A UV-regulated Theory of nonlinear Diffusion

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In collaboration with

Matthias Kaminski (University of Alabama)

Omid Tavakol (University of California, Irvine)

0. Outline

- A few words about diffusion
- The main problem with Fick's law
- Kadanoff-Martin resolution of the problem
- In experiment
- Nonlinear version of the Kadanoff-Martin resolution
- In experiment
- Long-time tail
- Discussion (QGP)

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diffusion

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Fick's law: Diffusion of a U(1) charge in a non-fluctuating background

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$$\mathbf{J} + D\, \boldsymbol{\nabla} n = 0$$

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fail to satisfy the sum-rules:

1. $$-\int \frac{d\omega}{\pi}\,\frac{\mathrm{Im}G^R_{nn}(\omega,\mathbf{k})}{\omega} = \chi$$

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[Kadanoff, Martin, Annals of Physics (1963)]

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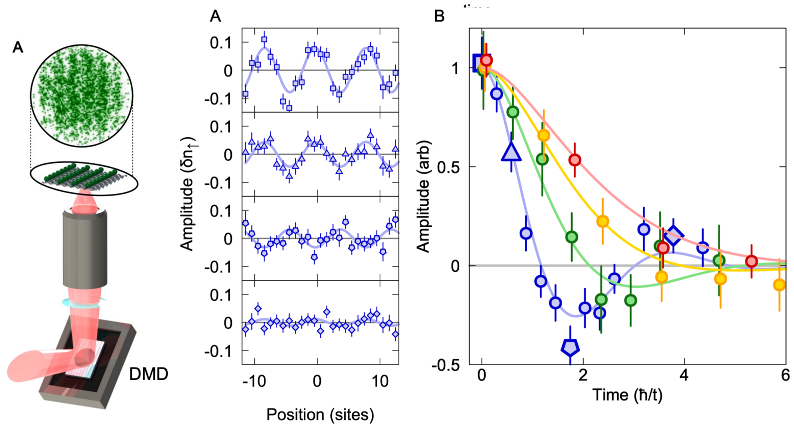
Measuring transport in the Hubbard model: ultracold 6Li in a 2D optical lattice

[Brown et al, Science, (2019) 1802.09456]



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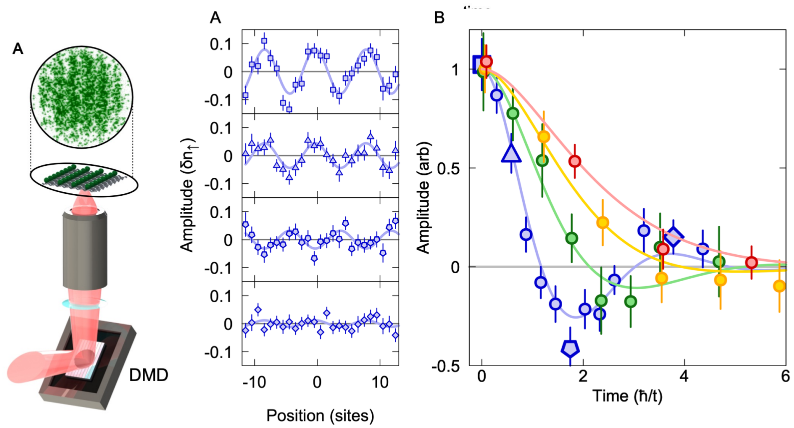

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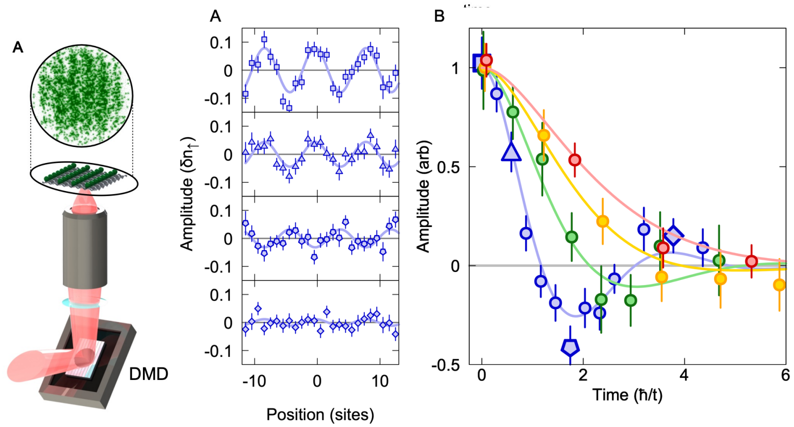

