# The (modified) Koide formula from flavor nonet scalars

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# The Koide formula

## A relation of charged lepton mass ratios

- The Koide formula: (Koide '82, '83)

$$K = \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2} = \frac{2}{3}.$$

  - Predicting $m_\tau = 1777$MeV from $m_e$ and $m_\mu$ data in 1980's.
  - $m_\tau = 1783^{+3}_{-4}$MeV, DELCO '78 (1783.5±4.2MeV in PDG '80) until $m_\tau = 1776.9^{+0.4}_{-0.5} \pm 0.2$MeV, BES '92.
- PDG '22 data of charged lepton masses:
  - $m_e = 0.51099895000 \pm 0.00000000015$MeV.
  - $m_\mu = 105.6583755 \pm 0.0000023$MeV.
  - $m_\tau = 1776.86 \pm 0.12$MeV ($1776.91 \pm 0.12^{+0.10}_{-0.13}$MeV, BES3 '14).
- The Koide's character from PDG '22 ($10^{-5}$ precision and 1-$\sigma$):

$$K = 0.6666610 \pm 0.0000068 = \frac{2}{3} \times (0.999991 \pm 0.000011).$$

# Developments

## Geometric visualization

- Consider $\vec{M} = \left(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau}\right)$, $\vec{I} = (1,1,1)$, then

$$K^{-1} = \left(\vec{I} \cdot \vec{M}/\|\vec{M}\|\right)^2 = 3\cos^2\theta(\vec{I}, \vec{M}).$$

- $K = 2/3$ sets the angle $\theta(\vec{I}, \vec{M}) = \pi/4$. (Foot '94)

## Radiative corrections to running masses

- QED correction shifts $K_{\mathrm{run}}$ by $10^{-3}$. (Li&Ma '06, Xing&Zhang '06)
- Gauging the flavor symmetry introduces another correction which may cancel the QED correction to $K_{\mathrm{run}}$. (Sumino '08)

## Analogy in quark and neutrino sectors

- Empirical formulas for masses and mixing are conjectured.
- Not as convincing as in the charged lepton sector.

# Explanation

## Coincidence?

- Around $K = 2/3$ is not statistically favored from random distribution of masses or Yukawa couplings.
- $K_{\text{run}} \neq 2/3$ considering radiative corrections to running masses, the known cancellation need some tuning.
- Use pole masses, running masses or on-shell masses?

## Sign of new physics?

- Notice the square roots in the Koide formula, try to build a model giving the charged lepton mass matrix $M \propto \Phi\Phi$, and then $K = \text{Tr}(\Phi\Phi)/(\text{Tr}\,\Phi)^2 \equiv [\Phi\Phi]/[\Phi]^2$.
- Promote $\Phi$ to be a Hermitian nonet scalar in $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{8} \oplus \mathbf{1}$ of the U(3) flavor symmetry, try to build a model setting $\langle\Phi\rangle$ ($\langle\Phi\rangle \equiv \Phi$ if there is no ambiguity) and then $K$.
- Try to naturally explain $K = 2/3$ from hidden symmetries.
- Candidates: the see-saw type model and the Yukawaon model.

# The seesaw-type model

## Seesaw to $M \propto \Phi\Phi$

- Introduce flavor nonet and singlet scalars: $\Phi^i_j$, $S$ and new heavy fermions: $L^i_L = (N^i_L, E^i_L)$, $E^i_R$. (Koide '90)
- The dimension-five effective operators:

$$\mathcal{L}^{(5)} = -\frac{y_0}{\Lambda}\left(\bar{l}_{Li}\Phi^i_j H E^j_R + \bar{L}_{Li}\Phi^i_j H e^j_R + \bar{L}_{Li} S H E^i_R\right) + \text{h.c.}.$$

