All-orders evolution of PDFs and proton spin

J. Rodríguez-Quintero

Strong QCD from Hadron Structure Experiments, May 14th - 17th, 2024.

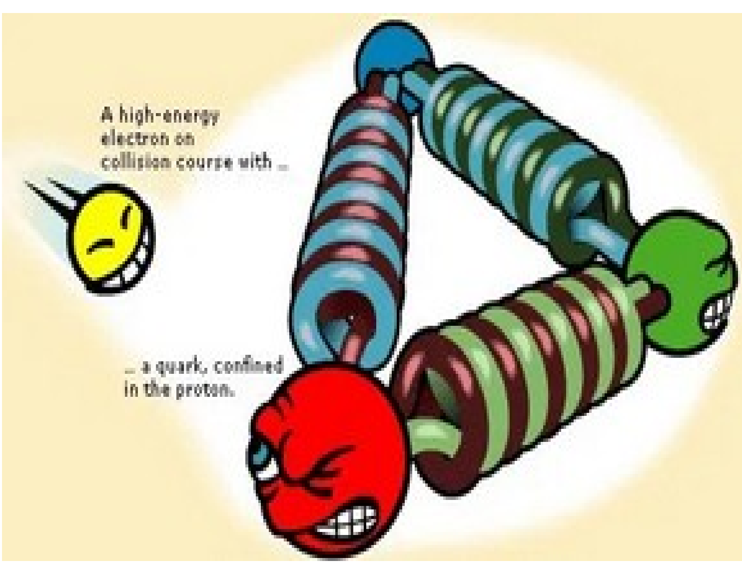
QCD: Basic Facts

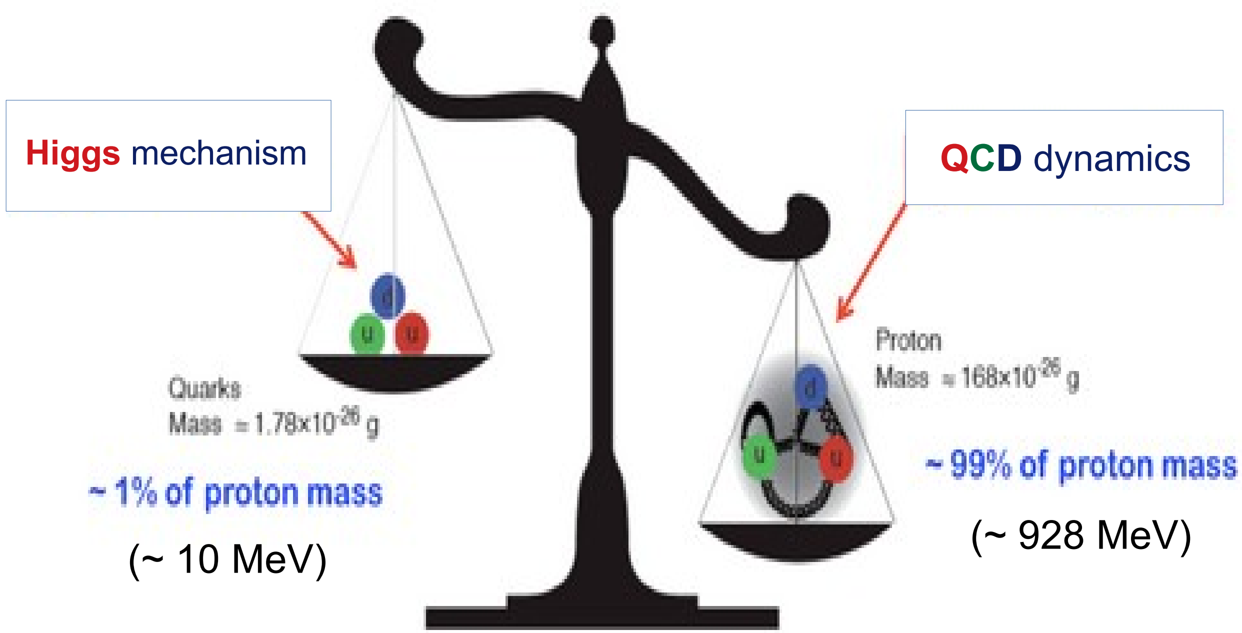
- **QCD** is characterized by two **emergent** phenomena: confinement and dynamical generation of mass (DGM).

$$\mathcal{L}_{\rm QCD} = \sum_{j=u,d,s,\ldots} \bar{q}_j[\gamma_\mu D_\mu + m_j]q_j + \tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu},$$
$$D_\mu = \partial_\mu + ig\tfrac{1}{2}\lambda^a A^a_\mu,$$
$$G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - \underline{g f^{abc} A^b_\mu A^c_\nu},$$

- Quarks and gluons not *isolated* in nature.
  - Formation of colorless bound states: "**Hadrons**"
  - **1-fm scale** size of hadrons?

- Emergence of hadron masses (EHM) from QCD **dynamics**





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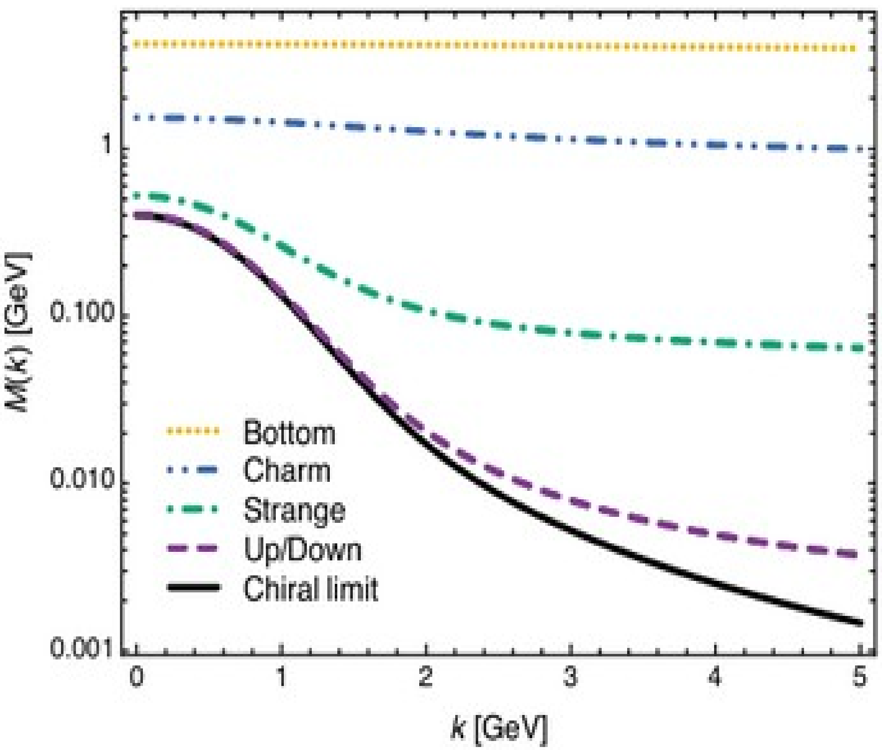
Can we trace them down to fundamental d.o.f ?

- Emergence of hadron masses (EHM) from QCD **dynamics**

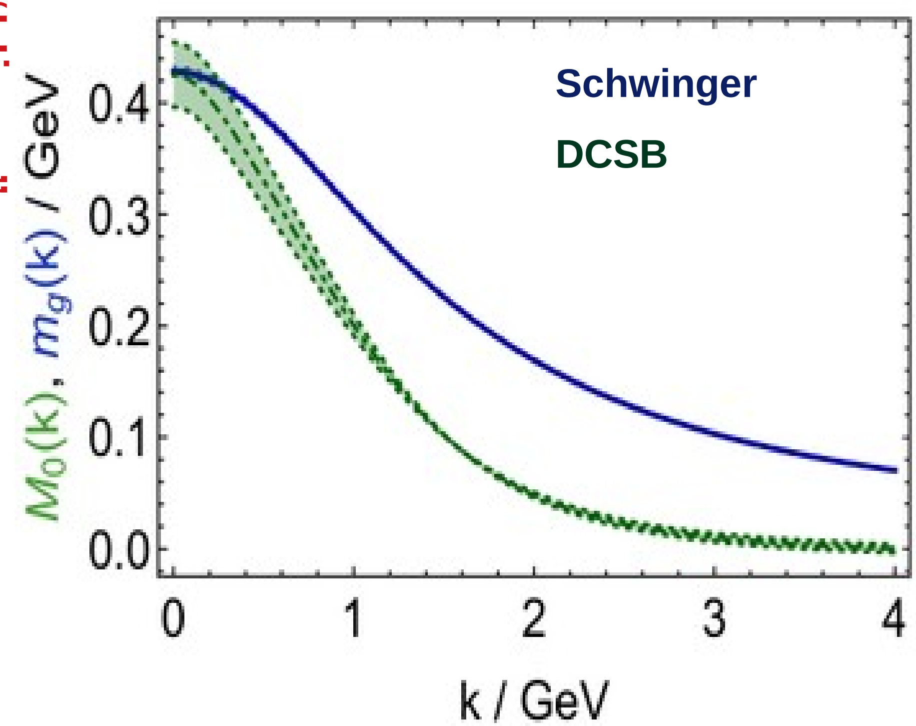
Dynamical masses

(**D**ynamical **C**hiral **S**ymmetry **B**reaking)



$$S_f^{-1}(p) = Z_f^{-1}(p^2)(i\gamma\cdot p + \mathrm{M_f(p^2)})$$

"Higgs" masses

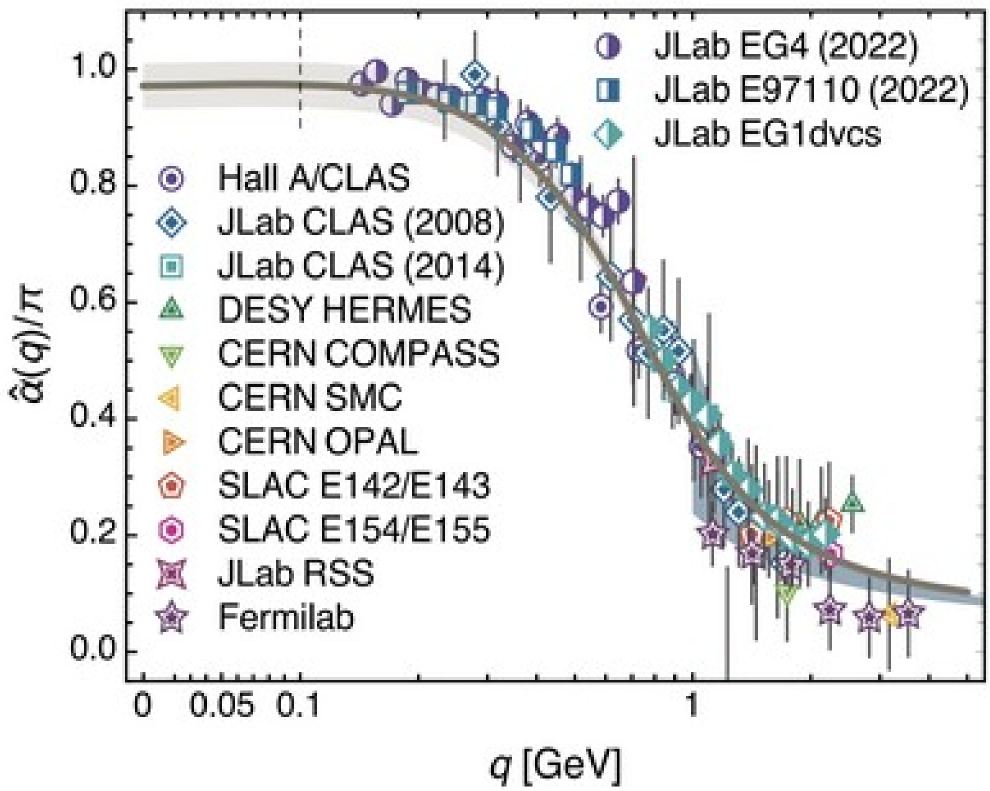


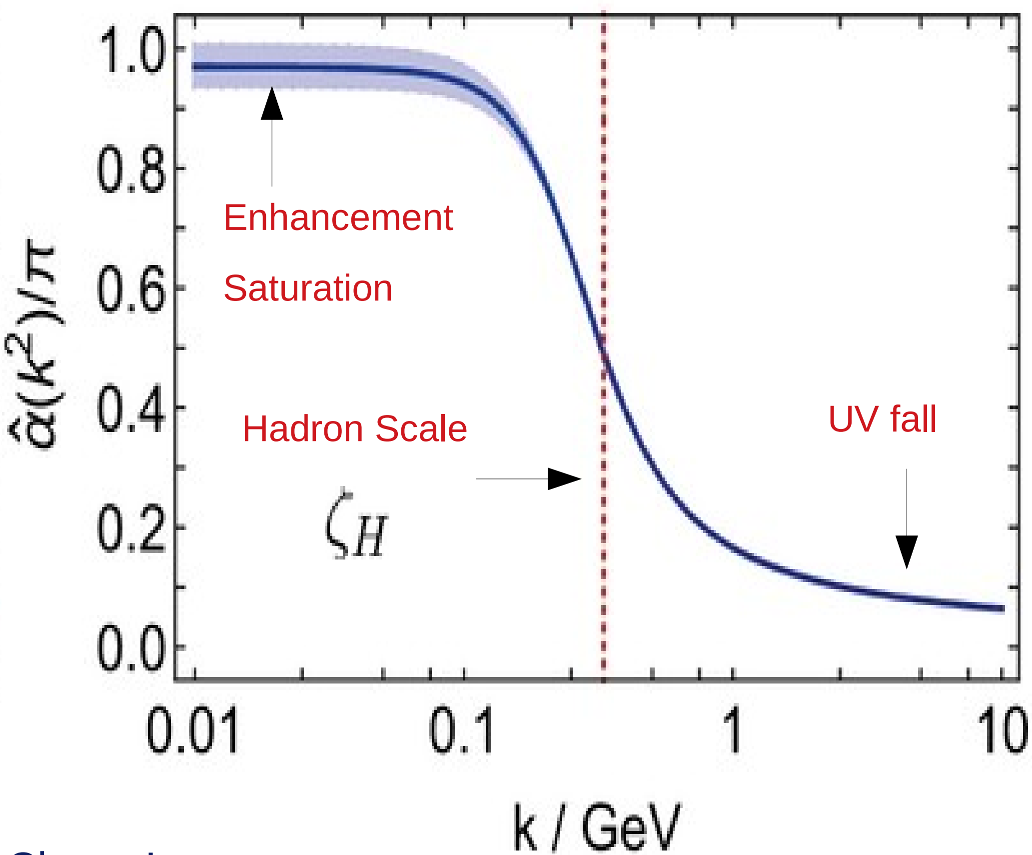
Gluon and quark *running masses*

QCD: Basic Facts

➢ **Confinement** and the **EHM** are tightly connected with **QCD's running coupling**.

(figure: D. Binosi's courtesy!)





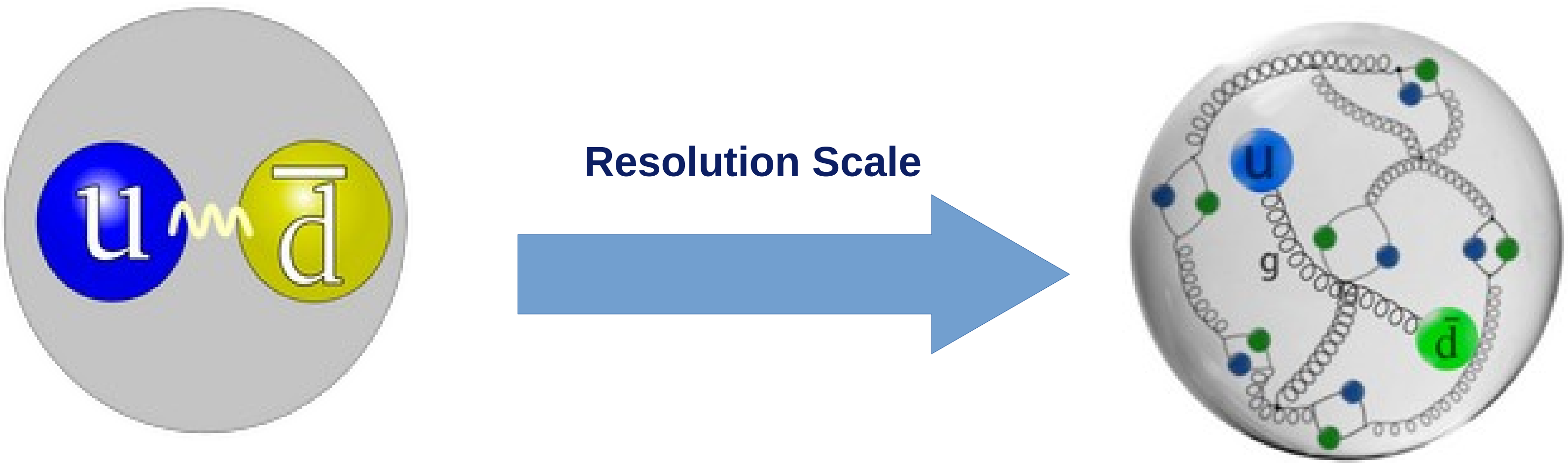
Modern picture of **QCD** coupling. 'Effective Charge'

Combined continuum + QCD lattice analysis

ζ_H : Fully **dressed valence** quarks express all hadron's properties

Parton distributions: energy scales

ζ_H $\qquad\qquad$ $\zeta > \zeta_H$



- **Fully-dressed valence quarks**

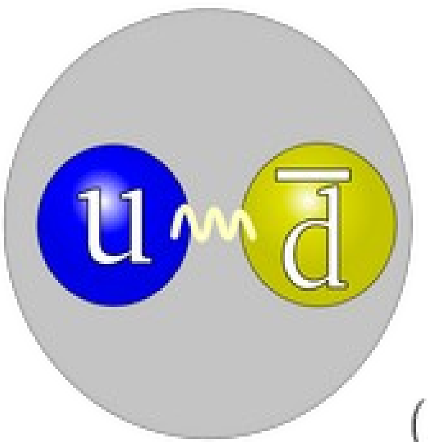
  (quasiparticles)

- **Unveiling of glue and sea d.o.f.**

  (partons)

Parton distributions: energy scales

ζ_H

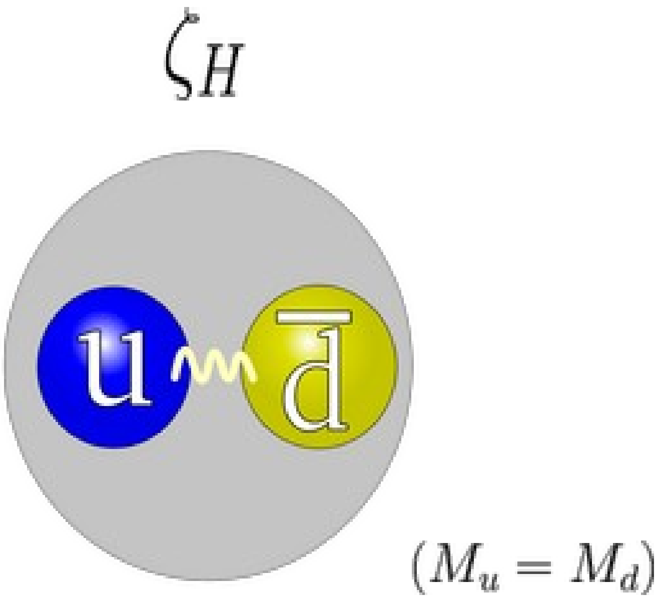


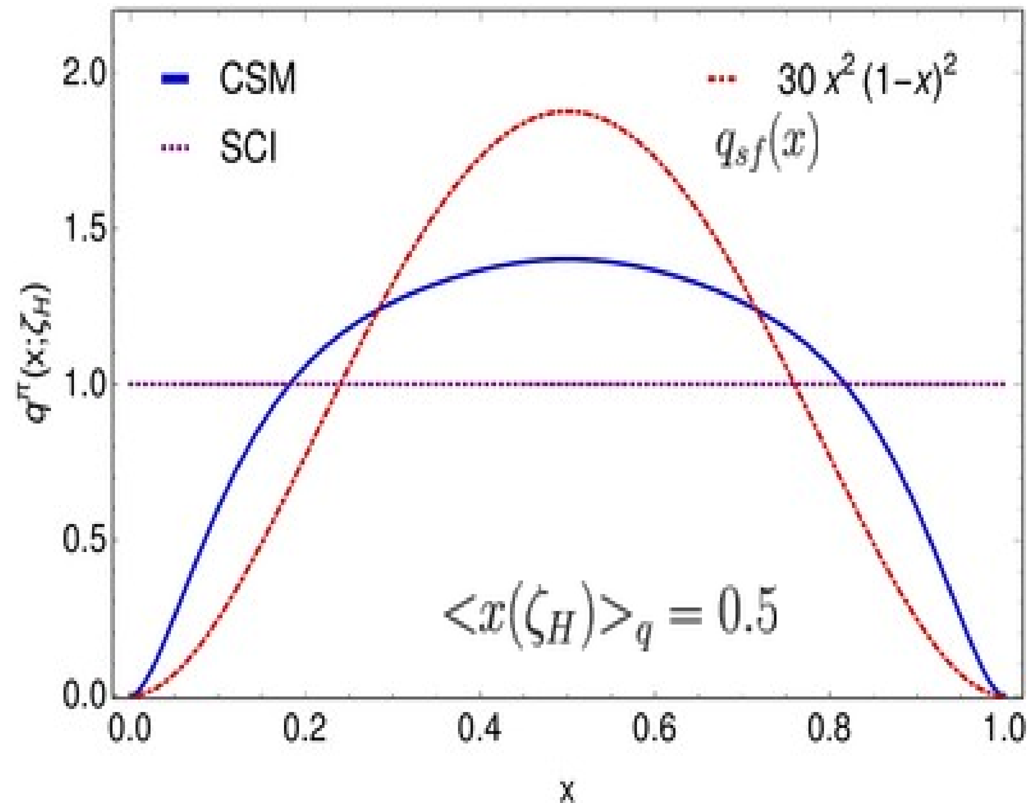
$(M_u = M_d)$

- **Fully-dressed valence quarks**
  - At this scale, **all properties** of the hadron are contained within their valence quarks.
  - **QCD constraints** are defined from here (e.g. large-x behavior of the PDF)

$$u^\pi(x;\zeta) \overset{x\simeq 1}{\approx} (1-x)^{\beta\,=\,2+\gamma(\zeta)}$$

