N-particle irreducible actions for stochastic fluids

Jingyi Chao

Jiangxi Normal University

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Outline and Motivations

- To provide a field-theoretical justification for stochastic hydrodynamics
- To present their own equations of motion for n-particle correlators
- To reveal some unusual behavior (critical phenomena) through the framework

Fluid Dynamics for Relativistic QCD Matter

Fluid dynamics is a universal effective field theory (EFT) of non-equilibrium many-body systems with a stable equation of state and

- Conservation of charge: $\partial_\mu J^\mu = 0$
- Conservation of energy and momentum: $\partial_\mu T^{\mu\nu} = 0$

$$
\begin{aligned}
J^\mu &= n\, u^\mu + \nu^\mu \\
T^{\mu\nu} &= \varepsilon\, u^\mu u^\nu + p\Delta^{\mu\nu} + \pi^{\mu\nu} \\
\Delta^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu \qquad \nu^\mu = -\kappa T \Delta^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) \\
\pi^{ij} &= -\eta\left(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\nabla\cdot\mathbf{u}\right) - \zeta\delta^{ij}\nabla\cdot\mathbf{u}
\end{aligned}
$$

The dissipation terms are described by the shear viscosity η, bulk viscosity ζ and charge conductivity κ

Dynamical Model in RHIC Beam Energy Scan

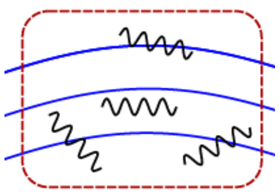
The real world is more complicated than the predictions in the first order. Additional factors must be considered, such as:

- Finite size and finite expansion rate effects
- Freeze-out, resonances, global charge conservation, and others
- Non-dissipation effects
  - The role of fluctuations is enhanced in nearly perfect fluids (long time tails)
  - Fluctuations are dominant near critical points

FLUCTUATIONS IN HYDRO

- The deterministic hydro equations do not lead to spontaneous fluctuations
- The occurrence of fluctuations is a consequence of the microscopic dynamics and must persist at the coarse-grained hydro-level

Introducing non-linear dissipation with temperature-dependent transport coefficients and random noises:



$$J^\mu \rightarrow J^\mu + \theta^\mu$$
$$T^{\mu\nu} \rightarrow T^{\mu\nu} + \theta^{\mu\nu}$$

$$\langle \theta^\mu \rangle = 0 \quad \left\langle (\theta^\mu)^2 \right\rangle \sim L_J(x)\delta(x-x')(t-t')$$
$$\langle \theta^{\mu\nu} \rangle = 0 \quad \left\langle (\theta^{\mu\nu})^2 \right\rangle \sim L_T(x)\delta(x-x')(t-t')$$

REPRESENTATION IN MSRJD FIELD THEORY

In terms of the slow variable (a conserved density), the free energy of the fluid:

$$\mathcal{F}[\psi] = \int d^3x \left\{ \frac{1}{2}(\vec{\nabla}\psi)^2 + \frac{r}{2}\,\psi(x,t)^2 + \frac{\lambda}{3!}\,\psi(x,t)^3 + \ldots + h(x,t)\psi(x,t) \right\}$$

The diffusion equation:

$$\partial_t \psi(x,t) = \vec{\nabla}\left\{ \kappa(\psi)\,\vec{\nabla}\left(\frac{\delta\mathcal{F}[\psi]}{\delta\psi}\right)\right\} + \theta(x,t)$$

where the Gaussian noise term $\theta(x,t)$ has a distribution

$$P[\theta] \sim \exp\left(-\frac{1}{4}\int d^3x\,dt\,\theta(x,t)L(\psi)^{-1}\theta(x,t)\right)$$

Representation in MSRJD Field Theory, cont.

