PRACTIOTIONER-UNBIASED INTERPOLATION AND EXTRAPOLATION OF (PRECISE) DATA

DANIELE BINOSI

ECT* - FONDAZIONE BRUNO KESSLER

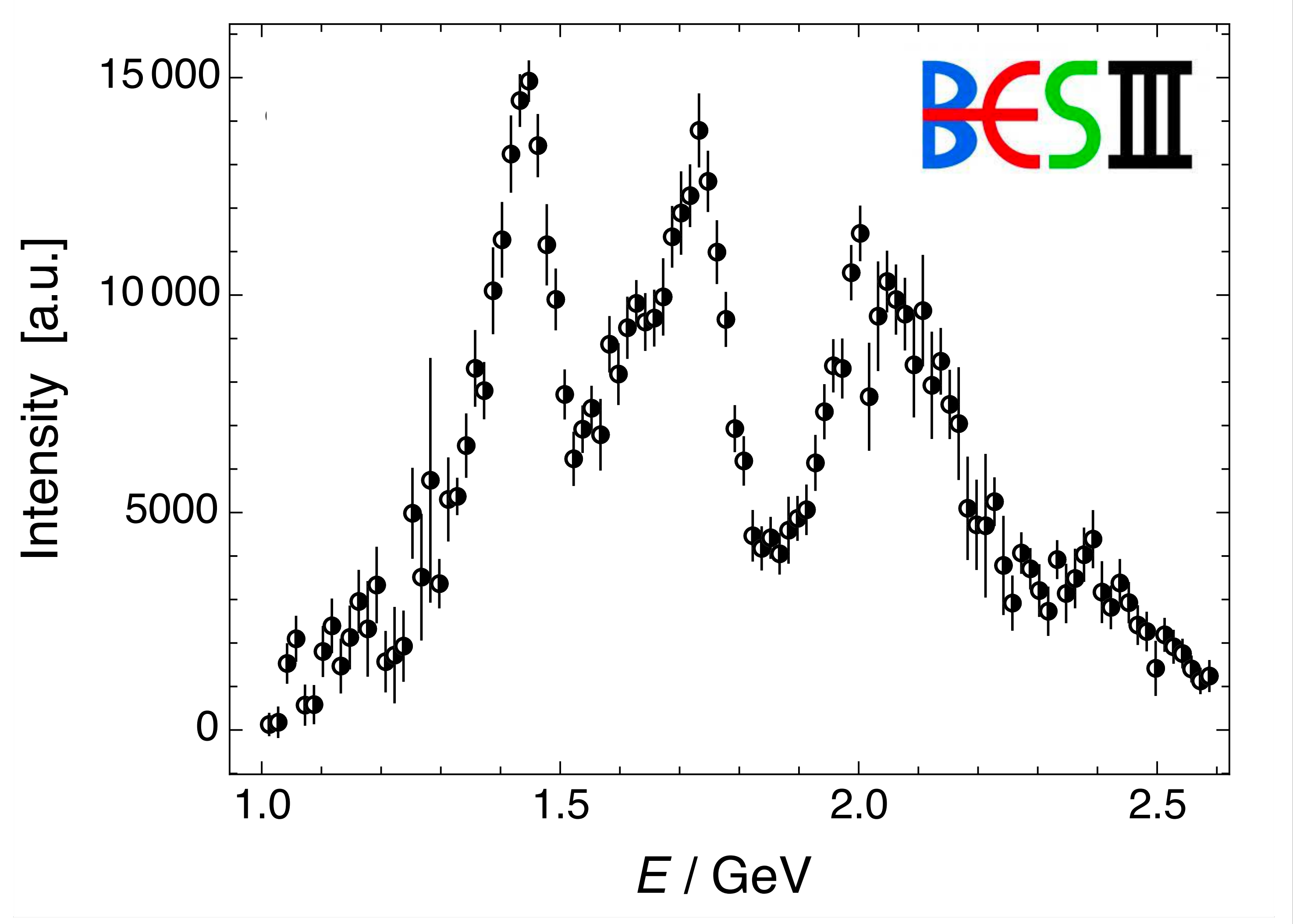
Strong QCD from Hadron Structure Experiments - VI

MAY 14 - 17 2024, NANJING

$J/\psi \to \gamma\pi^0\pi^0$

S-WAVE INTENSITY

BESIII data



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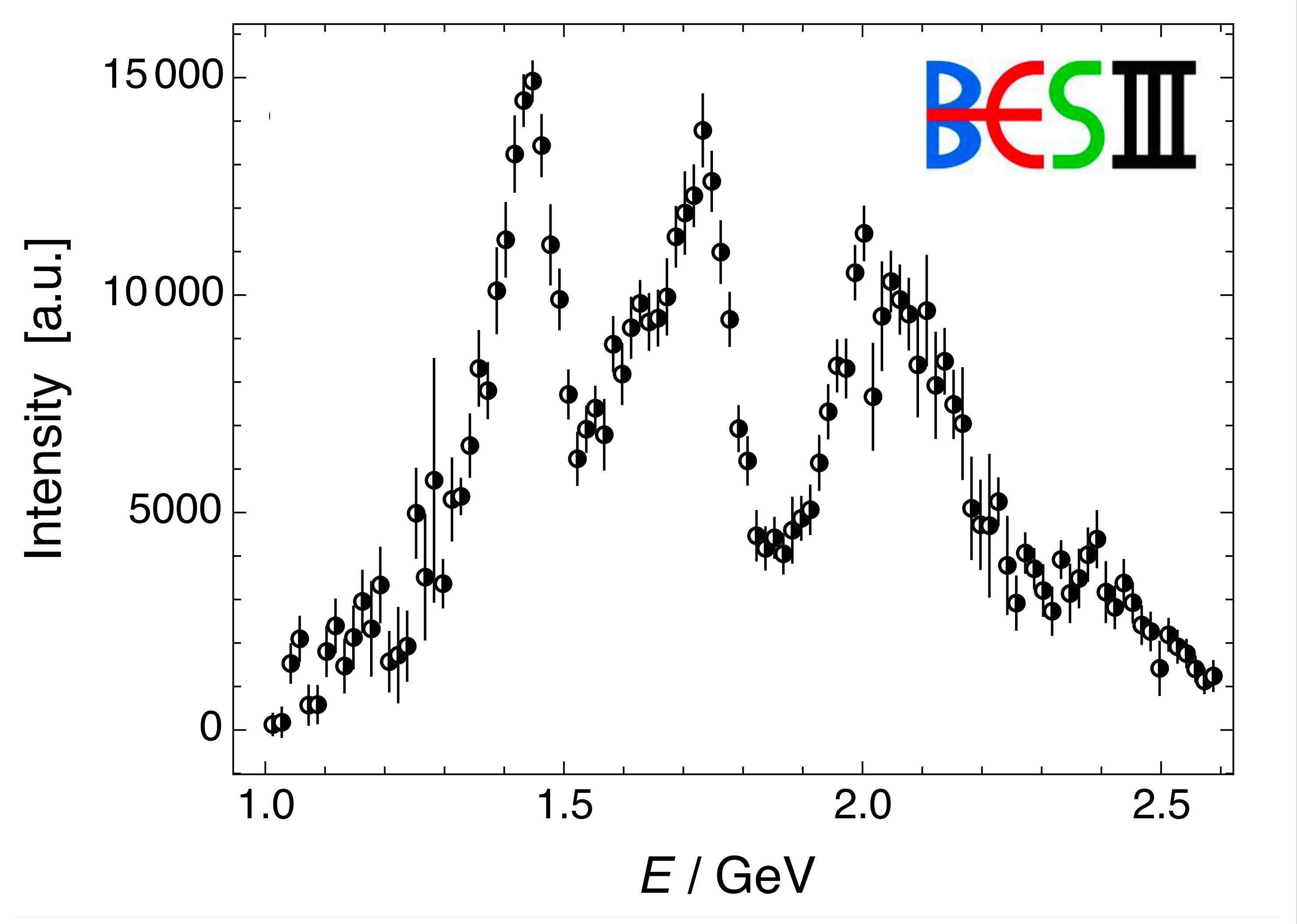