Parton distributions: energy scales





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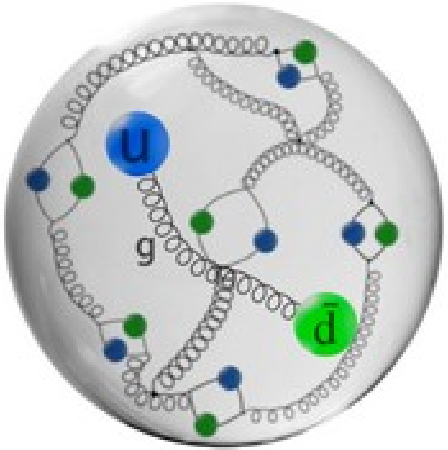
- **CSM** results produce:
  - **EHM-induced** dilated distributions
  - Soft end-point behavior

Cui:2020tdf

Parton distributions: energy scales

$$\zeta > \zeta_H$$



- **Unveiling of glue and sea d.o.f.**

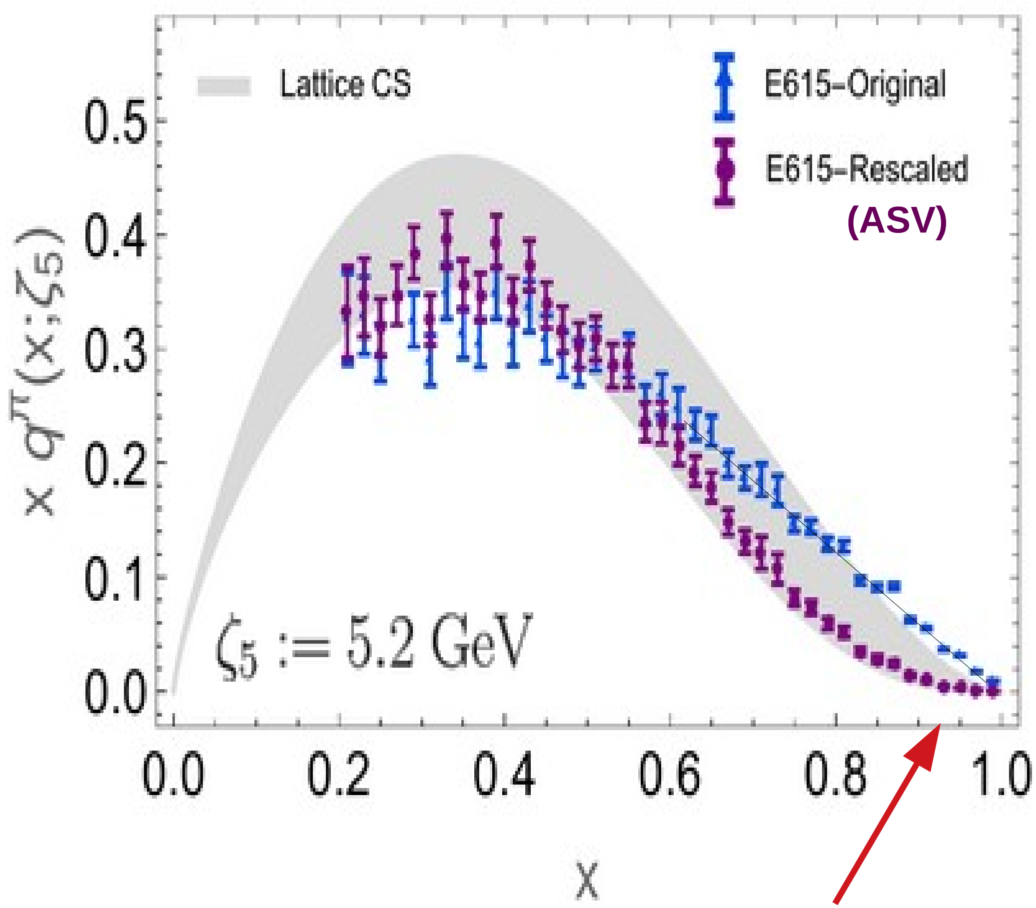
➢ **Experimental** data is given **here**.

➢ The interpretation of parton distributions from cross sections demands **special care**.

➢ In addition, the synergy with **lattice QCD** and phenomenological approaches is welcome.

Parton distributions: energy scales

Conway:1989fs Aicher:2010cb Sufian:2020vzb



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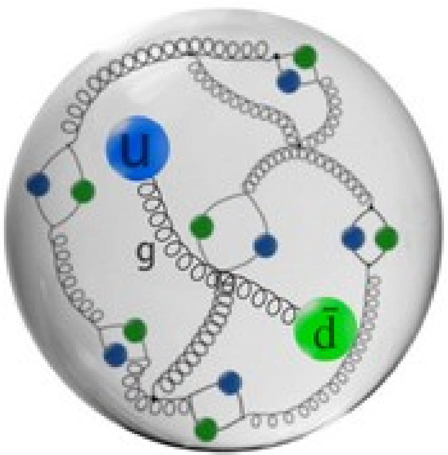

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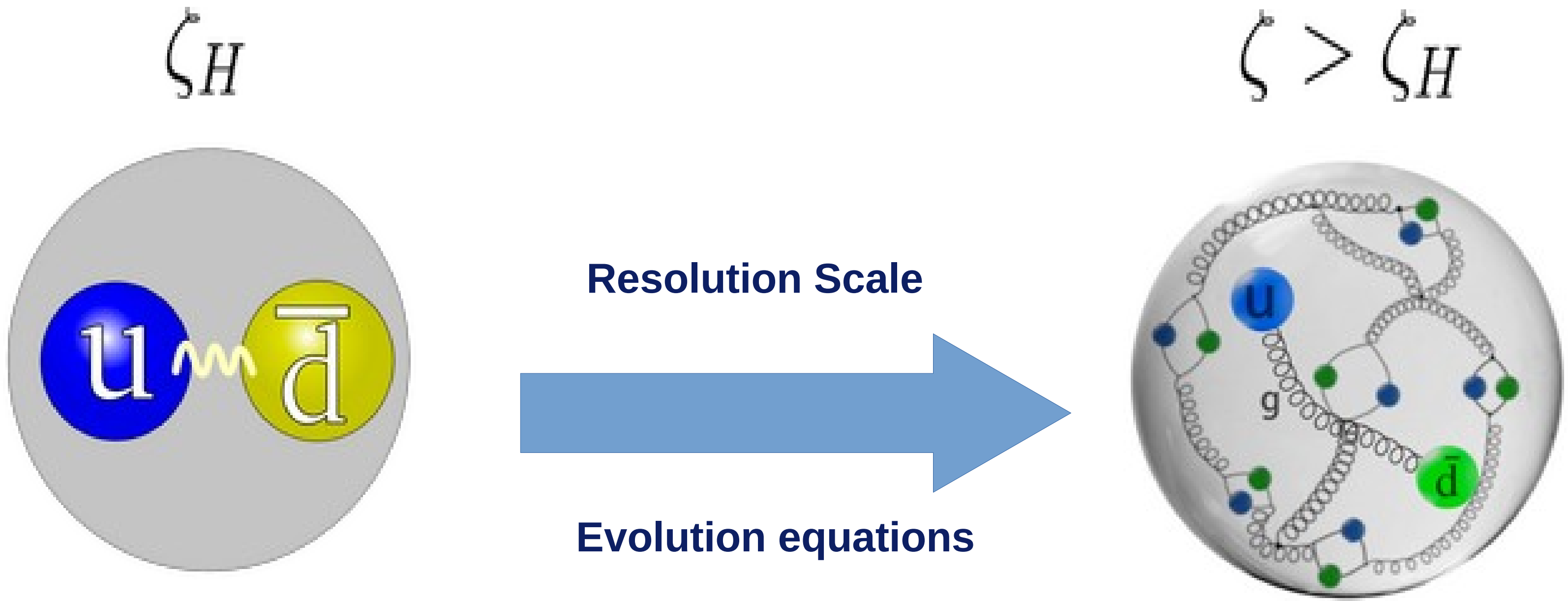

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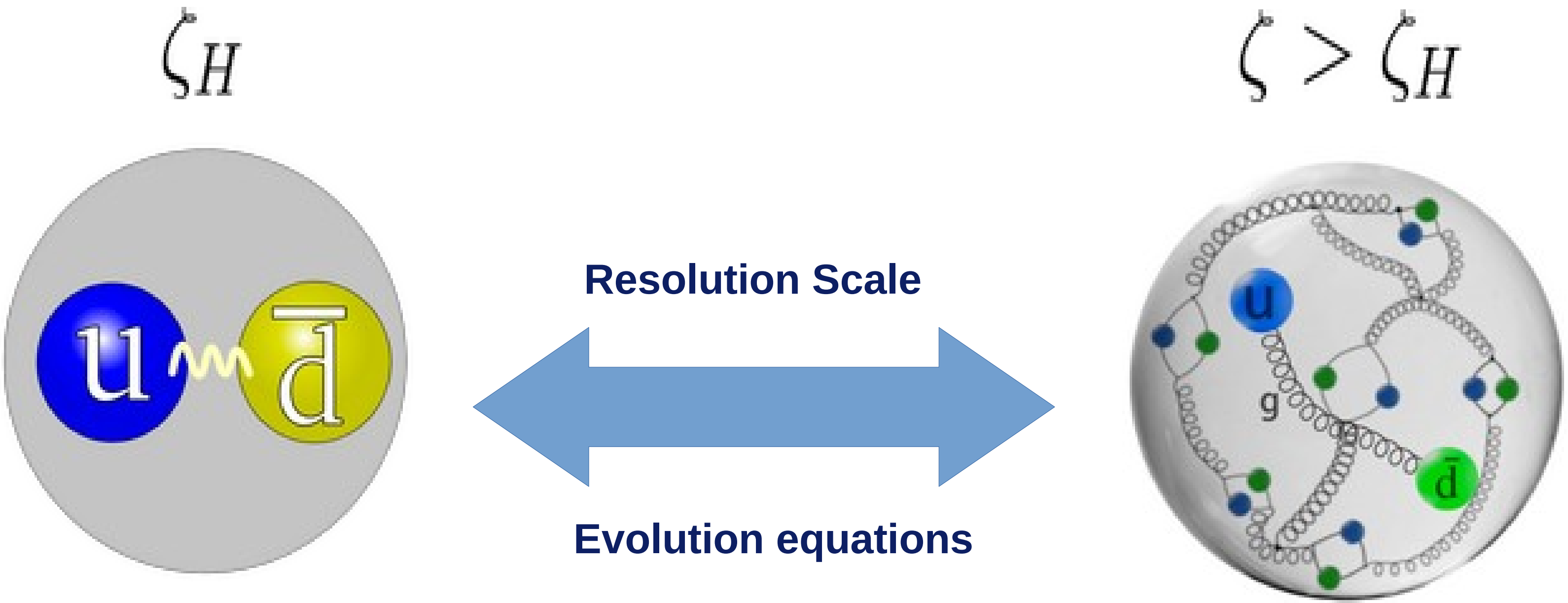

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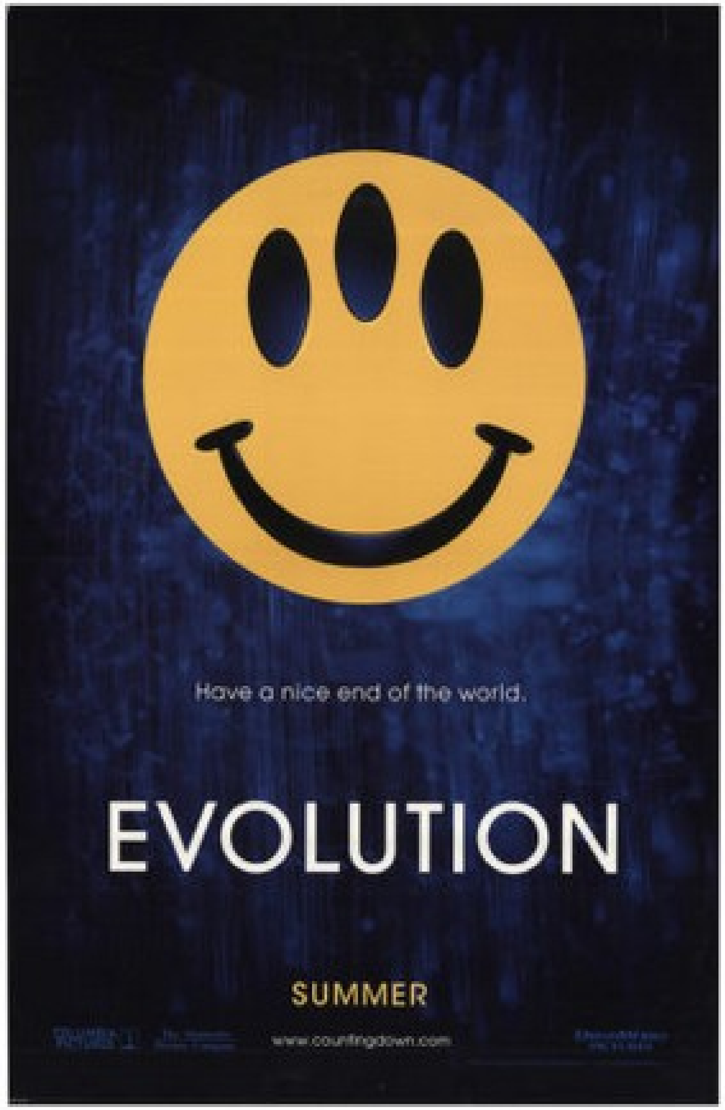
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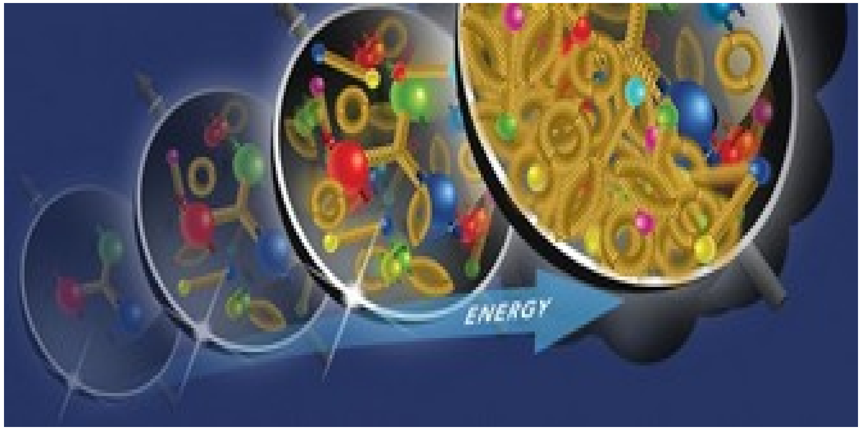
DGLAP: All orders evolution

Raya:2021zrz

Cui:2020tdf

$$\left\{\zeta^2 \frac{d}{d\zeta^2}\int_0^1 dy\delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi}\int_x^1 \frac{dy}{y}\begin{pmatrix} P_{qq}^{\rm NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\rm S}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{{\rm NS},+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\rm S}(y,t;\zeta)\end{pmatrix} = 0$$

DGLAP leading-order evolution equations



DGLAP: All orders evolution

Assumption: define an **effective** charge such that

Raya:2021zrz
Cui:2020tdf

Starting from fully-dressed **quasiparticles**, at ζ_H → **Sea** and **Gluon** content unveils, as prescribed by **QCD**

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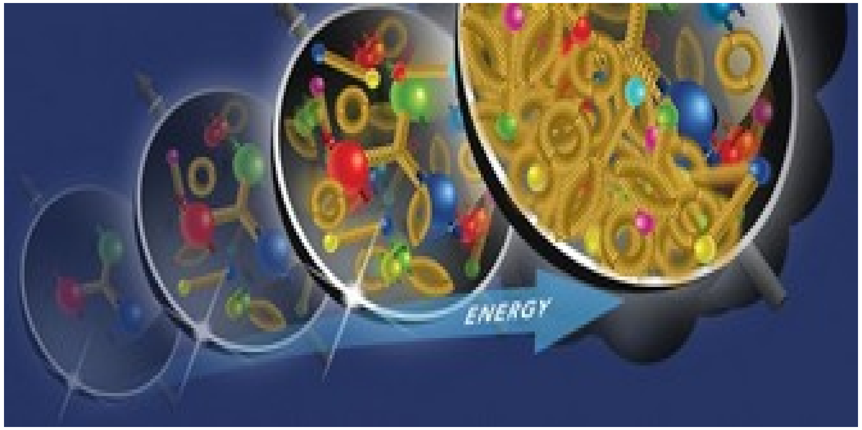

- <u>Not</u> the **LO** QCD coupling but an **effective** one.
- Making this equation <u>exact</u>.
- Connecting with the <u>hadron scale</u>, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)

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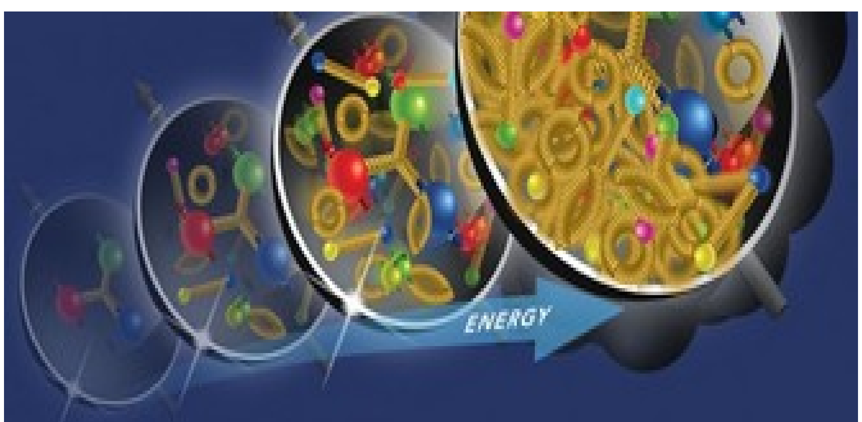
$$\left\{\zeta^2\frac{d}{d\zeta^2}\mathbb{1}+\frac{\alpha(\zeta^2)}{4\pi}\begin{pmatrix}\gamma_{qq}^{(n)}&0&0\\0&\gamma_{qq}^{(n)}&2n_f\gamma_{qg}^{(n)}\\0&\gamma_{gq}^{(n)}&\gamma_{gg}^{(n)}\end{pmatrix}\right\}\begin{pmatrix}\langle x^n\rangle_{\text{NS}}(\zeta)\\ \langle x^n\rangle_{\text{S}}(\zeta)\\ \langle x^n\rangle_g(\zeta)\end{pmatrix}=0$$

DGLAP ~~leading order~~ evolution equations

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$



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PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma_{\rm H}^{q}(x) = q_{\rm H}(x) + \bar{q}_{\rm H}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\rm H}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\rm H}(y)$$

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Valence-quark PDF in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_H}$$

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DGLAP: All orders evolution

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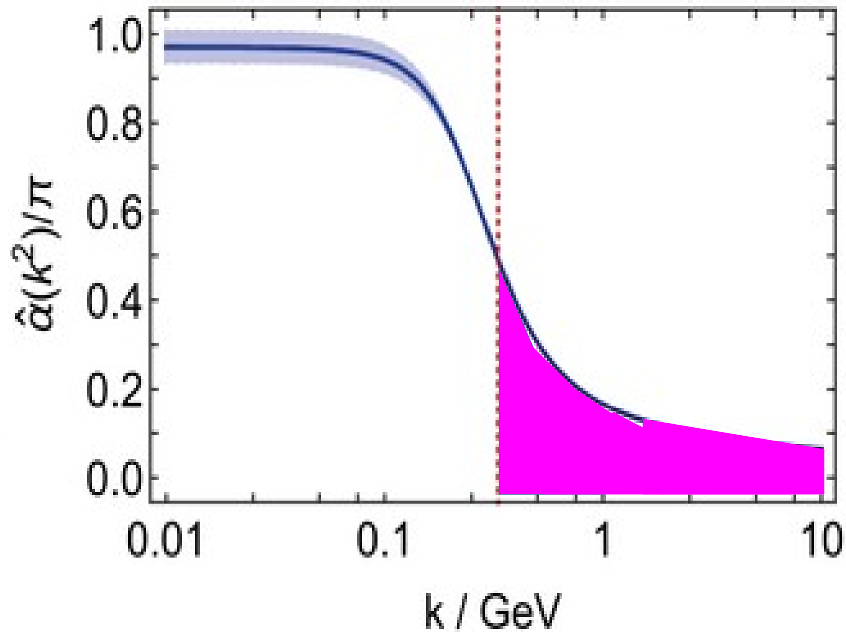

Moments' evolution is controlled by the integrated "strength" of the coupling beyond the hadron scale

DGLAP: All orders evolution

Cui:2020tdf

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$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

The ratio of lightcone momentum fractions encodes the required information of the charge

DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n \rangle_{q_H}^{\zeta} = \langle x^n \rangle_{q_H}^{\zeta_H} \exp\left(-\frac{\gamma_{qq}^n}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dz}{z}\alpha(z^2)\right) = \langle x^n \rangle_{q_H}^{\zeta_H}\left[\frac{\langle x \rangle_{q_H}^{\zeta}}{\langle x \rangle_{q_H}^{\zeta_H}}\right]^{\gamma_{qq}^n/\gamma_{qq}}$$

This ratio encodes the information of the charge

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve "all" the Mellin moments up and reconstruct the PDF.