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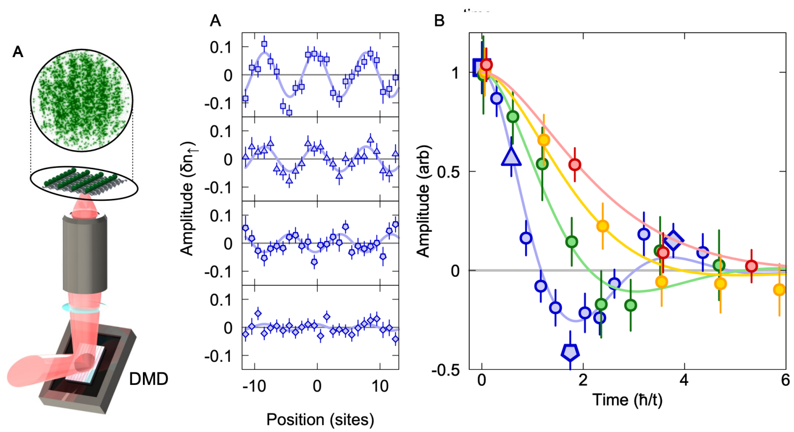

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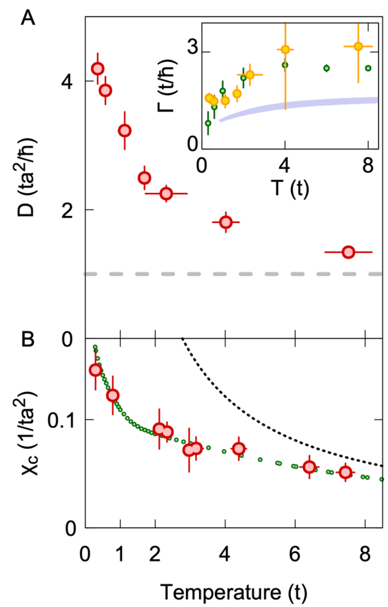

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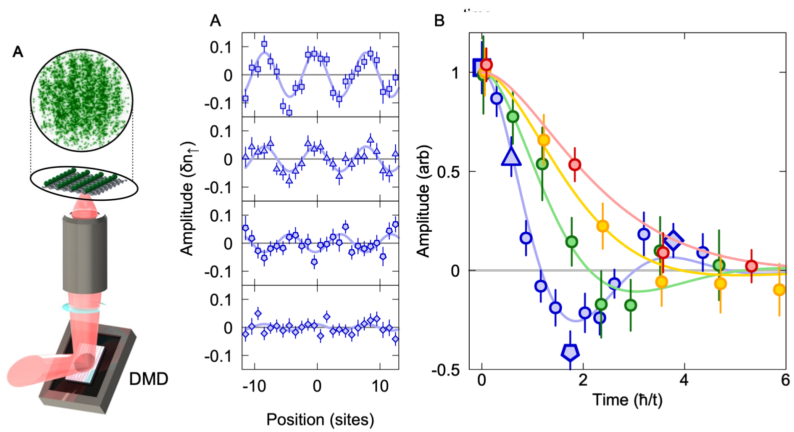

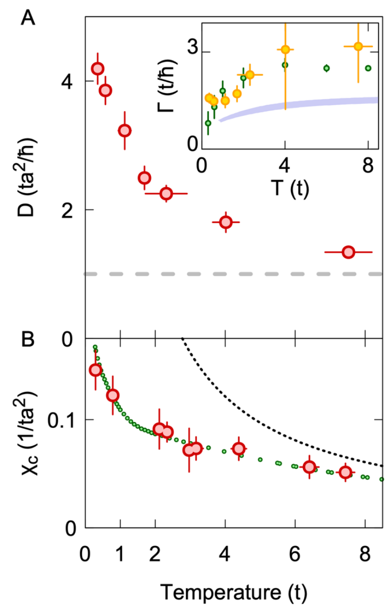

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- The lesson we learned:

$\tau \equiv \Gamma^{-1}$ can be measured experimentally.

