Nuclear Science
Computing Center at CCNU

Microscopic Encoding of Macroscopic Universality: Scaling Properties of Dirac Eigenspectra near QCD Chiral Phase Transition

Wei-Ping Huang (黄玮平)

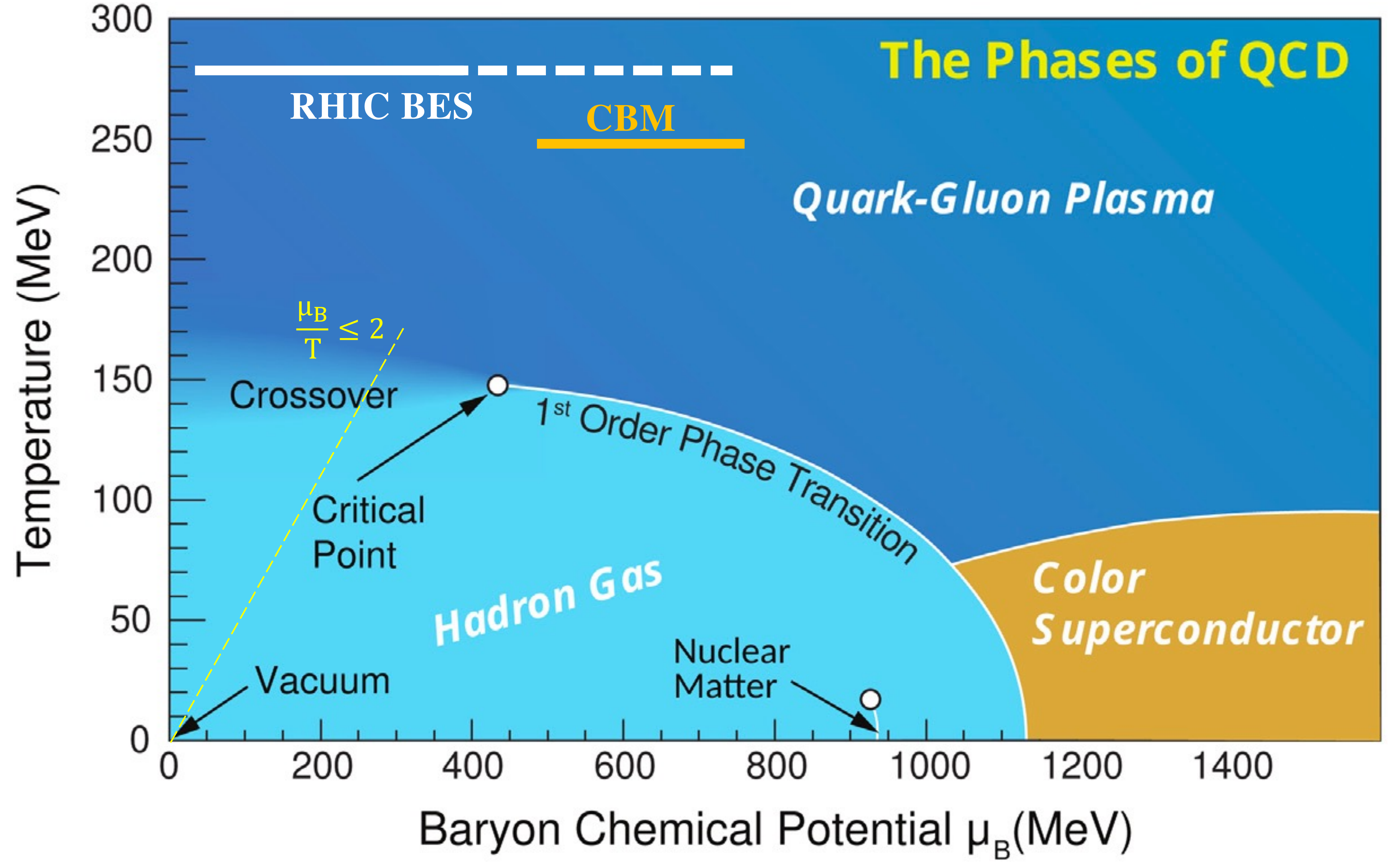
Central China Normal University

based on Phys. Rev. Lett. 131 (2023) 16, 161903,
in collaboration with
Heng-Tong Ding (丁亨通), Swagato Mukherjee, Peter Petreczky

The 15th Workshop on QCD Phase Transition and Relativistic Heavy Ion Physics
Dec 15 -19, 2023 @ Zhuhai

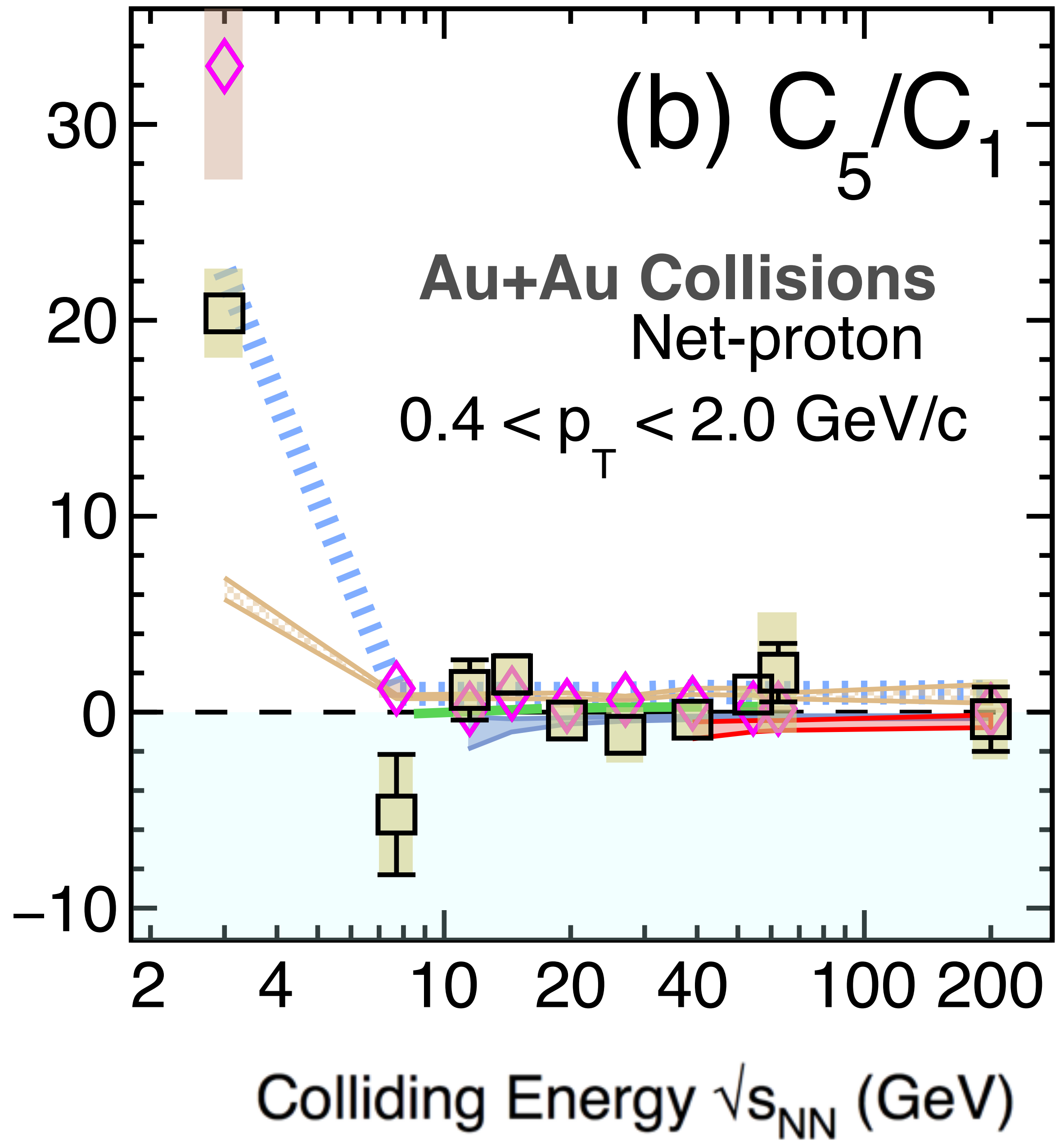
Search for Criticality in QCD

Exploration of QCD phase diagram



D. Almaalol et al., arXiv:2209.05009

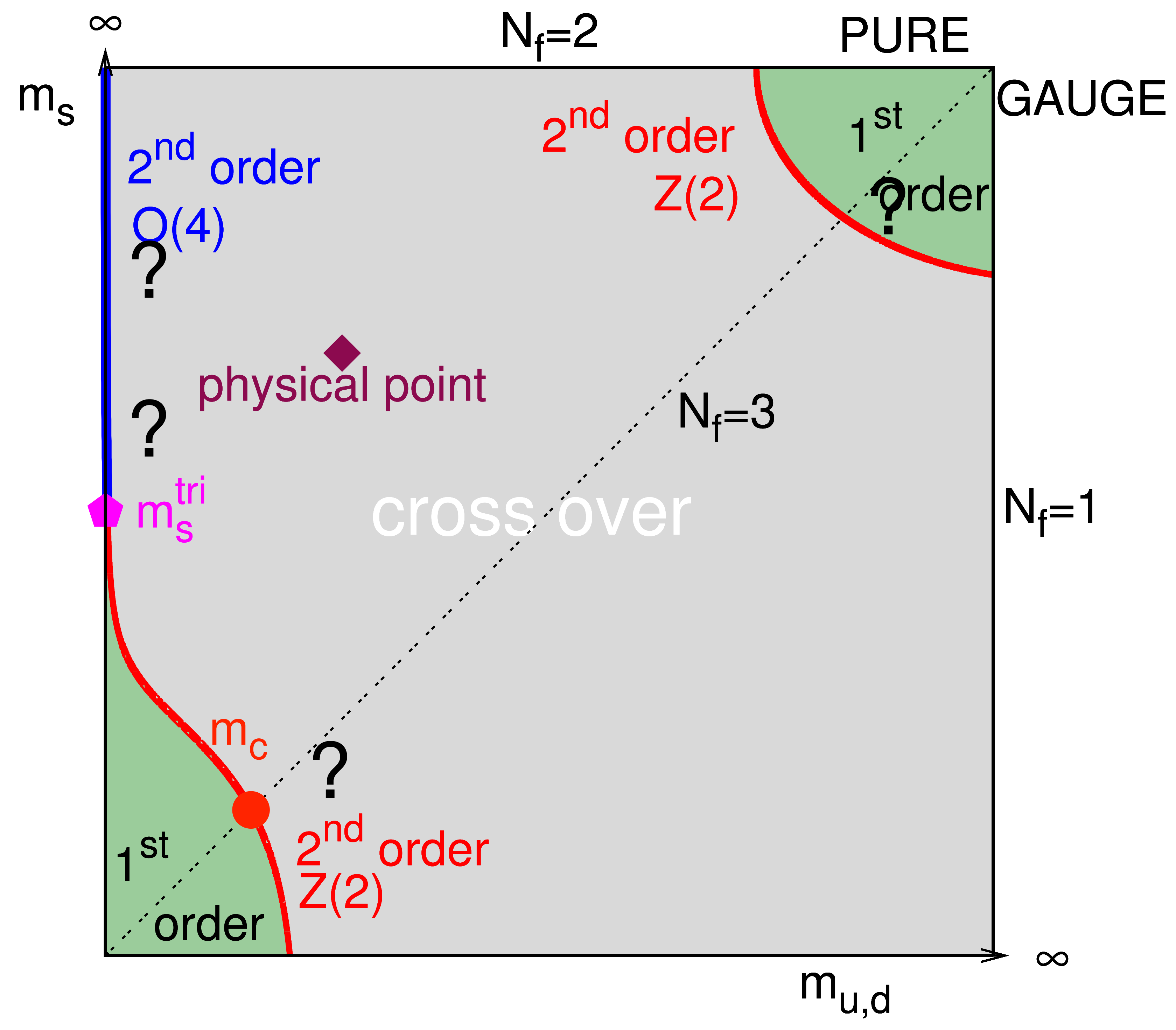
Searching for signatures of criticality using Macroscopic quantities



STAR, Phys. Rev. Lett. 130, 082301 (2023)

How criticality at Macroscale arises from Microscopic *d.o.f.* of QCD ?