DGLAP: All orders evolution

Cui:2020tdf

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Direct connection bridging from hadron to experimental scale: only one input is needed to evolve "all" the Mellin moments up and reconstruct the PDF.

This ratio encodes the information of the charge and use isospin symmetry (pion case)

$$\langle x \rangle_{u_\pi}^{\zeta_H} = \langle x \rangle_{d_\pi}^{\zeta_H} = \frac{1}{2}$$

DGLAP: All orders evolution

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Capitalizing on the Mellin moments of asymptotically large order:

$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)}\,(1+\mathcal{O}(1-x))$$

$$\beta(\zeta) = \beta(\zeta_H) + \frac{3}{2}\ln\frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle}$$

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Capitalizing on the Mellin moments of asymptotically large order:

$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)} \left(1 + \mathcal{O}(1-x)\right)$$

$$\beta(\zeta) = \beta(\zeta_H) + \frac{3}{2} \ln \frac{\langle x(\zeta_H) \rangle}{\langle x(\zeta) \rangle}$$

Under a sensible assumption at large momentum scale:

$$q(x;\zeta) \underset{x\to 0}{\sim} x^{\alpha(\zeta)} \left(1 + \mathcal{O}(x)\right)$$

$$1 + \alpha(\zeta) = \frac{3}{2} \langle x(\zeta) \rangle \ln \frac{\langle x(\zeta_H) \rangle}{\langle x(\zeta) \rangle} + \beta(\zeta_H) \langle x(\zeta) \rangle + \mathcal{O}\left(\frac{\langle x(\zeta) \rangle}{\ln \frac{\langle x(\zeta_H) \rangle}{\langle x(\zeta) \rangle}} \right)$$

DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n \rangle^{\zeta}_{q_\pi} = \langle x^n \rangle^{\zeta_H}_{q_\pi} \exp\left(-\frac{\gamma^n_{qq}}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dz}{z}\alpha(z^2)\right) = \langle x^n \rangle^{\zeta_H}_{q_\pi}\left[\langle 2x \rangle^{\zeta}_{q_\pi}\right]^{\gamma^n_{qq}/\gamma_{qq}}$$

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve "all" the Mellin moments up and reconstruct the PDF.

This ratio encodes the information of the charge and use isospin symmetry (pion case)

$$\langle x \rangle^{\zeta_H}_{u_\pi} = \langle x \rangle^{\zeta_H}_{d_\pi} = \frac{1}{2}$$

Capitalizing on the Mellin moments of asymptotically large order:

$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)}\,(1+\mathcal{O}(1-x))$$

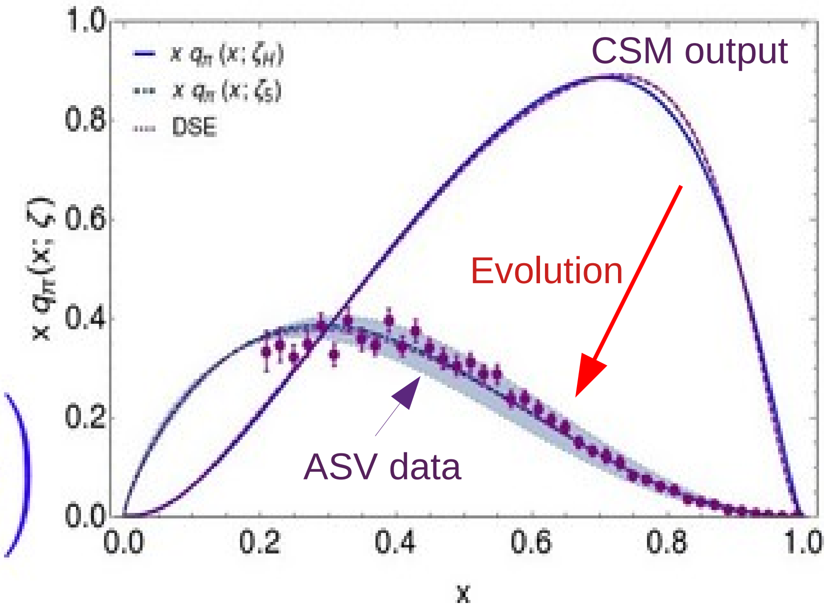
$$\beta(\zeta) = \beta(\zeta_H) + \frac{3}{2}\ln\frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle}$$

Under a sensible assumption at large momentum scale:

$$q(x;\zeta) \underset{x\to 0}{\sim} x^{\alpha(\zeta)}\,(1+\mathcal{O}(x))$$

$$1+\alpha(\zeta) = \frac{3}{2}\langle x(\zeta)\rangle \ln\frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle} + \beta(\zeta_H)\langle x(\zeta)\rangle + \mathcal{O}\left(\frac{\langle x(\zeta)\rangle}{\ln\frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle}}\right)$$

Reconstruction after evolving:



DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle_{u_\pi}^{\zeta_H} = \frac{1}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle_{u_\pi}^{\zeta_H}$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1} .$$

- Since **isospin symmetry** limit implies:

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DGLAP: All orders evolution

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$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1}.$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1}.$$

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Reported lattice moments agree very well with the recursion formula

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7		0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1}.$$

- Since **isospin symmetry** limit implies:

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- **Odd** moments can be expressed in terms of previous **even** moments.
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Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1}.$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

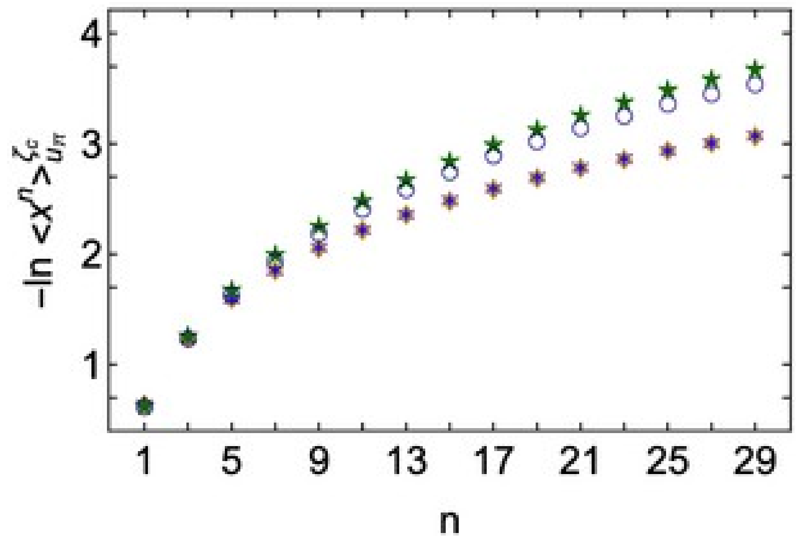
Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

Moments from global fits can be also compared to the estimated from recursion !

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

Moments computed from: P. Barry et al., PRL127(2021)232001



DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\langle x^n \rangle^{\zeta}_{u_\pi} (\langle 2x \rangle^{\zeta}_{u_\pi})^{-\gamma_0^n/\gamma_0^1}$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1}$$

$$q(x;\zeta_H) = \delta(x - 1/2)$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x - 1/2) \qquad q(x;\zeta_H) = 1$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

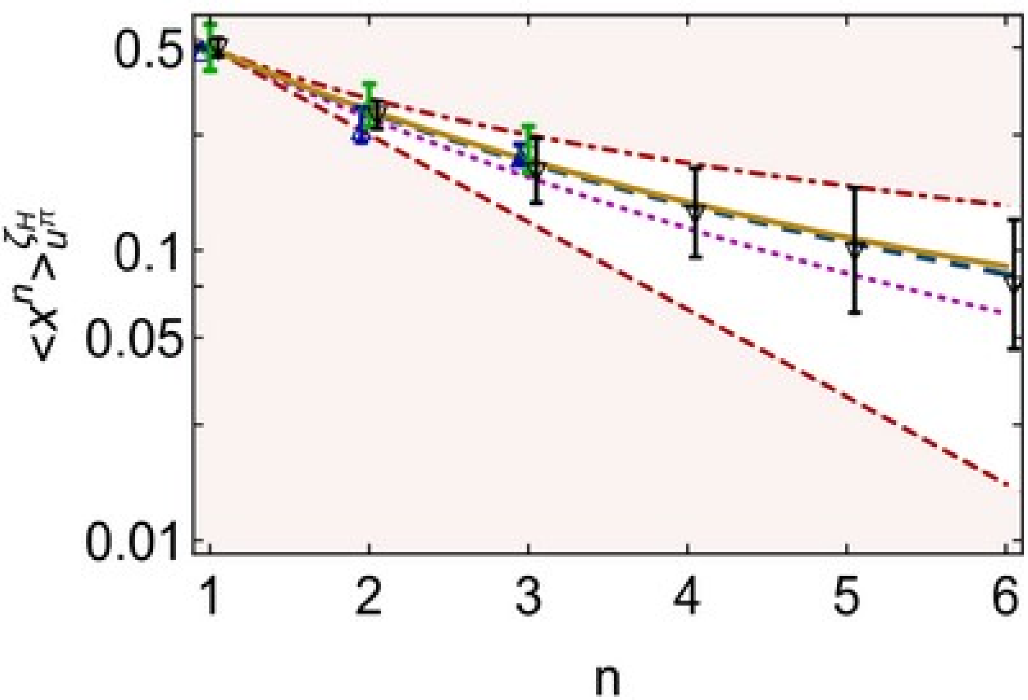
- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.
- **Upper bound** comes out from considering the opposite limit of a weekly interacting system of two (then fully decorrelated) partons: all the momentum fractions are equally probable.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case).

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x - 1/2) \qquad q(x;\zeta_H) = 1$$



- Keeping **isospin symmetry,** implying:

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- **Upper bound** comes out from considering the opposite limit of a weekly interacting system of two (then fully decorrelated) partons: all the momentum fractions are equally probable.

n	[61] Joo:2019bzr	[62] Sufian:2019bol	[63] Alexandrou:2021mmi
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4			0.023(05)(06)
5			0.014(04)(05)
6			0.009(03)(03)

Lattice moments verifying the **recurrence relation** too.

DGLAP: All orders evolution

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathsf{H}}(x) = q_{\mathsf{H}}(x) + \bar{q}_{\mathsf{H}}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathsf{H}}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + 2P^\zeta_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^\zeta_q = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^\zeta_{\Sigma^q_H} + 2\mathcal{P}^\zeta_q \gamma^n_{qg} \langle x^n \rangle^\zeta_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^\zeta_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^\zeta_{g_H} \right\}$$

DGLAP: All orders evolution

Cui:2020tdf

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$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + 2P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^{\zeta}_q = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + 2\mathcal{P}^{\zeta}_q \gamma^n_{qg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

Sea-quark PDF

$$\langle x^n \rangle^{\zeta}_{S^q_H} = \langle x^n \rangle^{\zeta}_{\Sigma^q_H} - \langle x^n \rangle^{\zeta}_{q_H}$$

DGLAP: All orders evolution

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathrm{H}}(x) = q_{\mathrm{H}}(x) + \bar{q}_{\mathrm{H}}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathrm{H}}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + 2P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^{\zeta}_q = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + 2n_f \mathcal{P}^{\zeta}_q \gamma^n_{ug} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

Sea-quark PDF

$$\langle x^n \rangle^{\zeta}_{S^q_H} = \langle x^n \rangle^{\zeta}_{\Sigma^q_H} - \langle x^n \rangle^{\zeta}_{q_H}$$

Full singlet and sea

$$\langle x^n \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x^n \rangle^{\zeta}_{\Sigma^q_H}\,,\quad \langle x^n \rangle^{\zeta}_{S_H} = \sum_q \langle x^n \rangle^{\zeta}_{S^q_H}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H\,,\ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \left(\begin{array}{c} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{array} \right) = \left(\begin{array}{cc} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{array} \right) \left(\begin{array}{c} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{array} \right)$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = [S(\zeta_H, \zeta)]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

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All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ 0 \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \; \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) \end{pmatrix} \sum_q \langle x^n \rangle^{\zeta_H}_q$$

In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta)\right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H ,\ \forall q$$

All quarks active

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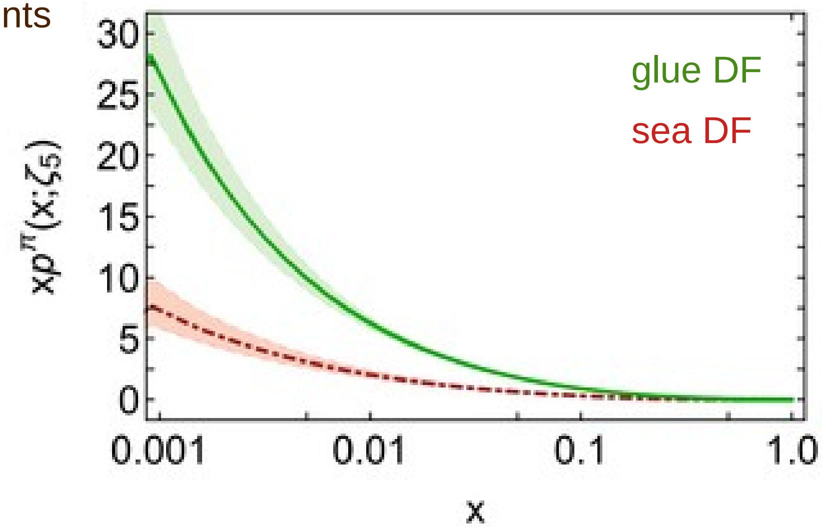
Compute all the moments and reconstruct:



DGLAP: All orders evolution

Implication 4: glue and sea from valence

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All quarks active

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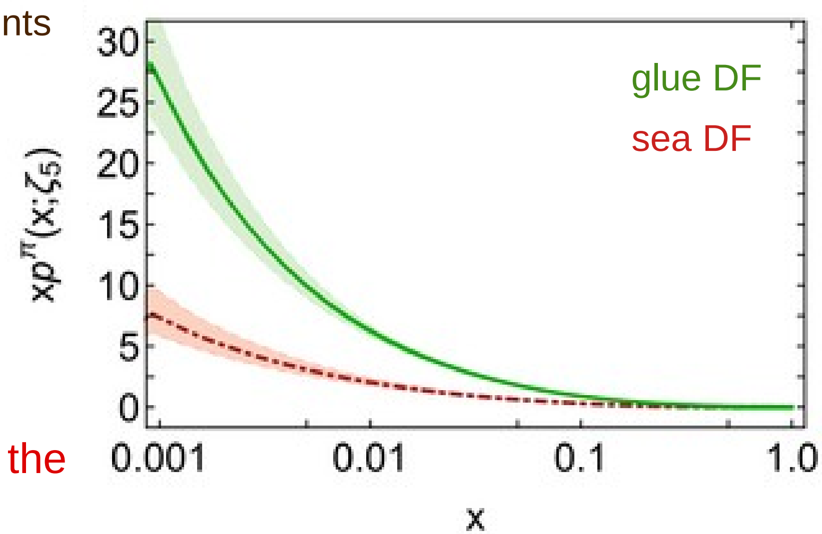
Compute all the moments and reconstruct:

$$S^n_{\pm} = \left[S(\zeta_H, \zeta)\right]^{\lambda^n_{\pm}/\gamma_{uu}} = \left[\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}}\right]^{\lambda^n_{\pm}/\gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$



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Implication 4: glue and sea from valence

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All quarks active

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n=1 case
$n_f = 4$

$$\langle x \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x \rangle^{\zeta}_{q_H} + \langle x \rangle^{\zeta}_{S_H} = \frac{3}{7} + \frac{4}{7} \left[S(\zeta_H, \zeta) \right]^{7/4}$$

$$\langle x \rangle^{\zeta}_{g_H} = \frac{4}{7} \left(1 - \left[S(\zeta_H, \zeta) \right]^{7/4} \right)$$

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DGLAP: All orders evolution

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ζ_5	$\langle 2x \rangle^{\pi}_q$	$\langle x \rangle^{\pi}_g$	$\langle x \rangle^{\pi}_{\text{sea}}$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

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Z-F. Cui et al., arXiv:2006.1465

DGLAP: All orders evolution

Implication 4: glue and sea from valence

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R.S. Sufian et al., arXiv:2001.04960

DGLAP: All orders evolution

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DGLAP: All orders evolution

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n=1 case

$n_f = 4$

$$\langle x \rangle^{\zeta}_{\Sigma_H} \underset{\zeta^2 \to \infty}{=} \langle x \rangle^{\zeta}_{S_H} \underset{\zeta^2 \to \infty}{=} \frac{3}{7}$$

$$\langle x \rangle^{\zeta}_{g_H} \underset{\zeta^2 \to \infty}{=} \frac{4}{7}$$

Asymptotic limit: G. Altarelli, Phys. Rep. 81, 1 (1982)

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DGLAP: All orders evolution

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All quarks active

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n=1 case
$n_f = 4$

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$$\langle x^n \rangle^{\zeta}_{\Sigma_H} \underset{\zeta^2 \to \infty}{=} \langle x^n \rangle^{\zeta}_{S_H} \underset{\zeta^2 \to \infty}{=} \langle x^n \rangle^{\zeta}_{g_H} \underset{\zeta^2 \to \infty}{=} 0\,, \quad \text{for } n > 1$$

owing to $\lambda^n_\pm > 0$

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n=1 case
$n_f = 4$

$$\Sigma_H(x) \underset{\zeta^2 \to \infty}{=} \frac{3}{7} \frac{\delta(x)}{x}$$

$$g_H(x) \underset{\zeta^2 \to \infty}{=} \frac{4}{7} \frac{\delta(x)}{x}$$

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DGLAP: All orders evolution

Implication 5: correlating glue and sea

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g}\left(S^n_- - S^n_+\right) \\ \beta^n_{g\Sigma}\left(S^n_- - S^n_+\right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

DGLAP: All orders evolution

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All quarks active

$$\begin{pmatrix} \alpha_+^n \left[S_-^n\right]^{-1} + \alpha_-^n \left[S_+^n\right]^{-1} & \beta_{\Sigma g}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) \\ \beta_{g\Sigma}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) & \alpha_-^n \left[S_-^n\right]^{-1} + \alpha_+^n \left[S_+^n\right]^{-1} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta_H} \\ \langle x^n \rangle_{g_H}^{\zeta_H} \end{pmatrix}$$

The equation can be easily inverted

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$$\begin{pmatrix} \alpha_+^n \left[S_-^n\right]^{-1} + \alpha_-^n \left[S_+^n\right]^{-1} & \beta_{\Sigma g}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) \\ \beta_{g\Sigma}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) & \alpha_-^n \left[S_-^n\right]^{-1} + \alpha_+^n \left[S_+^n\right]^{-1} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta_H} \\ 0 \end{pmatrix}$$

The equation can be easily inverted and, relying on the hadronic scale definition, delivers a constraint for all Mellin moments of glue and sea at any experimental scale:

$$\frac{\langle x^n \rangle_{\Sigma_\pi}^{\zeta}}{\langle x^n \rangle_{g_\pi}^{\zeta}} = \frac{\langle x^n \rangle_{\mathcal{S}_\pi}^{\zeta} + \langle 2x^n \rangle_{u_\pi}^{\zeta}}{\langle x^n \rangle_{g_\pi}^{\zeta}} = \frac{\alpha_-^n S_+^n + \alpha_+^n S_-^n}{\beta_{g\Sigma}^n \left(S_-^n - S_+^n\right)}$$

DGLAP: All orders evolution

Implication 5: correlating glue and sea

$$M_q = \zeta_H ,\ \forall q$$

All quarks active

$$\begin{pmatrix} \alpha_+^n \left[S_-^n\right]^{-1} + \alpha_-^n \left[S_+^n\right]^{-1} & \beta_{\Sigma g}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) \\ \beta_{g\Sigma}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) & \alpha_-^n \left[S_-^n\right]^{-1} + \alpha_+^n \left[S_+^n\right]^{-1} \end{pmatrix} \begin{pmatrix} \langle x^n\rangle_{\Sigma_H}^{\zeta} \\ \langle x^n\rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \langle x^n\rangle_{\Sigma_H}^{\zeta_H} \\ 0 \end{pmatrix}$$

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	$\langle 2x\rangle_{u_\pi}^{\zeta}$	$\langle x\rangle_{\mathcal{S}_\pi}^{\zeta}$	$\langle x\rangle_{g_\pi}^{\zeta}$	$n_f = 4$ $\langle x\rangle_{\Sigma_\pi}^{\zeta_H}$	$\langle x\rangle_{g_\pi}^{\zeta_H}$
NLO	0.53(2)	0.14(4)	0.34(6)	1.15(14)	-0.14(13)
NLL-Cos	0.47(2)	0.14(5)	0.39(6)	1.11(16)	-0.11(16)
NLL-Exp	0.46(2)	0.16(5)	0.38(6)	1.15(12)	-0.14(13)
NLL-dM	0.46(3)	0.15(7)	0.40(5)	1.12(22)	-0.11(18)

All-orders DGLAP: hard-wall thresholds

Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^\zeta_{q_H}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^\zeta_{\Sigma^q_H} + 2\theta(\zeta - M_q) \gamma^n_{ug} \langle x^n \rangle^\zeta_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \sum_q \langle x^n \rangle^\zeta_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^\zeta_{g_H} \right\}$$

Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$

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$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma^{u+d}_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma^n_{uu} & 4\gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma^{u+d}_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} - \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} 0 \\ \gamma^n_{gu} \langle x^n \rangle^{\zeta}_{\Sigma^s_H} \end{pmatrix}$$

Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors.