5. What about fluctuations?

Theory of diffusive fluctuations: [Chen-Lin, Delacretaz, Hartnoll, PRL (2019)]

$$\mathcal{L} = iT^2\kappa(\nabla\varphi_a)^2 - \varphi_a\left(\dot{\varepsilon} - D\nabla^2\varepsilon\right) + \nabla^2\varphi_a\left[\frac{1}{2}\lambda\varepsilon^2 + \frac{1}{3}\lambda'\varepsilon^3\right] + icT^2(\nabla\varphi_a)^2\left[\widetilde{\lambda}\varepsilon + \widetilde{\lambda}'\varepsilon^2\right] + \cdots$$

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- 1-loop renormalization gives: $\frac{\delta D}{D} = \frac{f_d}{c\,\ell_{\text{th}}^d}\lambda_D \qquad \frac{\delta\sigma}{\sigma} = \frac{f_d}{\ell_{\text{th}}^d}\frac{T}{\chi\mu^2}\lambda_\sigma$

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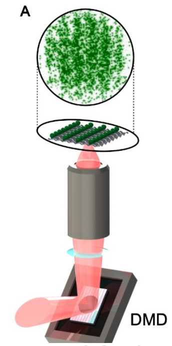
ultracold atom realization of the Hubbard model



[Brown et al, Science, (2019) 1802.09456]

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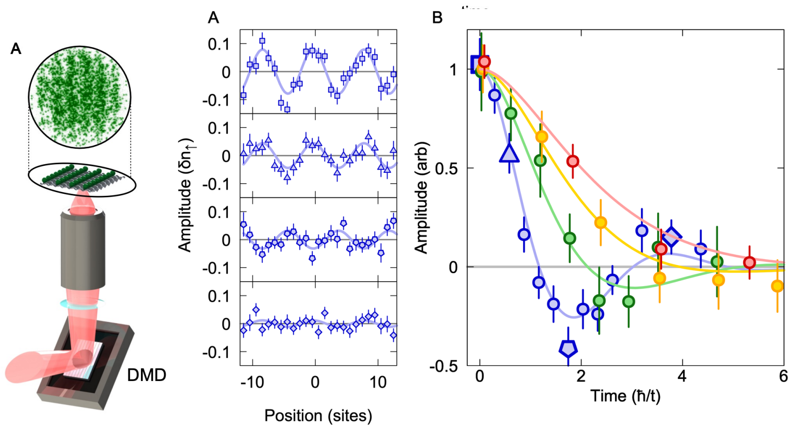
[NA, Kaminski, Tavakol, 2212.11499]

Instead of fitting data with the linear equation $$\tau\partial_t^2 n + \partial_t n - D\nabla^2 n = 0$$



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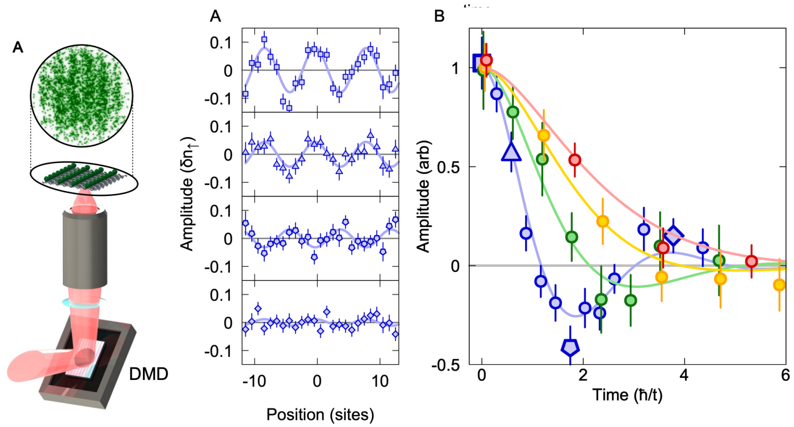