Phase Structure of QCD Chiral Transition



Columbia plot:
QCD phase diagram in quark mass plane

H.-T. Ding et al., arXiv:1504.05274

✦ Symmetry of QCD:

In the classical and chiral limit : $SU(2)_L \otimes SU(2)_R \otimes U(1)_V \otimes U(1)_A$

Spontaneous chiral symmetry breaking $\Downarrow$ *Quantum axial anomaly*

Quantized : $SU(2)_V \otimes U(1)_V$

✦ Restoration of chiral symmetry:

S. Borsanyi et al., PRL 125 (2020) 052001
A. Bazavov et al., [HotQCD], PLB 795 (2019) 15
H.-T. Ding et al., [HotQCD], PRL 123 (2019) 062002

— At physical point : crossover

— In limits of zero quark masses : true chiral phase transition

✦ Role of $U(1)_A$ symmetry in chiral phase transition:

Pelissetto & Vicari, PRD 88 (2013) 105018
Pisarski & Wilczek, PRD 29 (1984) 338
Grahl, PRD 90 (2014) 117904

— Broken: $2^{\rm nd}$ order O(4) phase transition

— Effective restored: $1^{\rm st}$ or $2^{\rm nd}$ order $U(2)_L \otimes U(2)_R / U(2)_V$ phase transition

Universal Scaling in QCD Chiral Transition

Dimensionality & Symmetries ← **determined by** — Scaling behaviors in 2nd order phase transition — **irrespective of** → Microscopic *d.o.f.* & Interactions

Universal O(2) scaling behaviors of chiral phase transition

Staggered discretization scheme ⟷ (same) Three-dimensional O(2) spin model

Order parameter :

$$M(t,h) = h^{1/\delta} f_1(z)$$

Order parameter susceptibility :

$$\chi_M(t,h) = \partial M/\partial H = h_0^{-1} h^{1/\delta-1} f_2(z)$$

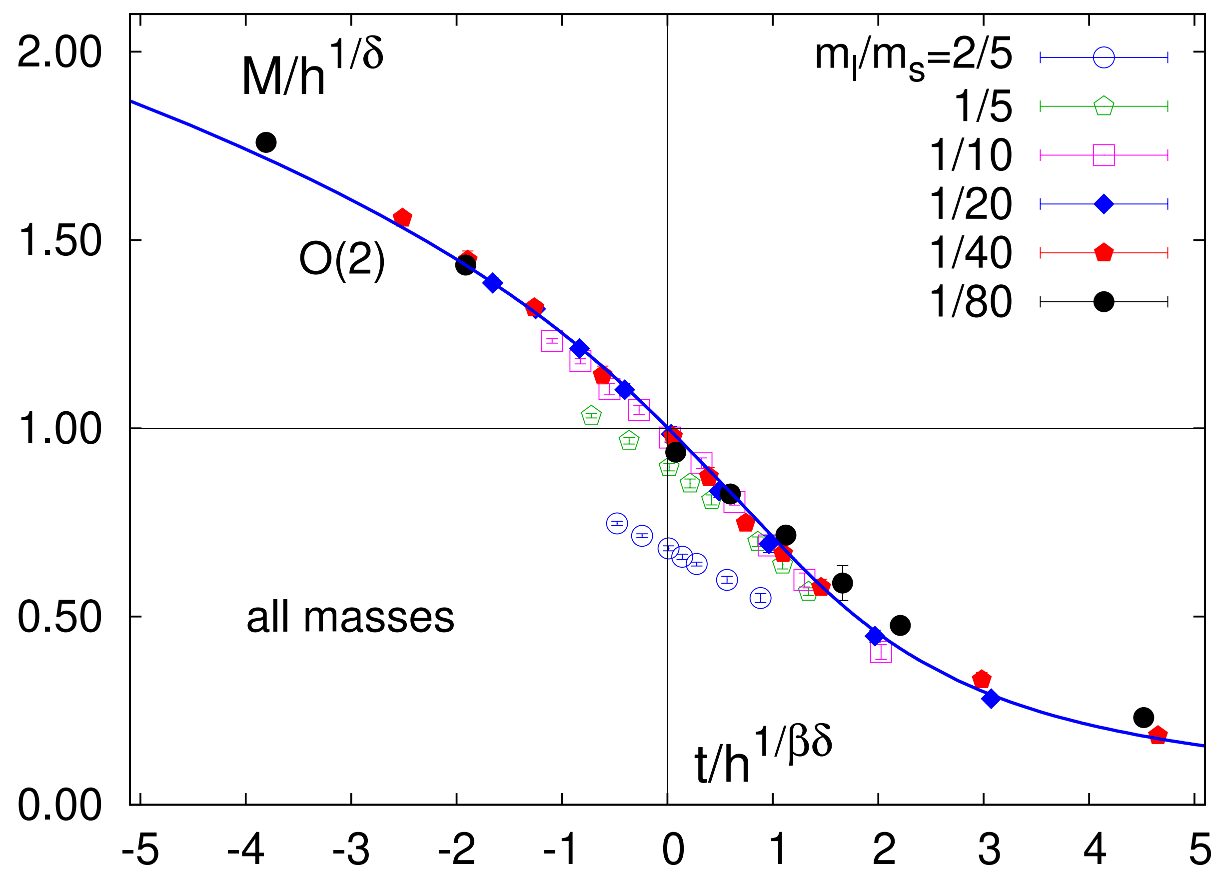
n-th order susceptibility :

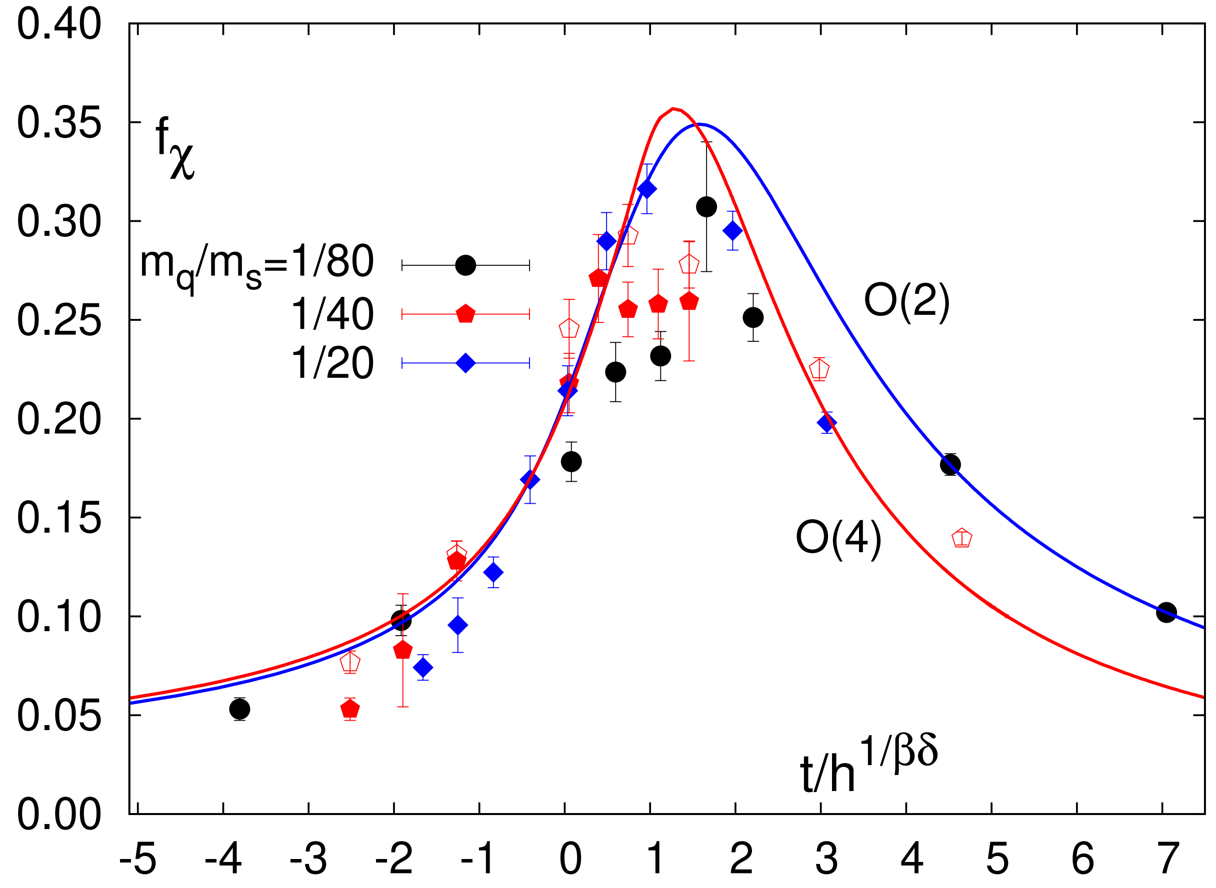
e.g., $\partial \chi_M/\partial H = h_0^{-2} h^{1/\delta-2} f_3(z)$

$$z = t/h^{1/\beta\delta}$$

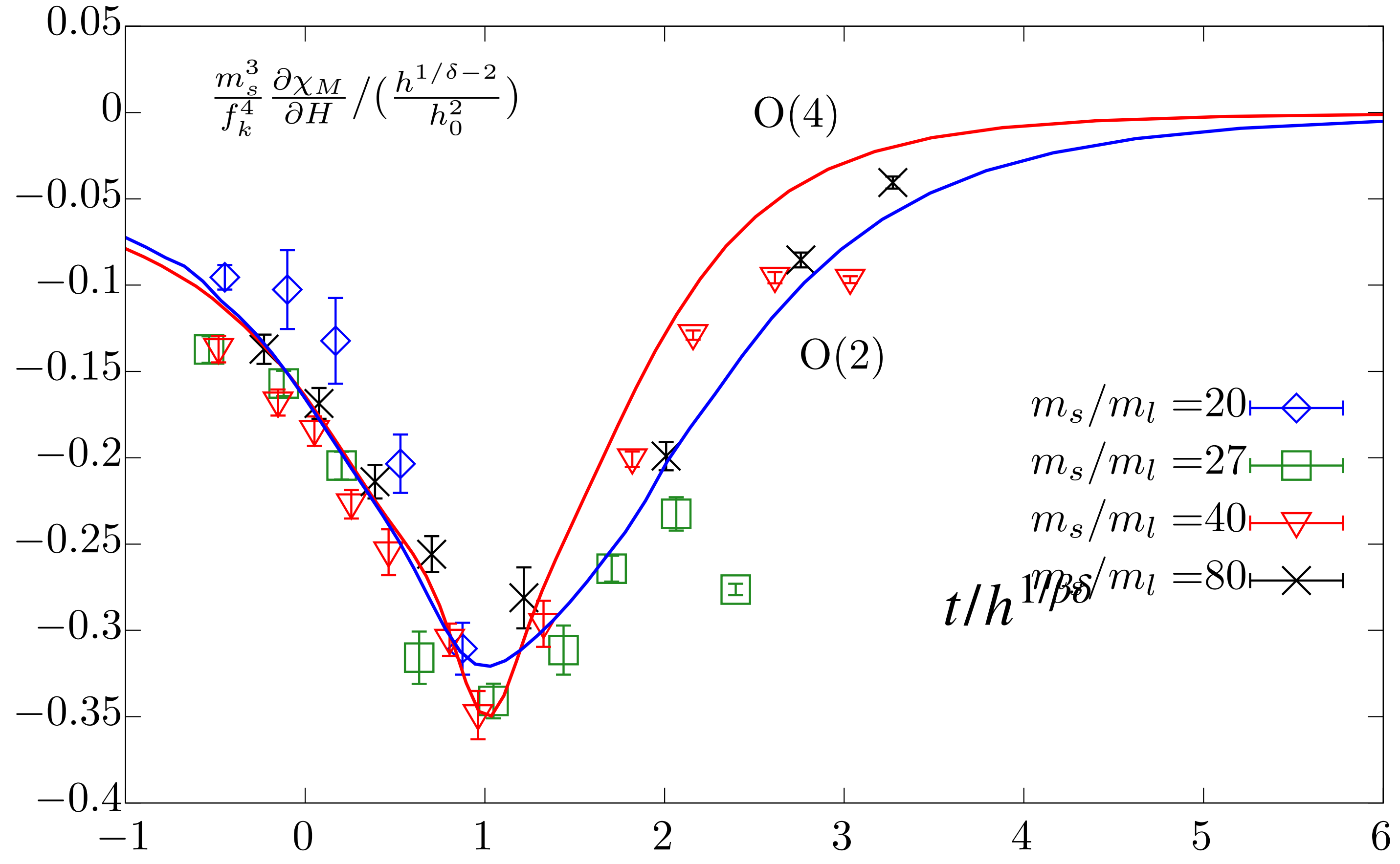
$$t = \frac{T - T_c}{t_0 T_c}$$

$$h = \frac{m_l}{h_0 m_s}$$





S. Ejiri et al., PRD 80, 094505 (2009)



W.-P. Huang et al., arXiv: 2112.00318

Hard to Reach Microscopic Origin of Criticality

Dimensionality
&
Symmetries

Microscopic *d.o.f.*
&
Interactions

determined by

irrespective of

Scaling behavior around
2nd order phase transition

resort to

Effective theories : ignoring microscopic complexities

— succeed in understanding macro properties

— failed to unveil the microscopic origin due to the blindness to micro complexities

Challenging to understand how criticality arises from micro level of d.o.f. !!