- The seesaw-type mass terms from $\langle H \rangle = \left(0, v/\sqrt{2}\right)$:

$$\begin{aligned}\mathcal{L} &= -\bar{e}_L m_L E_R - \bar{E}_L m_R e_R - \bar{E}_L M_E E_R + \text{h.c.} \\ &= -\begin{pmatrix}\bar{e}_{Li} & \bar{E}_{Lj}\end{pmatrix}\begin{pmatrix}0 & m^i_{Ll} \\ m^j_{Rk} & M_E\delta^j_l\end{pmatrix}\begin{pmatrix}e^k_R \\ E^l_R\end{pmatrix} + \text{h.c.},\end{aligned}$$

  with $m^i_{Ll} = m^i_{Rl} = \frac{y_0 v}{\sqrt{2}\Lambda}\Phi^i_j$, $M_E = \frac{y_0 v}{\sqrt{2}\Lambda}S$.
- Giving $M_E \gg \|m_L\| = \|m_R\|$, the block-diagonalized mass matrix: $M^i_{Ej} \approx M_E\delta^i_j$, $M^i_{ej} \approx m^i_{Lk} M_E^{-1} m^k_{Rj} = \frac{y_0 v}{\sqrt{2}\Lambda S}\Phi^i_k\Phi^k_j$.

# The Yukawaon model

## SUSY Yukawaons to $M \propto \Phi\Phi$

- Introduce two flavor nonet scalars: the Yukawaon $Y^i_j$ replacing the Yukawa coupling coefficients and the ur-Yukawaon $\Phi^i_j$, both are promoted to chiral superfields. (Koide '08)
- The dimension-five effective operators and the superpotential:

$$\mathcal{L}^{(5)} = -\frac{y_0}{\Lambda} \bar{l}_{Li} Y^i_j H e^j_R + \text{h.c.},$$

$$W_0 = \lambda_A [\Phi\Phi A] + \mu_A [Y A] + W(\Phi, \phi_a),$$

  with another flavor nonet $A^i_j$ and more chiral superfields $\phi_a$.
- At a SUSY vacuum, the F-term equations $\partial_A W_0 = \partial_Y W_0 = 0$ set $A^i_j = 0$ and $Y^i_j = -\frac{\lambda_A}{\mu_A} \Phi^i_k \Phi^k_j$.
- $\langle H \rangle = \left(0, v/\sqrt{2}\right)$ leads to the charged lepton mass matrix: $M^i_{ej} = \frac{y_0 v}{\sqrt{2}\Lambda} Y^i_j = -\frac{y_0 \lambda_A v}{\sqrt{2}\Lambda\mu_A} \Phi^i_k \Phi^k_j$.
- The F-term equations $\partial_\Phi W = \partial_{\phi_a} W = 0$ fix $\Phi$.

# The superpotential from symmetries

## $W(\Phi, \phi_a)$ for $\langle\Phi\rangle$

- The superpotential: (Liang&Sun '20)

$$
\begin{aligned}
W &= W_1 + W_2, \\
W_1 &= \frac{1}{2}\begin{pmatrix}\phi'_1 & \phi'_2\end{pmatrix}\begin{pmatrix}\mu'_{11} & \mu'_{12} \\ \mu'_{12} & \mu'_{22}\end{pmatrix}\begin{pmatrix}\phi'_1 \\ \phi'_2\end{pmatrix} \\
&\quad + \begin{pmatrix}\phi'_1 & \phi'_2\end{pmatrix}\begin{pmatrix}b'_{11} & b'_{12} \\ b'_{21} & b'_{22}\end{pmatrix}\begin{pmatrix}[\Phi_8\Phi_8] \\ [\Phi]^2\end{pmatrix}, \\
W_2 &= \mu_0[\Phi\Phi] = \mu_0[\Phi_8\Phi_8] + \frac{1}{3}\mu_0[\Phi]^2.
\end{aligned}
$$

- Introducing an R-symmetry, $\phi'_1$ and $\phi'_2$ has charge 1, the nonet $\Phi = \Phi_8 + \frac{1}{3}[\Phi]\mathbb{I}_{3\times3} = \Phi_8^a t^a + [\Phi]t^0$ has charge 1/2.
- All renormalizable terms respecting the flavor symmetry and the R-symmetry are included in $W_1$.
- $W_2$ breaks the R-symmetry with a nonzero $\mu_0$.

