\gamma^5 s\rangle+\langle\bar s i\gamma^5 u\rangle\\ K_d &= \langle\bar\psi i\gamma^5\lambda^6\psi\rangle=\langle\bar d i\gamma^5 s\rangle+\langle\bar s i\gamma^5 d\rangle\end{aligned}$$

$K_u=K_d=0$

$$S=\begin{pmatrix}S_{uu} & S_{ud} & 0\\ S_{du} & S_{dd} & 0\\ 0 & 0 & S_{0s}\end{pmatrix}=\begin{pmatrix}S_{uu} & -S_{uu}\Delta^-S_{0d} & 0\\ -S_{dd}\Delta^+S_{0u} & S_{dd} & 0\\ 0 & 0 & S_{0s}\end{pmatrix}$$

$$\Delta^+=\Delta^-=i\Delta\gamma^5.$$

$$\Delta=\left(G_{\rm S}+2G_{\rm IS}-K\sigma_s\right)\pi.$$

$$\begin{aligned}S_{uu} &= \frac{1}{S_{0u}^{-1}-\Delta^-S_{0d}\Delta^+},\\ S_{dd} &= \frac{1}{S_{0d}^{-1}-\Delta^+S_{0u}\Delta^-},\\ S_{0q}^{-1}(p) &= \gamma^0\left(p_0-E_q^-\right)\Lambda_+^q(\vec p)+\gamma^0\left(p_0+E_q^+\right)\Lambda_-^q(\vec p),\\ S_{0q}(p) &= \frac{\gamma^0\tilde\Lambda_-^q(\vec p)}{p_0-E_q^-}+\frac{\gamma^0\tilde\Lambda_+^q(\vec p)}{p_0+E_q^+}=\frac{\Lambda_+^q(\vec p)\gamma^0}{p_0-E_q^-}+\frac{\Lambda_-^q(\vec p)\gamma^0}{p_0+E_q^+},\end{aligned}$$

$$\begin{aligned}S_{uu}(k) &= \frac{i\omega_n+\xi_{\mathbf k}^-}{(i\omega_n)^2-(E_{\mathbf k}^-)^2}\Lambda_+^l\gamma_0+\frac{i\omega_n-\xi_{\mathbf k}^+}{(i\omega_n)^2-(E_{\mathbf k}^+)^2}\Lambda_-^l\gamma_0,\\ S_{dd}(k) &= \frac{i\omega_n-\xi_{\mathbf k}^-}{(i\omega_n)^2-(E_{\mathbf k}^-)^2}\Lambda_-^l\gamma_0+\frac{i\omega_n+\xi_{\mathbf k}^+}{(i\omega_n)^2-(E_{\mathbf k}^+)^2}\Lambda_+^l\gamma_0,\\ S_{ud}(k) &= \frac{i\Delta}{(i\omega_n)^2-(E_{\mathbf k}^-)^2}\Lambda_+^l\gamma_5+\frac{i\Delta}{(i\omega_n)^2-(E_{\mathbf k}^+)^2}\Lambda_-^l\gamma_5,\\ S_{du}(k) &= \frac{i\Delta}{(i\omega_n)^2-(E_{\mathbf k}^-)^2}\Lambda_-^l\gamma_5+\frac{i\Delta}{(i\omega_n)^2-(E_{\mathbf k}^+)^2}\Lambda_+^l\gamma_5,\end{aligned}$$

$$\begin{aligned}\Lambda_\pm^l &= \frac{1}{2}\left[1\pm\frac{\gamma_0(\boldsymbol\gamma\cdot\mathbf k+M_l)}{E_{\mathbf k}^l}\right],\\ \Lambda_\pm^s &= \frac{1}{2}\left[1\pm\frac{\gamma_0(\boldsymbol\gamma\cdot\mathbf k+M_s)}{E_{\mathbf k}^s}\right].\end{aligned}$$

M. Huang, P. Zhuang, and W. Chao, PRD (2002).
L. He and P. Zhuang, PRD (2005).
T. Xia, L. He, and P. Zhuang, PRD (2013).

Pion condensate: quark propagator

Rigorous results with

$M_u \neq M_d,\ \sigma_u \neq \sigma_d$

$$\mathcal{H}(\vec{p}) = -\begin{pmatrix} \frac{\tilde{\mu}_I}{2} - M_u & |\vec{p}| & 0 & -\Delta \\ |\vec{p}| & \frac{\tilde{\mu}_I}{2} + M_u & \Delta & 0 \\ 0 & \Delta & -\frac{\tilde{\mu}_I}{2} - M_d & |\vec{p}| \\ -\Delta & 0 & |\vec{p}| & -\frac{\tilde{\mu}_I}{2} + M_d \end{pmatrix}$$

$$\mathcal{S}_{uu}(p) = \sum_{k=1}^{4} g_{uu}(\lambda'_k)\frac{1}{p_0 - \lambda'_k},$$

$$\mathcal{S}_{dd}(p) = \sum_{k=1}^{4} g_{dd}(\lambda'_k)\frac{1}{p_0 - \lambda'_k},$$

$$\mathcal{S}_{ud}(p) = \sum_{k=1}^{4} g_{ud}(\lambda'_k)\frac{1}{p_0 - \lambda'_k},$$

$$\mathcal{S}_{du}(p) = \sum_{k=1}^{4} g_{du}(\lambda'_k)\frac{1}{p_0 - \lambda'_k},$$