Banks-Casher Relation: Connect Macro to Micro in Chiral Limit

Chiral order parameter : $\langle\bar{\psi}\psi\rangle = \int_0^\infty \frac{4m\,\rho(\lambda,m)}{\lambda^2+m^2}\,d\lambda \xrightarrow{m\to 0} \pi\rho(0)$

Banks & Casher, NPB 169(1980) 103

$$\rho(\lambda,m) = \frac{T}{V}\langle\rho_U(\lambda)\rangle = \frac{T}{V}\langle\sum_j \delta(\lambda-\lambda_j)\rangle \quad \text{with} \quad \not{D}\psi_j = i\lambda_j\psi_j$$

Calculable by Lattice QCD !!

Connect Macroscopic quantity to Microscopic *d.o.f.* : $\langle\bar{\psi}\psi\rangle \Longleftrightarrow \rho_U(\lambda)$: ***energy spectra of massless quarks***

More generally

This talk: A first lattice QCD-based understanding of Microscopic Origin of Criticality in QCD !

From Quark Energy Spectra to its Cumulants

$$\langle \bar\psi\psi \rangle = \int_0^\infty \frac{4m\ \rho(\lambda, m)}{\lambda^2 + m^2}\, d\lambda = \frac{T}{V}\int_0^\infty \left\langle \frac{4m\ \rho_U(\lambda)}{\lambda^2+m^2} \right\rangle d\lambda$$

H.-T. Ding et al., PRL 131, 161903

1st order cumulant :

$$\langle \bar\psi\psi \rangle = \frac{T}{V}\langle 2\ \mathrm{Tr}(\not{D}[U] + m)^{-1} \rangle$$

1-point correlation of quark energy spectra $\dfrac{4m\ \rho_U(\lambda)}{\lambda^2+m^2}$

Generalization : from ***generating functional*** of cumulants

— ***Generating functional*** : $\mathbb{G}(m;\epsilon) = \ln\left\langle \exp\{-m\ \bar\psi\psi(\epsilon)\} \right\rangle_0 = \ln\left\langle \exp\left\{ -m\int_0^\infty P_U(\lambda;\epsilon)\, d\lambda \right\}\right\rangle_0$

$\langle\cdots\rangle_0$: average over QCD partition function in the chiral limit

Probe operator with valance quark mass ϵ :
$\bar\psi\psi(\epsilon) \equiv 2\ \mathrm{Tr}(\not{D}[U] + \epsilon)^{-1}$

Defining $P_U(\lambda;\epsilon) \equiv \dfrac{4\epsilon\rho_U(\lambda)}{\lambda^2+\epsilon^2}$

— ***n-th order cumulant of $\bar\psi\psi$*** : $\mathbb{K}_n[\bar\psi\psi] = \frac{T}{V}(-1)^n \frac{\partial^n \mathbb{G}(m;\epsilon)}{\partial m^n}\Big|_{\epsilon=m}$

NEW!!

$$\mathbb{K}_n[\bar\psi\psi(m)] = \int_0^\infty K_1[P_U(\lambda_1;m), P_U(\lambda_2;m), \cdots, P_U(\lambda_n;m)] \prod_{i=1}^{n} d\lambda_i$$

$K_1[X_1, X_2, \cdots, X_n]$ denotes 1st order joint cumulant of n-variables

Microscopic Encoding of Macroscopic Criticality

H.-T. Ding et al., PRL 131, 161903

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1;m), P_U(\lambda_2;m), \cdots, P_U(\lambda_n;m)] \prod_{i=1}^{n} d\lambda_i \equiv \int_0^\infty P_n(\lambda)\, d\lambda$$

n-th order cumulant of the chiral order parameter

n-point correlation of the quark energy spectra

Chiral condensate :
$$\mathbb{K}_1[\bar{\psi}\psi] = \frac{T}{V}\langle\bar{\psi}\psi(m)\rangle = \int_0^\infty P_1(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m^{1/\delta} f_1(z)$$

Disconnected susceptibility :
$$\mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V}\left\langle\left[\bar{\psi}\psi(m) - \langle\bar{\psi}\psi(m)\rangle\right]^2\right\rangle = \int_0^\infty P_2(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m^{1/\delta-1} f_2(z)$$

......

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta-n+1} f_n(z)$$

How criticality of $\mathbb{K}_n[\bar{\psi}\psi]$ arises from $P_n(\lambda)$?

Microscopic Encoding of Macroscopic Criticality

Hints from the chiral limit :

$$P_U(\lambda; m) \equiv \frac{4m\rho_U(\lambda)}{\lambda^2 + m^2}$$
$$P_U(\lambda; m \to 0) = 2\pi\rho_U(\lambda)\delta(\lambda)$$

Generalized Banks-Casher relation :

$$\lim_{m\to 0} P_n(\lambda) = (2\pi)^n K_1[\rho_U(\lambda), \underbrace{\rho_U(0), \cdots, \rho_U(0)}_{(n-1)\ \text{terms}}]\delta(\lambda)$$

$$\Longrightarrow \lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]$$

$n = 1$ back to Banks-Casher relation !

Criticality in $\lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi]$ must arise from universal behaviors of ***λ-independent*** $\mathbb{K}_n[\rho_U(0)]$

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta - n + 1} f_n(z)$$

Conjecture: $P_n(\lambda) = m^{1/\delta - n + 1} f_n(z) g_n(\lambda)$

Scaling arises from $P_n(\lambda)$ at ***deep infrared*** λ region

Includes ***all*** system-specific λ-dependence

H.-T. Ding et al., PRL 131, 161903

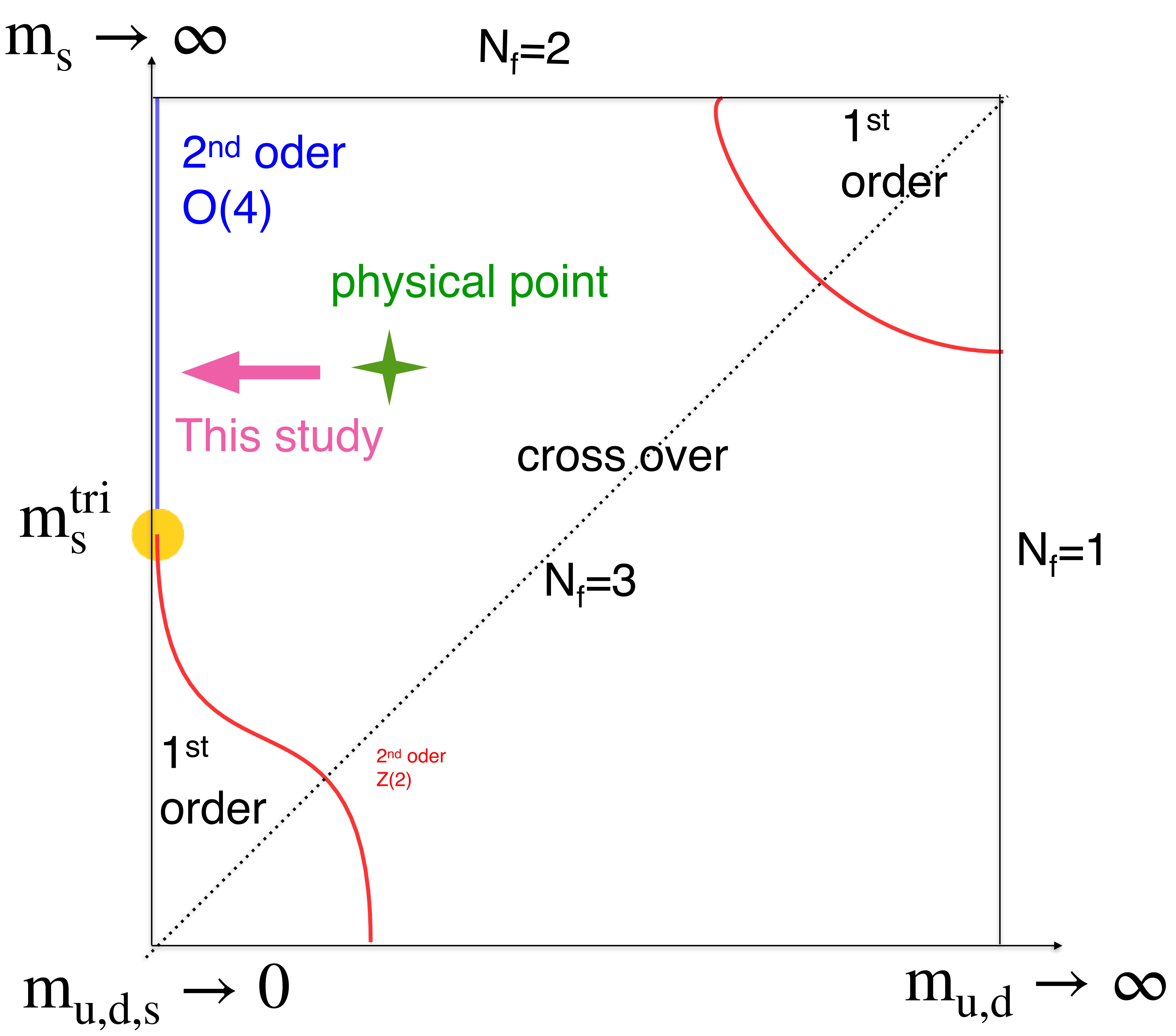
Lattice Setup

- Actions: Highly improved staggered quarks and tree-level Symanzik gauge action
- Lattice size: $N_\tau = 8, \quad N_\sigma = 32, 40, 56$
- Quark mass: $m_s^{\rm phy}/m_l = 27, 40, 80, 160$

  $(m_\pi \approx 140, 110, 80, 55 \text{ MeV})$
- Temperatures: $T \in (135,176)$ MeV
- $\rho_U(\lambda)$ computed via Chebyshev filtering technique

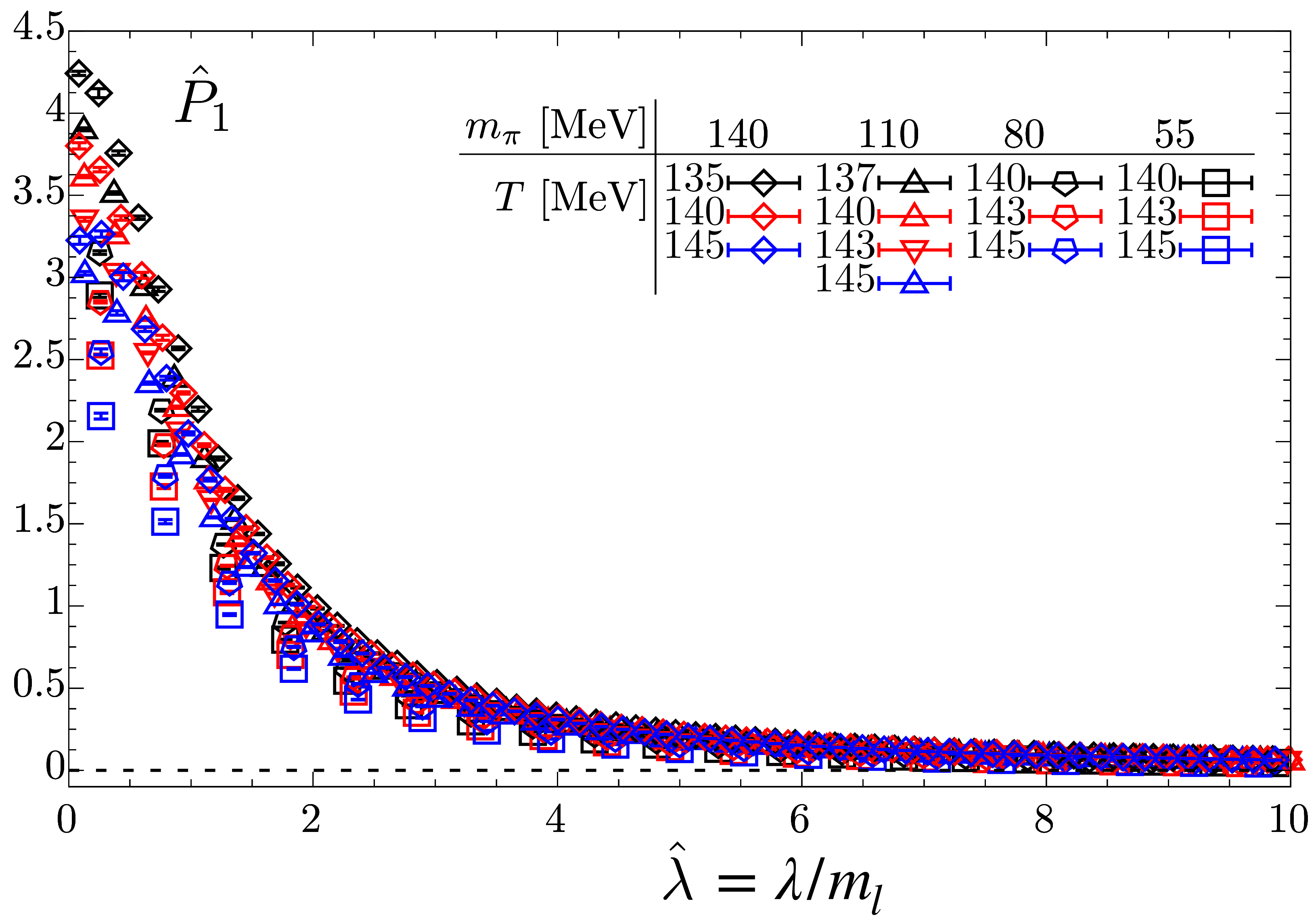
  H.-T. Ding et al., PRL 126 (2021) 8, 082001
- HotQCD configurations; Measurements carried out on NSC^3 at CCNU & BNL