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The intensity is related to the amplitude via

$$I(E) = \rho(E)\,|f(E)|^2$$

f : right hand cut at threshold and possibly other branch cuts (opening of channels)

Analytic continuation can access the unphysical Riemann sheet where resonant poles are found

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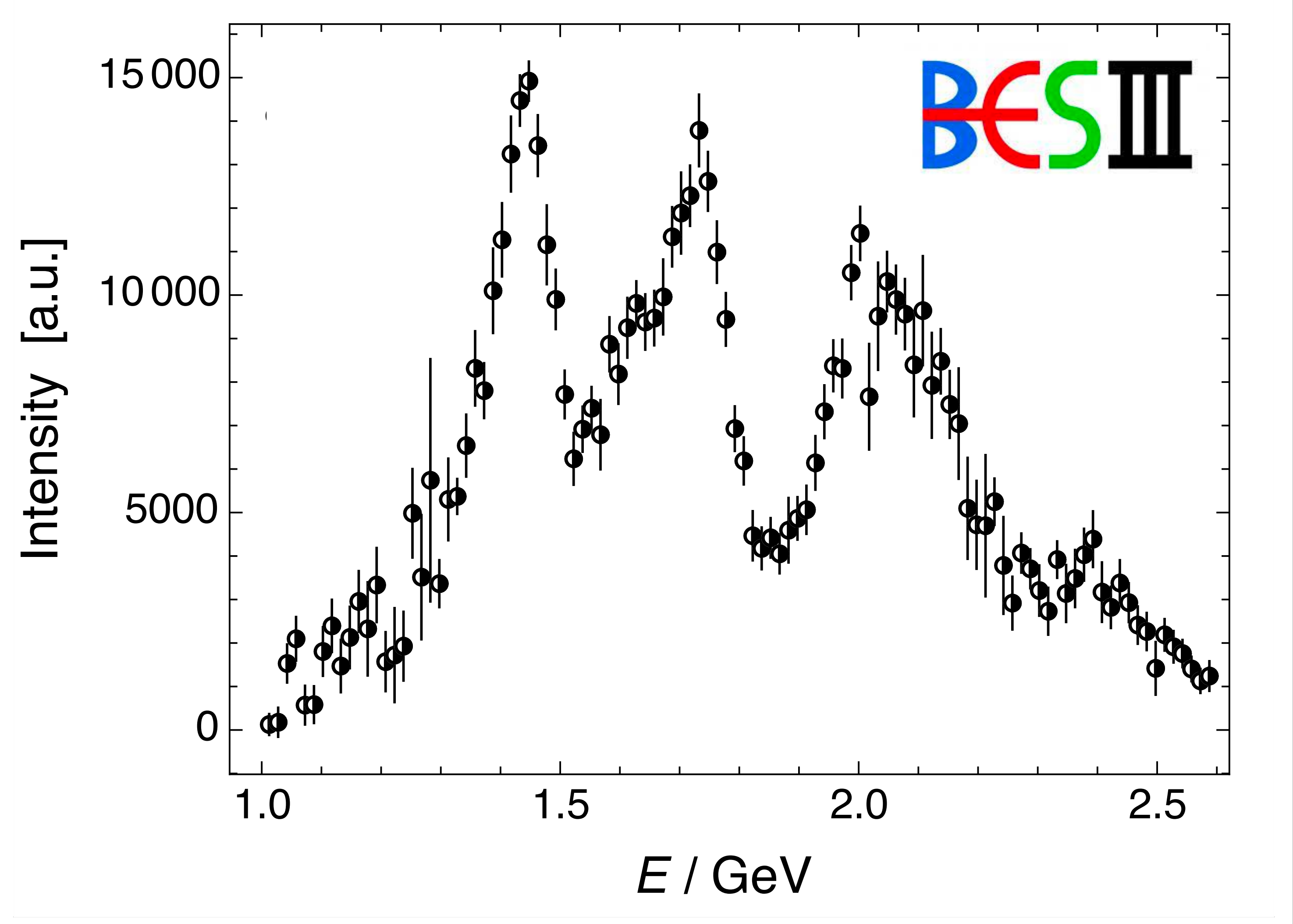

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$$C_M(E) = \frac{P(E)}{Q(E)}$$

and simply let E take on complex values!

Pole structure will provide an approximation to the one of the original intensity

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Propagate **uncertainties** through **resampling**

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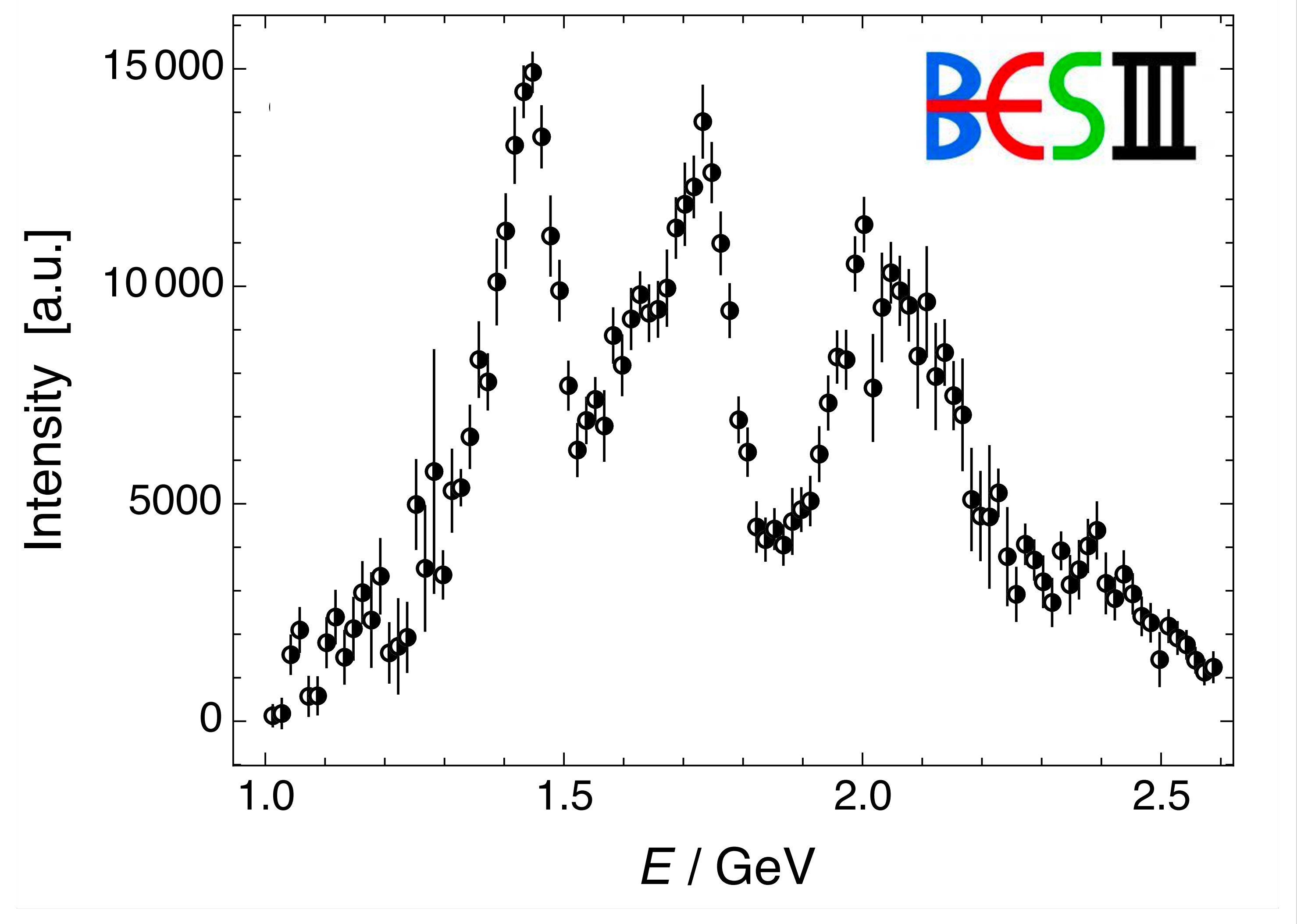