The conductivity, $\kappa(\psi)$, is field-dependent: $\kappa(\psi) = \kappa_0\,(1 + \lambda_D \psi)$

The partition function is given as: MSR, PhysRevA.8:423(1973)

$$
\begin{aligned}
Z &= \int \mathscr{D}\psi\, P[\theta] \exp\left(-i\tilde{\psi}\;(\mathrm{e.o.m}[\psi,\,\theta\,])\right) \\
&= \int \mathscr{D}\psi\, \mathscr{D}\tilde{\psi}\;\exp\left(-\int d^3x\, dt\, \mathscr{L}(\psi, \tilde{\psi})\right)
\end{aligned}
$$

The effective Lagrangian of this theory is:

$$
\mathscr{L}(\psi, \tilde{\psi}) = \tilde{\psi}\left(\partial_t - D_0 \nabla^2\right)\psi - \frac{D_0 \lambda'}{2}\left(\nabla^2 \tilde{\psi}\right)\psi^2 - \tilde{\psi} L(\psi) \tilde{\psi}
$$

Note: $D_0 = r\kappa_0$ and $\lambda' = \lambda/r + \lambda_D$.

The noise kernel is chosen as $L(\psi) = \overleftarrow{\nabla}\left[k_B T \kappa(\psi)\right]\overrightarrow{\nabla}$

Time Reversal Symmetry

Stochastic theories must describe the detailed balance condition:

$$\frac{P\left(\psi_1 \longrightarrow \psi_2\right)}{P\left(\psi_2 \longrightarrow \psi_1\right)} = e^{-\Delta \mathscr{F}/k_B T}$$

which is related to time-reversal symmetry:

$$\Psi(t) \longrightarrow \psi(-t)$$

$$\tilde{\Psi}(t) \longrightarrow -\left[\tilde{\psi}(-t) + \frac{\delta F}{\delta \psi}\right]$$

$$\mathscr{L} \rightarrow \mathscr{L} + \frac{d}{dt}F$$

Janssen, ZPhyB.23:377(1976)

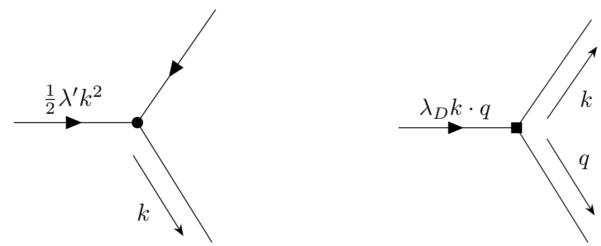
The Ward identity is revised to

$$\left\langle \psi(x_1, t_1)\left[\overleftarrow{\nabla}\kappa(\psi)\vec{\nabla}\tilde{\psi}\right](x_2, t_2)\right\rangle = \Theta(t_2 - t_1)\left\langle \psi(x_1, t_1)\dot{\psi}(x_2, t_2)\right\rangle$$

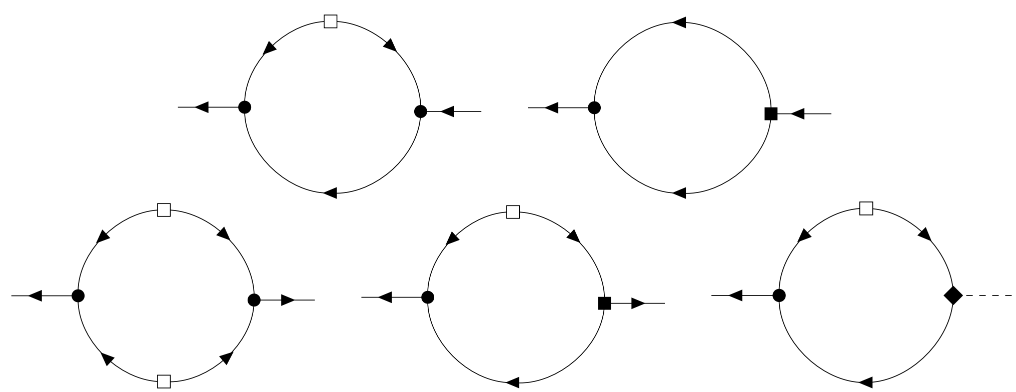
Simpler Example of Model B

- Linearized propagator:

- Vertex and new vertices:



- Loop contributions:



Analytical Results of One-Loop

The retarded function

$$G^{-1}(\omega, k) = \frac{1}{-i\omega + D_0 k^2 + \Sigma(\omega, k)}$$

with
$$\Sigma(\omega, k) = \frac{\lambda'}{32\pi}\left(i\lambda'\omega k^2 + \lambda_D\left[i\omega - D_0 k^2\right] k^2\right)\sqrt{k^2 - \frac{2i\omega}{D_0}}$$

The charge (thermal) conductivity in this system becomes a scale-dependent term in the hydro theory

The field-dependent fluctuation-dissipation relation becomes:

$$2\,\mathrm{Im}\,\left\{G(\omega, k)\left[D_0 k^2 + \Gamma_D(\omega, k)\right]\right\} = \omega C(\omega, k)$$

with the vertex function of composite operator $\lambda_D[\psi\vec{\nabla}\tilde{\psi}]$ is given by

$$\Gamma_D(\omega, k) \equiv (-i\omega + D_0 k^2)\left\langle D_0\lambda_D[\psi\vec{\nabla}\tilde{\psi}]\vec{\nabla}\psi\right\rangle_{\omega,k}$$

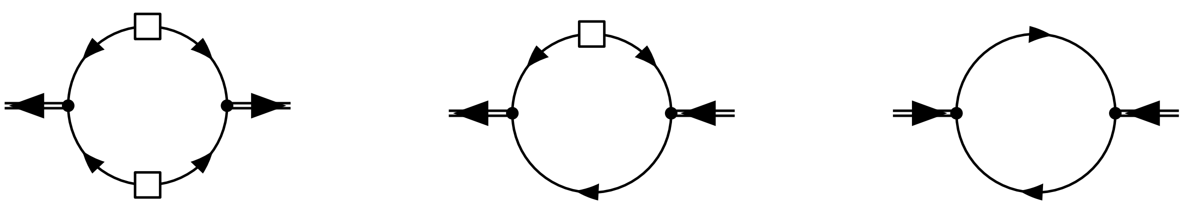
1PI Effective Action

Consider the generating functional with local source J, $\tilde{J}$:

$$W[J,\tilde{J}] = -\ln \int \mathscr{D}\psi \mathscr{D}\tilde{\psi}\; e^{-\int dt\, d^3x \left\{\mathscr{L} + J\psi + \tilde{J}\tilde{\psi}\right\}}$$

Performing a Legendre transform to the 1PI effective action via background field method with $\psi = \Psi + \delta\psi$:

$$\Gamma[\Psi,\tilde{\Psi}] = W[J,\tilde{J}] - \int dt\, d^3x \left(J\Psi + \tilde{J}\tilde{\Psi}\right)$$



Taking the derivative of the 1PI effective action w.r.t. the classical field Ψ yields the e.o.m. encoded the fluctuation effects:

$$(\partial_t - D\nabla^2)\Psi - \frac{\kappa\lambda_3^2}{2}\nabla^2\Psi^2 + \int d^3x'\, dt'\, \Psi(x',t')\Sigma(x,t;x',t') = 0$$

Double Legendre Transformation

☞ nPI effective action $\Longrightarrow$ e.o.m. for n-point functions

✓ Couple a bi-local source $\frac{1}{2}\psi_a K_{ab} \psi_b$ to the system Cornwall, Jackiw and Tomboulis, PhysRevD.10:2428 (1974)

✓ Plug in the 1-loop 1PI effective action

✓ Sum beyond 1-loop terms

✓ Apply the stationary conditions:

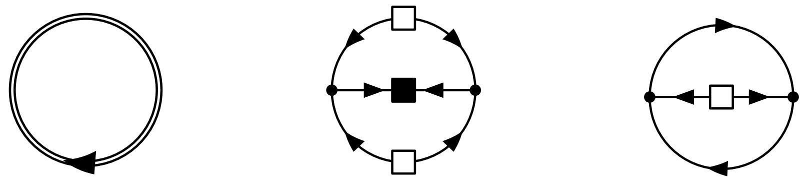
$$\frac{\delta W}{\delta J_a} = \langle \psi_a \rangle = \Psi_a \,, \qquad \frac{\delta W}{\delta K_{ab}} = \frac{1}{2}\langle \psi_a \psi_b \rangle = \frac{1}{2}\left[\Psi_a \Psi_b + G_{ab}\right]$$