NuclearScience Computing Center at CCNU

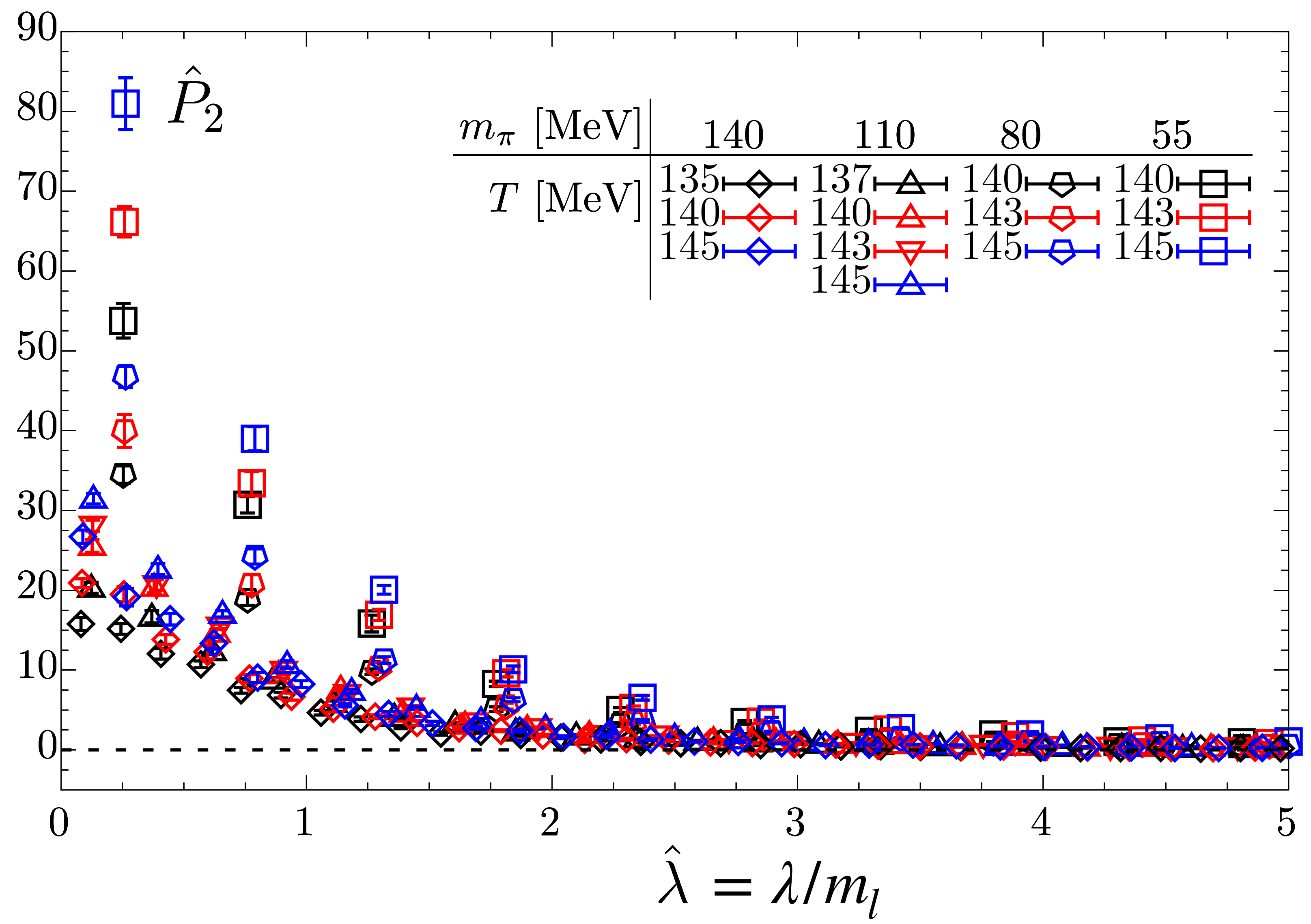


$P_n(\lambda)$ around T_c

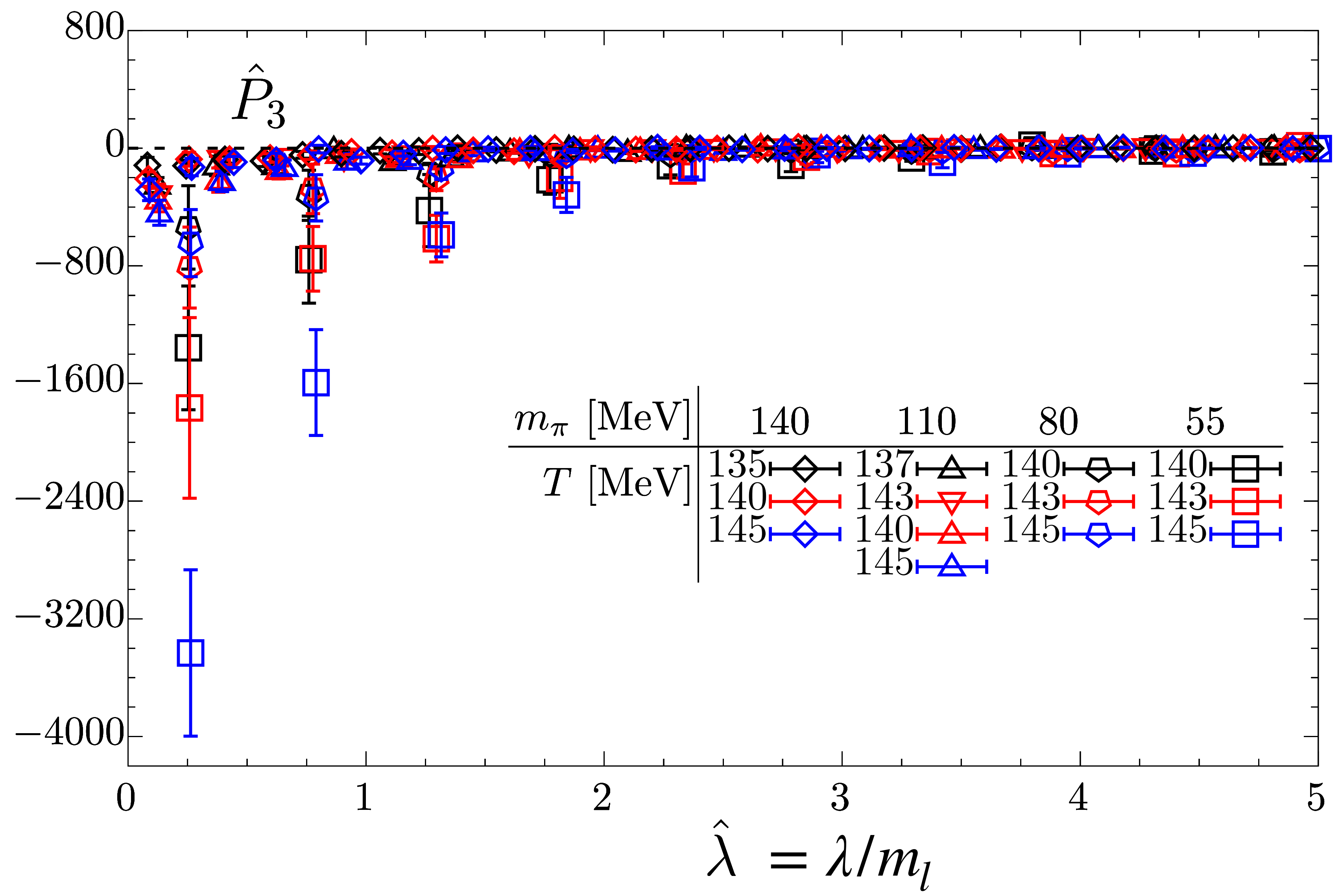
$\hat{P}_1(\hat{\lambda}) = m_s^2(m_l/m_s)P_1(\lambda)/T_c^4$



$\hat{P}_2(\hat{\lambda}) = m_s^3(m_l/m_s)P_2(\lambda)/T_c^4$



$\hat{P}_3(\hat{\lambda}) = m_s^4(m_l/m_s)P_3(\lambda)/T_c^4$



$\hat{P}_1(\hat{\lambda}), \hat{P}_2(\hat{\lambda})$ and $\hat{P}_3(\hat{\lambda})$

— Infrared lambda region dominates;

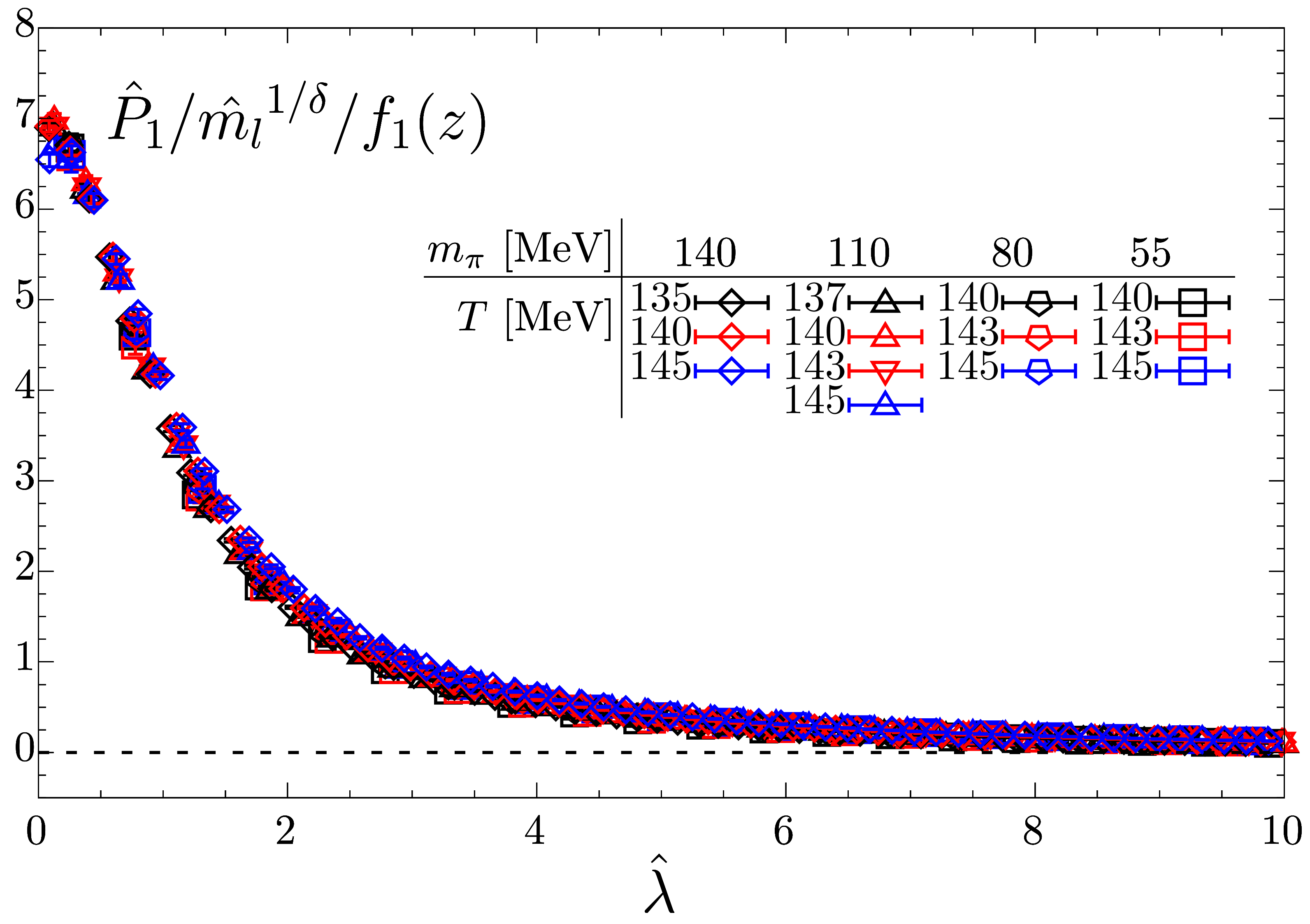
— Significant dependence on quark mass and temperature