$$g_{uu}(\lambda_k) = \frac{\left[\left(\lambda_k - \frac{\tilde{\mu}_I}{2}\right)^2 - E_d^2\right]\left[-\vec{\gamma}\cdot\vec{p} + \left(\lambda_k + \frac{\tilde{\mu}_I}{2}\right)\gamma^0 + M_u\right] - \Delta^2\left[-\vec{\gamma}\cdot\vec{p} + \left(\lambda_k - \frac{\tilde{\mu}_I}{2}\right)\gamma^0 + M_d\right]}{\prod_{j=1,j\neq k}^{4}\left(\lambda_k - \lambda_j\right)\times \mathbb{1}_4}.$$

$$g_{dd}(\lambda_k) = \frac{\left[\left(\lambda_k + \frac{\tilde{\mu}_I}{2}\right)^2 - E_u^2\right]\left[-\vec{\gamma}\cdot\vec{p} + \left(\lambda_k - \frac{\tilde{\mu}_I}{2}\right)\gamma^0 + M_d\right] - \Delta^2\left[-\vec{\gamma}\cdot\vec{p} + \left(\lambda_k + \frac{\tilde{\mu}_I}{2}\right)\gamma^0 + M_u\right]}{\prod_{j=1,j\neq k}^{4}\left(\lambda_k - \lambda_j\right)\times \mathbb{1}_4}.$$

$$g_{ud}(\lambda_k) = \frac{\left[-\vec{\gamma}\cdot\vec{p} + \left(\lambda_k - \frac{\tilde{\mu}_I}{2}\right)\gamma^0 + M_d\right]\left[-\vec{\gamma}\cdot\vec{p} + \left(\lambda_k + \frac{\tilde{\mu}_I}{2}\right)\gamma^0 - M_u\right] i\Delta\gamma^5 - i\Delta^3\gamma^5}{\prod_{j=1,j\neq k}^{4}\left(\lambda_k - \lambda_j\right)\times \mathbb{1}_4},$$

$$g_{du}(\lambda_k) = \frac{\left[-\vec{\gamma}\cdot\vec{p} + \left(\lambda_k + \frac{\tilde{\mu}_I}{2}\right)\gamma^0 + M_u\right]\left[-\vec{\gamma}\cdot\vec{p} + \left(\lambda_k - \frac{\tilde{\mu}_I}{2}\right)\gamma^0 - M_d\right] i\Delta\gamma^5 - i\Delta^3\gamma^5}{\prod_{j=1,j\neq k}^{4}\left(\lambda_k - \lambda_j\right)\times \mathbb{1}_4}, \quad \text{(B.60)}$$

$$\mathcal{S}_{uu} = \frac{1}{S_{0u}^{-1} - \Delta^- S_{0d}\Delta^+},$$

$$\mathcal{S}_{dd} = \frac{1}{S_{0d}^{-1} - \Delta^+ S_{0u}\Delta^-},$$

$$S_{0q}^{-1}(p) = \gamma^0\left(p_0 - E_q^-\right)\Lambda_+^q(\vec{p}) + \gamma^0\left(p_0 + E_q^+\right)\Lambda_-^q(\vec{p}),$$

$$S_{0q}(p) = \frac{\gamma^0\tilde{\Lambda}_-^q(\vec{p})}{p_0 - E_q^-} + \frac{\gamma^0\tilde{\Lambda}_+^q(\vec{p})}{p_0 + E_q^+} = \frac{\Lambda_+^q(\vec{p})\gamma^0}{p_0 - E_q^-} + \frac{\Lambda_-^q(\vec{p})\gamma^0}{p_0 + E_q^+},$$

$$\mathcal{S}_{uu}(k) = \frac{i\omega_n + \xi_{\mathbf{k}}^-}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2}\Lambda_+^l\gamma_0 + \frac{i\omega_n - \xi_{\mathbf{k}}^+}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2}\Lambda_-^l\gamma_0,$$

$$\mathcal{S}_{dd}(k) = \frac{i\omega_n - \xi_{\mathbf{k}}^-}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2}\Lambda_-^l\gamma_0 + \frac{i\omega_n + \xi_{\mathbf{k}}^+}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2}\Lambda_+^l\gamma_0,$$

$$\mathcal{S}_{ud}(k) = \frac{i\Delta}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2}\Lambda_+^l\gamma_5 + \frac{i\Delta}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2}\Lambda_-^l\gamma_5,$$

$$\mathcal{S}_{du}(k) = \frac{i\Delta}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2}\Lambda_-^l\gamma_5 + \frac{i\Delta}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2}\Lambda_+^l\gamma_5,$$

$$\Lambda_\pm^l = \frac{1}{2}\left[1 \pm \frac{\gamma_0(\boldsymbol{\gamma}\cdot\mathbf{k} + M_l)}{E_{\mathbf{k}}^l}\right],$$

$$\Lambda_\pm^s = \frac{1}{2}\left[1 \pm \frac{\gamma_0(\boldsymbol{\gamma}\cdot\mathbf{k} + M_s)}{E_{\mathbf{k}}^s}\right].$$