✓ Perform a Legendre transform to yield the 2PI effective action:

$$\Gamma[\Psi_a, G_{ab}] = S_{bck}[\Psi_a] + \frac{1}{2}\frac{\delta^2 S_{bck}}{\delta \Psi_A \delta \Psi_B} G_{AB} - \frac{1}{2}\mathrm{Tr}\left[\log(G)\right] + \Gamma_F[\Psi_a, G_{ab}]$$

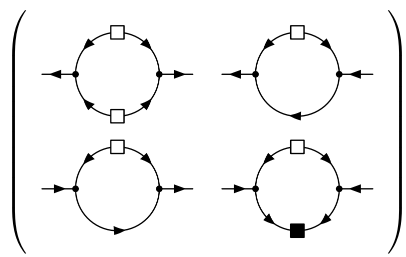
DSE in Mixed Representation

The loop diagrams generated by Γ_F use the full propagator G_{ab}:



Taking the derivative w.r.t G, obtain the DS equation ($\tilde{\psi}$, ψ = 1, 2):

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} =$$

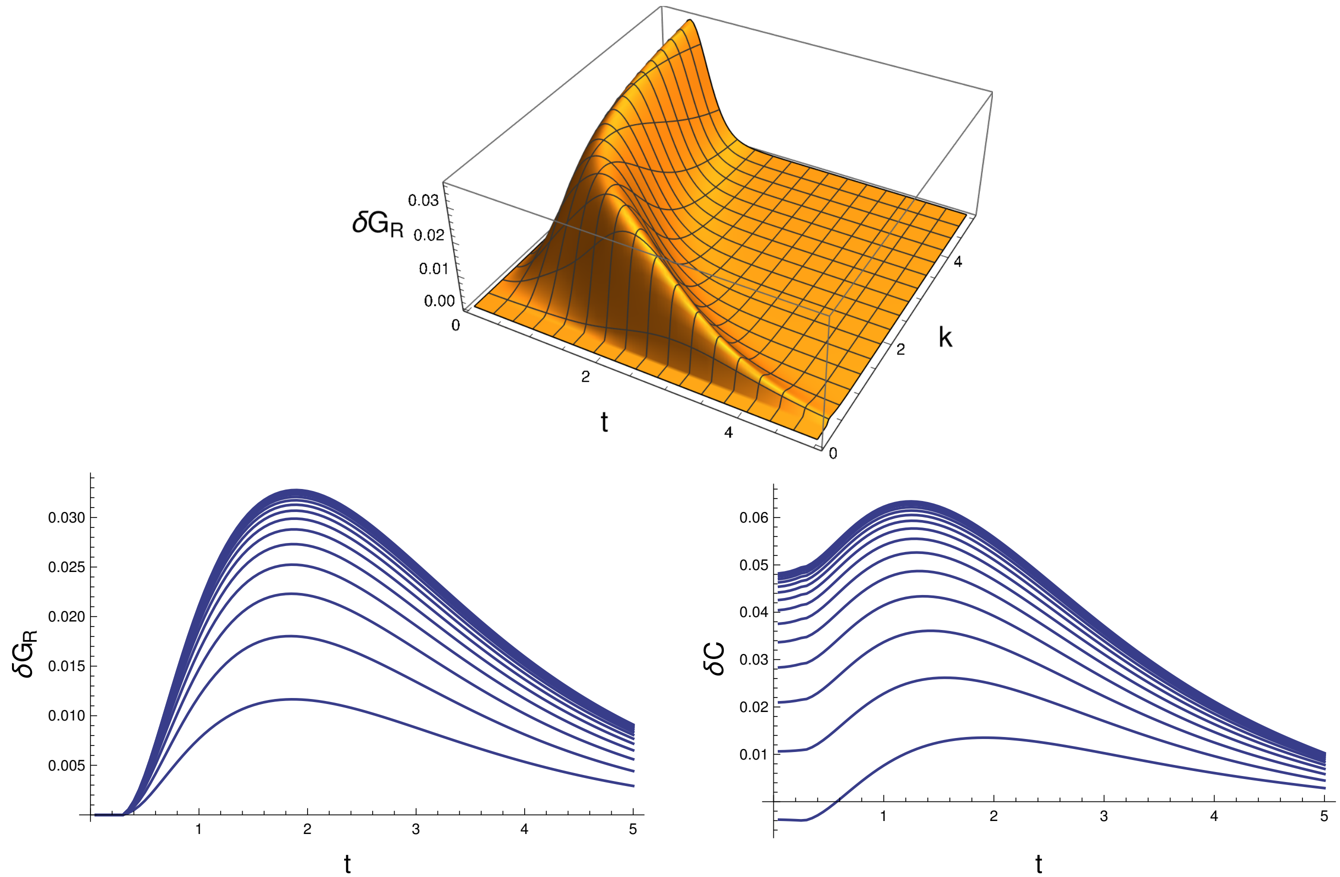


In time-momentum mixed representation