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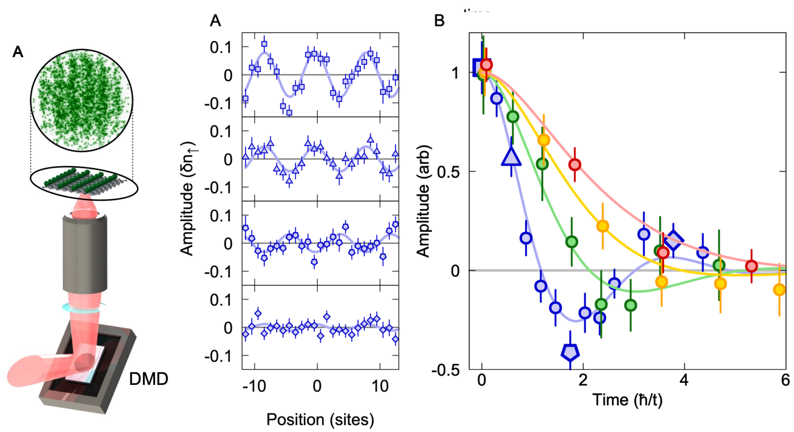

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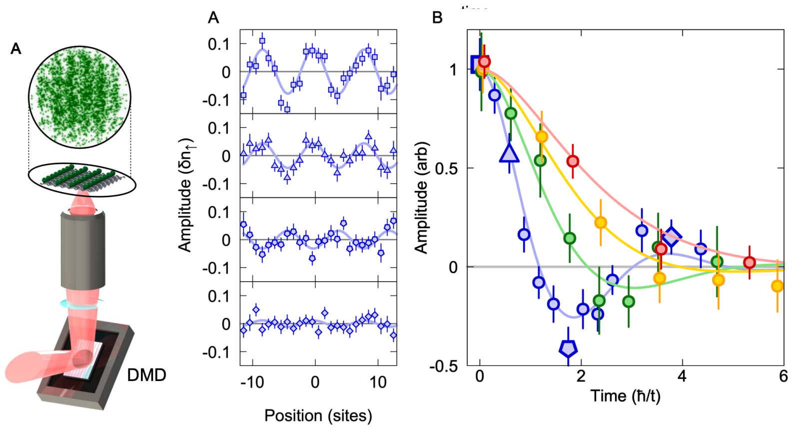

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- Self-interactions: $D(n) = D + \lambda_D \, n + \frac{\lambda'_D}{2} \, n^2 + \cdots$

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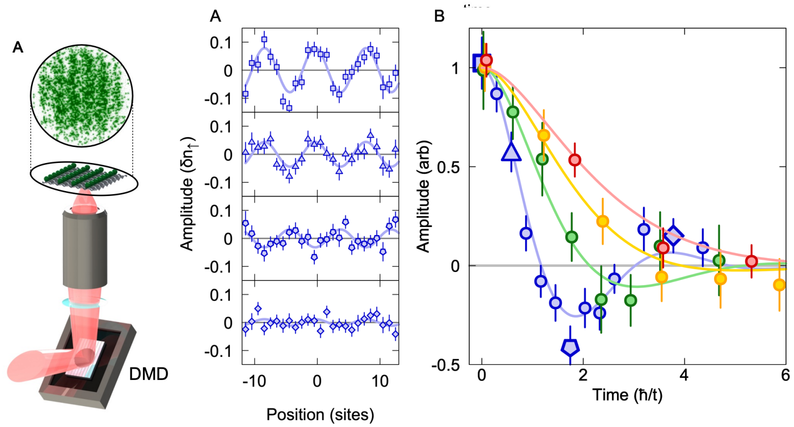

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$$\mathrm{E}[n] \equiv \tau\partial_t^2 n + \partial_t n - \nabla^2\left(D\,n + \frac{\lambda_D}{2}n^2 + \frac{\lambda'_D}{6}n^3\right) = 0$$