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n + 1} f_n(z) g_n(\hat{\lambda})$

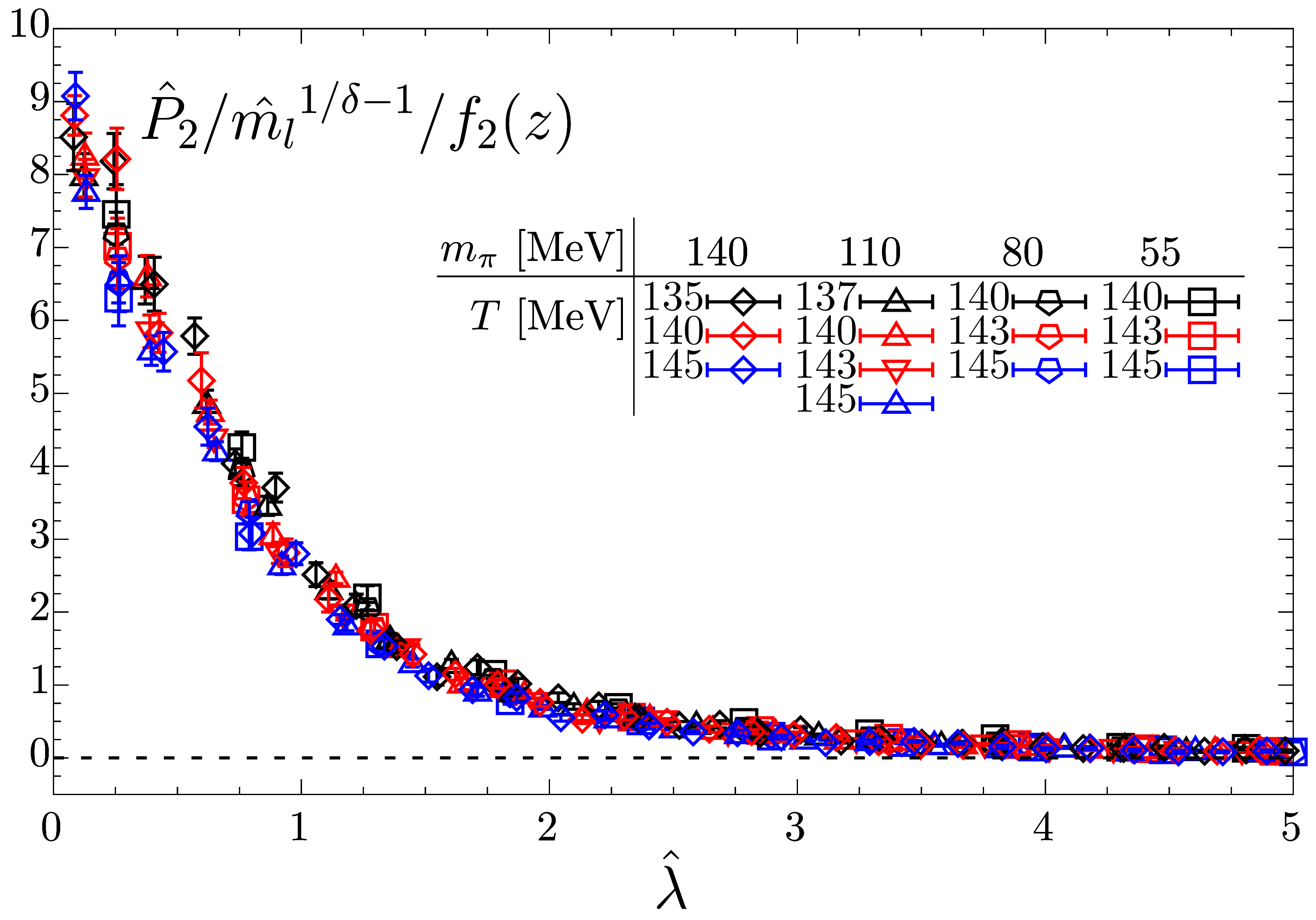
Rescaled $P_n(\lambda)$ around T_c

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n + 1} f_n(z) g_n(\hat{\lambda})$

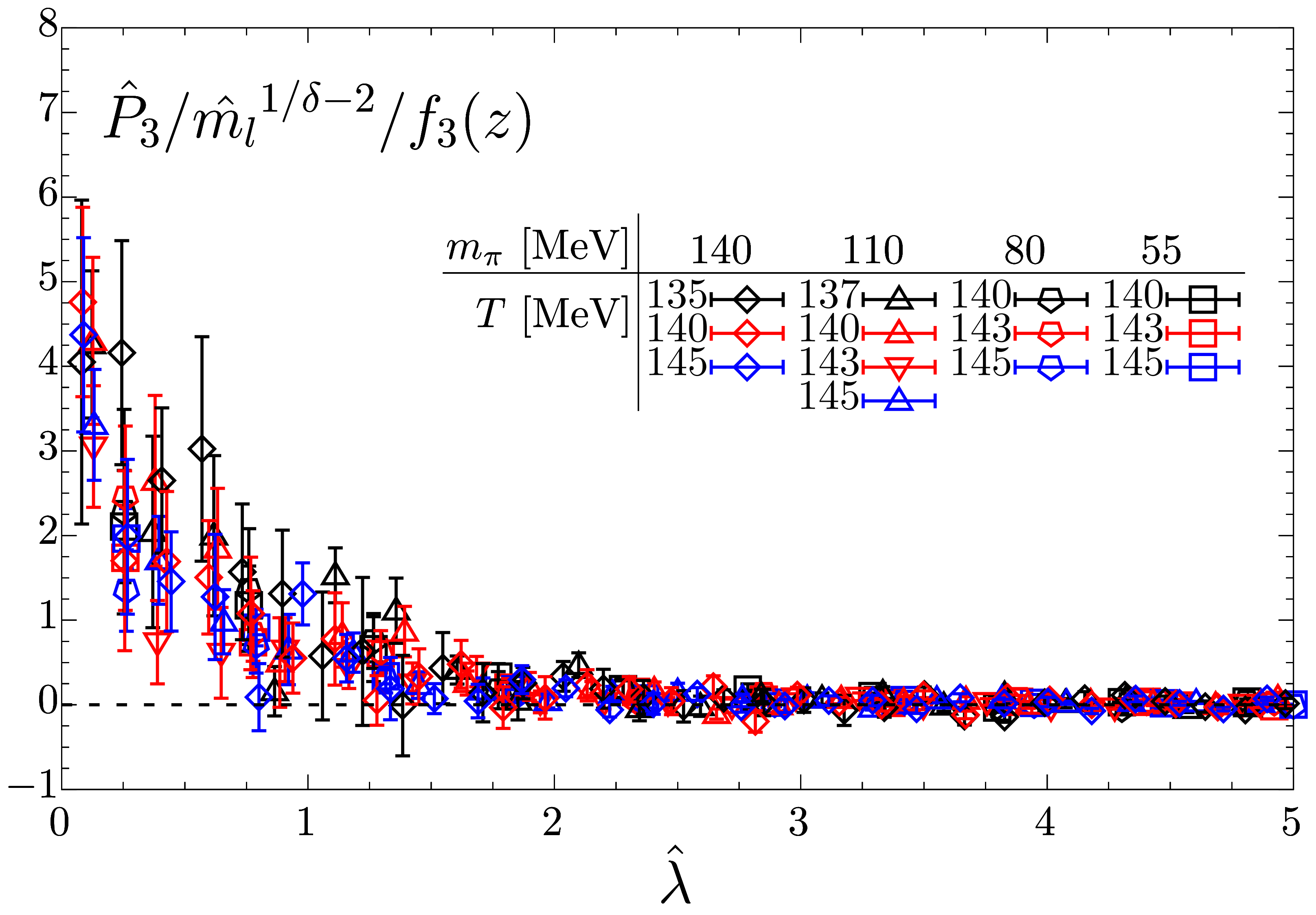
$\hat{P}_1(\hat{\lambda})/(m_l/m_s)^{1/\delta}/f_1(z)$



$\hat{P}_2(\hat{\lambda})/(m_l/m_s)^{1/\delta-1}/f_2(z)$



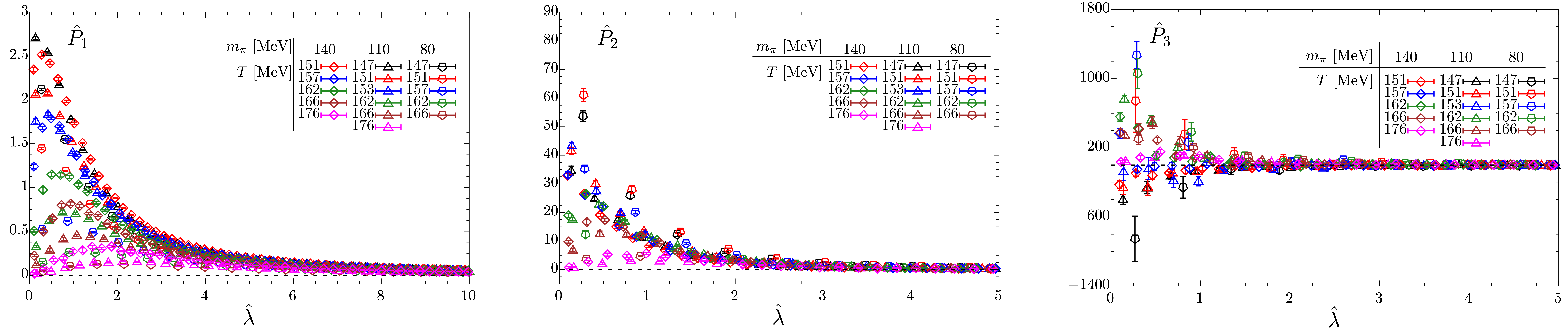
$\hat{P}_3(\hat{\lambda})/(m_l/m_s)^{1/\delta-2}/f_3(z)$



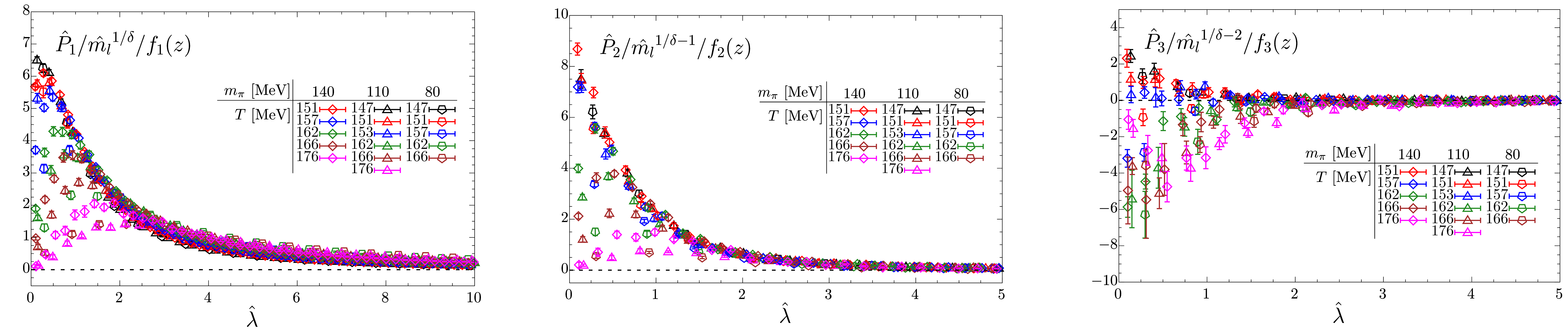
$z = z_0(m_l/m_s)^{-\frac{1}{\beta\delta}}(T - T_c)/T_c$: $O(2)$ scaling parameters adopted from [S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)]