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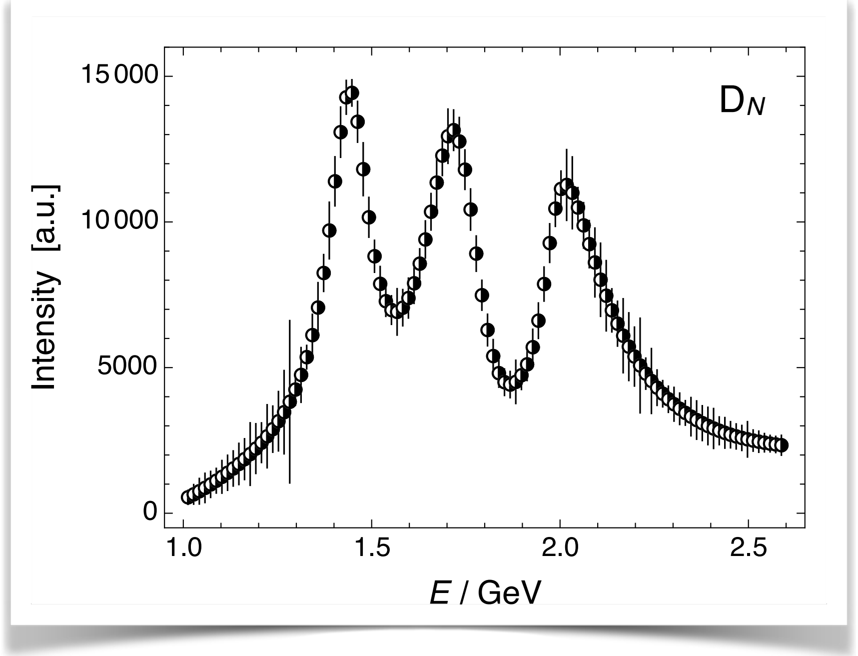

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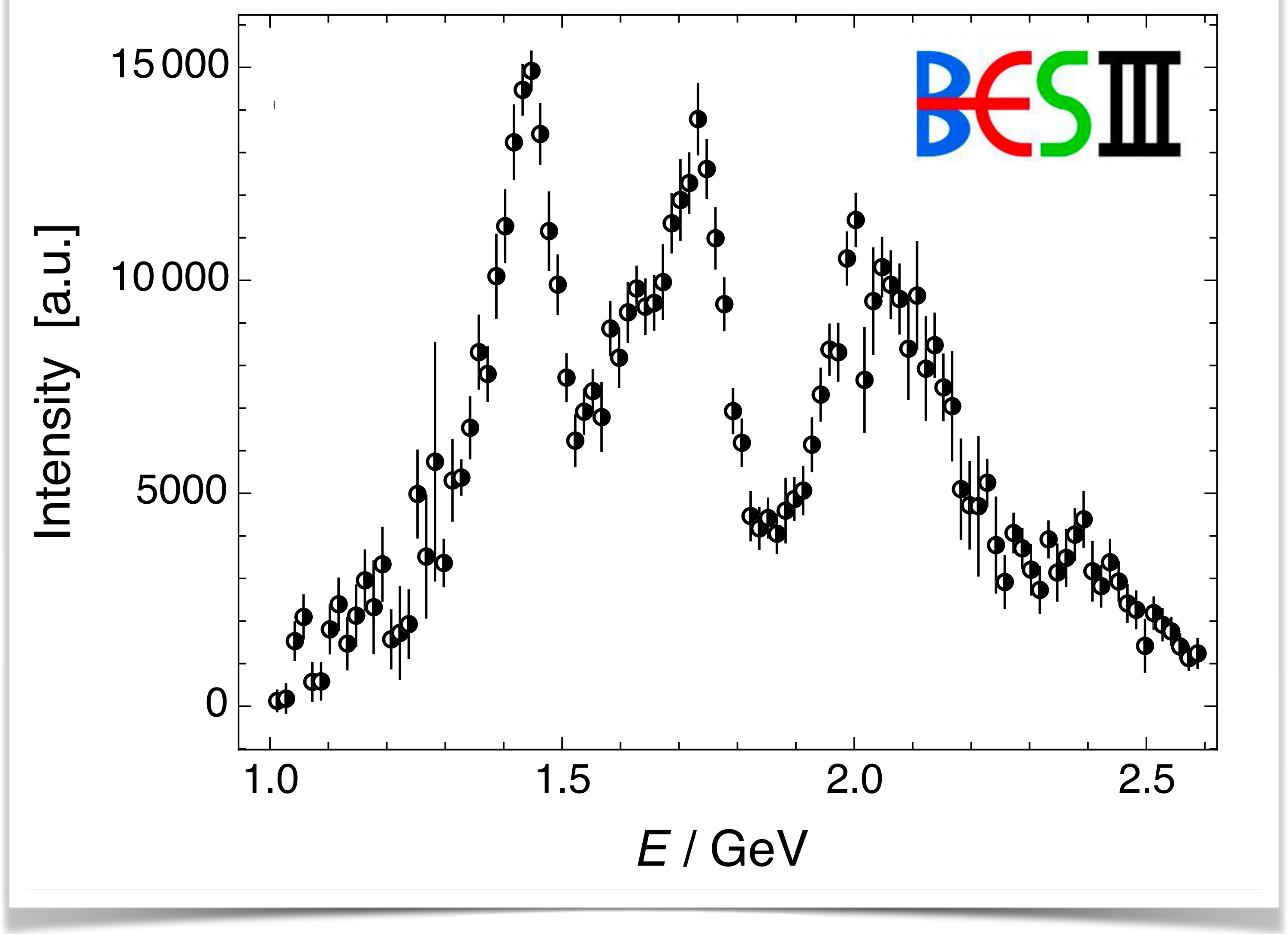