UV-regulated nonlinear theory of diffusion

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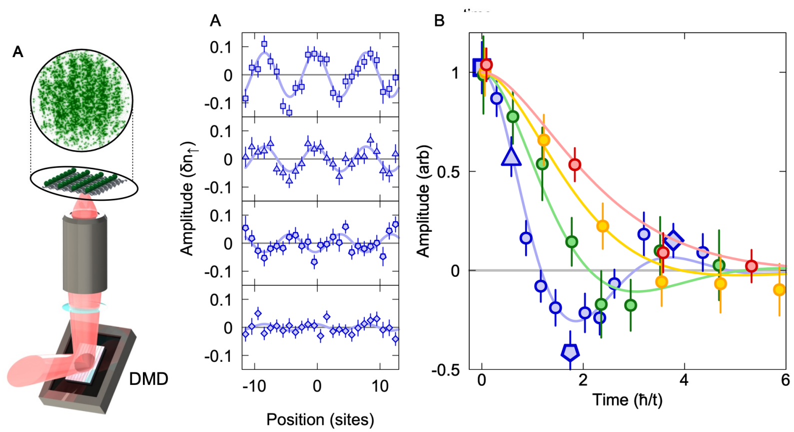

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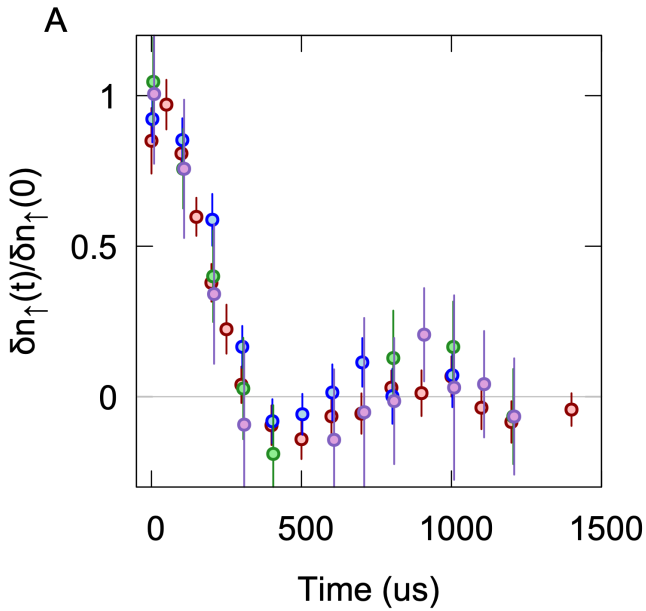

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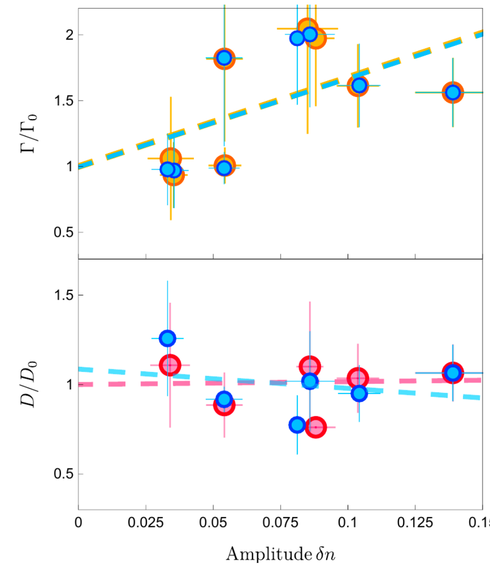