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Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors. Specializing for the averaged momentum fraction.

In pion's (proton's) case $\langle x \rangle^{\zeta_H}_{s_\pi} = 0$

$$\langle x \rangle^{\zeta}_{\Sigma^s_\pi} \equiv 0$$

$$\begin{pmatrix} \langle x \rangle^{\zeta}_{\Sigma^{u+d}_\pi} \\ \langle x \rangle^{\zeta}_{g_\pi} \end{pmatrix} = \begin{pmatrix} \frac{3}{11} + \frac{8}{11} [S(\zeta_H, \zeta]^{11/8} \\ \frac{8}{11} \left(1 - [S(\zeta_H, \zeta]^{11/8}\right) \end{pmatrix}$$

$$S(\zeta_H, \zeta) = \exp\left(-\frac{\gamma_{uu}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2)\right)$$

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In kaon's case (after some algebra) $\langle x \rangle^{\zeta_H}_{s_K} = s_0$

$$\langle x \rangle^{\zeta}_{\Sigma^s_K} = s_0\, S(\zeta_H, \zeta)$$

$$\begin{pmatrix} \langle x \rangle^{\zeta}_{\Sigma^{u+d}_K} \\ \langle x \rangle^{\zeta}_{g_K} \end{pmatrix} = \begin{pmatrix} \frac{3}{11} + \frac{8}{11} [S(\zeta_H, \zeta)]^{11/8} - \langle x \rangle^{\zeta}_{\Sigma^s_K} \\ \frac{8}{11} \left(1 - [S(\zeta_H, \zeta)]^{11/8}\right) \end{pmatrix}$$

$$S(\zeta_H, \zeta) = \exp\left(-\frac{\gamma_{uu}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2)\right)$$

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$$q = u, d, s$$

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$$\begin{aligned}
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{q_H}^{\zeta} &= -\frac{\alpha(\zeta^2)}{4\pi} \gamma_{uu}^n \langle x^n \rangle_{q_H}^{\zeta} \\
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{\Sigma_H^q}^{\zeta} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{uu}^n \langle x^n \rangle_{\Sigma_H^q}^{\zeta} + 2\theta(\zeta - M_q) \gamma_{ug}^n \langle x^n \rangle_{g_H}^{\zeta} \right\} \\
\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{g_H}^{\zeta} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{gu}^n \sum_q \langle x^n \rangle_{\Sigma_H^q}^{\zeta} + \gamma_{gg}^n \langle x^n \rangle_{g_H}^{\zeta} \right\}
\end{aligned}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

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$$\langle x \rangle^{\zeta}_{q_H} = \langle x \rangle^{\zeta_H}_{q_H} S(\zeta_H, \zeta) \qquad \langle x \rangle^{\zeta}_{g_H} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle^{\zeta}_{u_\pi} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-3/16} \left[\langle 2x \rangle^{M_s}_{u_\pi} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle^{M_c}_{u_\pi} \langle 2x \rangle^{M_s}_{u_\pi} \right]^{-3/16}$$

Capitalizing on the universality of the effective charge, **all hadrons'** momentum fraction averages can be expressed in terms of **pion's** ones.

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3 (always) active flavors

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3 (always) active flavors →

$$\tau(\zeta_H, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u} \right]^{-7/4} + \frac{16}{25} \left[\langle 2x \rangle^{M_c}_{u} \right]^{-3/16}$$

4 (always) active flavors →

$$\tau(\zeta_H, \zeta_H) = \frac{4}{7}$$

Thus recovering the previous result!

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$$\langle x \rangle^{\zeta}_{S^q_H} = \langle x \rangle^{\zeta}_{\Sigma^q_H} - \langle x \rangle^{\zeta}_{q_H} = \theta(\zeta - M_q) \frac{1}{3\pi} \int_{M_q}^{\zeta} \frac{dz}{z} \alpha(z^2) \langle x \rangle^{z}_{g_H} S(z, \zeta)$$

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$$q = u, d, s, c$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^\zeta_{q_H}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^\zeta_{\Sigma^q_H} + 2\theta(\zeta - M_q) \gamma^n_{ug} \langle x^n \rangle^\zeta_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \sum_q \langle x^n \rangle^\zeta_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^\zeta_{g_H} \right\}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x \rangle^\zeta_{q_H} = \langle x \rangle^{\zeta_H}_{q_H} S(\zeta_H, \zeta)$$

$$\langle x \rangle^\zeta_{g_H} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-3/16} \left[\langle 2x \rangle^{M_s}_{u_\pi} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle^{M_c}_{u_\pi} \langle 2x \rangle^{M_s}_{u_\pi} \right]^{-3/16}$$

$$\langle x \rangle^\zeta_{S^q_H} = \langle x \rangle^\zeta_{\Sigma^q_H} - \langle x \rangle^\zeta_{q_H} = \theta(\zeta - M_q) \frac{1}{3\pi} \int_{M_q}^{\zeta} \frac{dz}{z} \alpha(z^2) \langle x \rangle^z_{g_H} S(z, \zeta)$$

$$\sum_q \langle x \rangle^\zeta_{S^q_H} = \frac{3}{7} + \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4} - \sum_q \langle x \rangle^\zeta_{q_H}$$

All-orders DGLAP: hard-wall thresholds

Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s, c$$

$$\begin{aligned}
\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{q_H} &= -\frac{\alpha(\zeta^2)}{4\pi}\gamma^n_{uu}\langle x^n\rangle^{\zeta}_{q_H} \\
\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{\Sigma^q_H} &= -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma^n_{uu}\langle x^n\rangle^{\zeta}_{\Sigma^q_H} + 2\theta(\zeta - M_q)\gamma^n_{ug}\langle x^n\rangle^{\zeta}_{g_H}\right\} \\
\zeta^2 \frac{d}{d\zeta^2}\langle x^n\rangle^{\zeta}_{g_H} &= -\frac{\alpha(\zeta^2)}{4\pi}\left\{\gamma^n_{gu}\sum_q \langle x^n\rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg}\langle x^n\rangle^{\zeta}_{g_H}\right\}
\end{aligned}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x\rangle^{\zeta}_{q_H} = \langle x\rangle^{\zeta_H}_{q_H} S(\zeta_H, \zeta)$$

$$\langle x\rangle^{\zeta}_{g_H} = \frac{4}{7} - \tau(M_s, M_c)\left[\langle 2x\rangle^{\zeta}_{u_\pi}\right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175}\left[\langle 2x\rangle^{M_c}_{u_\pi}\right]^{-7/4} - \frac{24}{275}\left[\langle 2x\rangle^{M_c}_{u_\pi}\right]^{-3/16}\left[\langle 2x\rangle^{M_s}_{u_\pi}\right]^{-25/16} + \frac{8}{11}\left[\langle 2x\rangle^{M_c}_{u_\pi}\langle 2x\rangle^{M_s}_{u_\pi}\right]^{-3/16}$$

$$\langle x\rangle^{\zeta}_{S^q_H} = \langle x\rangle^{\zeta}_{\Sigma^q_H} - \langle x\rangle^{\zeta}_{q_H} = \theta(\zeta - M_q)\frac{1}{3\pi}\int_{M_q}^{\zeta}\frac{dz}{z}\alpha(z^2)\langle x\rangle^{z}_{g_H} S(z,\zeta)$$

$$\sum_q \langle x\rangle^{\zeta}_{S^q_H} = \frac{3}{7} + \tau(M_s, M_c)\left[\langle 2x\rangle^{\zeta}_{u_\pi}\right]^{7/4} - \sum_q \langle x\rangle^{\zeta}_{q_H}$$

Momentum conservation

All-orders DGLAP: Pauli blocking

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathrm{H}}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + 2 P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

Modeling the Pauli-blocking contribution:

$$P^{\zeta}_{q\leftarrow g}(z) = \left[P_{q\leftarrow g}(z) + \delta_q \sqrt{3}(1-2z)\mathcal{D}\left(\frac{\zeta}{\zeta_H}\right)\right] \theta(\zeta - M_q)$$

$$\mathcal{D}(t) = \frac{1}{1+(t-1)^2}$$

All-orders DGLAP: Pauli blocking

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^\zeta_{q_\pi}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{\Sigma^q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^\zeta_{\Sigma^q_\pi} + 2\theta(\zeta - M_q) \left[\gamma^n_{qg} + \delta_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x^n \rangle^\zeta_{g_\pi} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{g_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^\zeta_{\Sigma^q_\pi} + \gamma^n_{gg} \langle x^n \rangle^\zeta_{g_\pi} \right\} ;$$

Modeling the Pauli-blocking contribution:

$$a_n = \frac{\sqrt{3}n}{2 + 3n + n^2}$$

$$\mathcal{D}(t) = \frac{1}{1 + (t-1)^2}$$

Momentum conservation

$$\gamma_{qq} + \gamma_{gq} = 2\sum_q \gamma_{qg} + \gamma_{gg} = \sum_q \delta_q = 0$$

All-orders DGLAP: Pauli blocking

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_\pi}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + 2\theta(\zeta - M_q) \left[\gamma^n_{qg} + \delta_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x^n \rangle^{\zeta}_{g_\pi} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_\pi} \right\} ;$$

Modeling the Pauli-blocking contribution:

$$a_n = \frac{\sqrt{3}n}{2 + 3n + n^2}$$

$$\mathcal{D}(t) = \frac{1}{1 + (t-1)^2}$$

Momentum conservation

$$\gamma_{qq} + \gamma_{gq} = 2\sum_q \gamma_{qg} + \gamma_{gg} = \sum_q \delta_q = 0$$

Equations and solutions for $\sum_q \langle x \rangle^{\zeta}_{S^q_H}$ and $\langle x \rangle^{\zeta}_{g_H}$ remain the same, while:

$$\langle x^n \rangle^{\zeta}_{S_q} = -\frac{1}{\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \left[\gamma^n_{ug} + g_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x^n \rangle^{z}_{g_\pi} \left[S(z,\zeta) \right]^{\gamma^n_{uu}/\gamma_{uu}}$$

All-orders DGLAP: Pauli blocking

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_\pi}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + 2\theta(\zeta - M_q) \left[\gamma^n_{qg} + \delta_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x^n \rangle^{\zeta}_{g_\pi} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_\pi} \right\} ;$$

Modeling the Pauli-blocking contribution:

$$a_n = \frac{\sqrt{3}n}{2 + 3n + n^2}$$

$$\mathcal{D}(t) = \frac{1}{1 + (t-1)^2}$$

Momentum conservation

$$\gamma_{qq} + \gamma_{gq} = 2\sum_q \gamma_{qg} + \gamma_{gg} = \sum_q \delta_q = 0$$

Particularly, for the pion momentum fractions, and with:

$$\delta_u = \delta_d = \delta = -\delta_s/2 \,; \delta_c = 0 \,.$$

$$\langle x \rangle^{\zeta}_{\mathcal{S}^{u+d}_\pi} = \frac{2}{3\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \left[1 - \frac{\sqrt{3}}{2} \delta \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x \rangle^{z}_{g_\pi} S(z,\zeta) \,,$$

$$\langle x \rangle^{\zeta}_{\mathcal{S}^{s}_\pi} = \theta(\zeta - M_s) \frac{1}{3\pi} \int_{M_s}^{\zeta} \frac{dz}{z} \alpha(z^2) \left[1 + \sqrt{3} \delta \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x \rangle^{z}_{g_\pi} S(z,\zeta)$$

$$\langle x \rangle^{\zeta}_{\mathcal{S}^{c}_\pi} = \theta(\zeta - M_c) \frac{1}{3\pi} \int_{M_c}^{\zeta} \frac{dz}{z} \alpha(z^2) \langle x \rangle^{z}_{g_\pi} S(z,\zeta) \,.$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

In general, at any momentum scale $\zeta \geq M_c$:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \,,$$

$$\Delta G^{\zeta}_H = \langle x^0 \rangle^{\zeta}_{\tilde{g}^q_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \left\{ \frac{12}{29} \left([S(\zeta_H, M_s)]^{-87/32} - 1 \right) [S(M_s, M_c)]^{-81/32} [S(M_c, \zeta)]^{-75/32} \right.$$
$$\left. + \frac{4}{9} \left([S(M_s, M_c)]^{-81/32} - 1 \right) [S(M_c, \zeta)]^{-75/32} + \frac{12}{25} \left([S(M_c, \zeta)]^{-75/32} - 1 \right) \right\}$$

$$S(\zeta_H, \zeta) = \frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

In general, at any momentum scale $M_s \leq \zeta \leq M_c$:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \,,$$

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$$S(\zeta_H, \zeta) = \frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

In general, at any momentum scale $\zeta_H \leq \zeta \leq M_s$:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} ,$$

$$\Delta G^{\zeta}_{H} = \langle x^0 \rangle^{\zeta}_{\tilde{g}^q_H} = \frac{12}{29} \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \left([S(\zeta_H, \zeta)]^{-87/32} - 1 \right)$$

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All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

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In general, at any momentum scale $\zeta_H \leq \zeta$, and neglecting the mass thresholds:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H}$$

$$\Delta G^{\zeta}_H = \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \begin{cases} \frac{12}{25} \left([S(\zeta_H, \zeta)]^{-75/32} - 1 \right) & n_f = 4 \\ \frac{4}{9} \left([S(M_s, \zeta)]^{-81/32} - 1 \right) & n_f = 3 \end{cases}$$

$$S(\zeta_H, \zeta) = \frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

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A [CT18]+ no thresholds

B [Ya2022]+no thresholds

C [Ya2022]+[Chen2022]

D [Ya2022]+ thresholds

	A	B	C	D
a^{ζ}_{0p}	0.74(11)	0.74(11)	0.65(02)	0.65(02)
ΔG^{ζ}_{p}	2.27(30)	1.50(25)	1.33(15)	1.41(16)
$\tilde{a}^{\zeta}_{0p}$	0.20(11)	0.38(11)	0.33(04)	0.32(04)

Abelian anomaly corrected:

$$\tilde{a}^{\zeta}_{0p} = a^{\zeta}_{0p} - n_f \frac{\hat{\alpha}(\zeta)}{2\pi} \Delta G^{\zeta}_{p}$$

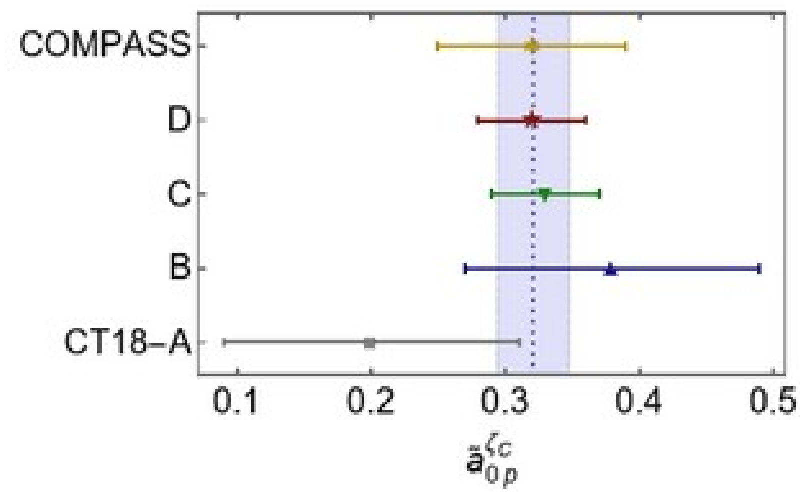
$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0$$



A [CT18]+ no thresholds

B [Ya2022]+no thresholds

C [Ya2022]+[Chen2022]

D [Ya2022]+ thresholds

In general, at any momentum scale $\zeta_H \leq \zeta$, and neglecting the mass thresholds:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H}$$

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	A	B	C	D
a^{ζ}_{0p}	0.74(11)	0.74(11)	0.65(02)	0.65(02)
ΔG^{ζ}_p	2.27(30)	1.50(25)	1.33(15)	1.41(16)
$\tilde{a}^{\zeta}_{0p}$	0.20(11)	0.38(11)	0.33(04)	0.32(04)

Abelian anomaly corrected:

$$\tilde{a}^{\zeta}_{0p} = a^{\zeta}_{0p} - n_f \frac{\hat{\alpha}(\zeta)}{2\pi} \Delta G^{\zeta}_p$$

$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

Reverse engineering the PDF data

Pion **PDF**

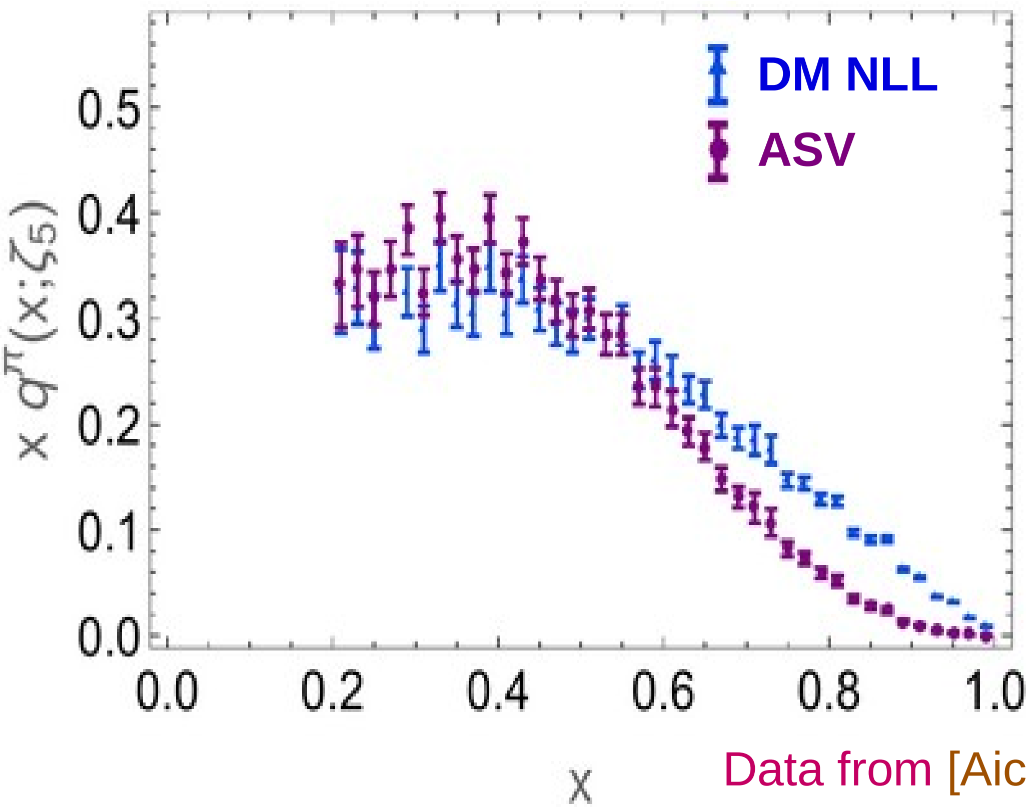
- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^\pi(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Normalization

$$\{\alpha_i^\zeta | i = 1,2,3\}$$

Free parameters



- Then, we proceed as follows:

1) Determine the **best values α_i** via least-squares fit to the data.

2) Generate new **values α_i,** distributed randomly around the best fit.

3) Using the latter set, evaluate:

$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

Data point with error

4) Accept a replica with probability:

$$\mathcal{P} = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}, \quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

5) Evolve back to ζ_H

Repeat (2-5).