M. Huang, P. Zhuang, and W. Chao, PRD (2002).
L. He and P. Zhuang, PRD (2005).
T. Xia, L. He, and P. Zhuang, PRD (2013).

Pion condensate: gap equation

Rigorous results with $M_u \neq M_d,\ \sigma_u \neq \sigma_d$

- Chiral condensates

$$\sigma_u = 4N_c \sum_{k=1}^{4} \int \frac{d^3p}{(2\pi)^3} g_{\sigma u}(\lambda_k') f(\lambda_k'),$$

$$\sigma_d = 4N_c \sum_{k=1}^{4} \int \frac{d^3p}{(2\pi)^3} g_{\sigma d}(\lambda_k') f(\lambda_k'),$$

$$\sigma_s = 2N_c \int \frac{d^3p}{(2\pi)^3} \frac{M_s}{E_s} \left[f(E_s^-) + f(E_s^+) - 1 \right],$$

- Net-quark densities

$$\rho_u = 4N_c \sum_{k=1}^{4} \int \frac{d^3p}{(2\pi)^3} g_{\rho u}(\lambda_k') \left[-\frac{1}{2} + f(\lambda_k') \right],$$

$$\rho_d = 4N_c \sum_{k=1}^{4} \int \frac{d^3p}{(2\pi)^3} g_{\rho d}(\lambda_k') \left[-\frac{1}{2} + f(\lambda_k') \right],$$

$$\rho_s = 2N_c \int \frac{d^3p}{(2\pi)^3} \left[f(E_s^-) - f(E_s^+) \right].$$

- Pion condensate

$$\pi = 4N_c \sum_{k=1}^{4} \int \frac{d^3p}{(2\pi)^3} g_\pi(\lambda_k') f(\lambda_k')$$

- g functions

$$g_{\sigma u}(\lambda_k') = \frac{\left[(\lambda_k' + \tilde{\mu}_d)^2 - E_d^2 \right] M_u - \Delta^2 M_d}{\prod_{j=1, j\neq k}^{4} (\lambda_k' - \lambda_j')},$$

$$g_{\sigma d}(\lambda_k') = \frac{\left[(\lambda_k' + \tilde{\mu}_u)^2 - E_u^2 \right] M_d - \Delta^2 M_u}{\prod_{j=1, j\neq k}^{4} (\lambda_k' - \lambda_j')},$$

$$g_{\rho u}(\lambda_k') = \frac{\left[(\lambda_k' + \tilde{\mu}_d)^2 - E_d^2 \right] (\lambda_k' + \tilde{\mu}_u) - \Delta^2 (\lambda_k' + \tilde{\mu}_d)}{\prod_{j=1, j\neq k}^{4} (\lambda_k' - \lambda_j')},$$

$$g_{\rho d}(\lambda_k') = \frac{\left[(\lambda_k' + \tilde{\mu}_u)^2 - E_u^2 \right] (\lambda_k' + \tilde{\mu}_d) - \Delta^2 (\lambda_k' + \tilde{\mu}_u)}{\prod_{j=1, j\neq k}^{4} (\lambda_k' - \lambda_j')},$$

$$g_\pi(\lambda_k') = 2\frac{\left[\vec{p}^2 + M_u M_d - (\lambda_k' + \tilde{\mu}_u)(\lambda_k' + \tilde{\mu}_d) \right] \Delta + \Delta^3}{\prod_{j=1, j\neq k}^{4} (\lambda_k' - \lambda_j')},$$

$$\sum_{k=1}^{4} g_{\sigma u}(\lambda_k') = \sum_{k=1}^{4} g_{\sigma d}(\lambda_k') = \sum_{k=1}^{4} g_\pi(\lambda_k') = 0,$$

$$\sum_{k=1}^{4} g_{\rho u}(\lambda_k') = \sum_{k=1}^{4} g_{\rho d}(\lambda_k') = 1,$$

$$g_{\rho u}(\lambda_k') + g_{\rho d}(\lambda_k') = \frac{1}{2}.$$

- 2-flavor NJL model

$$\sigma_u = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{N_c M_q}{E_k} \Big[f_f(E_-^-) + f_f(-E_+^-) - f_f(E_-^+) - f_f(-E_+^+) + \frac{E_k - \mu_I/2}{E_k^-} (f_f(E_-^-) - f_f(-E_+^-)) + \frac{E_k + \mu_I/2}{E_k^+} (f_f(E_-^+) - f_f(-E_+^+)) \Big],$$

$$\sigma_d = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{N_c M_q}{E_k} \Big[-f_f(E_-^-) - f_f(-E_+^-) + f_f(E_-^+) + f_f(-E_+^+) + \frac{E_k - \mu_I/2}{E_k^-} (f_f(E_-^-) - f_f(-E_+^-)) + \frac{E_k + \mu_I/2}{E_k^+} (f_f(E_-^+) - f_f(-E_+^+)) \Big],$$

$$\pi = -4N_c G\pi \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[\frac{1}{E_k^-} (f_f(E_-^-) - f_f(-E_+^-)) + \frac{1}{E_k^+} (f_f(E_-^+) - f_f(-E_+^+)) \Big],$$

L. He and P. Zhuang, PRD (2005)

$M_u = M_d = M_l$

$\sigma_u = \sigma_d = \sigma_l$

- 3-flavor NJL model at $\mu_B = 0$

$$\sigma_s = -2N_c M_s \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1 - 2f(E_\mathbf{k}^s)}{E_\mathbf{k}^s},$$

$$\sigma_l = -N_c M_l \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{E_\mathbf{k}^l} \sum_{\alpha=\pm} \frac{\xi_\mathbf{k}^\alpha}{E_\mathbf{k}^\alpha} (1 - 2f(E_\mathbf{k}^\alpha))$$

$$\phi_{ud} = 2N_c \Delta \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\alpha=\pm} \frac{1 - 2f(E_\mathbf{k}^\alpha)}{E_\mathbf{k}^\alpha},$$