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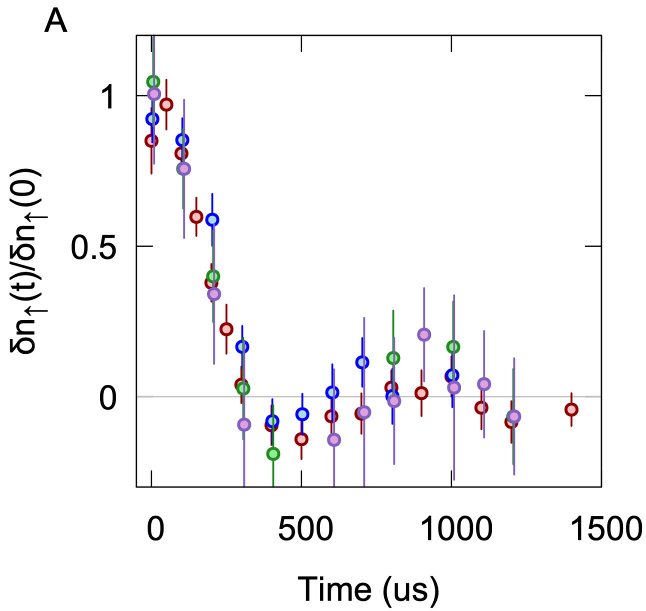

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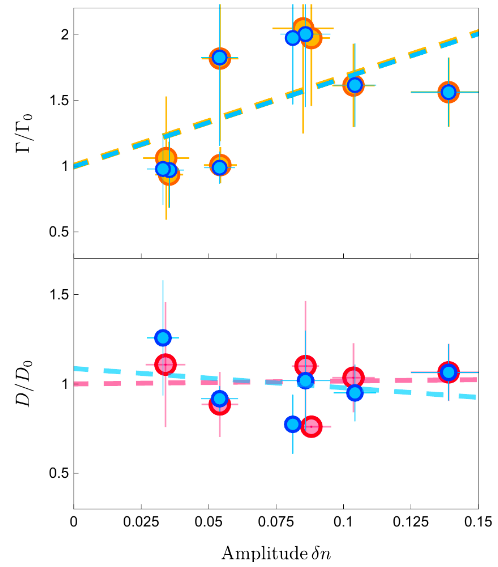

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By including the nonlinear effect

- Gamma is effectively unchanged.
- D is significantly affected.

8. Towards the EFT of UV-regulated diffusion

MSR effective action:
[Martin, Siggia, Rose, PRA (1973)]

$$
\begin{aligned}
\mathcal{L} &= iT\sigma(\boldsymbol{\nabla} n_a)^2 - n_a\left(\tau\partial_t^2 n + \partial_t n - D\boldsymbol{\nabla}^2 n\right) \\
&+ iT\chi\lambda_\sigma\, n(\boldsymbol{\nabla} n_a)^2 + \frac{\lambda_D}{2}\,\boldsymbol{\nabla}^2 n_a\, n^2 \\
&+ \frac{1}{2} iT\chi\lambda'_\sigma\, n^2(\boldsymbol{\nabla} n_a)^2 + \frac{\lambda'_D}{6}\,\boldsymbol{\nabla}^2 n_a\, n^3
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Equation of motion for n:

$$\boxed{\mathrm{E}[n] \equiv \tau\partial_t^2 n + \partial_t n - \nabla^2\left(D\, n + \frac{\lambda_D}{2} n^2 + \frac{\lambda'_D}{6} n^3\right) = 0}$$

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- Leading effects caused by the nonlinear term in E[n]=0

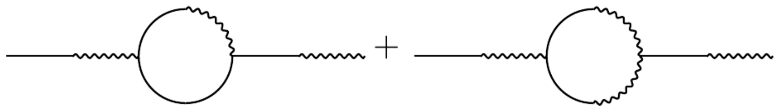
~ the effect of 1-loop corrections on the $G^R_{nn}(\omega, \mathbf{k})$

9. Loop calculations

[NA, Kaminski, Tavakol, 2212.11499]