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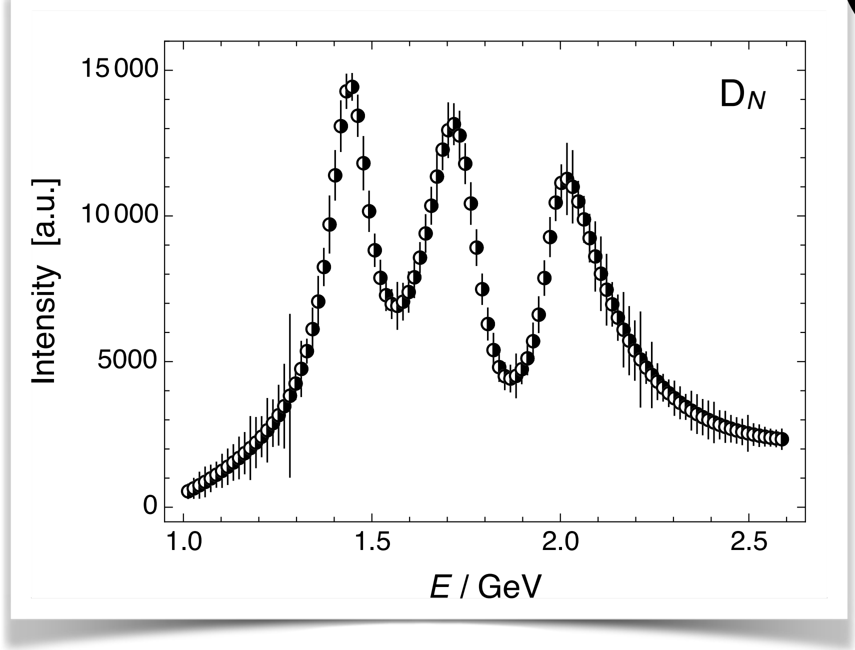

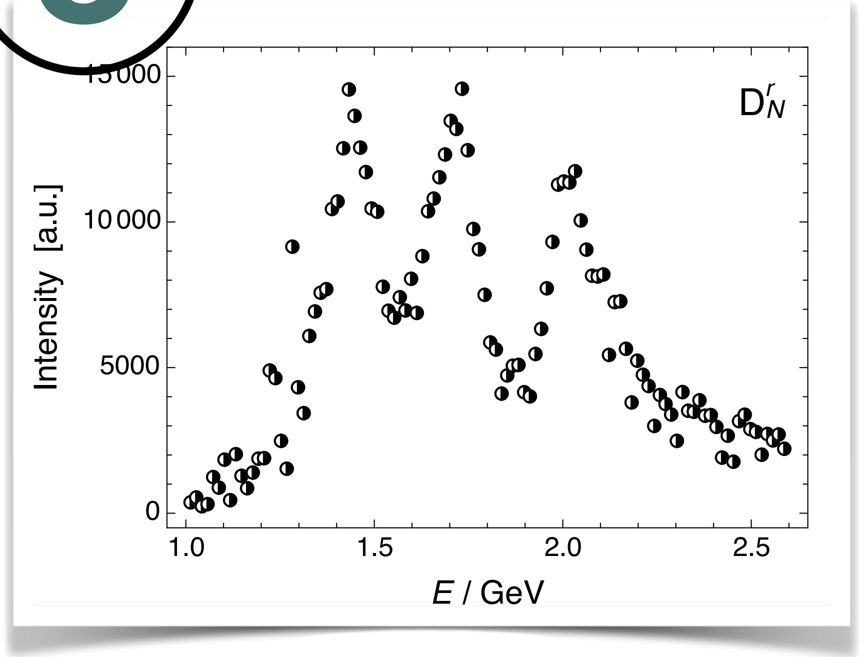

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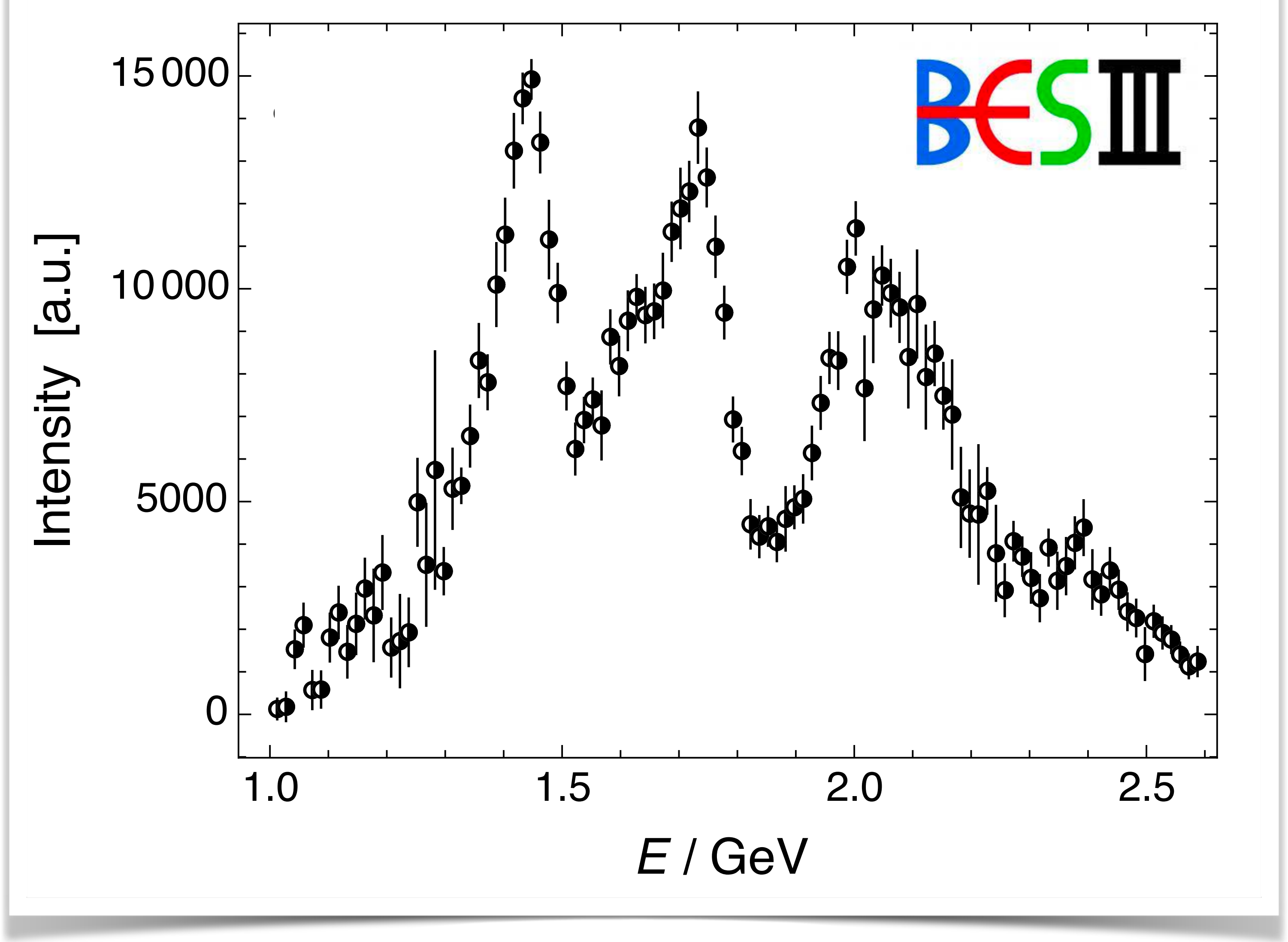