T. Xia, L. He, and P. Zhuang, PRD (2013)

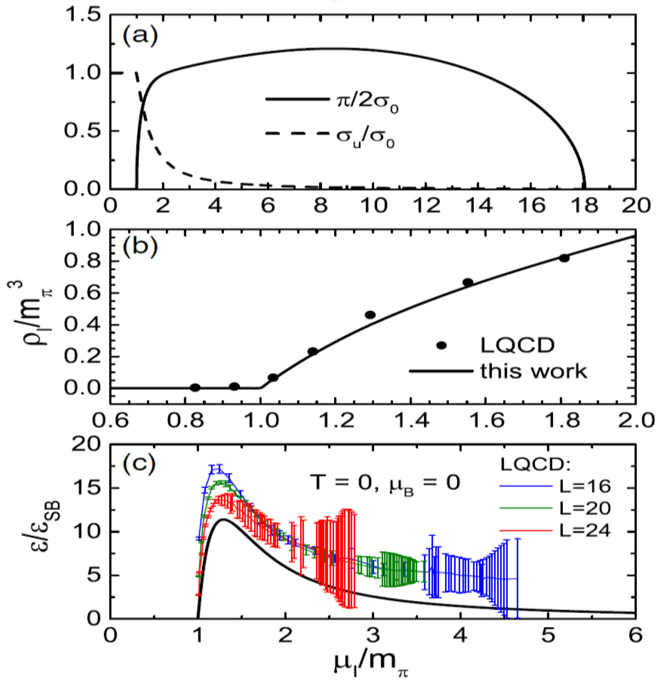
Pion condensate: results

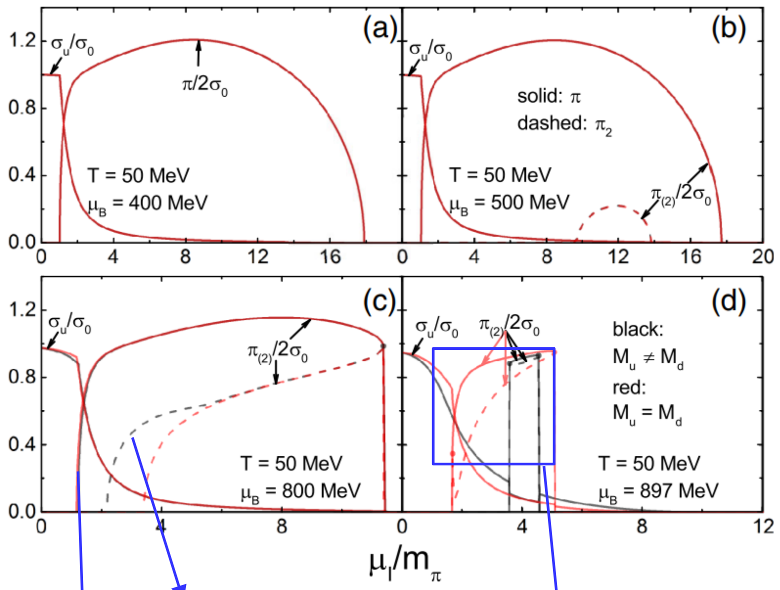
$$\mathcal{L}_{\rm IS} = G_{\rm IS}\sum_{a=1}^{3}[(\bar\psi\lambda^a\psi)^2+(\bar\psi i\gamma^5\lambda^a\psi)^2],$$

$$\mathcal{L}_{\rm IV} = -G_{\rm IV}\sum_{a=1}^{3}[(\bar\psi\gamma^\mu\lambda^a\psi)^2+(\bar\psi\gamma^5\gamma^\mu\lambda^a\psi)^2],$$