- In the vicinity of T_c , $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n + 1} f_n(z) g_n(\hat{\lambda})$
- Scaling behaviors in $\hat{P}_n(\hat{\lambda})$ extend up to physical light quark mass

$P_n(\lambda)$ and Rescaled $P_n(\lambda)$ away from T_c



Away from T_c , no scaling behaviors are observed in $\hat{P}_n(\hat{\lambda})$



Summary

- We establish a novel relation

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1; m), P_U(\lambda_2; m), \cdots, P_U(\lambda_n; m)] \prod_{i=1}^{n} d\lambda_i \equiv \int_0^\infty P_n(\lambda)\ d\lambda$$

n-th order cumulant of the chiral condensate

n-point correlation of the quark energy spectra

- A generalization of the Banks-Casher relation is obtained:

$$\lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]$$

- Microscopic encoding of macroscopic criticality

$$P_n(\lambda) = m_l^{1/\delta - n + 1} f_n(z) g_n(\lambda)$$

- Universal behaviors manifested in microscopic energy levels of QCD extend up to physical light quark masses

Backup

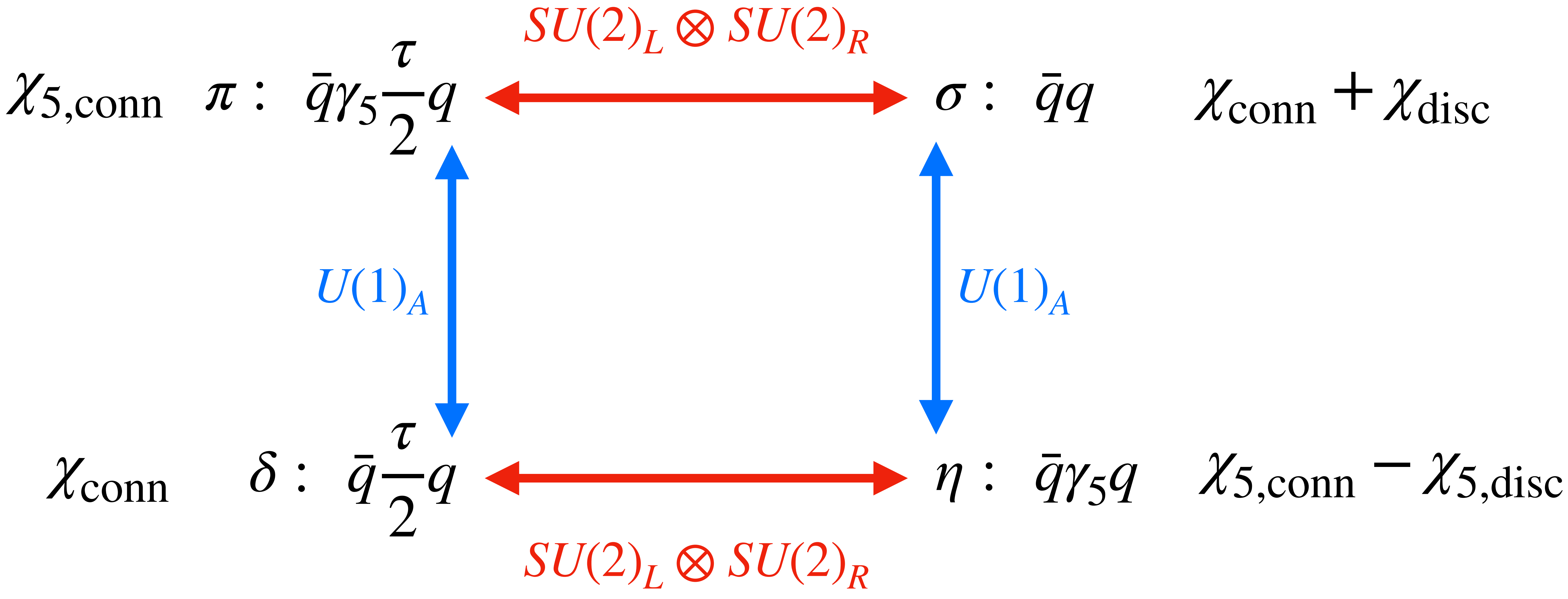
Signatures of symmetry restorations

Susceptibilities defined as integrated two point correlation functions of quark bilinear $J_M(x) = \bar{q}(x)\Gamma_M q(x)$

$$\chi_M = \int d^4x \left\langle J_M(x) J_M^\dagger(0) \right\rangle$$

A. Bazavov et al., [HotQCD], PRD 86 (2012) 094503

N. Carabba et al., [HotQCD], PRD 105 (2022) 5, 054034



Restoration of $SU(2)_L \otimes SU(2)_R$

$$\chi_\pi - \chi_\sigma = 0$$
$$\chi_\delta - \chi_\eta = 0$$
$$\Rightarrow \chi_\pi - \chi_\delta = \chi_{\mathrm{disc}} = \chi_{5,\mathrm{disc}}$$

Effective restoration of $U(1)_A$

$$\chi_\pi - \chi_\delta = 0$$
$$\chi_\sigma - \chi_\eta = 0$$
$$\Rightarrow \chi_\pi - \chi_\delta = \chi_{\mathrm{disc}} = \chi_{5,\mathrm{disc}} = 0$$

Related to Dirac eigenvalues:

$$\langle\bar{\psi}\psi\rangle = \int_0^\infty \mathrm{d}\lambda \frac{4m_l \rho(\lambda, m_l)}{\lambda^2 + m_l^2} \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8m_l^2 \rho(\lambda, m_l)}{\left(\lambda^2 + m_l^2\right)^2} \qquad \chi_{\mathrm{disc}} = \int_0^\infty \mathrm{d}\lambda \frac{4m_l \cdot \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

Microscopic origin in Dirac eigenvalues

$$\langle\bar{\psi}\psi\rangle_l = \int_0^\infty \mathrm{d}\lambda \frac{4m_l\rho(\lambda, m_l)}{\lambda^2 + m_l^2} \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8m_l^2\rho(\lambda, m_l)}{\left(\lambda^2 + m_l^2\right)^2} \qquad \chi_{\rm disc} = \int_0^\infty \mathrm{d}\lambda \frac{4m_l \cdot \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

- Restoration of $SU(2)_L \otimes SU(2)_R$ symmetry:

  $\rho(0) = 0$ from Banks-Casher formula $\lim_{m_l\to 0}\lim_{V\to\infty}\langle\bar{\psi}\psi\rangle_l = = \lim_{m_l\to 0}\lim_{V\to\infty} 2\pi\rho\left(0,m_l\right)$

  Banks and Casher, NPB 169 (1980) 103

- Effective restoration of $U(1)_A$ symmetry:

  A sizable gap in the near-zero mode Cohen, arXiv:nucl-th/9801061

- Underlying structure of $\rho(\lambda, m_l)$ responsible for symmetry restorations:

$$\rho(\lambda, m_l) = \mathrm{c}_0 + \mathrm{c}_1\lambda + \mathrm{c}_2 m_l^2\delta(\lambda) + \mathrm{c}_3 m_l + \mathrm{c}_4 m_l^2 + \cdots$$

$$\Rightarrow \quad \langle\bar{\psi}\psi\rangle = 2\mathrm{c}_0\pi - 4\mathrm{c}_1 m_l \ln(m_l) + 2\mathrm{c}_2 m_l + 2\mathrm{c}_3\pi + 2\pi\mathrm{c}_4 m_l^2$$

$$\chi_\pi - \chi_\delta = 2\mathrm{c}_0\pi/m_l + 4\mathrm{c}_1 + 4\mathrm{c}_2 + 2\mathrm{c}_3\pi + 2\mathrm{c}_4\pi m_l$$

c_0 & c_1 term: break both symmetries — Gross et al., RMP 81'

c_2 term: dilute instanton gas predicts — HotQCD, PRD86(2012)094503

c_3 term: break $U(1)_A$ symmetry — Aoki et al., PRD86(2012)114512

c_4 term: make $U(1)_A$ anomaly unmanifested in 2-pt correlators

Mass derivatives of ρ are needed to determine the microscopic origin

Calculation of eigenvalue spectrum

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues

- Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number

Mode number : $$n_{[s,t]} \approx \frac{1}{n_v} \sum_{k=1}^{n_v} \left[\sum_{j=0}^{p} g_j^p \gamma_j v_k^T T_j(A) v_k \right]$$

T_j : Chebyshev polynomial

γ_j & g_j^p : expansion coefficients

n_v : number of random vectors

p : number of polynomial orders

eigenvalue spectrum : $$\rho_U(\lambda) = \frac{1}{4} \frac{n_{[\lambda-\delta/2,\lambda+\delta/2]}}{2\delta\lambda}$$

$1/4$: Staggered Fermion Discretization Scheme

$1/2$: positive and negative eigenvalue pairs

$\delta\lambda$: bin-size

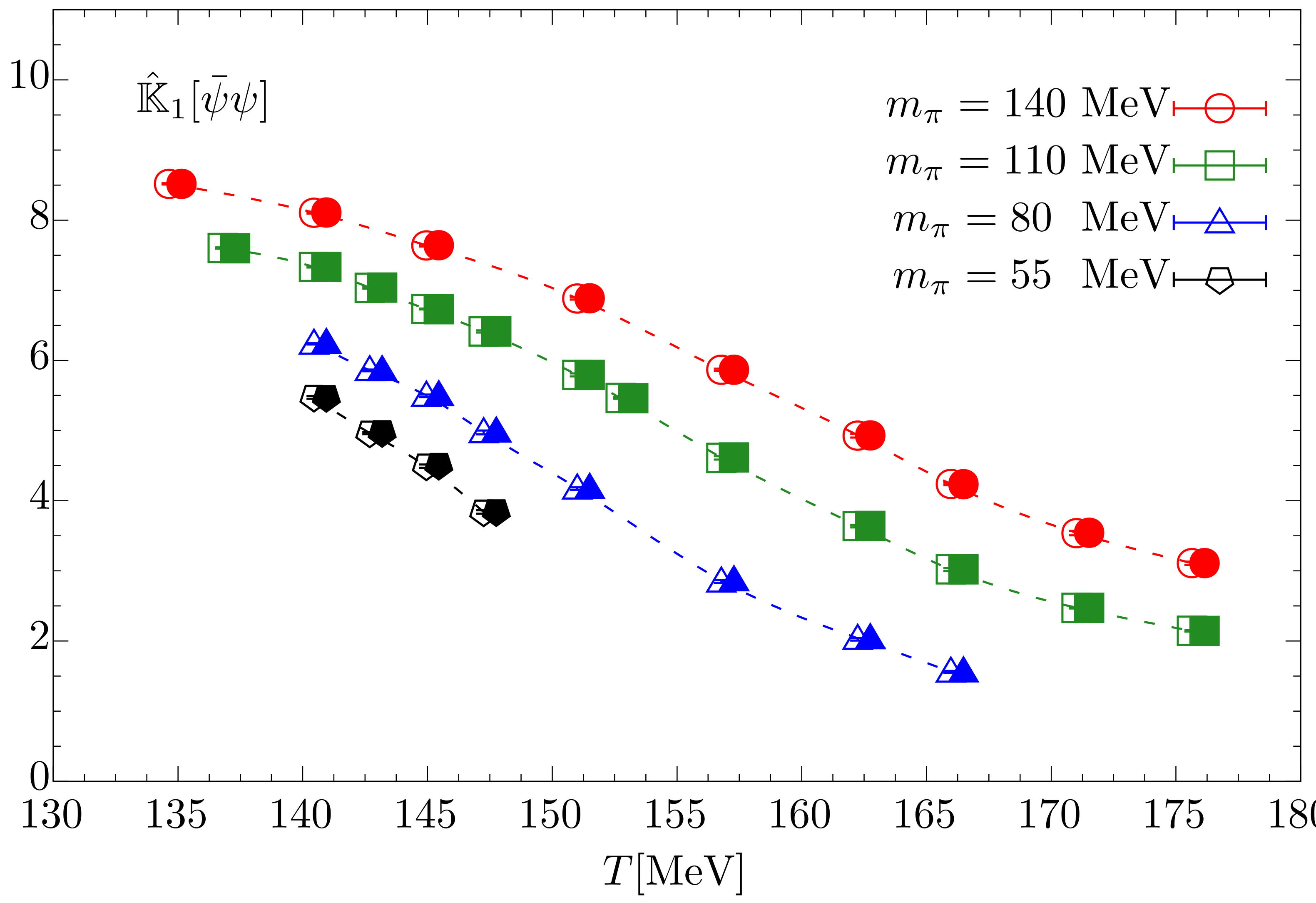
H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)