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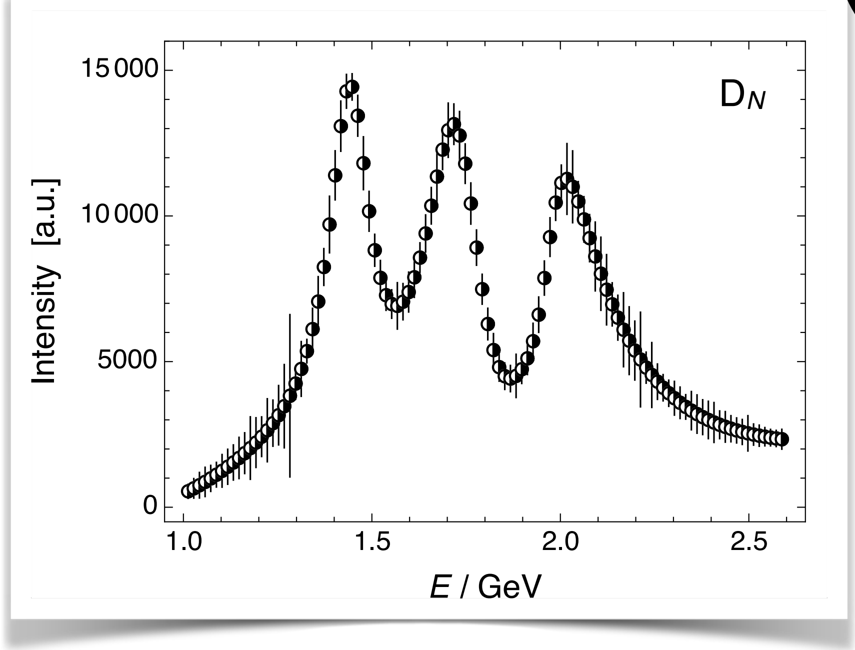