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

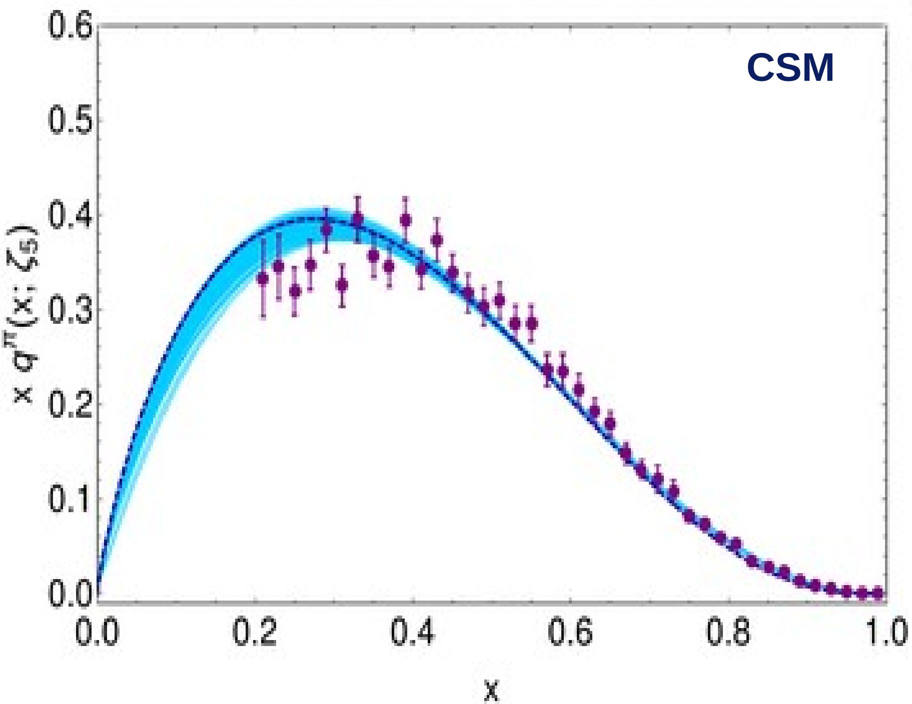
Pion PDF: ASV analysis of E615 data

➢ Applying this algorithm to the **ASV** **data** yields:

(average)

Mean values (of moments) and errors

{{0.5, 2.75144 × 10^-17}, {0.299833, 0.00647045}, {0.199987, 0.00735448}, {0.142895, 0.0068623},
{0.107274, 0.00608759}, {0.0835168, 0.00532834}, {0.0668711, 0.0046596},
{0.0547511, 0.00409028}, {0.0456496, 0.00361041}, {0.0386394, 0.00320609}}



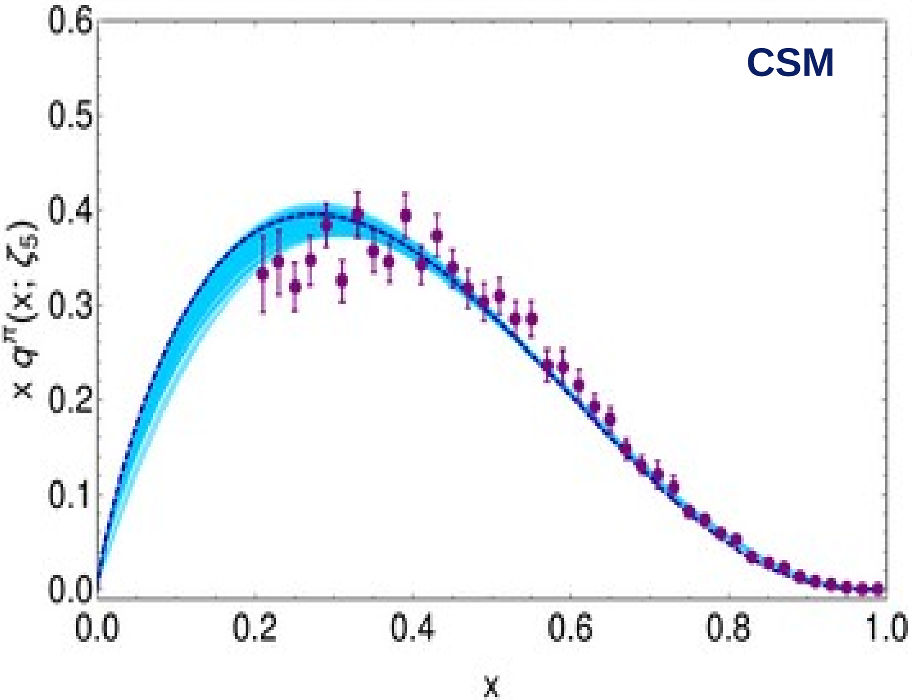
✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✔ It seems it favors a **soft end-point** behavior just like the **CSM result**.

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Pion PDF: ASV analysis of E615 data

➢ Applying this algorithm to the **ASV data** yields:

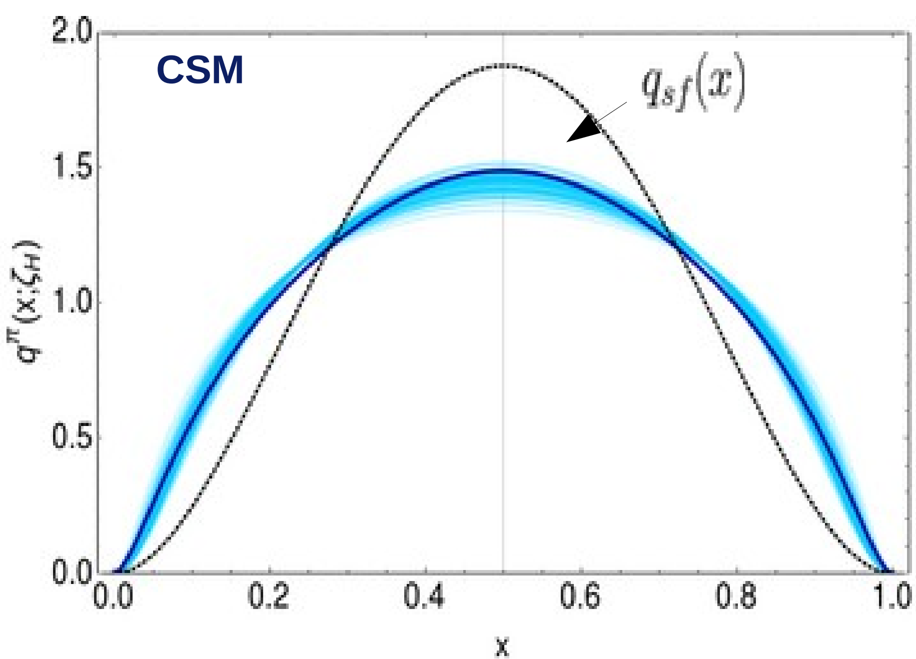


Mean values (of moments) and errors

{{0.5, 2.75144 × 10^-17}, {0.299833, 0.00647045}, {0.199907, 0.00735448}, {0.142895, 0.0068623}, {0.107274, 0.00608759}, {0.0835168, 0.00532834}, {0.0668711, 0.0046596}, {0.0547511, 0.00409028}, {0.0456496, 0.00361041}, {0.0386394, 0.00320609}}

✔ Then, we can **reconstruct** the moments produced by each replica, using the **single-parameter Ansatz**:

$$u^{\pi}(x;\zeta_{\mathcal{H}}) = n_0 \ln(1 + x^2(1-x)^2/\rho^2)$$

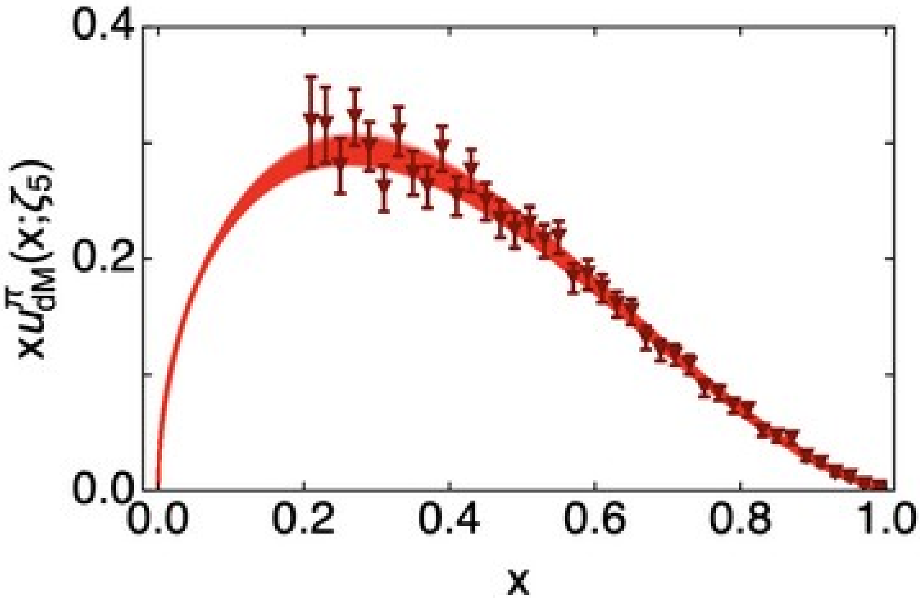


✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✔ It seems it favors a **soft end-point** behavior… just like the **CSM result**.

Pion PDF: dM NLL analysis of E615 data

- Applying this algorithm to the original data yields:



(average)

lean values (of moments) and errors, ζ_H

{{0.5, 2.52187 × 10^-17}, {0.331527, 0.00803273}, {0.247615, 0.0110893},
{0.19784, 0.0121977}, {0.165066, 0.0124911}, {0.141928, 0.0124198},
{0.124755, 0.0121811}, {0.111521, 0.0118683}, {0.101021, 0.0115275},
{0.0924926, 0.0111824}, {0.085431, 0.010845}, {0.0794897, 0.0105214},
{0.0744232, 0.0102142}, {0.0700521, 0.00992435}, {0.0662432, 0.00965182}}

(SCI)

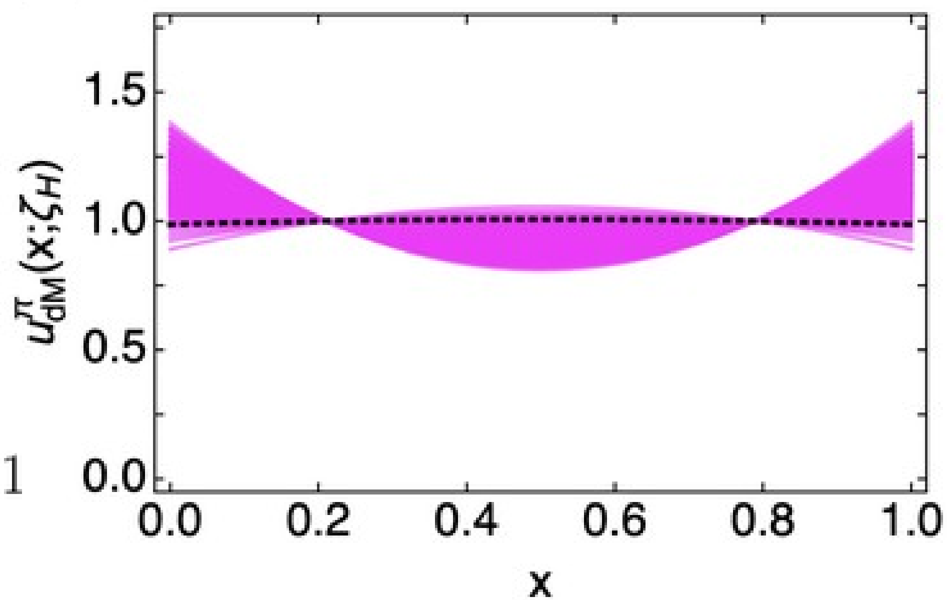
loments from SCI, ζ_H

0.5, 0.332885, 0.249327, 0.199231, 0.165865, 0.142056, 0.124215, 0.11035,
0.0992657, 0.090203, 0.0826552, 0.0762721, 0.0708035, 0.0660661, 0.0619225}

✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✗ But also exhibit agreement with the **SCI results**.

$$q_{\rm SCI}(x;\zeta_H) \approx 1$$



Kaon PDF

- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^{K,\pi}(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Pion's free parameters: $\{\alpha_i^\zeta | i = 1,2,3\}$

Kaon's : α_{3K}^ζ

- Then, we proceed as follows:

1) Determine the **best values α_i** via least-squares fit to the ASV data for the pion.
2) Use u^K/u^π data to fix the only free parameter for the kaon
3) Generate new **values α_i,** distributed randomly around the best fit parameters
4) With these values, evaluate for the pion:

$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

$$\mathcal{P}_\pi = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}\,,\quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

5) And for the kaon in terms of data for

$$R_{K/\pi}\left(x;\left[\alpha_{3K}^{\zeta_5}\right];\zeta_5\right) = \frac{u^K(x;[\alpha_1^{\zeta_5},\alpha_2^{\zeta_5},\alpha_{3K}^{\zeta_5}];)}{u^\pi(x;[\alpha_i^{\zeta_5}])}$$

6) Accept replicas with probabilities

$$\mathcal{P}_{u_\pi}\,,\quad \mathcal{P}_{u_K} = \mathcal{P}_{R_{K/\pi}} \mathcal{P}_{u_\pi}$$

7) Evolve back to ζ_H and **repeat (2-7)**

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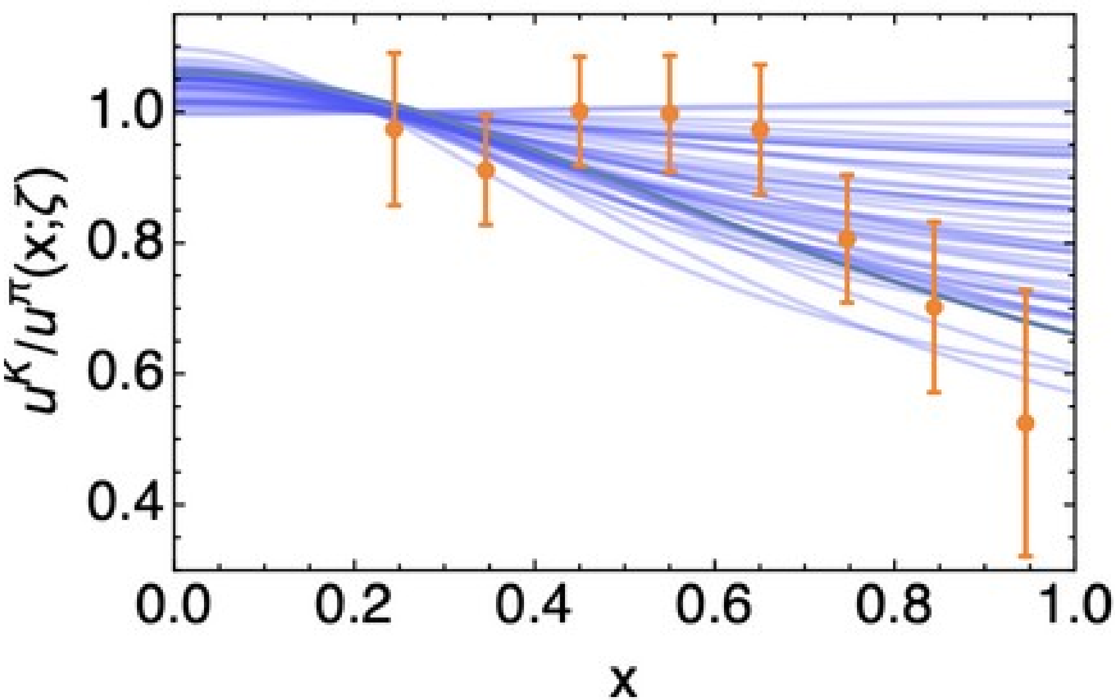

Data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

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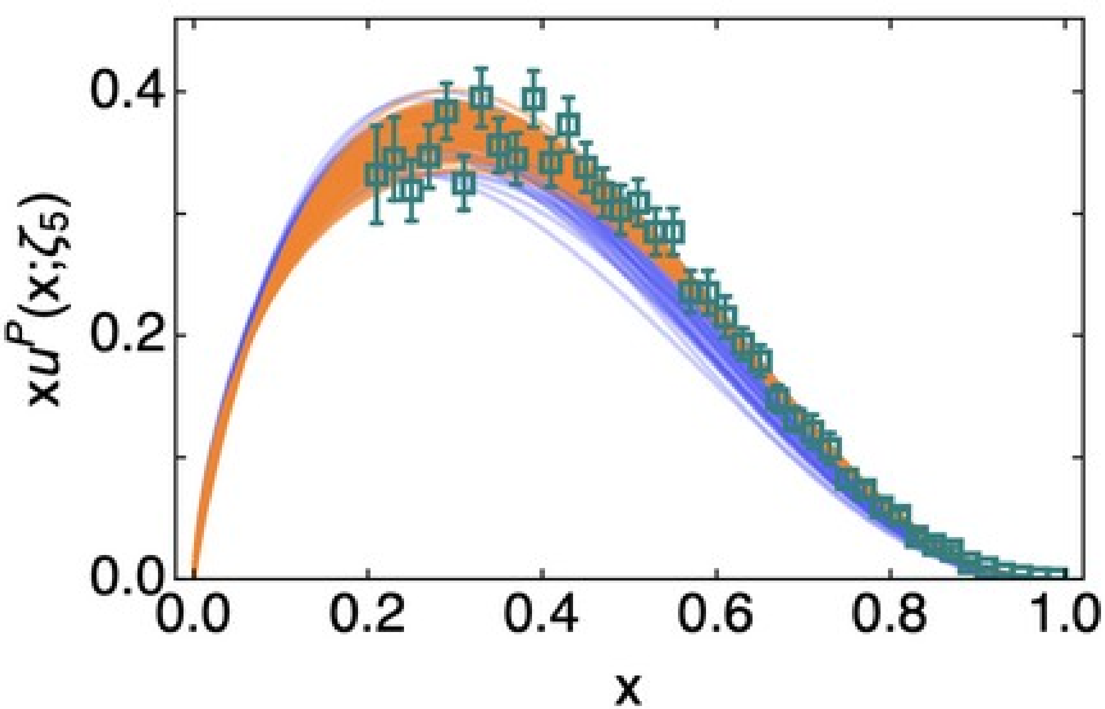

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6) **Accept** replicas with probabilities

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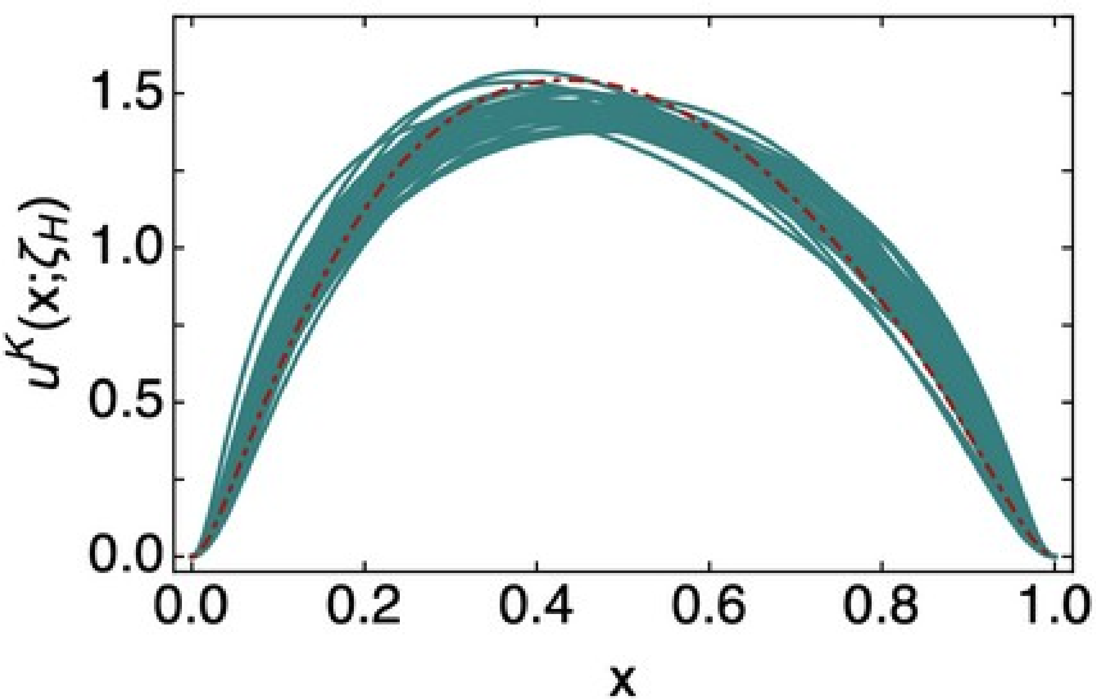

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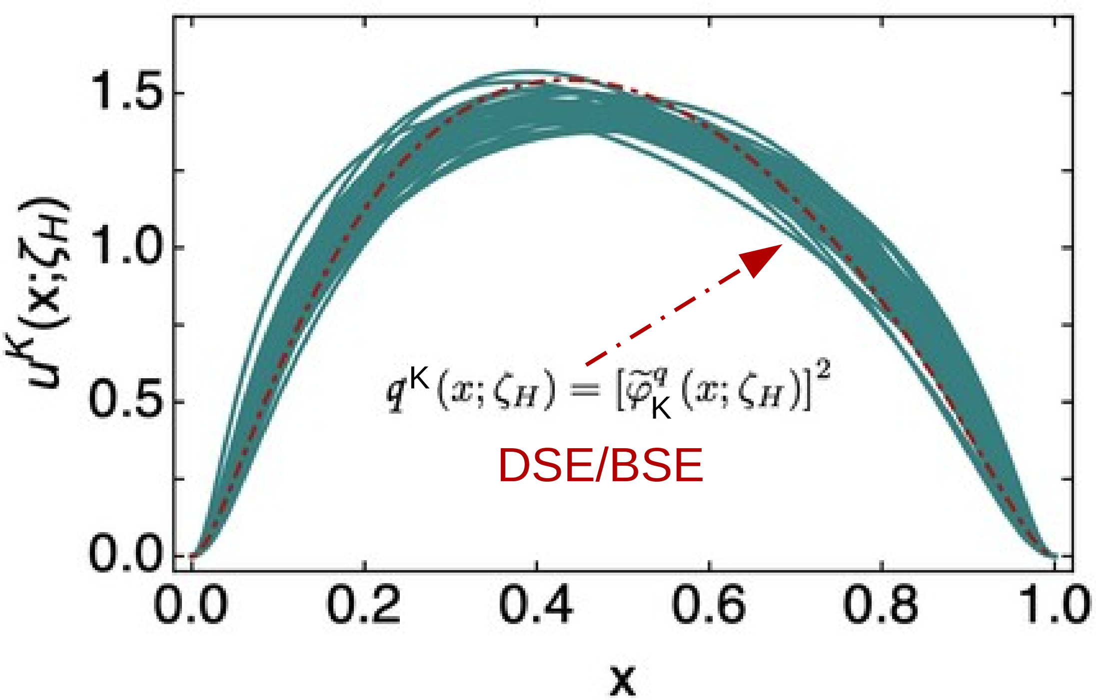