Yu Zhang, Lattice 19', arXiv: 2001.05217

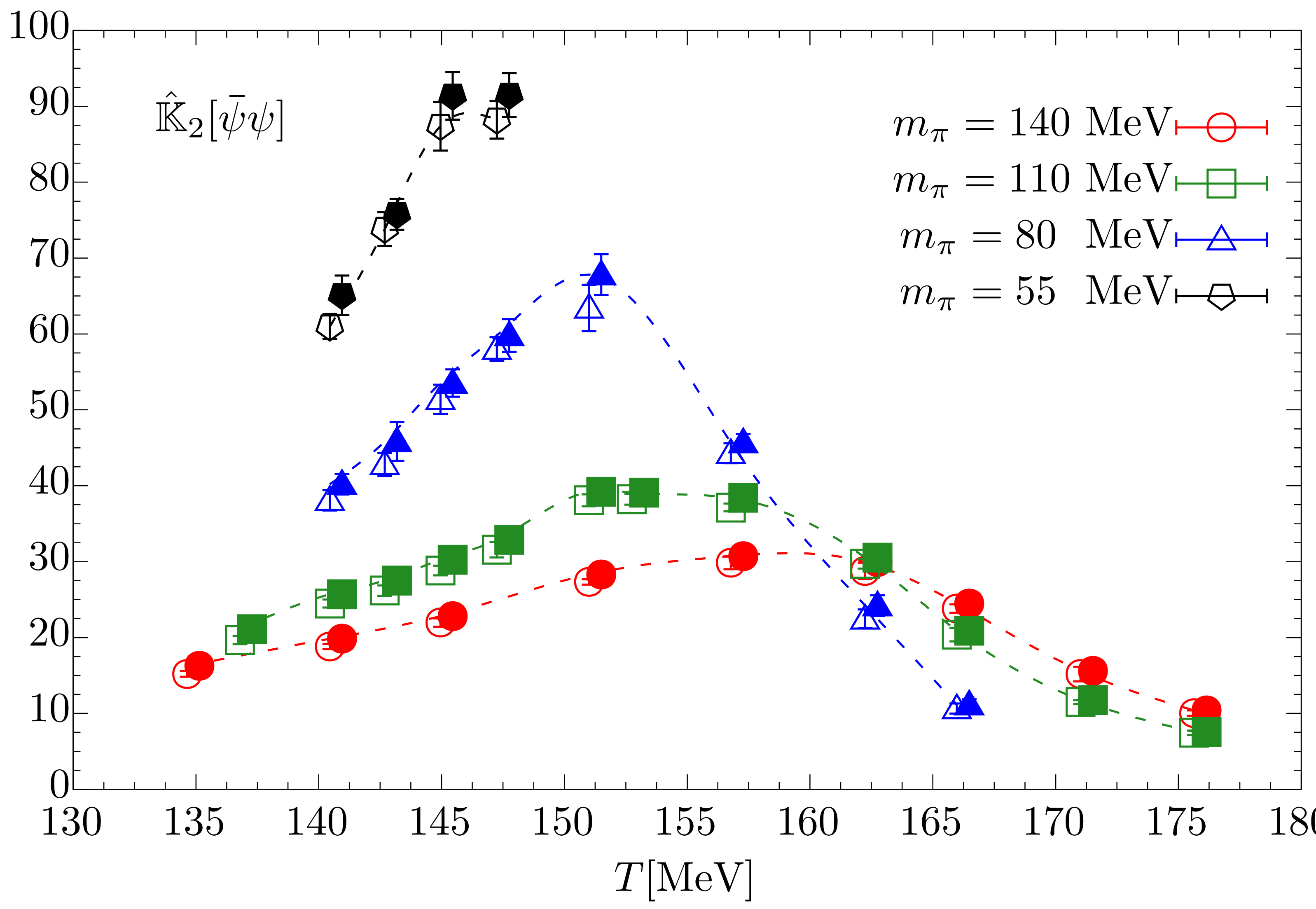
Cossu et al., arXiv: 1601.00744

Reproduction of Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$ via $P_n(\lambda)$

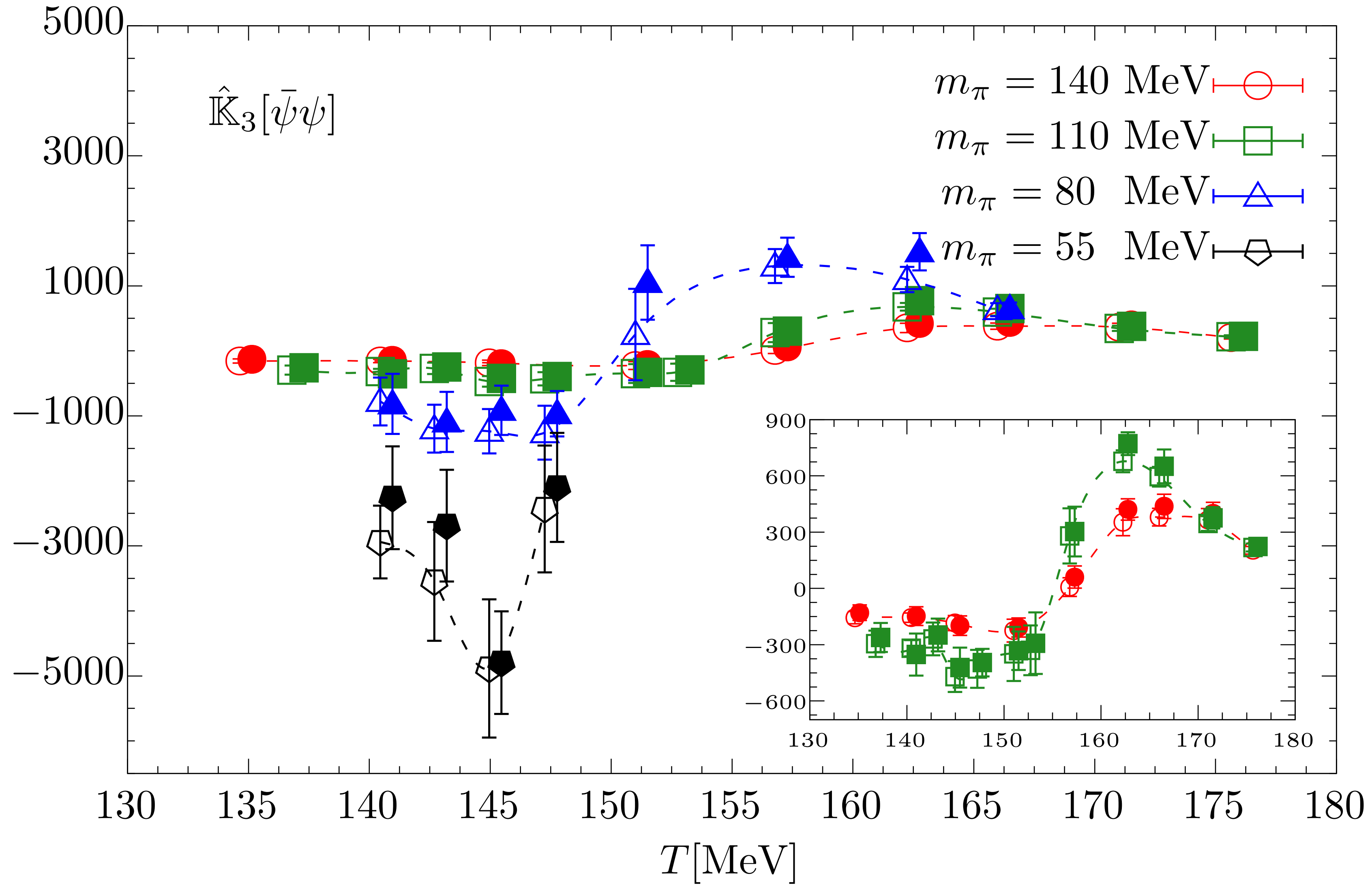
$$\mathbb{K}_1[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_1[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_1(\lambda)\ d\lambda$$



$$\mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_2[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_2(\lambda)\ d\lambda$$



$$\mathbb{K}_3[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_3[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_3(\lambda)\ d\lambda$$



Open symbols: computation via inversions of the fermion matrix $\mathrm{Tr}M^{-1}$

Filled symbols: reconstructed from $P_n(\lambda)$

Cumulants related to $P_n(\lambda)$ can successfully reproduce their corresponding results from inverse fermion matrix

Criticality in Macroscopic Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$

$O(2)$ scaling with $\beta = 0.349$, $\delta = 4.78$, $z_0 = 1.83(9)$, $T_c(N_\tau = 8) = 144.2(6)$ MeV

Parameters adopted from: S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)

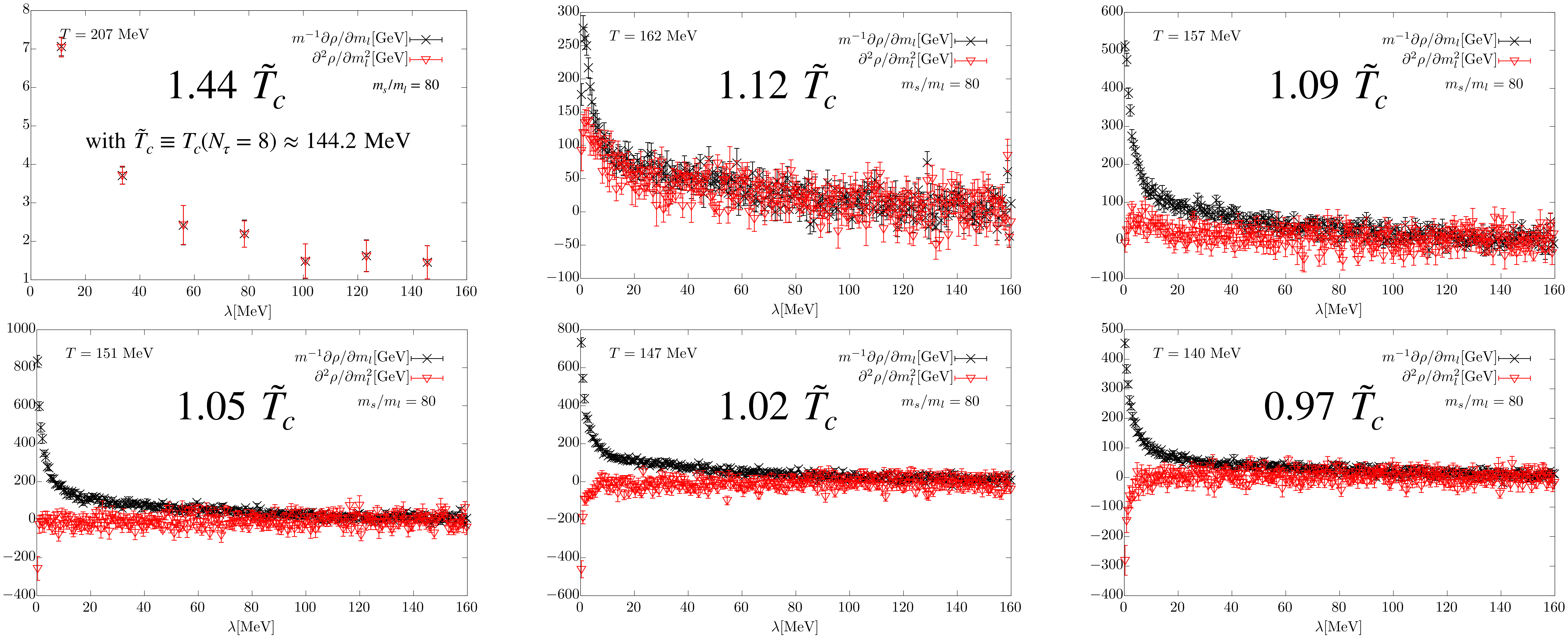
- For $|z/z_0| \lesssim 0.2$, $K_n(z)/K_n(z=0)$ with $n = 1,2,3$ can be well described by $O(2)$ scaling function $\dfrac{f_{n-1}(z)}{f_{n-1}(z=0)}$
- For $|z/z_0| \lesssim 0.2$, K_n rescaled by $H^{1/\delta-n+1}$ show small quark dependence