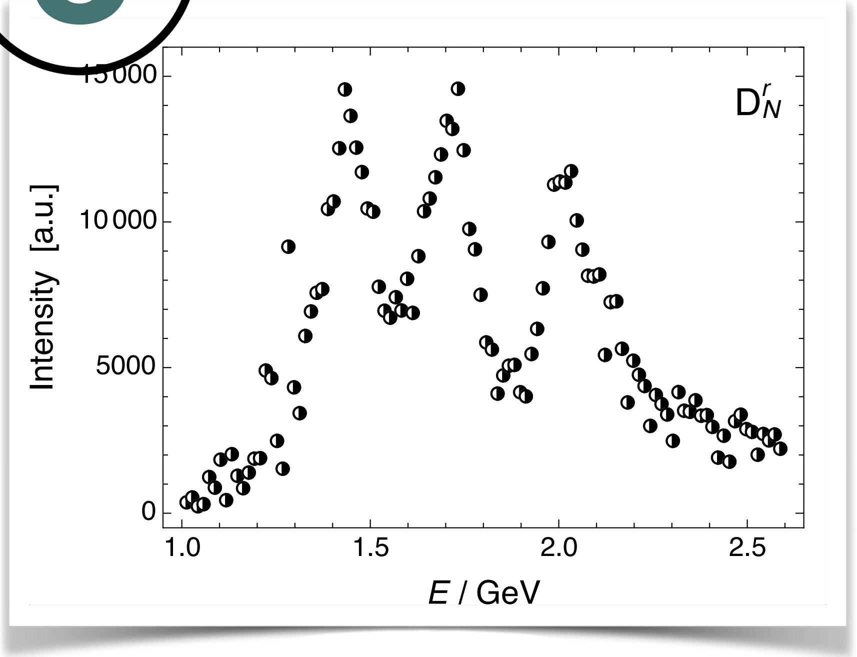

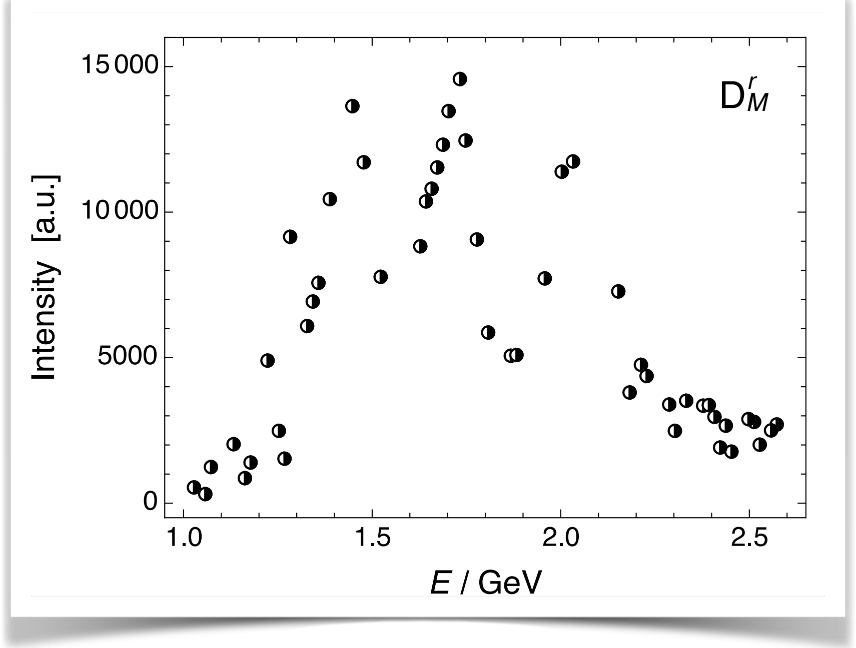

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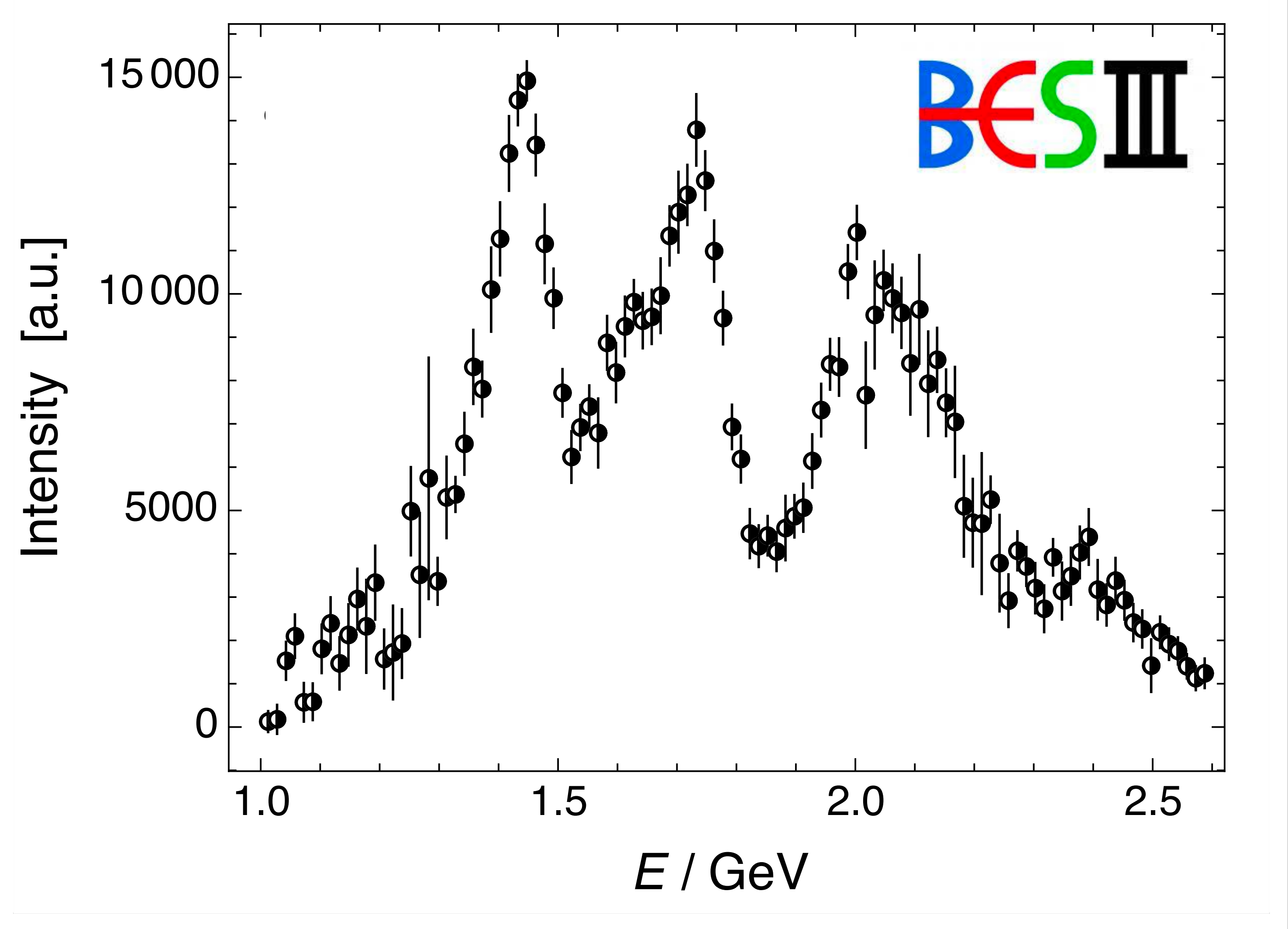

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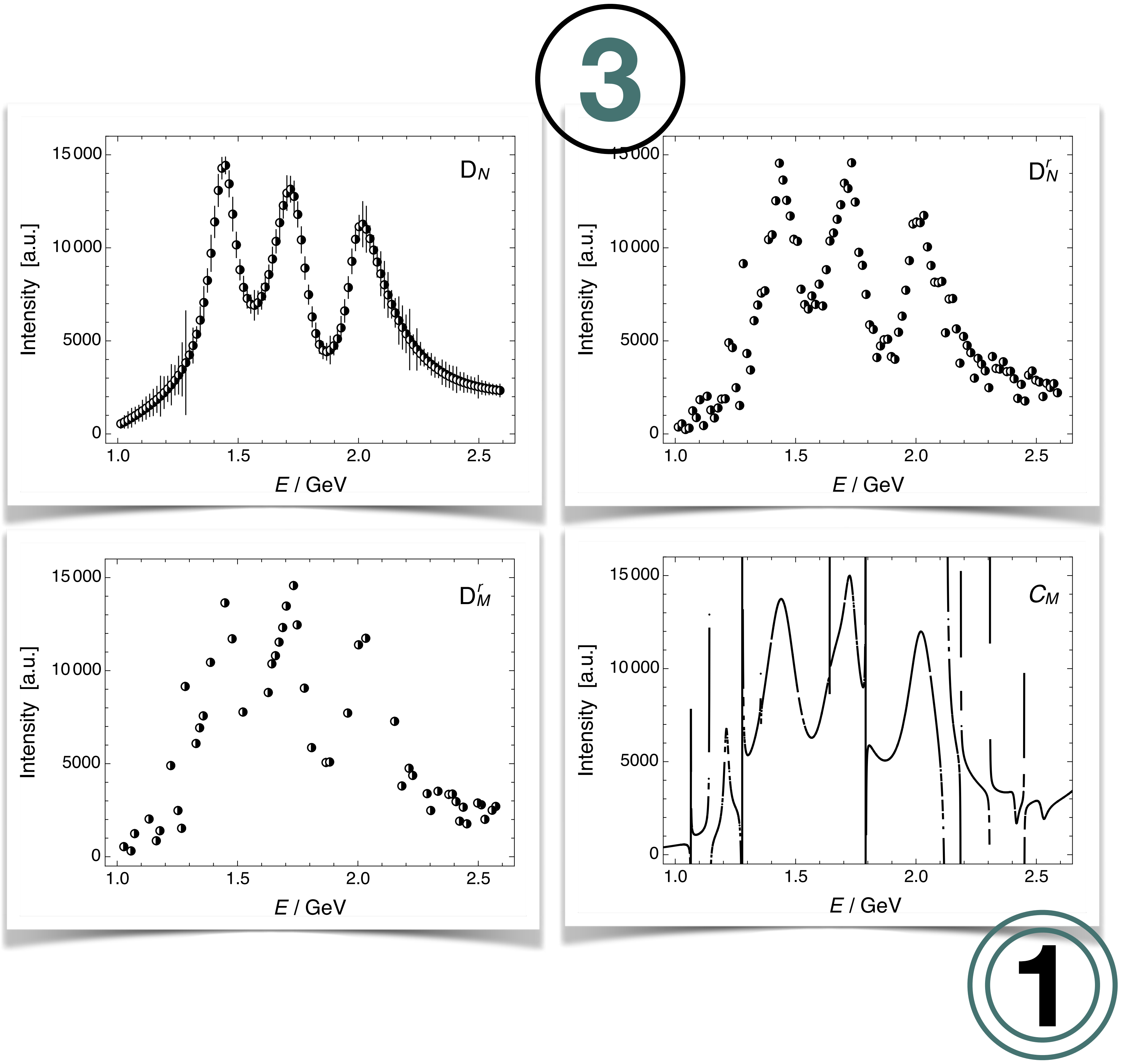