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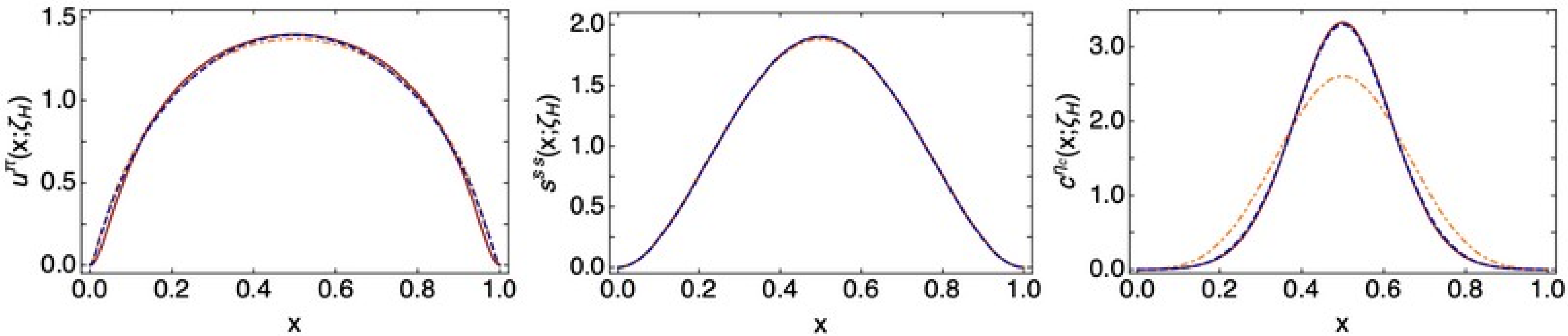
Kaon PDF: DSE/BSE



Symmetry-preserving DSE computation of the valence-quark DA and PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]
[M. Ding et al., Phys.Rev.D101(2020)054014]
[J. Xu et al., Work in progress]

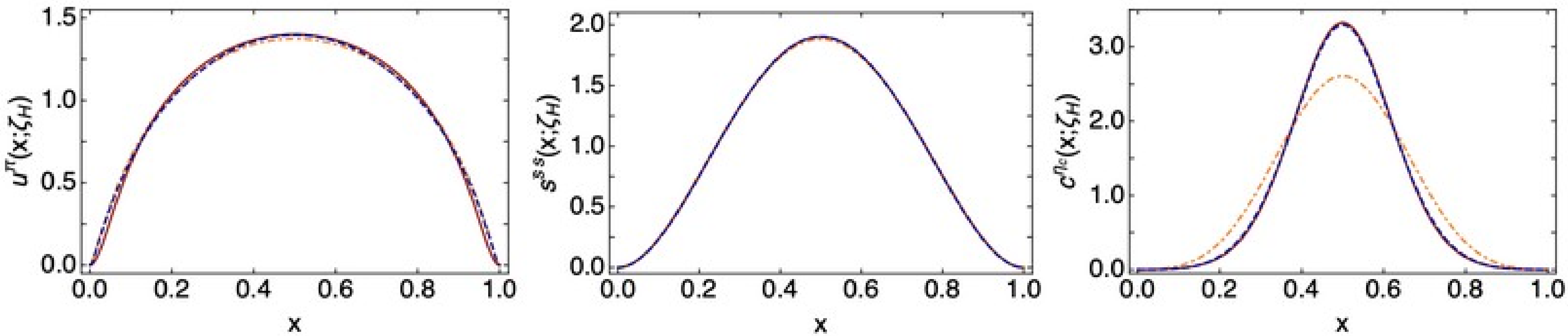
$$q^{H}(x;\zeta) = N_c \mathrm{tr}\int_{dk} \delta_n^x(k_\eta)\Gamma_H^P(k_{\bar{\eta}\eta};\zeta)\, S(k_{\bar{\eta}};\zeta) \times \left\{n\cdot \frac{\partial}{\partial k_\eta}\left[\Gamma_H^{-P}(k_{\eta\bar{\eta}};\zeta)S(k_\eta;\zeta)\right]\right\}.$$

$$\varphi_\pi^u(x;\zeta_H) = n_\varphi^\pi \ln\left(1+\frac{x(1-x)}{\rho_\varphi^\pi}\right),$$

$$u^\pi(x;\zeta_H) = n_u^\pi \ln\left(1+\frac{x^2(1-x)^2}{(\rho_u^\pi)^2}\right)$$

Computation the ~ 8 Mellin moments and parametrization

Kaon PDF: DSE/BSE



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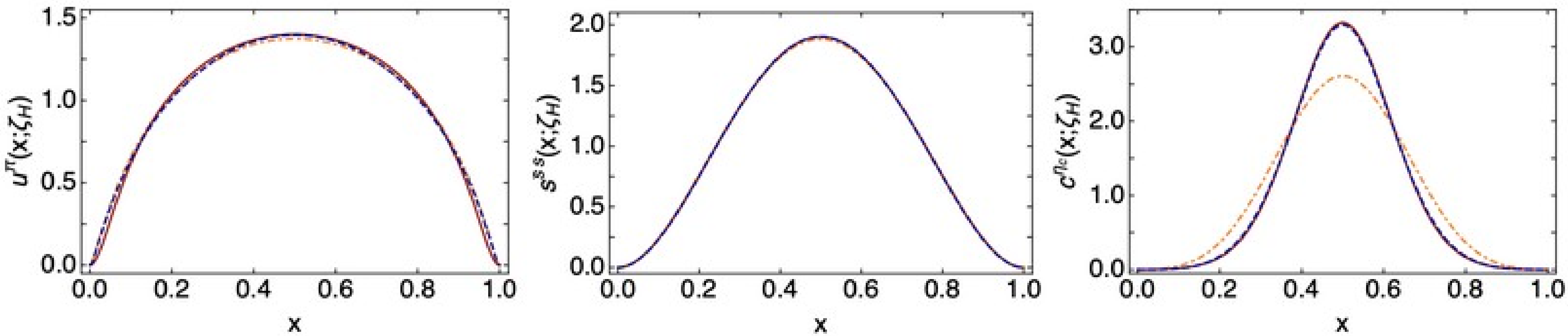
Computation the ~ 8 Mellin moments and parametrization

$$\varphi_\pi^u(x;\zeta_H) = n_\varphi\, x(1-x)\exp\left(\frac{x(1-x)}{\rho_\varphi^{\mathrm{H}}}\right),$$

$$q^\pi(x;\zeta_H) = n_q{}' x^2(1-x)^2 \exp\left(\frac{x(1-x)}{\rho_q^{\mathrm{H}}}\right)$$

Kaon PDF: DSE/BSE



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$$\varphi_{\mathrm{H}}^u(x;\zeta_H) = n_\varphi\, x(1-x) \exp\left(\frac{x(1-x)}{\rho_\varphi^{\mathrm{H}}}\right),$$

$$q^{\mathrm{H}}(x;\zeta_H) = n_q\, x^2(1-x)^2 \exp\left(\frac{x(1-x)}{\rho_q^{\mathrm{H}}}\right)$$

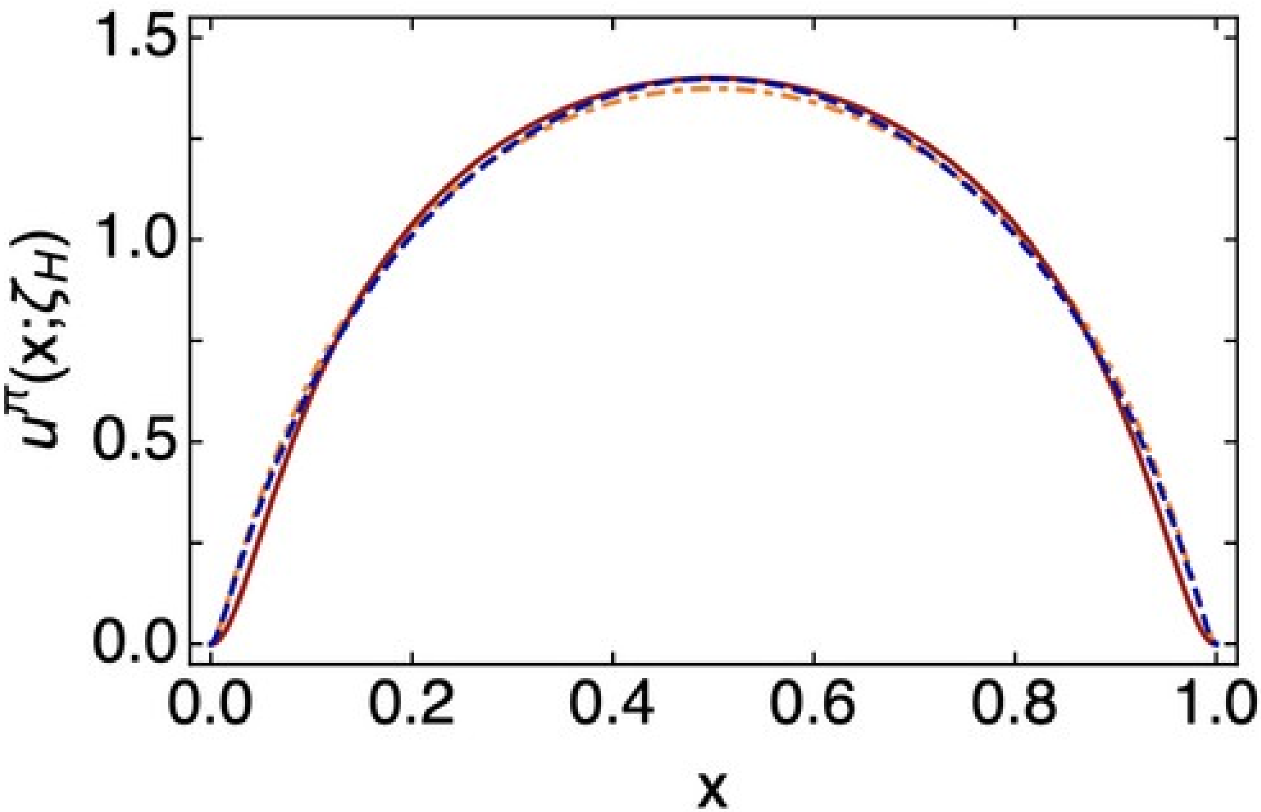
Then, capitalizing on the LFWF overlap representation:

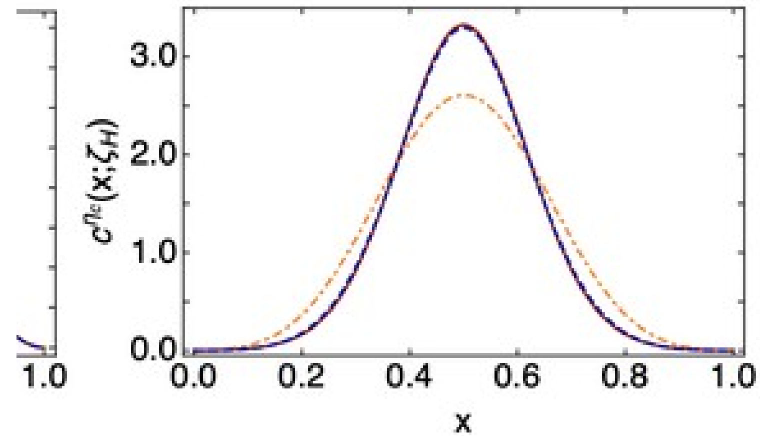
$$\psi_{\mathbf{P}}^q(x,k_\perp^2;\zeta_H) = 4\sqrt{3}\pi M_q \frac{M_q^2\,[1-\alpha_{\mathbf{P}} x(1-x)]}{\left(k_\perp^2 + M_q^2\,[1-\alpha_{\mathbf{P}} x(1-x)]\right)^2}\, \tilde{\varphi}_{\mathbf{P}}^q(x;\zeta_H)$$

$$\int \frac{d^2k_\perp}{16\pi^3} \psi_{\mathbf{P}}^q(x,k_\perp^2;\zeta_H) = \frac{\sqrt{3}}{4\pi} M_q \tilde{\varphi}_{\mathbf{P}}^q(x;\zeta_H) = f_{\mathbf{P}} r_{\mathbf{P}} \tilde{\varphi}_{\mathbf{P}}^q(x;\zeta_H)$$

$$q^{\mathbf{P}}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3} |\psi_{\mathbf{P}}^q(x;k_\perp^2;\zeta_H)|^2 = \frac{[\tilde{\varphi}_{\mathbf{P}}^q(x;\zeta_H)]^2}{1-\alpha_{\mathbf{P}} x(1-x)}$$

Kaon PDF: DSE/BSE





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$$(x;\zeta_H) = n_\varphi\, x(1-x)\exp\left(\frac{x(1-x)}{\rho_\varphi^{H}}\right),$$

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Then, capitalizing on the LFWF overlap representation:

$$\psi^q_{\mathbf{P}}(x,k_\perp^2;\zeta_H) = 4\sqrt{3}\pi M_q \frac{M_q^2\left[1-\alpha_{\mathbf{P}}x(1-x)\right]}{\left(k_\perp^2+M_q^2\left[1-\alpha_{\mathbf{P}}x(1-x)\right]\right)^2}\,\widetilde{\varphi}^q_{\mathbf{P}}(x;\zeta_H)$$

$$\int\frac{d^2k_\perp}{16\pi^3}\psi^q_{\mathbf{P}}(x,k_\perp^2;\zeta_H) = \frac{\sqrt{3}}{4\pi}M_q\widetilde{\varphi}^q_{\mathbf{P}}(x;\zeta_H) = f_{\mathbf{P}}r_{\mathbf{P}}\widetilde{\varphi}^q_{\mathbf{P}}(x;\zeta_H)$$

$$q^{\mathbf{P}}(x;\zeta_H) = \int\frac{d^2k_\perp}{16\pi^3}\left|\psi^q_{\mathbf{P}}(x;k_\perp^2;\zeta_H)\right|^2 = \frac{\left[\widetilde{\varphi}^q_{\mathbf{P}}(x;\zeta_H)\right]^2}{1-\alpha_{\mathbf{P}}x(1-x)}$$

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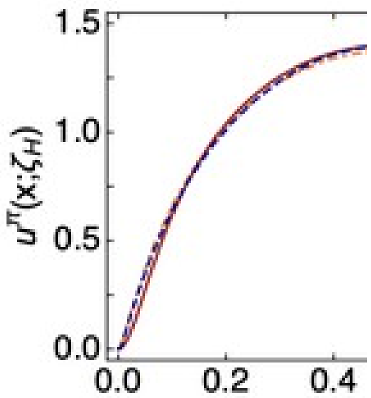
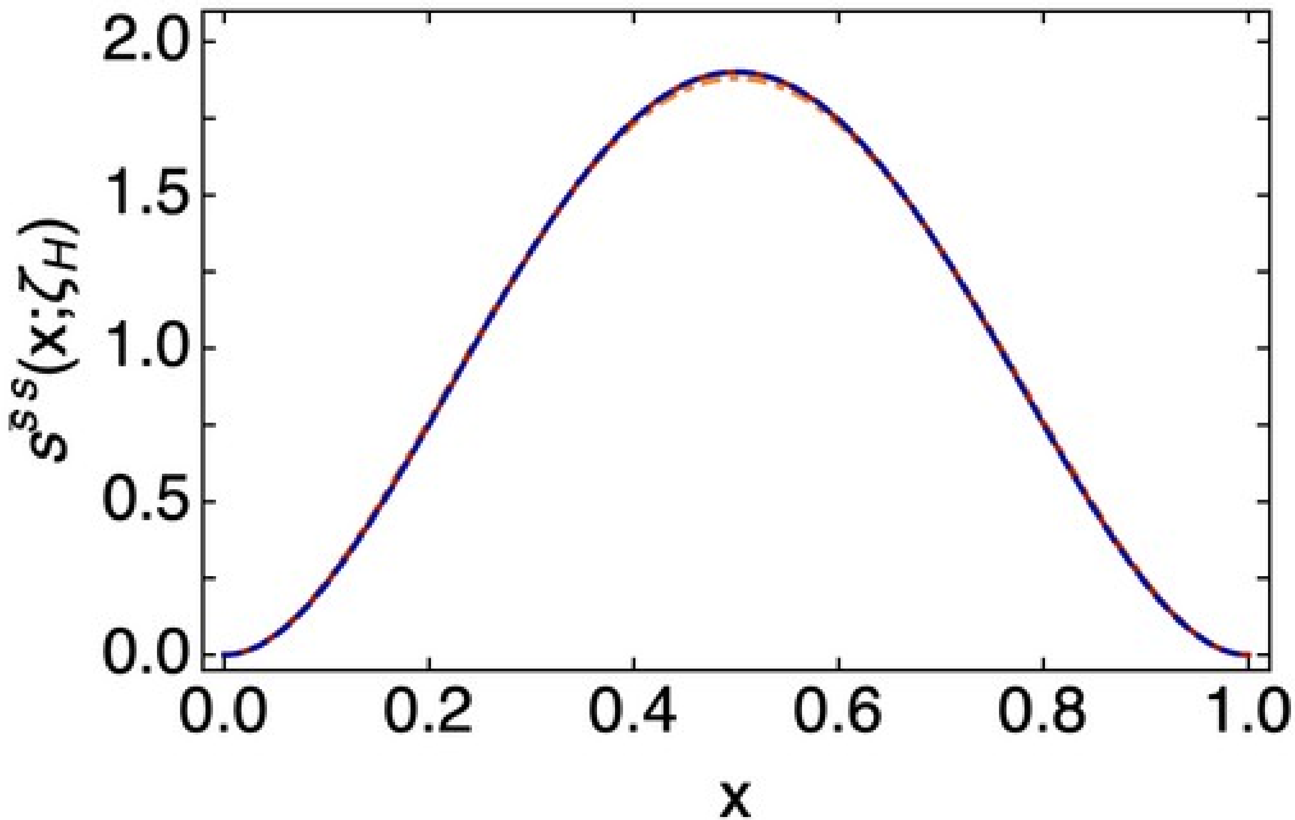


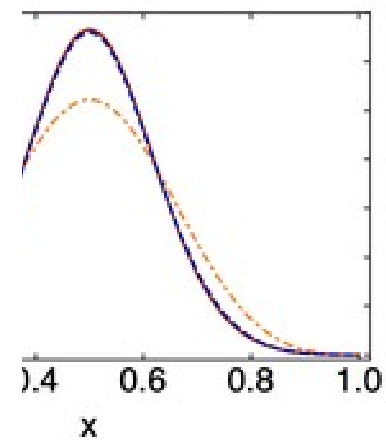
Symmetry-prese
and PDF:

$q^{\rm H}(x;\zeta) = N_c {\rm tr} \int$

$\times \{n \cdot \frac{}{\partial}$

Computation





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.D101(2020)054014]
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$$q^{\bf P}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\left|\psi_{\bf P}^q(x;k_\perp^2;\zeta_H)\right|^2 = \frac{[\widetilde{\varphi}_{\bf P}^q(x;\zeta_H)]^2}{1-\alpha_{\bf P}x(1-x)}$$

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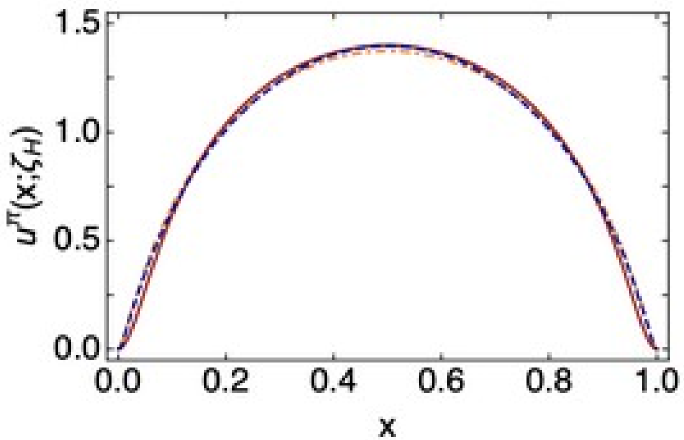

Symmetry-preserving DSE compu
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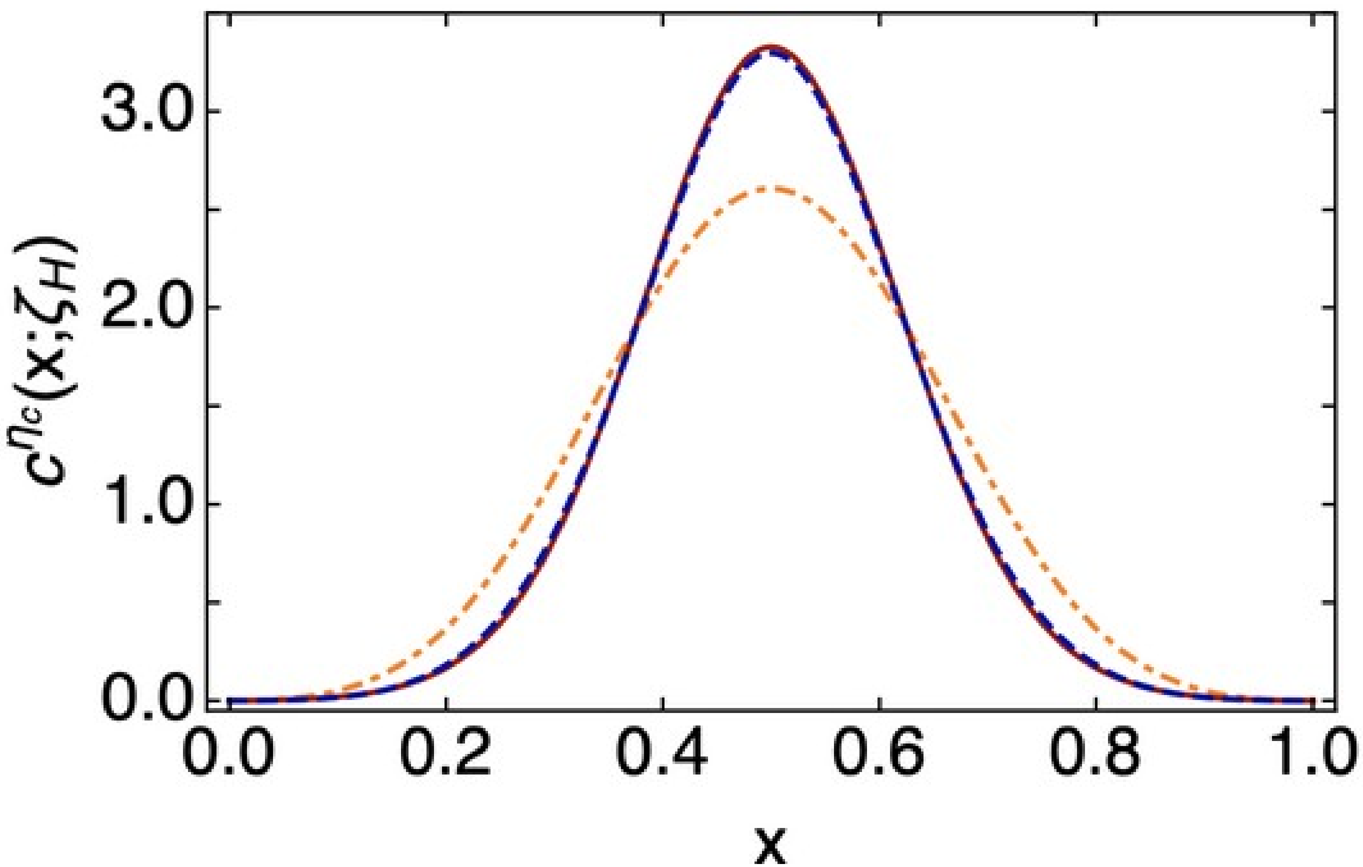
$$\times \{n \cdot \frac{\partial}{\partial k_\eta} [\Gamma_{\mathrm{H}}^{-P}(k_{\eta\bar{\eta}};\zeta) S($$

Computation the ~ 8 Mellin mo



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$$q^{\mathbf{P}}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3} |\psi_{\mathbf{P}}^q(x;k_\perp^2;\zeta_H)|^2 = \frac{[\widetilde{\varphi}_{\mathbf{P}}^q(x;\zeta_H)]^2}{1-\alpha_{\mathbf{P}} x(1-x)}$$

Summary

Summary

- The **EHM** is argued to be intimately connected to a **PI effective** charge which enters a conformal regime, below a given momentum scale, where gluons acquiring a dynamical mass decouple from interaction.