$$G_{\rm IS} = -0.002G_{\rm S} \qquad G_{\rm IV} = 0.25G_{\rm S}$$

Lattice fitting

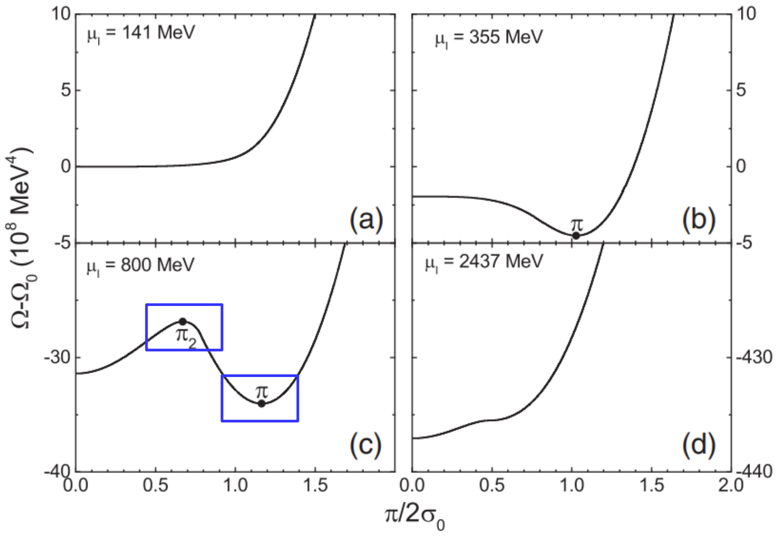


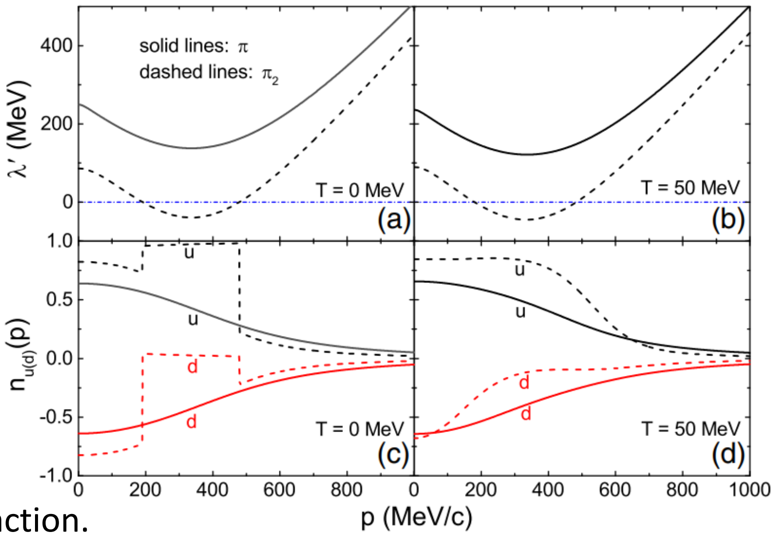


π_2: Samar phase

π: pion superfluid phase

Large errors within $\mu_B \in (500,900)$ MeV





$\mu_B = 800$ MeV, $\mu_I = 800$ MeV

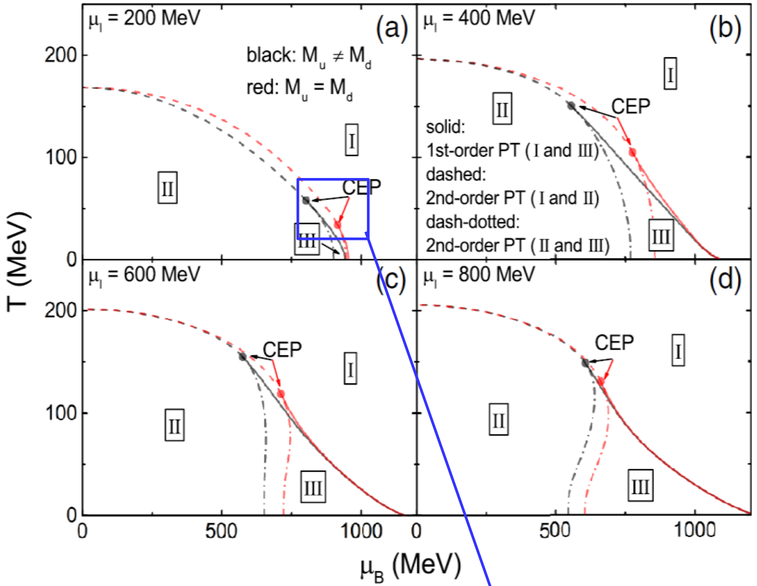
The cure of the instability of π_2:

- The free energy of a system with a fixed baryon density.
- A Fermi system with a finite-range momentum-dependent interaction.

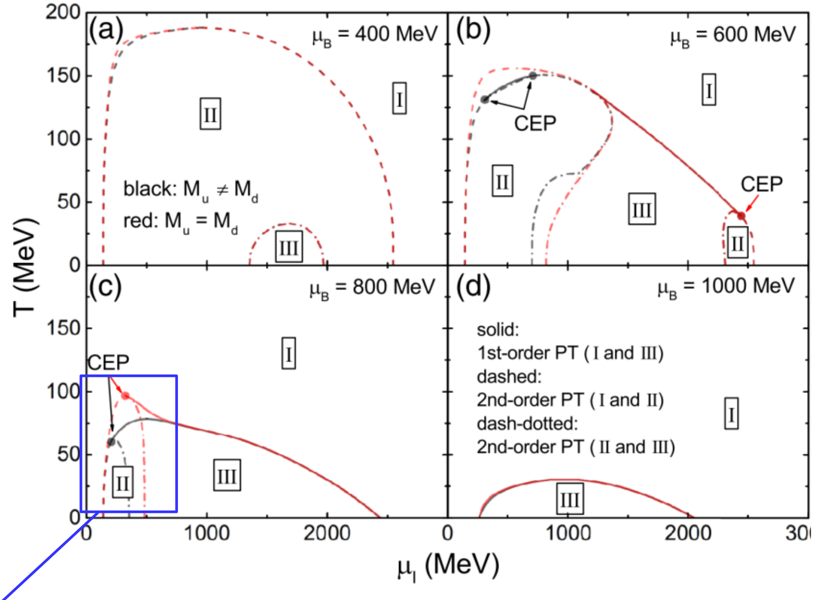
L. He et al., PRD (2006); M. M. Forbes et al., PRL (2005)