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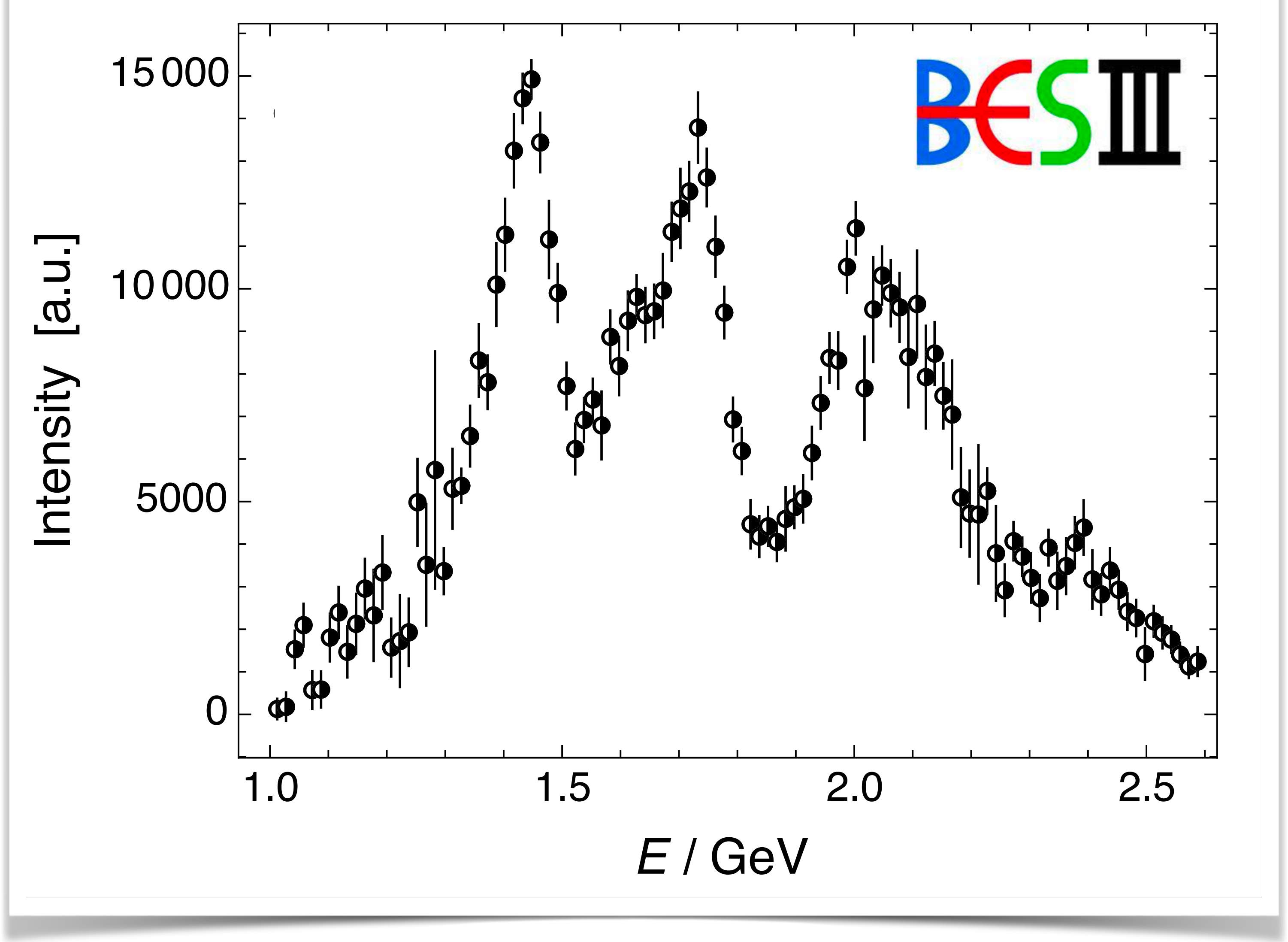

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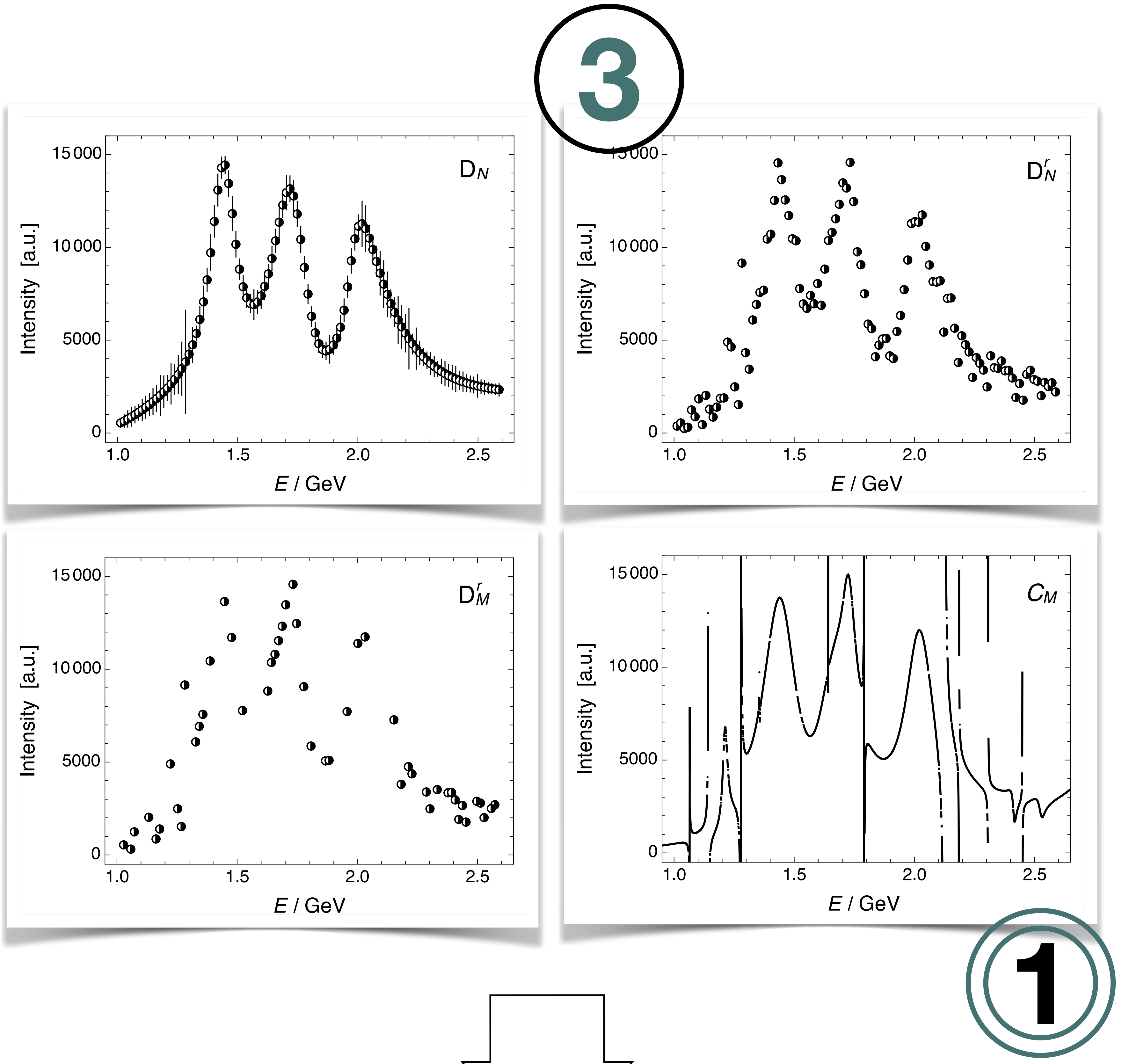