$$\Sigma(t, k^2) = (\kappa\lambda_3)^2 \int d^3k' \, k^2 (k + k')^2 \, C(t, k') \, G_R(t, k + k') \, ,$$

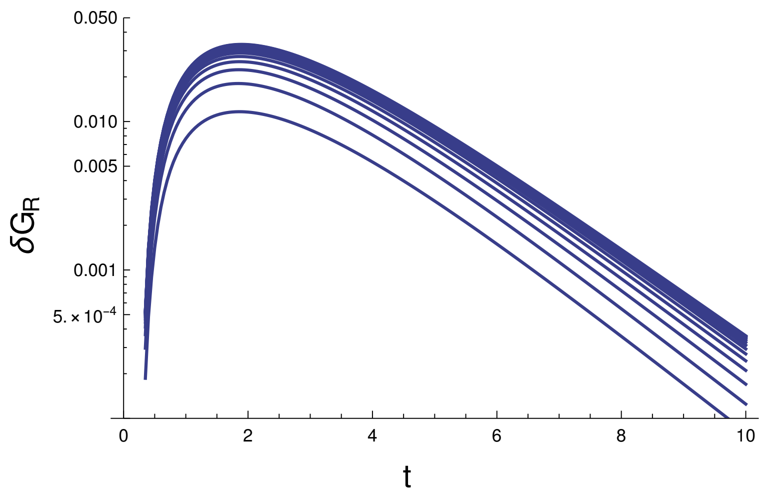
$$\delta D(t, k^2) = \frac{(\kappa\lambda_3)^2}{2} \int d^3k' \, k^4 C(t, k') \, C(t, k + k')$$

Naive Numerical Simulations

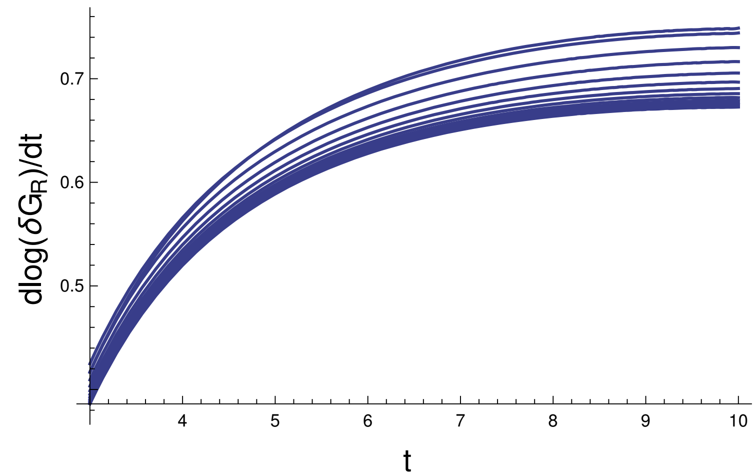


3D curve of $\delta G(t,k)$ and the iterative solutions of DSE taking advantage of convergence

LONG-TIME BEHAVIOR



A logarithmic plot of the loop corrections to the retarded $\delta G_R(k,t)$



The logarithmic derivative of $\delta G_R(k,t)$ w.r.t t

The long-time behavior of the diffusion cascade is conjectured to be $\sim n!\exp(-Dk^2t/n)$ because of the n-loop terms (shown but not reached).