Pion condensate: 3D phase structure

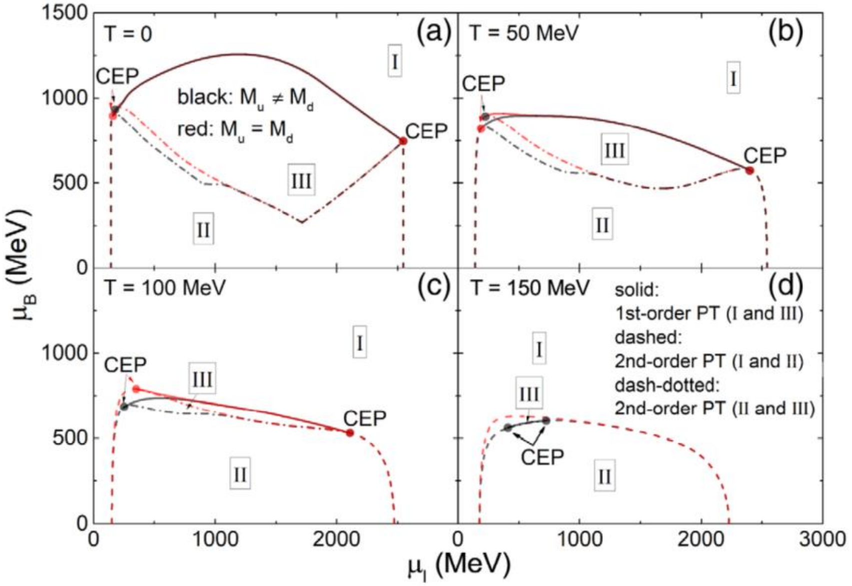
- $T - \mu_B$ plane



- $T - \mu_I$ plane



- $\mu_B - \mu_I$ plane

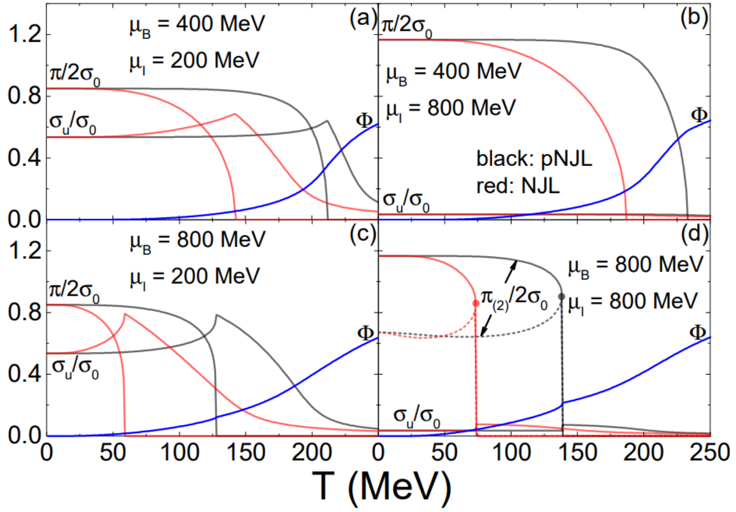


Affect the position of CEP

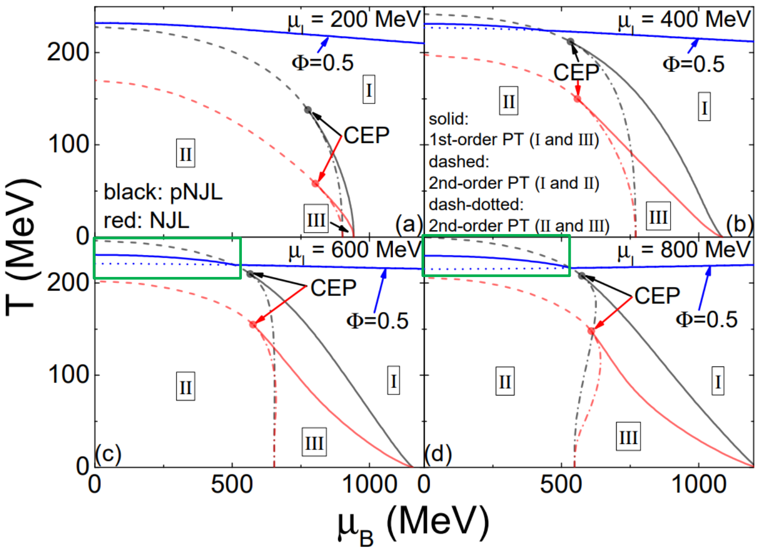
Calculations by assuming equal constituent mass of u and d quarks would lead to large errors of the QCD phase diagram, and affect the position of the critical end point

Pion condensate: Polyakov loop

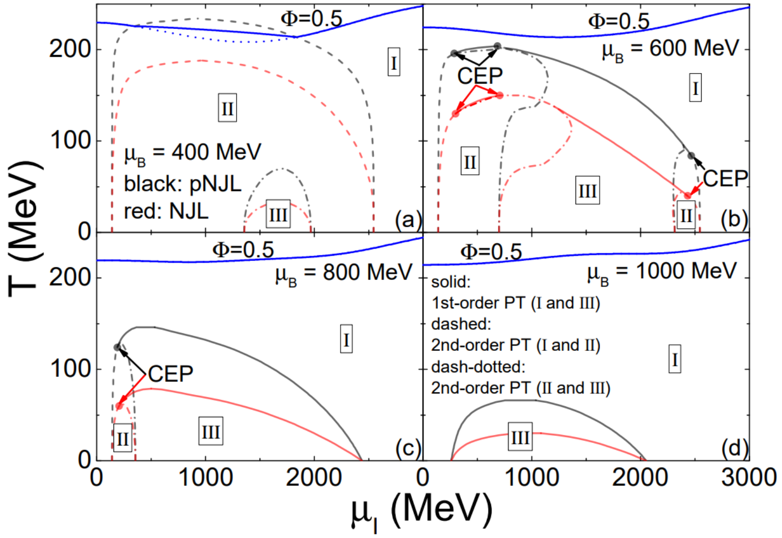
- Polyakov loop contribution $\mathcal{U}(\Phi,\bar{\Phi},T) = -b\cdot T\{54e^{-a/T}\Phi\bar{\Phi} + \ln[1-6\Phi\bar{\Phi} -3(\Phi\bar{\Phi})^2 + 4(\Phi^3+\bar{\Phi}^3)]\}.$



- $T-\mu_B$ plane



- $T-\mu_I$ plane



- The pNJL model leads to larger areas of the pion condensate phase and the CEP at higher temperatures or larger baryon chemical potentials, compared to the NJL model.
- The existence of pion condensate may also affect the deconfinement phase transition.