- The response function: $$G^R_{nn}(\omega, \mathbf{k}) = \frac{i\left(\sigma + \delta\sigma(\omega, \mathbf{k})\right)\mathbf{k}^2}{-i\tau\omega^2 + \omega + iD\mathbf{k}^2 + \Sigma(\omega, \mathbf{k})}$$

$$G^{(0)}_{nn_a}\Sigma(p)G^{(0)}_{nn_a} =$$



$$G^{(0)}_{nn_a} = \frac{1}{\omega + iD\mathbf{k}^2 - i\tau\omega^2}$$

$$G^{(0)}_{n_a n} = \frac{-1}{\omega - iD\mathbf{k}^2 + i\tau\omega^2}$$

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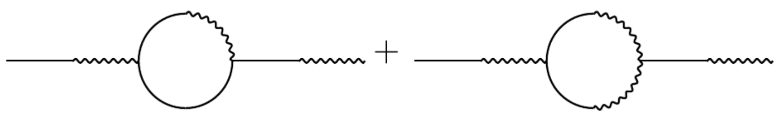

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[NA, Kaminski, Tavakol, 2212.11499]

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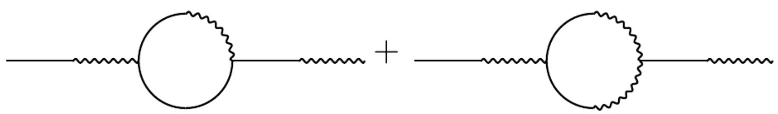

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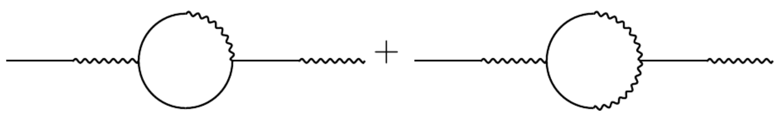

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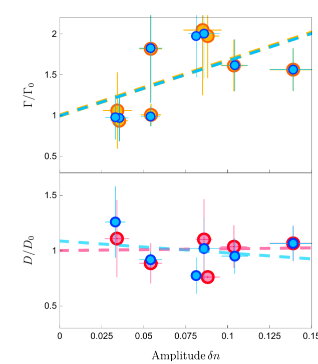

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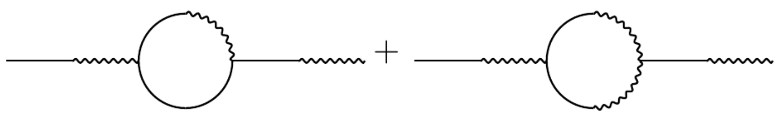

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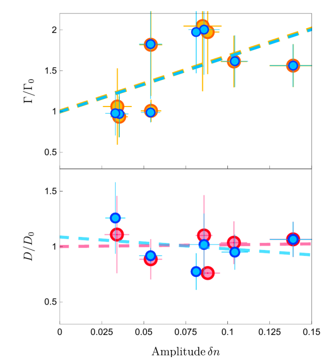

10. Long-time tails

In a simple diffusive system without a UV gapped mode

	linear response
$G_{nn}(\omega, \mathbf{k})$	$\dfrac{2T\,\chi\,D\mathbf{k}^2}{\omega^2 + (D\mathbf{k}^2)^2}$
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[Michailidis, Abanin, Delacretaz, 2310.10564]

Long-time tail

- The appearance of $t^{-d/2}$ in $G_{nn}(t, \mathbf{k})$

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- Branch points are produced due to the interactions.

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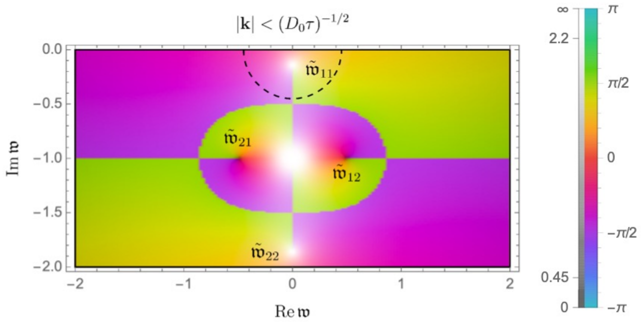
- All our calculations are analytic except for $\mathfrak{G}^{(1)}_{nn}(t,\mathbf{k})$

  we find the inverse Fourier transform numerically.

  because of this complicated analytic structure!