Delacretaz, SciPostPhys.9:034(2020)

MODE COUPLING THEORY (MCT)

- For non-critical fluids, use the gradient expansion method where $k\xi \ll 1$

- For critical fluids, their behaviors are characterized by the transport coefficients in the MCT (= Poisson bracket terms + the critical transport coeffcients)

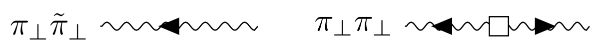
By applying an uncontrolled approximation within the MCT, the well known retarded function $G^{-1}(\omega, k) = i\omega - \Gamma_k$ of the diffusion mode is modified to:

$$\Gamma_k = \frac{T}{6\pi\eta_0\xi^3}K(k\xi) \quad \text{with} \quad K(k\xi = x) = \frac{3}{4}\left[1 + x^2 + (x^3 - x^{-1})\arctan(x)\right]$$

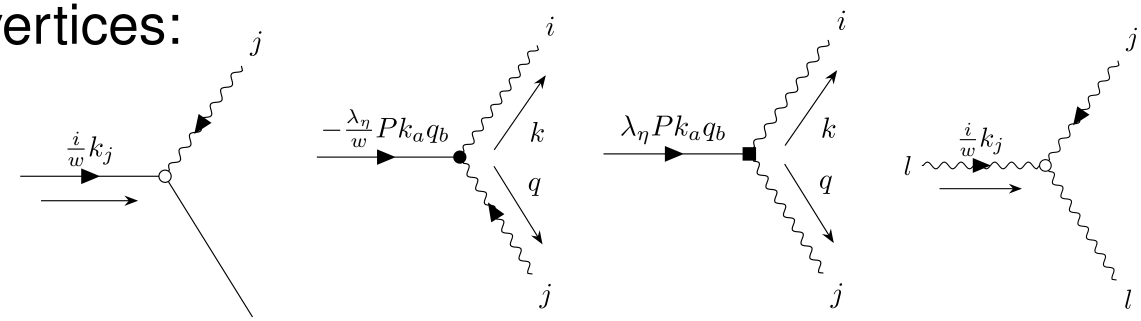
η_0 is the bare shear viscosity. Kawasaki, AnnPhys.61:1(1970); JC and T. Schaefer, work-in-process

Model H

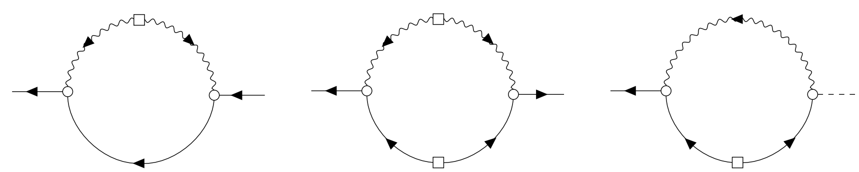
- Linearized propagator:



- Vertices and new vertices:

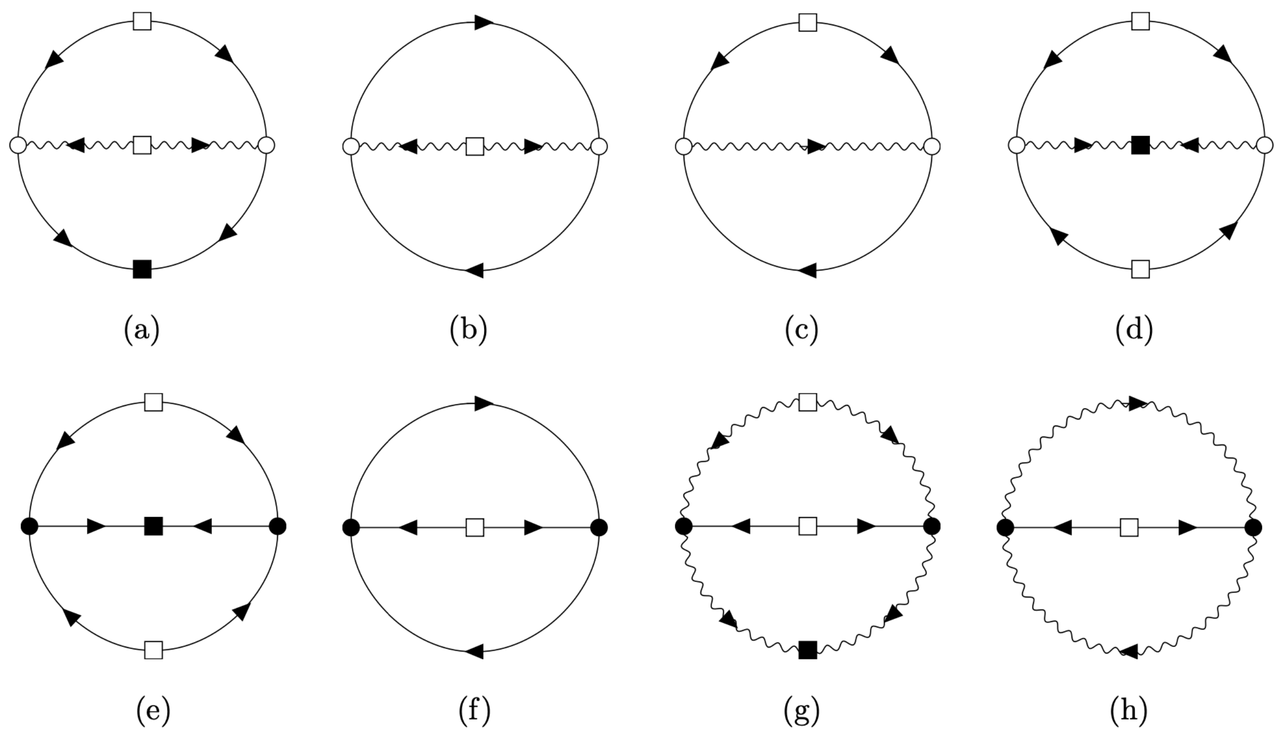


- Mode-coupling loop contributions:



☞ the multiplicative noise contribution to the tails is subleading compared to the contributions induced by mode couplings in hydro limit

2PI Effective Action in Model H



Above: The traditional contribution, which originates from the vertex of the Poisson bracket, is illustrated within the MCT

Below: Additional contributions are derived from the newer vertex

SCALING FORMS OF THE TRANSPORT COEFFICIENTS

The modified critical transport coefficients:

$$
\begin{aligned}
D \to D^c(\omega, k, \xi) &= D\,(k\xi)^{x_D} F_D\,(\omega\xi^z, k\xi) \\
\kappa \to \kappa^c(\omega, k, \xi) &= \kappa\,(k\xi)^{x_\kappa} F_\kappa\,(\omega\xi^z, k\xi) \\
\eta \to \eta^c(\omega, k, \xi) &= \eta\,(k\xi)^{x_\eta} F_\eta\,(\omega\xi^z, k\xi) \\
\gamma \to \gamma^c(\omega, k, \xi) &= \gamma\,(k\xi)^{x_\gamma} F_\gamma\,(\omega\xi^z, k\xi)
\end{aligned}
$$

- Contrary to the hydrodynamic limit, where $D = \kappa m^2$, $\eta = \gamma w$ and w is enthalpy
- The relaxation frequency scales as $\omega \sim k^z$
- The dynamical exponent z is determined as $z = 4 - \eta + x_D$ for the diffusion mode in the regime where $k \gg \xi^{-1}$

Critical Self-Energy Functions

The Ornstein-Zernike form is utilized, expressed as $\chi^{-1}(x) = g(x) = 1 + x^2$, with the static critical exponent set to $\eta = 0$. And then,

$$\begin{aligned}
\Sigma^c_{12}(s,x) &= D\,\xi^{z-2}\,x^{2+x_D} g(x) F_D(s,x)\,, \\
\Sigma^c_{11}(s,x) &= \kappa\,\xi^{2z-2}\,x^{2+x_\kappa} F_\kappa(s,x)\,, \\
\Delta^c_{12}(s,x) &= \gamma\,\xi^{z-2}\,x^{2+x_\gamma} F_\gamma(s,x)\,, \\
\Delta^c_{11}(s,x) &= \eta\,\xi^{2z-2}\,x^{2+x_\eta} F_\eta(s,x)\,.
\end{aligned}$$

where

$$F_i(s=0, x\to\infty) = F_i^\infty = \text{constant}$$

and

$$F_i(s=0, x\to 0) = F_i^0\, x^{-x_i}$$

with $i = D, \kappa, \gamma, \eta$.