- Capitalizing on the latter, two main ideas emerge: (I) the identification of that decoupling with a **hadronic scale** at which the structure of hadrons can be expressed only in terms of valence dressed partons; and (ii) the reliability of an **all-orders** evolution scheme to describe the splitting of valence into more partons, generating thus the glue and sea, when the resolution scale decreases.

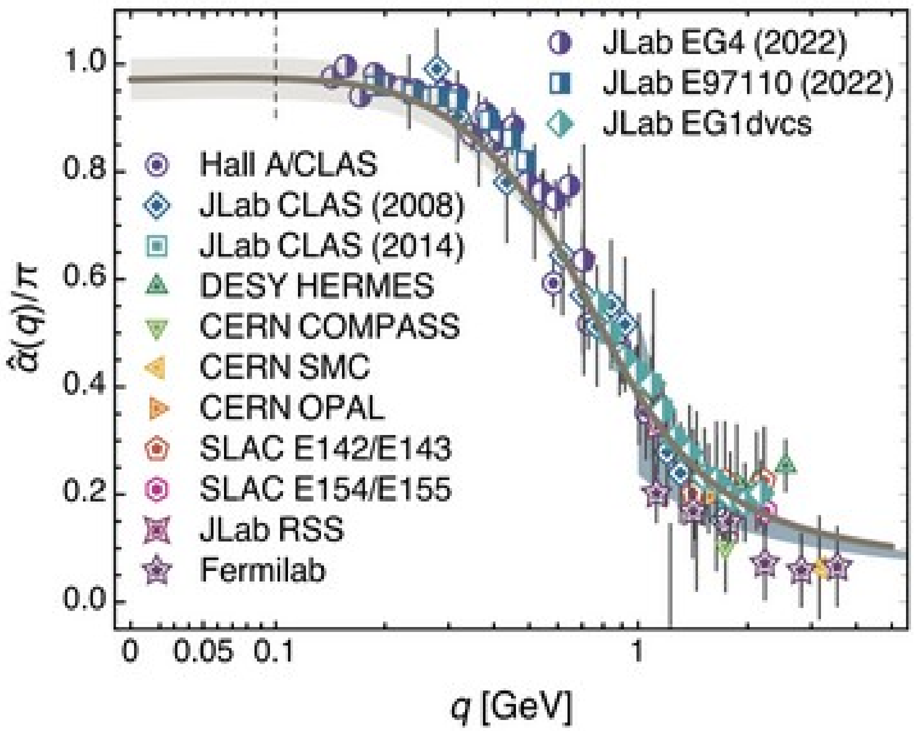
- Key implications stemming from both ideas have been derived and tested for the pion PDFs. Grounding on them, **Lattice QCD** and experimental data have been shown to confirm **CSM** results.

- The robustness of the approach based on **all-orders** evolution from **hadronic** to experimental scale has been proved with its application to the pion, kaon and proton cases. A model featuring massless evolution for quark flavors activated after a hard-wall threshold and accounrting for Pauli blocking has been solved analytically, and seen to expose some of the main results implied by the approach.

To be continued...

Backslides

QCD effective charge



Then, we define:

$$\alpha(k^2) = \frac{\gamma_m \pi}{\ln\left[\frac{\mathcal{M}^2(k^2)}{\Lambda^2_{QCD}}\right]}\ ;\ \alpha(0) = 0.97(4)$$

where

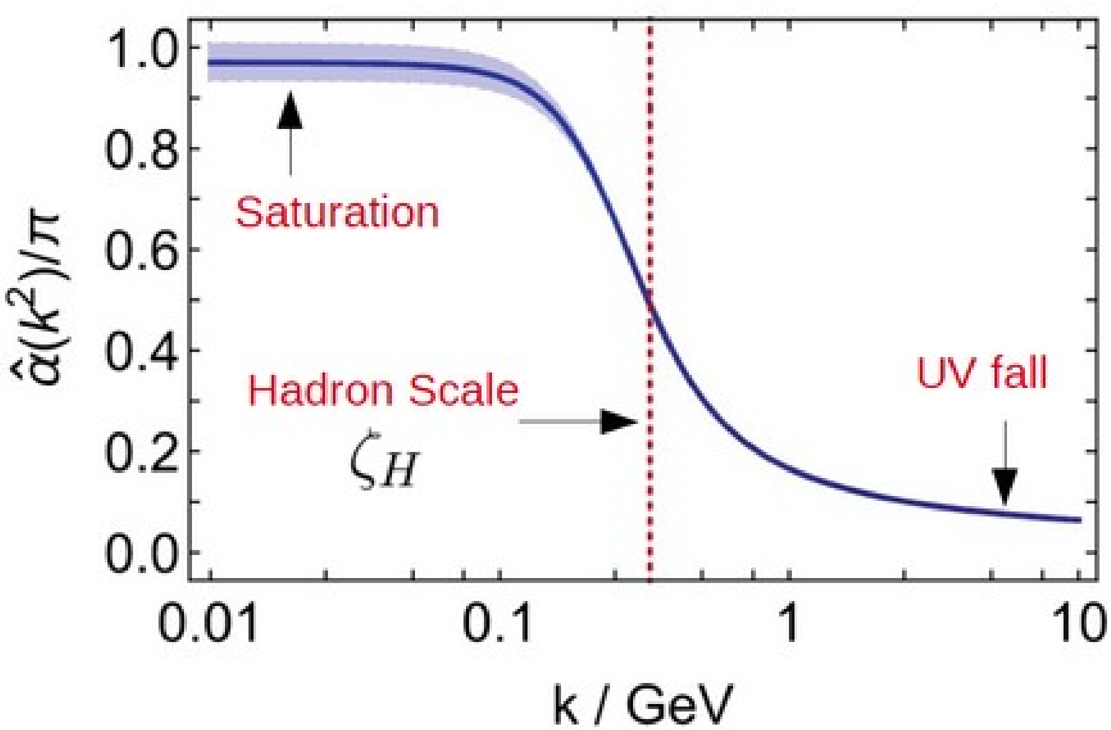
$$\mathcal{M}(k^2 = \Lambda^2_{QCD}) := m_G = 0.331(2)\ \text{GeV}$$

defines the screening mass and an associated wavelength, such that larger gluon modes decouple.

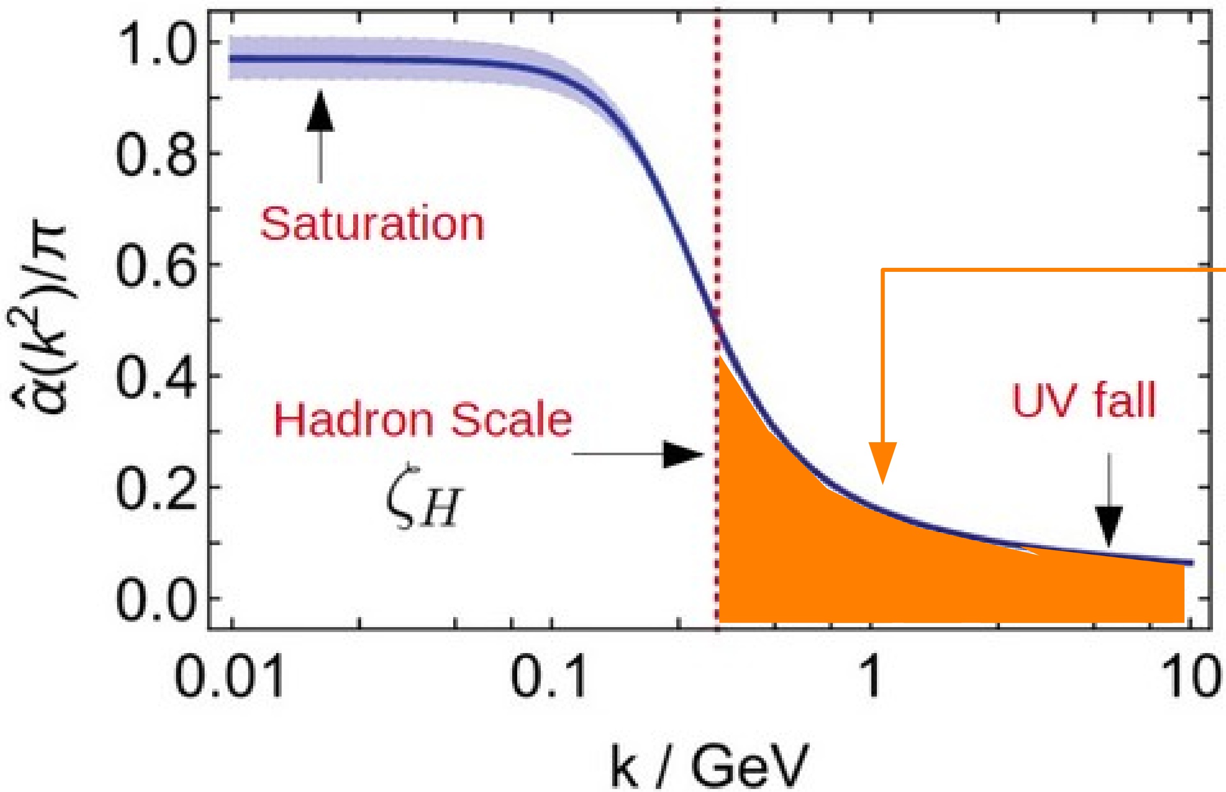
Then, we identify: $\zeta_H := m_G(1 \pm 0.1)$

Modern continuum & lattice QCD analysis in the gauge sector delivers an analogue "Gell-Mann-Low" running charge, from which one obtains a process-independent, parameter-free prediction for the low-momentum saturation

- No landau pole
- Below a given mass scale, the interaction become scale-independent and QCD practically conformal again (as in the lagrangian).



QCD effective charge



The strength of the charge defines de input for the evolution

$$S(\zeta_H, \zeta_f) = \int_{2\ln(\zeta_H/\Lambda_{QCD})}^{2\ln(\zeta_f/\Lambda_{QCD})} dt\, \widehat{\alpha}(t)$$

$$\langle x(\zeta_5)\rangle_q^\pi = \frac{1}{2}\exp\left(-\frac{8}{9\pi}S(\zeta_H, \zeta_5)\right) = 0.20(2)$$

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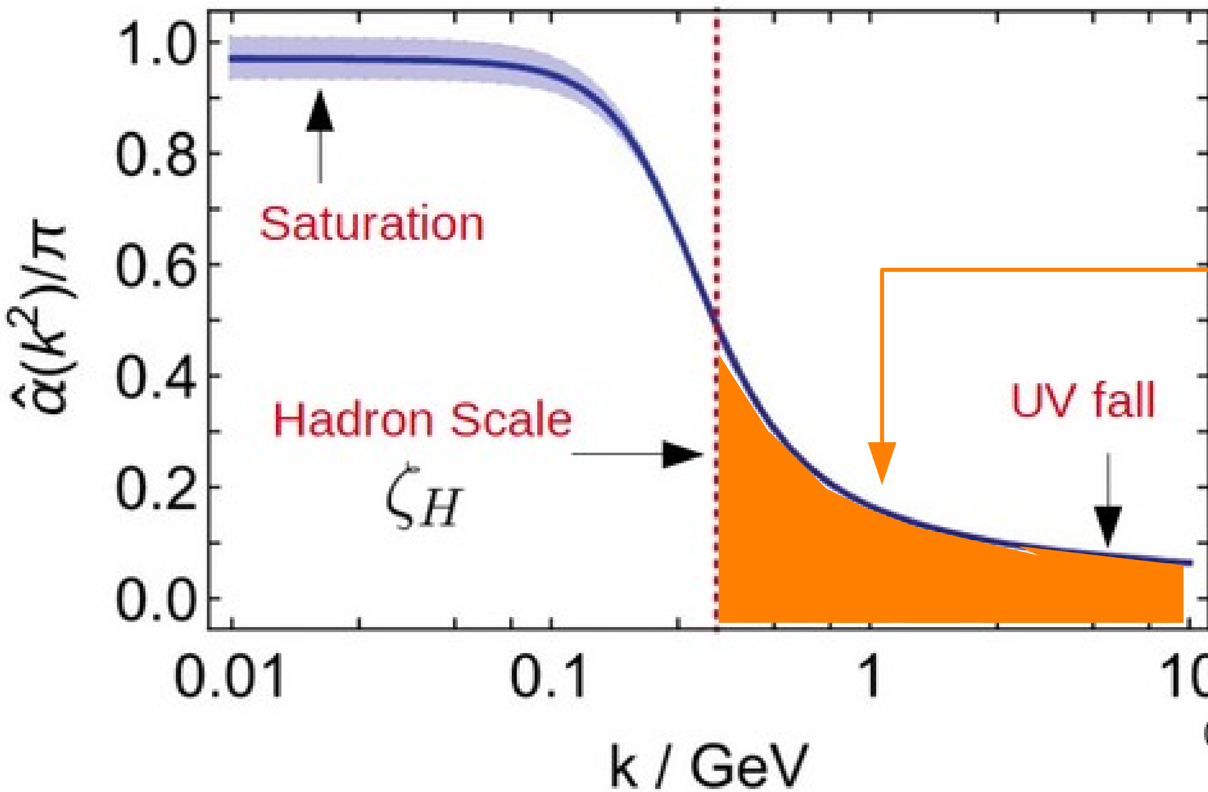

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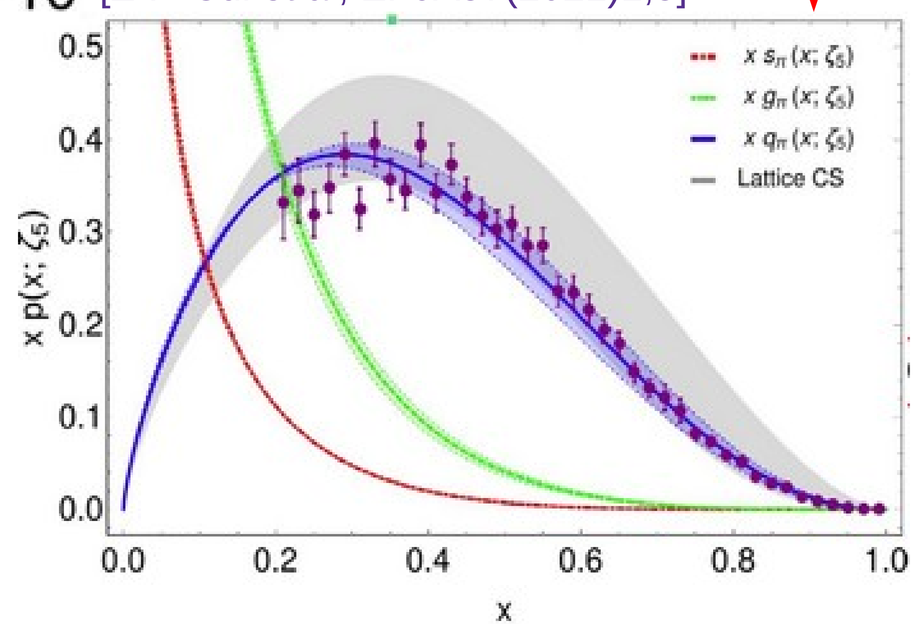
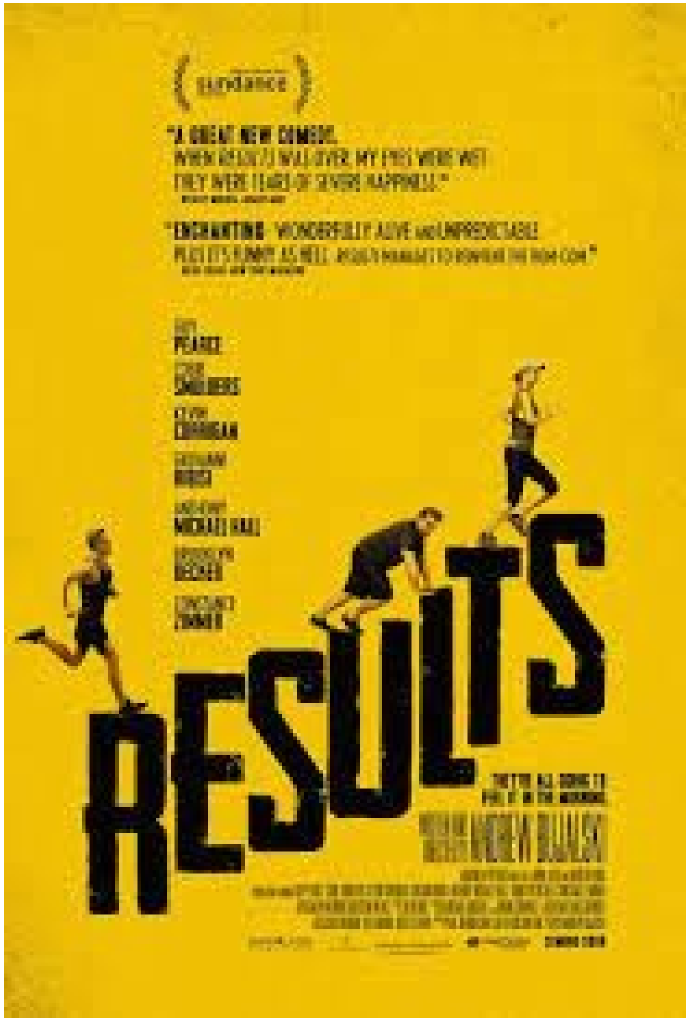
[Z-F. Cui et al, EPJC80(2020)11,1064]
[Z-F. Cui et al, EPJA57(2021)1,5]



Then, the glue, valence- and sea-quark DFs can be predicted, with no tuned parameter, on the ground of the effective charge definition, from the LFWF (or, equivalentely, from a symmetry-preserving DSE/BSE computation of the valence-quarks Mellin moments

[M. Ding et al, CPC44(2020)3,031002]


sundance
"A GREAT NEW COMEDY.
"ENCHANTING. WONDERFULLY ALIVE AND UNPREDICTABLE
PEARCE
SMULDERS
CORRIGAN
RESULTS
ANDREW BUJALSKI

Pion PDF: from CSM (DSEs) to the experiment

Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]
[M. Ding et al., Phys.Rev.D101(2020)054014]

$$q^\pi(x;\zeta) = N_c \mathrm{tr} \int_{dk} \delta_n^x(k_\eta) \Gamma_\pi^P(k_{\bar\eta\eta};\zeta)\, S(k_{\bar\eta};\zeta) \times \{ n \cdot \frac{\partial}{\partial k_\eta} [\Gamma_\pi^{-P}(k_{\eta\bar\eta};\zeta) S(k_\eta;\zeta)] \} .$$