Summary

- Rigorous analytical expressions for pion condensate in the three-flavor NJL model is derived comparing with the previous studies.
- Calculations by assuming $m_{0u} = m_{0d} = m_{0l},\ \sigma_u = \sigma_d = \sigma_l$ would lead to large errors of the QCD phase diagram, and affect the position of the critical end point.
- The pNJL model leads to larger areas of the pion condensate phase and the CEP at higher temperatures or larger baryon chemical potentials , compared to the NJL model.
- The existence of pion condensate may also affect the deconfinement phase transition.

Outlook

QCD phase structure

- Electromagnetic field and rotation.
- Pauli-Villars regularization scheme.

Thanks!

Backup

Pion condensate: mean-field approximation

$$
\begin{aligned}
(\bar\psi\gamma^i\psi) &\approx \langle\bar\psi\gamma^i\psi\rangle,\\
(\bar\psi\gamma^i\psi)^2 &\approx 2\bar\psi\gamma^i\psi\langle\bar\psi\gamma^i\psi\rangle - \langle\bar\psi\gamma^i\psi\rangle^2,\\
(\bar\psi\gamma^i\psi\bar\psi\gamma^j\psi) &\approx \bar\psi\gamma^i\psi\langle\bar\psi\gamma^j\psi\rangle + \bar\psi\gamma^j\psi\langle\bar\psi\gamma^i\psi\rangle - \langle\bar\psi\gamma^i\psi\rangle\langle\bar\psi\gamma^j\psi\rangle,\\
(\bar\psi\gamma^i\psi\bar\psi\gamma^j\psi\bar\psi\gamma^k\psi) &\approx (\bar\psi\gamma^i\psi)\langle\bar\psi\gamma^j\psi\rangle\langle\bar\psi\gamma^k\psi\rangle + (\bar\psi\gamma^j\psi)\langle\bar\psi\gamma^i\psi\rangle\langle\bar\psi\gamma^k\psi\rangle\\
&\quad +(\bar\psi\gamma^k\psi)\langle\bar\psi\gamma^i\psi\rangle\langle\bar\psi\gamma^j\psi\rangle - 2\langle\bar\psi\gamma^i\psi\rangle\langle\bar\psi\gamma^j\psi\rangle\langle\bar\psi\gamma^k\psi\rangle,\\
(\bar\psi\gamma^i\psi\bar\psi\gamma^j\psi)^2 &\approx \langle\bar\psi\gamma^i\psi\rangle^2(2\bar\psi\gamma^j\psi\langle\bar\psi\gamma^j\psi\rangle) + \langle\bar\psi\gamma^j\psi\rangle^2(2\bar\psi\gamma^i\psi\langle\bar\psi\gamma^i\psi\rangle)\\
&\quad -3\langle\bar\psi\gamma^i\psi\rangle^2\langle\bar\psi\gamma^j\psi\rangle^2, \qquad \text{(B.1)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\rm MF} &= \bar\psi(i\gamma^\mu\partial_\mu + \hat\mu\gamma^0 + \tilde\mu_{\rm V}\gamma^0 + \tilde\mu_{\rm IV}\gamma^0 - \hat m + \Sigma_{\rm S} + \Sigma_{\rm IS} + \Sigma_{\rm K})\psi\\
&\quad -\mathcal{V}_{\rm S} - \mathcal{V}_{\rm IS} - \mathcal{V}_{\rm K} - \mathcal{V}_{\rm V} - \mathcal{V}_{\rm IV}\\
&= \bar\psi(i\gamma^\mu\partial_\mu + \hat{\tilde\mu}\gamma^0 - \hat M)\psi - \mathcal{V}\\
&= \bar\psi S^{-1}\psi - \mathcal{V},
\end{aligned}
$$