UV Finite Self-Consistent Equations

Re-scale the frequency and the momentum as $(s, r) = (\omega\xi^z, \omega'\xi^z)$ and $(x, y) = (k\xi, k'\xi)$, the self-energies are:

$$
\begin{aligned}
\Sigma^c_{12}(s,x) \quad &= \quad \xi^{z-7}\int_{r,y}\left[\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{(\kappa\lambda_3)^2x^2(\vec{x}+\vec{y})^2}{i(s+r)+\Sigma^c_{12}(-s-r,x+y)}\right.\\
&-\frac{\Delta^c_{11}(r,y)}{r^2+|\Delta^c_{12}(r,y)|^2}\frac{\xi^2}{w^2y^2}\frac{x^2y^2-(\vec{x}\cdot\vec{y})^2}{i(s+r)+\Sigma^c_{12}(-s-r,x+y)}\\
&\left.-\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{x^2-y^2}{w(\vec{x}+\vec{y})^2}\frac{x^2(\vec{x}+\vec{y})^2-(x^2+\vec{x}\cdot\vec{y})^2}{i(s+r)+\Delta^c_{12}(-s-r,x+y)}\right],
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^c_{11}(s,x) \quad &= \quad \xi^{z-7}\int_{r,y}\left\{\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{(\kappa\lambda_3)^2}{2}\frac{x^4\,\Sigma^c_{11}(s+r,x+y)}{(s+r)^2+|\Sigma^c_{12}(s+r,x+y)|^2}\right.\\
&\left.+\frac{\Delta^c_{11}(r,y)}{r^2+|\Delta^c_{12}(r,y)|^2}\frac{\xi^2}{w^2y^2}\frac{\left(x^2y^2-(\vec{x}\cdot\vec{y})^2\right)\Sigma^c_{11}(s+r,x+y)}{(s+r)^2+|\Sigma^c_{12}(s+r,x+y)|^2}\right\},
\end{aligned}
$$

$$\Delta^c_{12}(s,x) = \xi^{z-7}\int_{r,y}\left\{\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{(\gamma\lambda_\eta)^2\mathscr{P}_t(x,y)(x^2+\vec{x}\cdot\vec{y})^2}{i(s+r)+\Delta^c_{12}(-s-r,x+y)}\right.$$
$$\left.-\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{x^2y^2-(\vec{x}\cdot\vec{y})^2}{wx^2}\frac{\left(x^2+2\,\vec{x}\cdot\vec{y}\right)}{i(s+r)+\Sigma^c_{12}(-s-r,x+y)}\right\},$$

$$\Delta^c_{11}(s,x) = \xi^{z-7}\int_{r,y}\left\{\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{(\gamma\lambda_\eta)^2\mathscr{P}_t(x,y)(x^2+\vec{x}\cdot\vec{y})^2\Delta^c_{11}(s+r,x+y)}{(s+r)^2+|\Delta^c_{12}(s+r,x+y)|^2}\right.$$
$$\left.+\frac{\Sigma^c_{11}(r,y)}{r^2+|\Sigma^c_{12}(r,y)|^2}\frac{x^2y^2-(\vec{x}\cdot\vec{y})^2}{\xi^2x^2}\frac{(\vec{x}+\vec{y})^2\left(x^2+2\,\vec{x}\cdot\vec{y}\right)\Sigma^c_{11}(s+r,x+y)}{(s+r)^2+|\Sigma^c_{12}(s+r,x+y)|^2}\right\},$$

where $\mathscr{P}_t(x,y)=1+(x^2+\vec{x}\cdot\vec{y})^2\,x^{-2}(\vec{x}+\vec{y})^{-2}$

stay tune for the numerical results and thank you!