Temperature dependence of $\partial^n \rho / \partial m_l^n$ with $n = 1,2$

W.-P. Huang, Lattice 21', arXiv: 2112.00318

H.-T. Ding et al., PRL 126, 082001

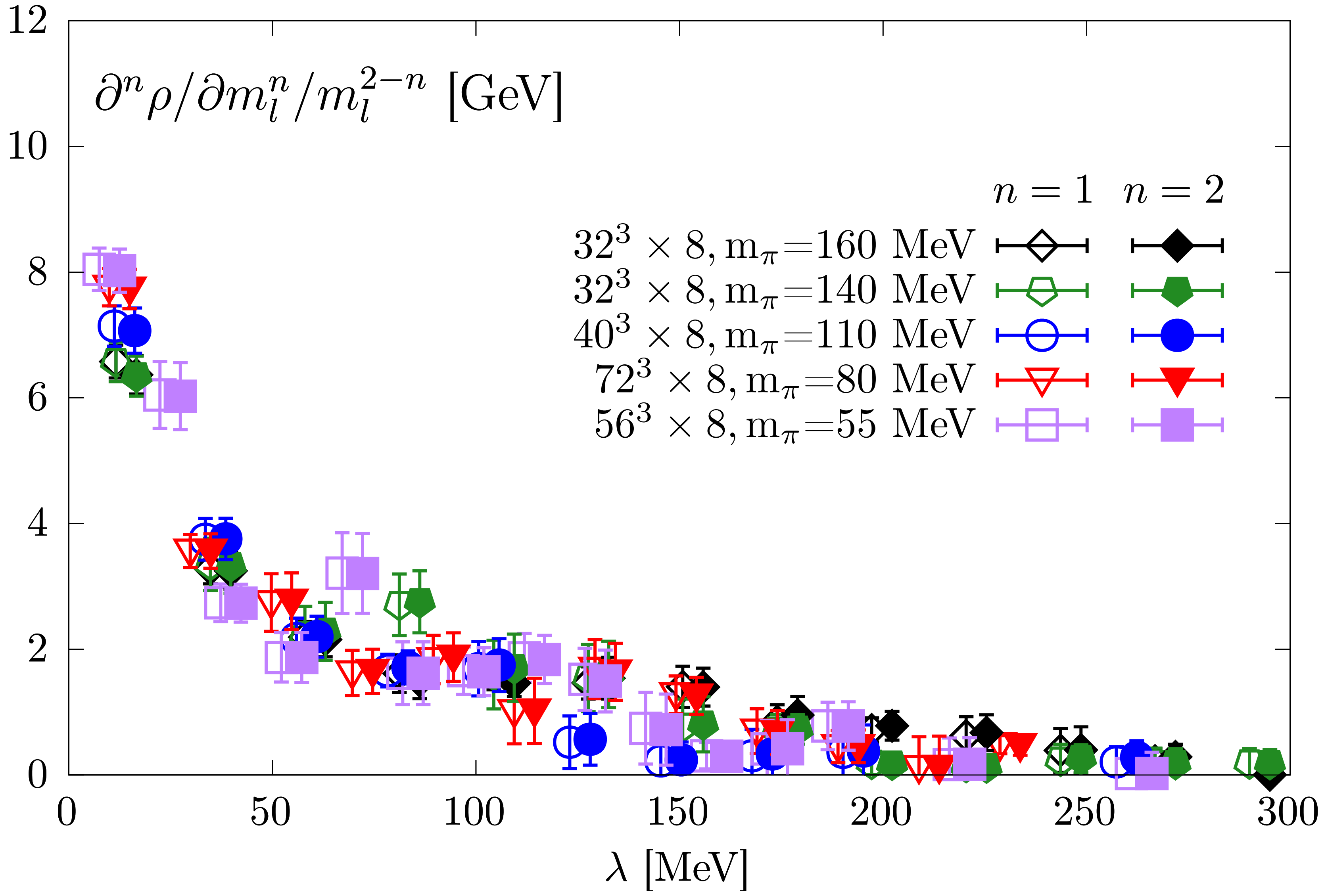
$$\frac{\partial \rho}{\partial m} = \frac{T}{V}\int_0^\infty \frac{4mK_1[\rho_U(\lambda),\rho_U(\lambda_2)]}{\lambda_2^2+m^2}d\lambda_2\ ,\quad \frac{\partial^2 \rho}{\partial m^2} = \frac{T}{V}\int_0^\infty \frac{(4m)^2K_1[\rho_U(\lambda),\rho_U(\lambda_2),\rho_U(\lambda_3)]}{(\lambda_2^2+m^2)(\lambda_3^2+m^2)}d\lambda_2 d\lambda_3 + \frac{T}{V}\int_0^\infty \frac{4(\lambda_2^2-m^2)K_1[\rho_U(\lambda),\rho_U(\lambda_2)]}{(\lambda_2^2+m^2)^2}d\lambda_2$$

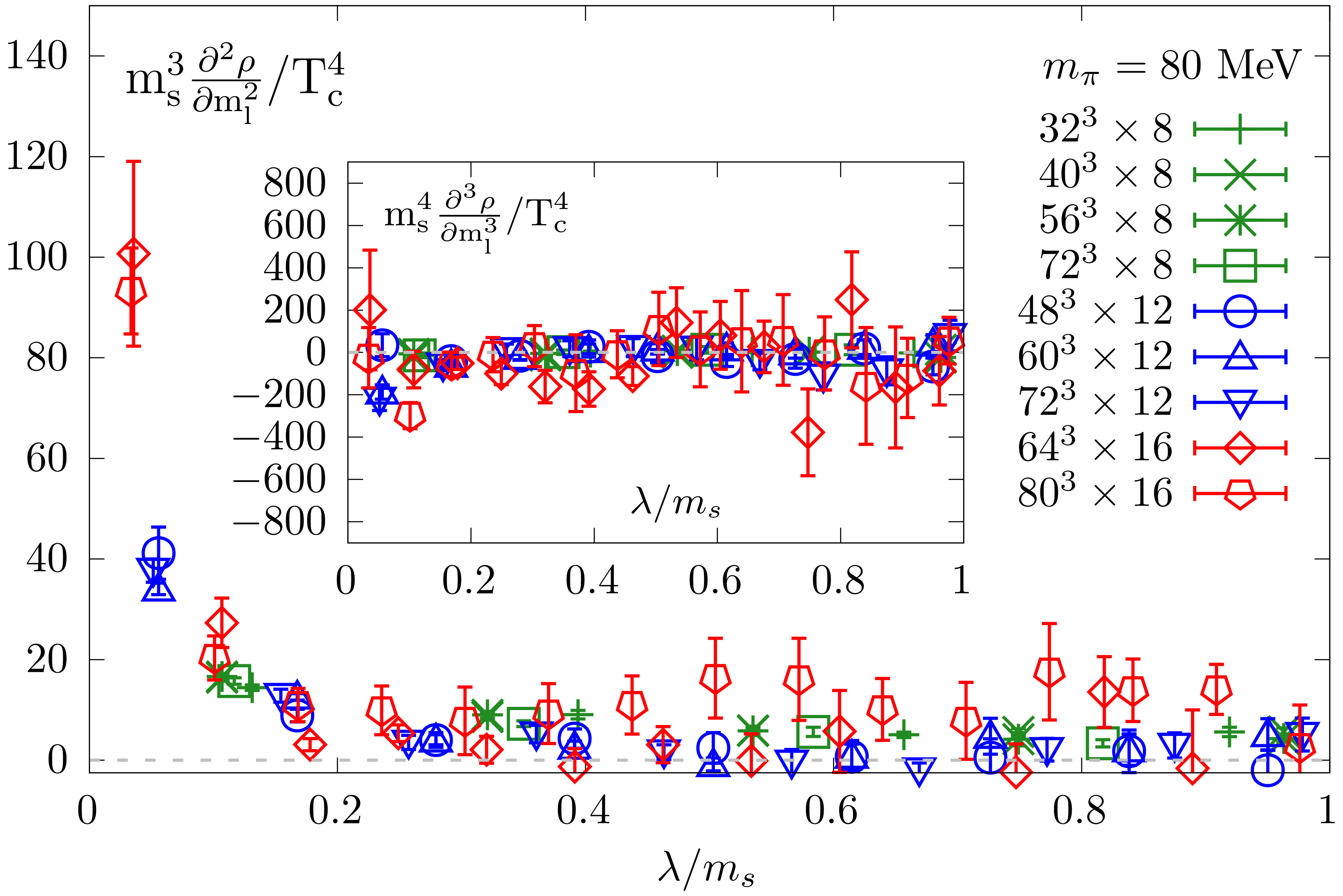


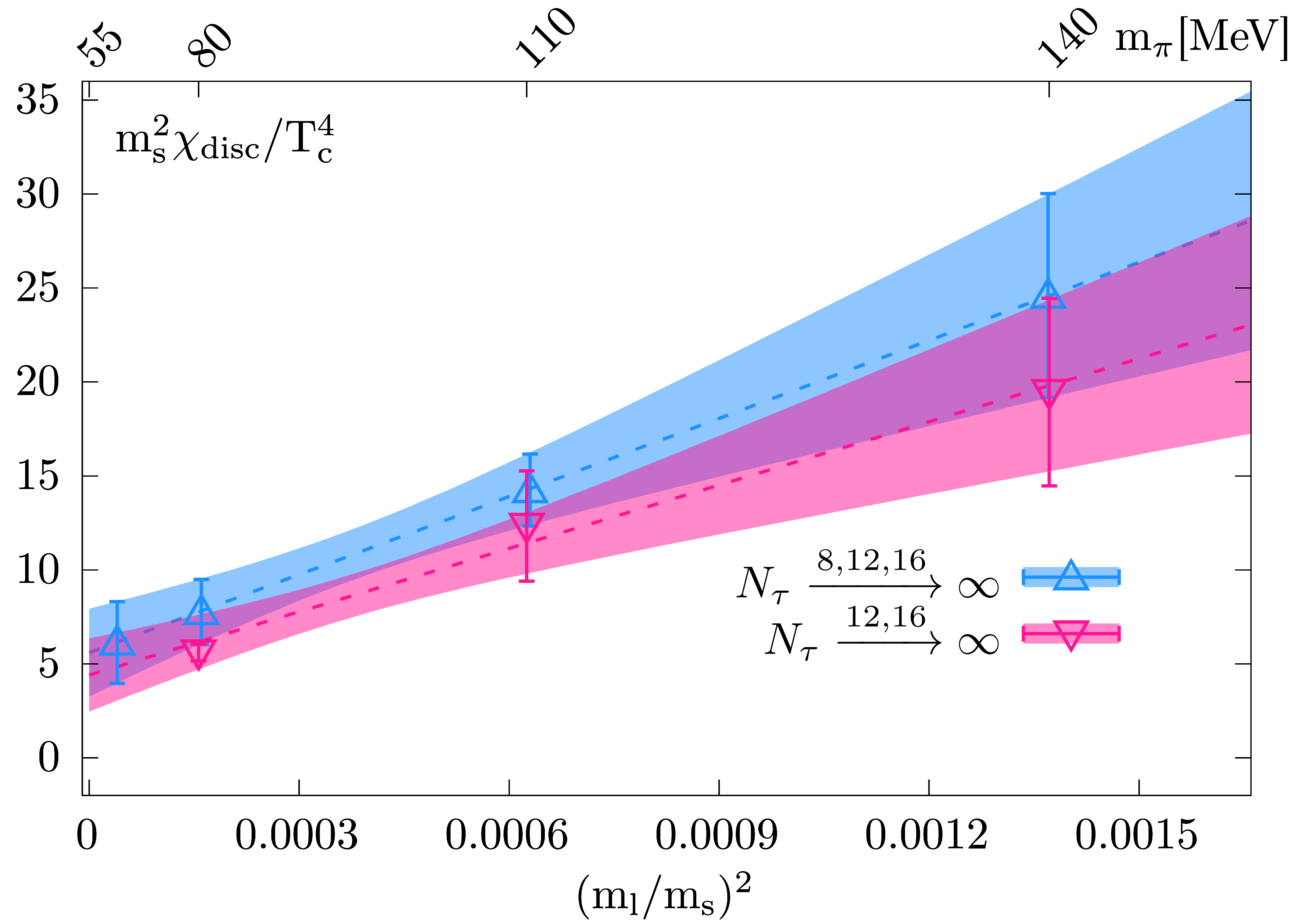
$\partial^2\rho/\partial m^2 \neq m^{-1}\partial\rho/\partial m$, Dilute Instanton Gas Approximation becomes invalid as getting closer to T_c

Recent Results at high temperature

In (2+1)-flavor QCD at $T \approx 1.6T_c$







$$m_l^{-1}\partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2 \\ \partial^3\rho/\partial m_l^3 \approx 0 \quad \Rightarrow \rho(\lambda\to 0,\, m_l\to 0) \propto m_l^2\delta(\lambda)$$

- Axial anomaly remains manifested in the $U_A(1)$ measures at a 2-3 sigma level
- Chiral phase transition should be 2^{nd} order belonging to O(4) universality class

H.-T. Ding et al., PRL 126, 082001