$$
\begin{aligned}
\pi^+ &= \langle\bar\psi i\gamma^5\lambda^1_+\psi\rangle = \sqrt2\langle\bar u i\gamma^5 d\rangle = \frac{\pi}{\sqrt2}e^{i\theta_{ud}},\\
\pi^- &= \langle\bar\psi i\gamma^5\lambda^1_-\psi\rangle = \sqrt2\langle\bar d i\gamma^5 u\rangle = \frac{\pi}{\sqrt2}e^{-i\theta_{ud}},\\
\pi &= \langle\bar\psi i\gamma^5\lambda^1\psi\rangle = \langle\bar u i\gamma^5 d\rangle + \langle\bar d i\gamma^5 u\rangle,\\
K^+ &= K_{us} = \langle\bar\psi i\gamma^5\lambda^4_+\psi\rangle = \sqrt2\langle\bar u i\gamma^5 s\rangle = \frac{K_u}{\sqrt2}e^{i\theta_{us}},\\
K^- &= K_{su} = \langle\bar\psi i\gamma^5\lambda^4_-\psi\rangle = \sqrt2\langle\bar s i\gamma^5 u\rangle = \frac{K_u}{\sqrt2}e^{-i\theta_{us}},\\
K_u &= \langle\bar\psi i\gamma^5\lambda^4\psi\rangle = \langle\bar u i\gamma^5 s\rangle + \langle\bar s i\gamma^5 u\rangle,\\
K^0 &= K_{ds} = \langle\bar\psi i\gamma^5\lambda^6_+\psi\rangle = \sqrt2\langle\bar d i\gamma^5 s\rangle = \frac{K_d}{\sqrt2}e^{i\theta_{ds}},\\
\bar K^0 &= K_{sd} = \langle\bar\psi i\gamma^5\lambda^6_-\psi\rangle = \sqrt2\langle\bar s i\gamma^5 d\rangle = \frac{K_d}{\sqrt2}e^{-i\theta_{ds}},\\
K_d &= \langle\bar\psi i\gamma^5\lambda^6\psi\rangle = \langle\bar d i\gamma^5 s\rangle + \langle\bar s i\gamma^5 d\rangle,
\end{aligned}
$$

Pion condensate: mean-field approximation

其中

$$S^{-1}(p) = \begin{pmatrix} \gamma^\mu p_\mu + \tilde{\mu}_u\gamma^0 - M_u & \Delta_\pi & \Delta_{K_u} \\ \Delta_\pi & \gamma^\mu p_\mu + \tilde{\mu}_d\gamma^0 - M_d & \Delta_{K_d} \\ \Delta_{K_u} & \Delta_{K_d} & \gamma^\mu p_\mu + \tilde{\mu}_s\gamma^0 - M_s \end{pmatrix} \quad \text{(B.15)}$$

是夸克传播子 $S(p)$ 的逆，是动量 p 的函数，并且

$$\begin{aligned} \Delta_\pi &= i\left(G_S + 2G_{IS} - K\sigma_s\right)\pi\gamma^5 + \frac{K}{2}K_u K_d, \\ \Delta_{K_u} &= i\left(G_S - K\sigma_d\right)K_u\gamma^5 + \frac{K}{2}\pi K_d, \\ \Delta_{K_d} &= i\left(G_S - K\sigma_u\right)K_d\gamma^5 + \frac{K}{2}\pi K_u. \end{aligned}$$

$$\begin{aligned} \mathcal{V} &= G_S\left(\sigma_u^2 + \sigma_d^2 + \sigma_s^2\right) + \frac{G_S}{2}(\pi^2 + K_u^2 + K_d^2) + G_{IS}(\sigma_u - \sigma_d)^2 + G_{IS}\pi^2 \\ &\quad -4K\sigma_u\sigma_d\sigma_s - K\left(\pi^2\sigma_s + K_u^2\sigma_d + K_d^2\sigma_u\right) - \frac{1}{3}G_V\left(\rho_u + \rho_d + \rho_s\right)^2 - G_{IV}(\rho_u - \rho_d)^2 \end{aligned} \quad \text{(B.16)}$$

是独立于夸克场的凝聚能。

$$\hat{M} = \begin{pmatrix} M_u & 0 & 0 \\ 0 & M_d & 0 \\ 0 & 0 & M_s \end{pmatrix}$$

是 Dirac 有效质量矩阵，并且

$$\begin{aligned} M_u &= m_u - 2G_S\sigma_u - 2G_{IS}\tau_u(\sigma_u - \sigma_d) + 2K\sigma_d\sigma_s + \frac{K}{2}K_d^2, \\ M_d &= m_d - 2G_S\sigma_d - 2G_{IS}\tau_d(\sigma_u - \sigma_d) + 2K\sigma_u\sigma_s + \frac{K}{2}K_u^2, \\ M_s &= m_s - 2G_S\sigma_s - 2G_{IS}\tau_s(\sigma_u - \sigma_d) + 2K\sigma_u\sigma_d + \frac{K}{2}\pi^2. \end{aligned}$$

有效化学势矩阵表示为

$$\hat{\mu} = \begin{pmatrix} \tilde{\mu}_u & 0 & 0 \\ 0 & \tilde{\mu}_d & 0 \\ 0 & 0 & \tilde{\mu}_s \end{pmatrix} = \begin{pmatrix} \frac{\tilde{\mu}_B}{3} + \frac{\tilde{\mu}_I}{2} & 0 & 0 \\ 0 & \frac{\tilde{\mu}_B}{3} - \frac{\tilde{\mu}_I}{2} & 0 \\ 0 & 0 & \frac{\tilde{\mu}_B}{3} - \tilde{\mu}_S \end{pmatrix}, \quad \text{(B.17)}$$

其中

$$\begin{aligned} \tilde{\mu}_u &= \frac{\mu_B}{3} + \frac{\mu_I}{2} - \frac{2}{3}G_V\rho - 2\,G_{IV}\tau_u(\rho_u - \rho_d), \\ \tilde{\mu}_d &= \frac{\mu_B}{3} - \frac{\mu_I}{2} - \frac{2}{3}G_V\rho - 2\,G_{IV}\tau_d(\rho_u - \rho_d), \\ \tilde{\mu}_s &= \frac{\mu_B}{3} - \mu_S - \frac{2}{3}G_V\rho - 2\,G_{IV}\tau_s(\rho_u - \rho_d). \end{aligned} \quad \text{(B.18)}$$