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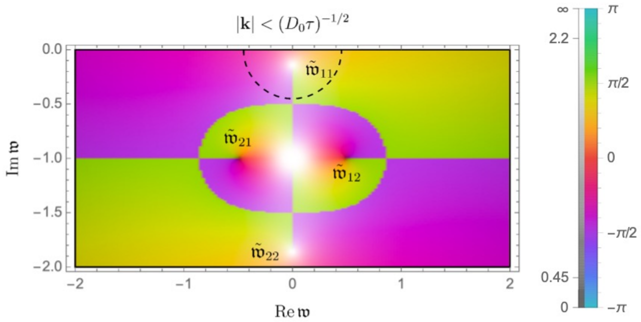

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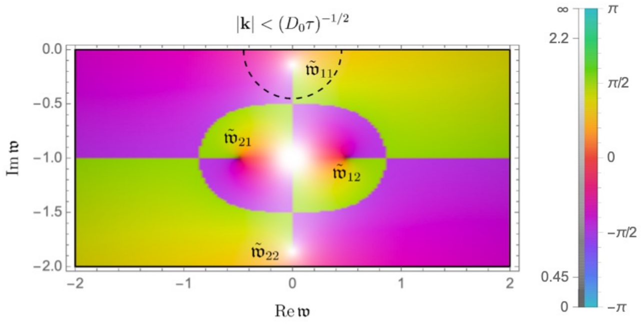

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- It is interesting to
  - Make the background dynamical
  - Repeat the calculation near the critical point

[Akamatsu, Mazeliauskas, Teaney, PRC (2017)]
[Martinez, Schafer, PRC (2019)]

Thank you for your attention

- [Kadanoff, Martin, Annals of Physics (1963)]
- [Kovtun, J.Phys.A (2012)]
- [Brown et al, Science, (2019) 1802.09456]
- [NA, Kaminski, Tavakol, 2212.11499]
- [Chen-Lin, Delacretaz, Hartnoll, PRL (2019)]
- [Kovtun, Yaffe, PRD (2003)]
- [Michailidis, Abanin, Delacretaz, 2310.10564]
- [Sogabe, Yin, JHEP (2022)]
- [Stephanov, Rajagopal, Shuryak, PRL (1998)]
- [Akamatsu, Mazeliauskas, Teaney, PRC (2017)]
- [Martinez, Schafer, PRC (2019)]
- [Fick, 1895]
- [Martin, Siggia, Rose, PRA (1973)]

11. Long-time tails in UV-regulated theory of diffusion

Our theory has two scales: $\tau\partial_t^2 n + \partial_t n - D\nabla^2 n = 0$

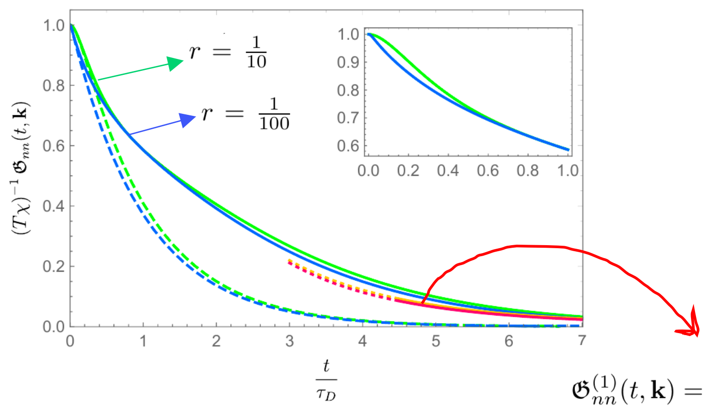
$$\omega_{1,2} = -\frac{i}{2\tau}\left(1 \mp \sqrt{1-4\tau D\mathbf{k}^2}\right)$$

$$\tau_{UV} = \frac{i}{\omega_1}\,, \qquad \tau_D = \frac{i}{\omega_2}$$

- When $\tau \ll (D\mathbf{k}^2)^{-1}$ ➡ $\tau_{UV} = \tau\,, \qquad \tau_D = (D\mathbf{k}^2)^{-1}$

- We numerically find:

$$r = \frac{\tau_{UV}}{\tau_D}$$



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