# Simplification

## Simplifying $W$ by a field redefinition

- Off-diagonalize the quadratic part of $W_1$:

$$P^{\mathsf{T}}\begin{pmatrix}\mu'_{11} & \mu'_{12}\\ \mu'_{12} & \mu'_{22}\end{pmatrix}P=\begin{pmatrix}0 & \mu_3\\ \mu_3 & 0\end{pmatrix},\quad P=\begin{pmatrix}\frac{\mu_3\mu_\pm}{2\Delta} & -\frac{\mu_{22}}{\mu_\pm}\\ -\frac{\mu_3\mu_{11}}{2\Delta} & 1\end{pmatrix},$$

  with $\Delta=\mu_{12}^2-\mu_{11}\mu_{22},\ \mu_\pm=\mu_{12}\pm\sqrt{\Delta}$.

- The field and coefficient redefinition:

$$\begin{pmatrix}\phi'_1\\ \phi'_2\end{pmatrix}=P\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\quad \begin{pmatrix}b'_{11} & b'_{12}\\ b'_{21} & b'_{22}\end{pmatrix}=\left(P^{\mathsf{T}}\right)^{-1}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}.$$

- $W$ is simplified after the redefinition:

$$W=\mu_0[\Phi\Phi]+\mu_3\phi_1\phi_2+\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}[\Phi\Phi]-\frac{1}{3}[\Phi]^2\\ [\Phi]^2\end{pmatrix}.$$

# Towards the modified Koide formula

## Setting $\Phi$ and $K$

- $\Phi \in \mathfrak{u}(3)_{\mathbb{C}} \cong \mathfrak{gl}(3,\mathbb{C}) = \mathbb{C}^{3\times 3}$, so the F-term equations for $\Phi$: $\partial_{\Phi_8} W = \partial_{[\Phi]} W = 0$ are equivalent to $\partial_{\Phi^i_j} W = 0$.
- Assuming $\Phi$ gets a non-zero Hermitian expectation value, the F-term equations $\partial_\Phi W = \partial_{\phi_a} W = 0$ lead to

$$K = \frac{[\Phi\Phi]}{[\Phi]^2} = \frac{2}{3} \times \left(1 - \frac{a_{02} - 9a_4}{2a_{02} - 3a_2}\right),$$

$$\mu_0\mu_3 = \left(2a_{02}\left(K - \frac{1}{3}\right) + a_2\right)[\Phi]^2,$$

  with $a_{02} = b_{11}b_{21},\ a_2 = b_{11}b_{22} + b_{12}b_{21},\ a_4 = b_{12}b_{22}$.
- $K$ is modified by two effective parameters $a_{02}/a_2$ and $a_4/a_2$.
- Nonzero $\mu_0$ and $\mu_3$ are generally needed for a nonzero $\Phi$.
- Building a scalar potential can also set a non-zero $\Phi$, but it is not protected by the SUSY non-renormalization theorem.

# Discussion

## Tuning or lack of prediction power?

- The Cauchy-Schwarz inequality and positive mass condition leads to $K \in [1/3, 1]$, corresponding to $\frac{a_{02}-9a_4}{2a_{02}-3a_2} \in [-1/2, 1/2]$.
- Adjusting parameters can fit any $K$ in the range.
- $K = 2/3$ corresponds to $a_{02} - 9a_4 = 0$.
- In particular, $a_{02} = a_4 = 0$ corresponds to the superpotential:

$$W = \mu_0[\Phi\Phi] + \mu_3\phi_1\phi_2 + b_{11}\phi_1[\Phi_8\Phi_8] + b_{22}\phi_2[\Phi]^2,$$

  which is used in previous literature. (Koide '18)
- Currently no satisfactory reason to choose such a $W$.

## Other pheno

- Pheno from the dimension-five operators or UV completion?
- Gauged flavor symmetry to cancel raidative corrections?