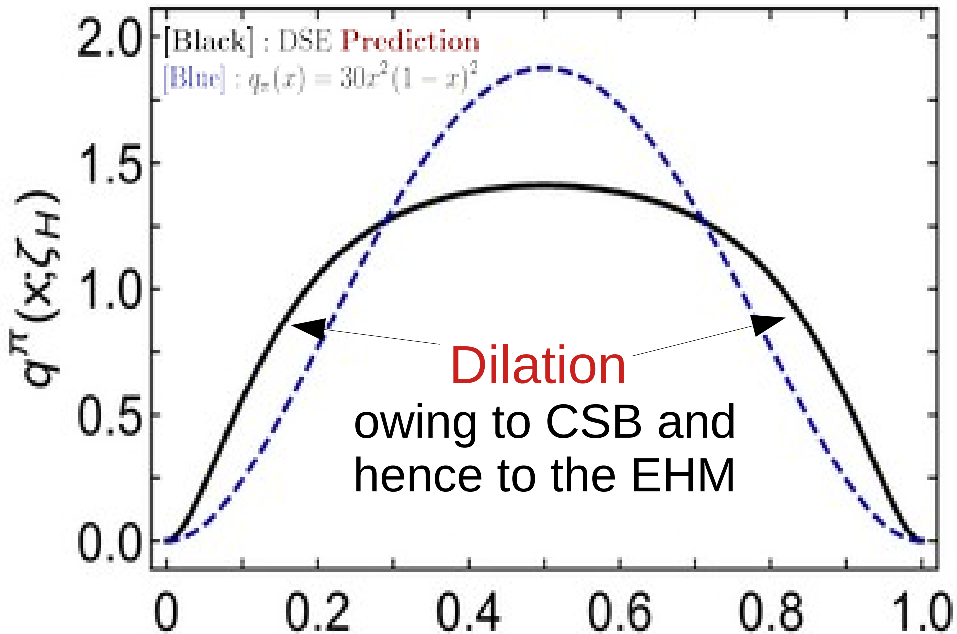
$$q_{\mathbb{O}}^\pi(x;\zeta_H) = 213.32\, x^2 (1-x)^2 \times [1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)]$$

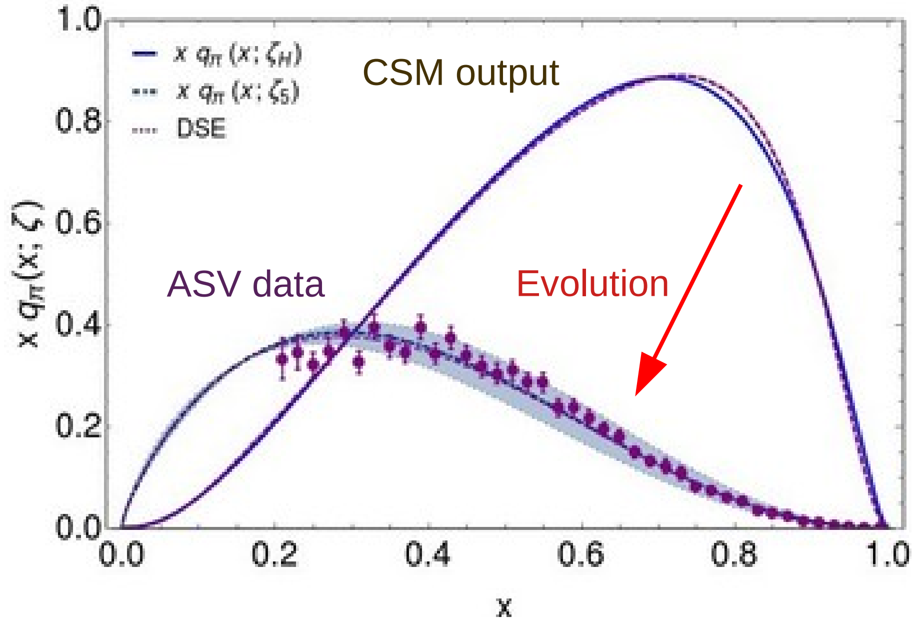
$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)} (1 + \mathcal{O}(1-x))$$

$$\beta(\zeta_H) = 2$$

Farrar, Jackson, Phys.Rev.Lett 35(1975)1416
Berger, Brodsky, Phys.Rev.Lett 42(1979)940

- The EHM-triggered broadening shortens the extent of the domain of convexity lying on the neighborhood of the endpoints, induced too by the QCD dynamics
- It cannot however spoil the asymptotic QCD behaviour at large-x (and, owing to isospin symmetry, at low-x)





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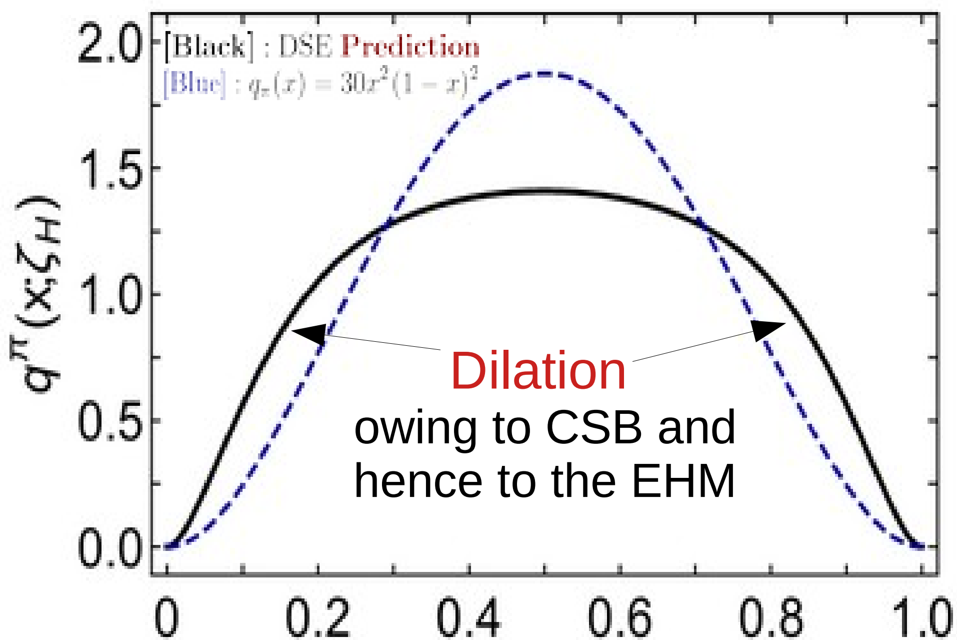
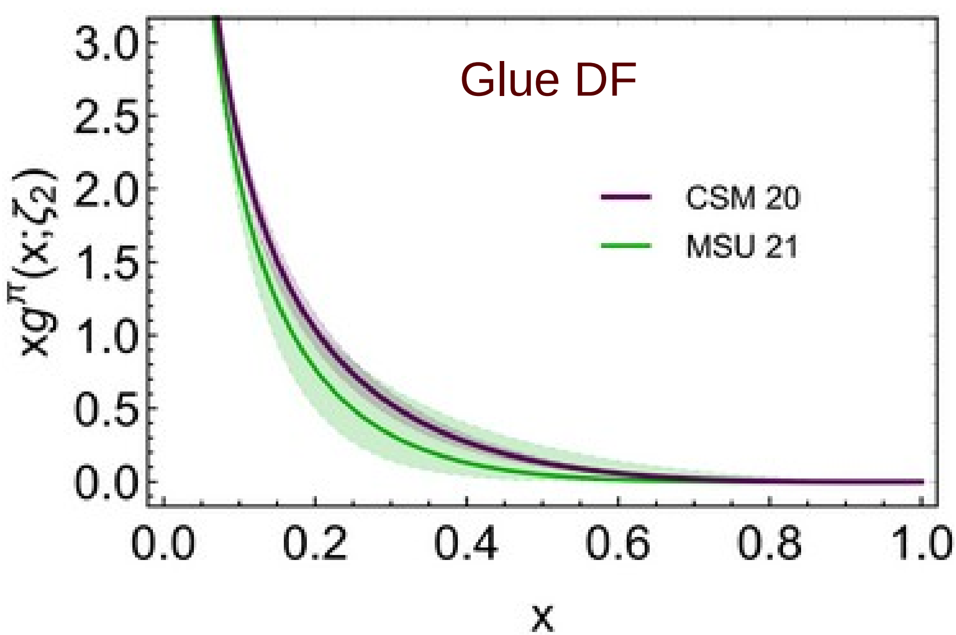
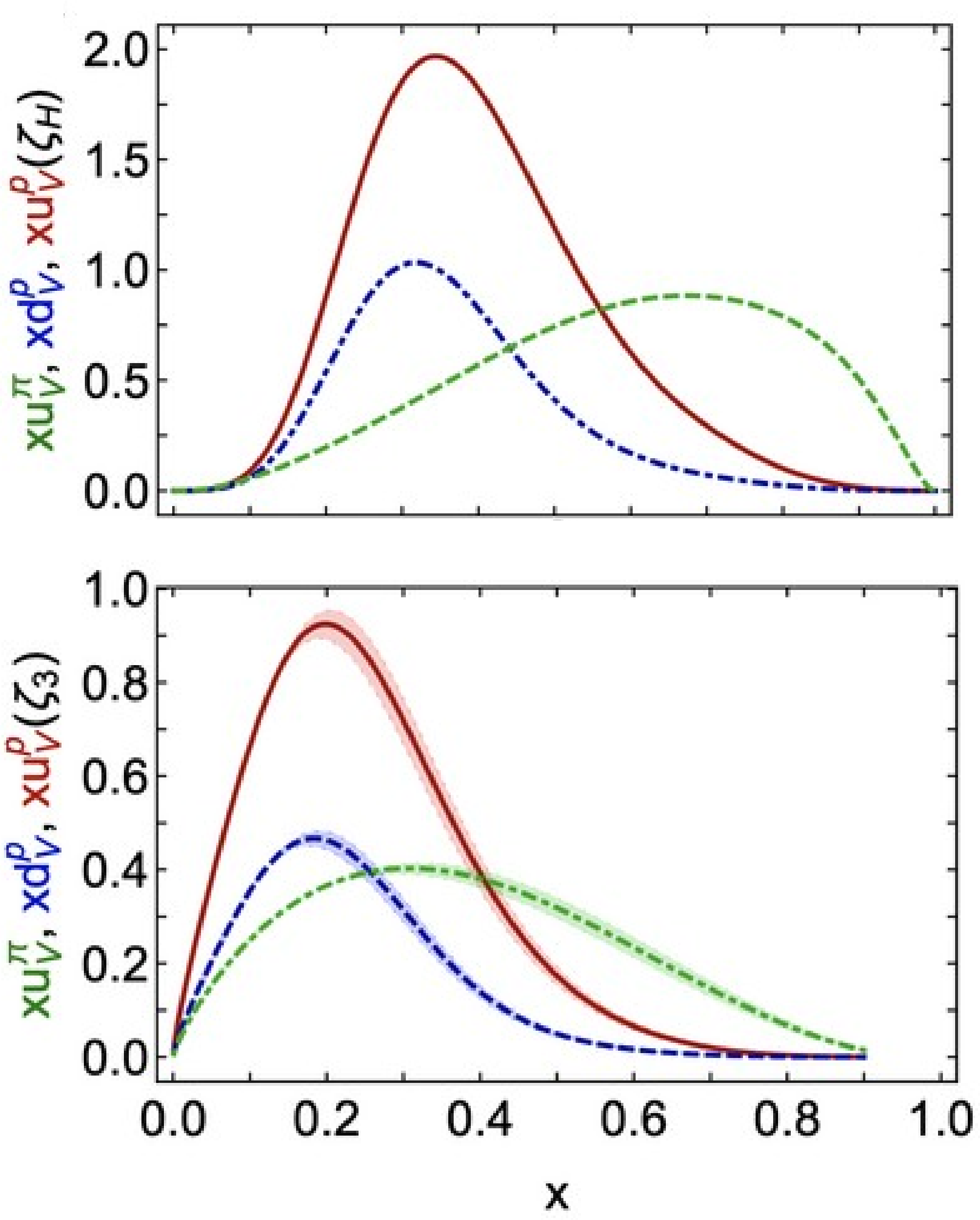




Proton PDF: from CSM (DSEs) to the experiment

An analogous symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach:
[L. Chang et al., Phys.Lett.B, arXiv:2201.07870]



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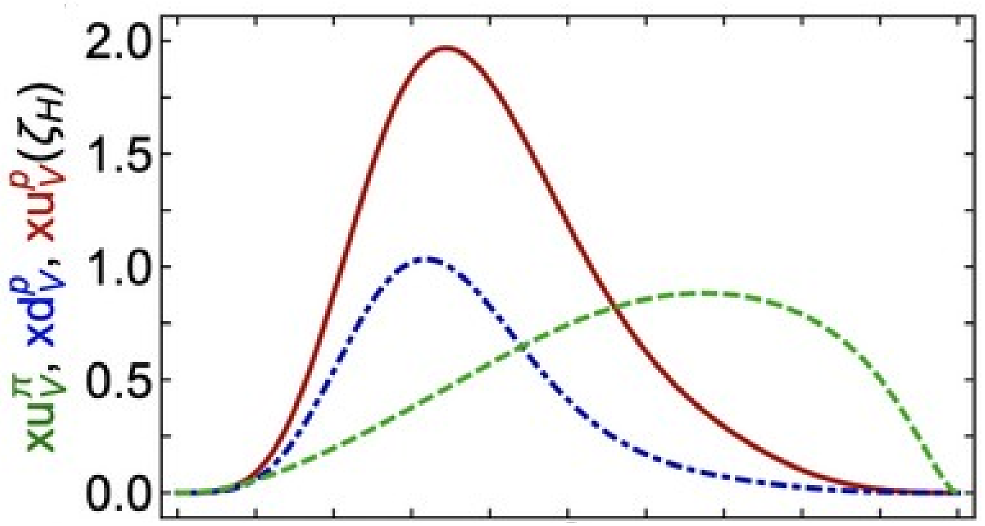
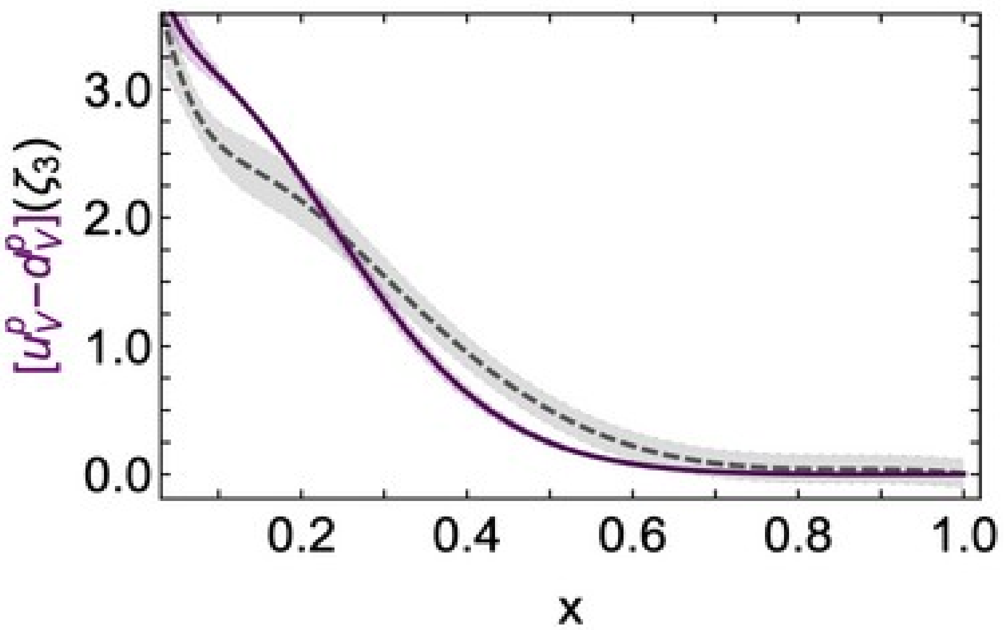
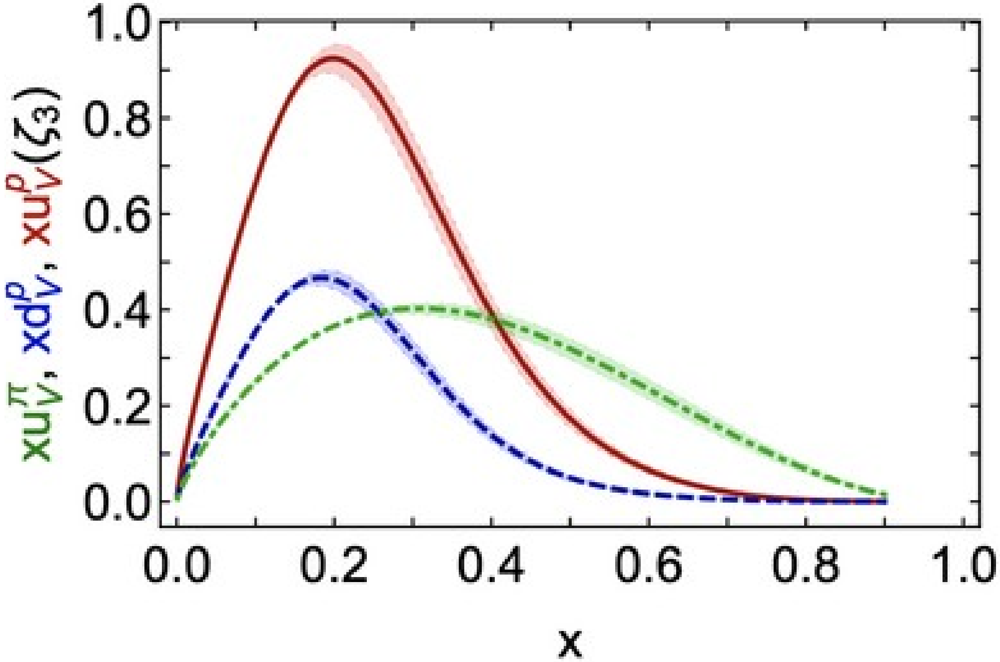
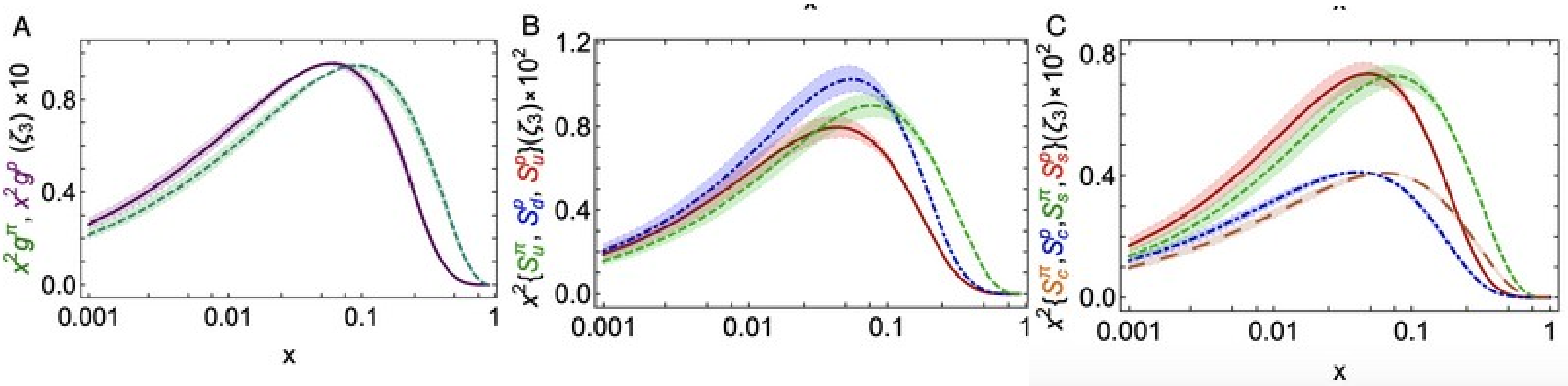


Producing an isovector distribution in fair agreement with lattice results
[H-W. Lin et al., arXiv:2011.14791]





Proton PDF: pion and proton in counterpoint



pion	u^π	$\bar{d}^\pi$	g^π	S_π^u	$S_\pi^{\bar{d}}$	S_π^s	S_π^c
$\langle x\rangle^{\zeta_2}$	24.0(1.1)	24.0(1.1)	41.0(1.2)	3.3(3)	3.3(3)	2.65(22)	1.33(5)
$\langle x^2\rangle^{\zeta_2}$	9.5(7)	9.5(7)	3.7(1)	0.27(1)	0.27(1)	0.21(1)	0.092(2)
$\langle x^3\rangle^{\zeta_2}$	4.7(4)	4.7(4)	0.92(6)	0.057(1)	0.057(1)	0.044(0)	0.018(1)
$\langle x\rangle^{\zeta_3}$	22.1(1.0)	22.1(1.0)	42.9(1.0)	3.7(3)	3.7(3)	3.0(2)	1.83(6)
$\langle x^2\rangle^{\zeta_3}$	8.4(6)	8.4(6)	3.5(1)	0.27(1)	0.27(1)	0.22(1)	0.120(3)
$\langle x^3\rangle^{\zeta_3}$	4.0(3)	4.0(3)	0.82(5)	0.056(0)	0.056(0)	0.044(0)	0.022(1)

proton	u^p	d^p	g^p	S_p^u	S_p^d	S_p^s	S_p^c
$\langle x\rangle^{\zeta_2}$	32.9(1.4)	15.0(0.7)	40.9(1.1)	2.9(2)	3.7(3)	2.64(22)	1.32(5)
$\langle x^2\rangle^{\zeta_2}$	8.7(6)	3.6(2)	2.4(1)	0.14(1)	0.21(1)	0.13(0)	0.059(2)
$\langle x^3\rangle^{\zeta_2}$	2.9(3)	1.1(1)	0.39(2)	0.019(0)	0.030(1)	0.019(0)	0.008(0)
$\langle x\rangle^{\zeta_3}$	30.4(1.3)	13.8(0.6)	42.8(1.0)	3.3(3)	4.1(3)	3.0(2)	1.82(6)
$\langle x^2\rangle^{\zeta_3}$	7.7(5)	3.2(2)	2.2(1)	0.15(1)	0.21(1)	0.14(0)	0.075(2)
$\langle x^3\rangle^{\zeta_3}$	2.5(2)	0.9(1)	0.35(2)	0.019(0)	0.028(0)	0.019(0)	0.010(1)