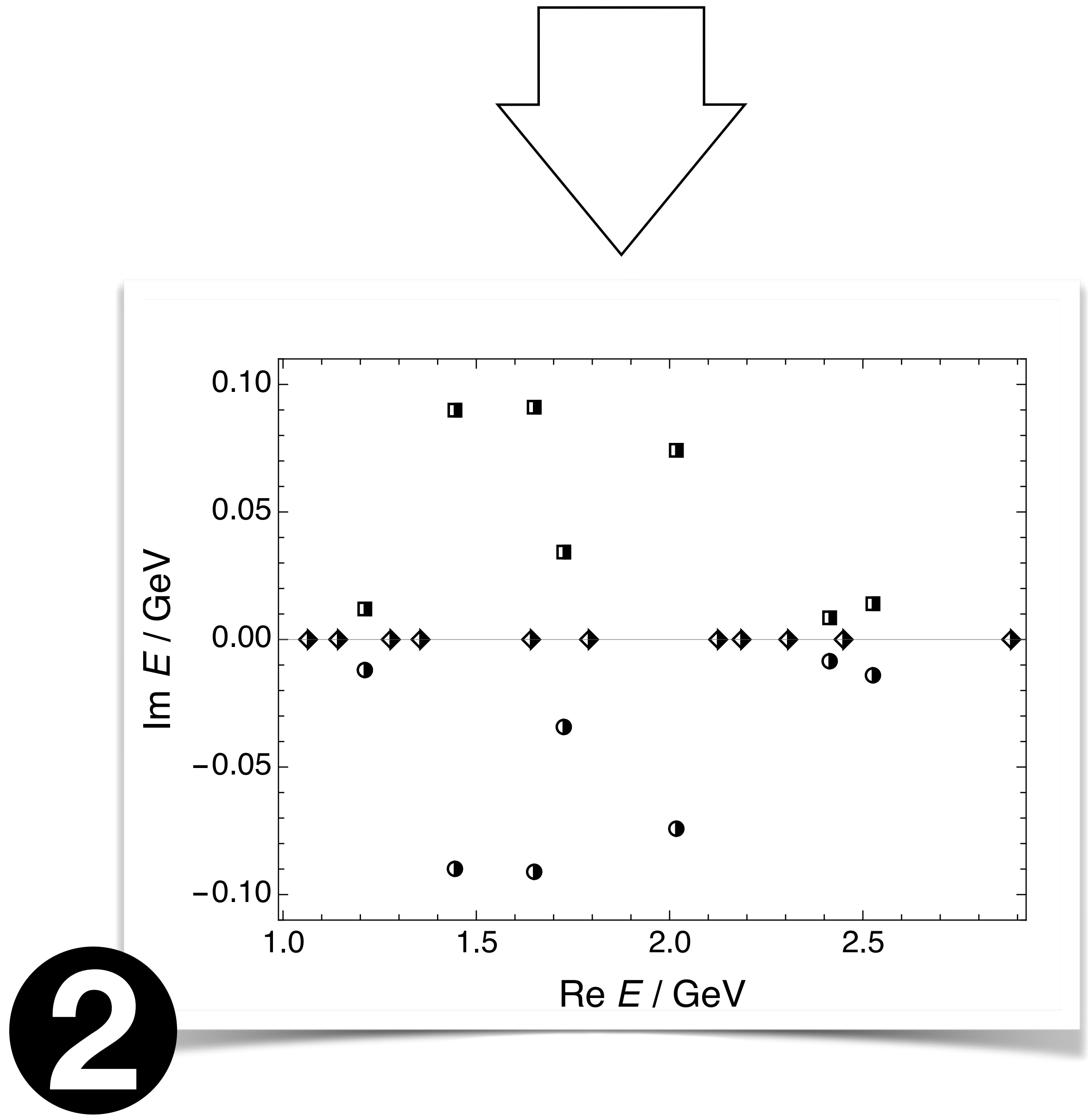

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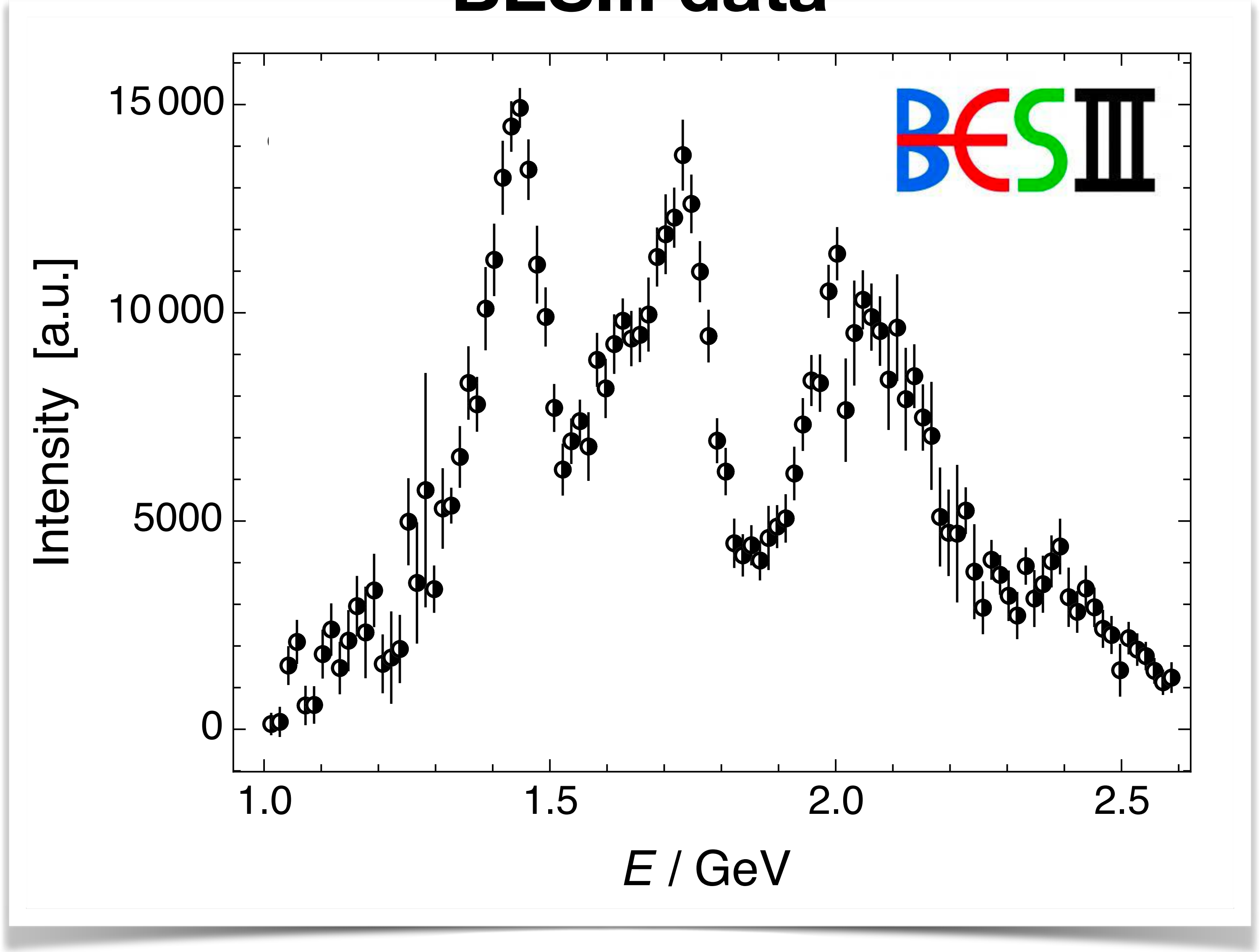

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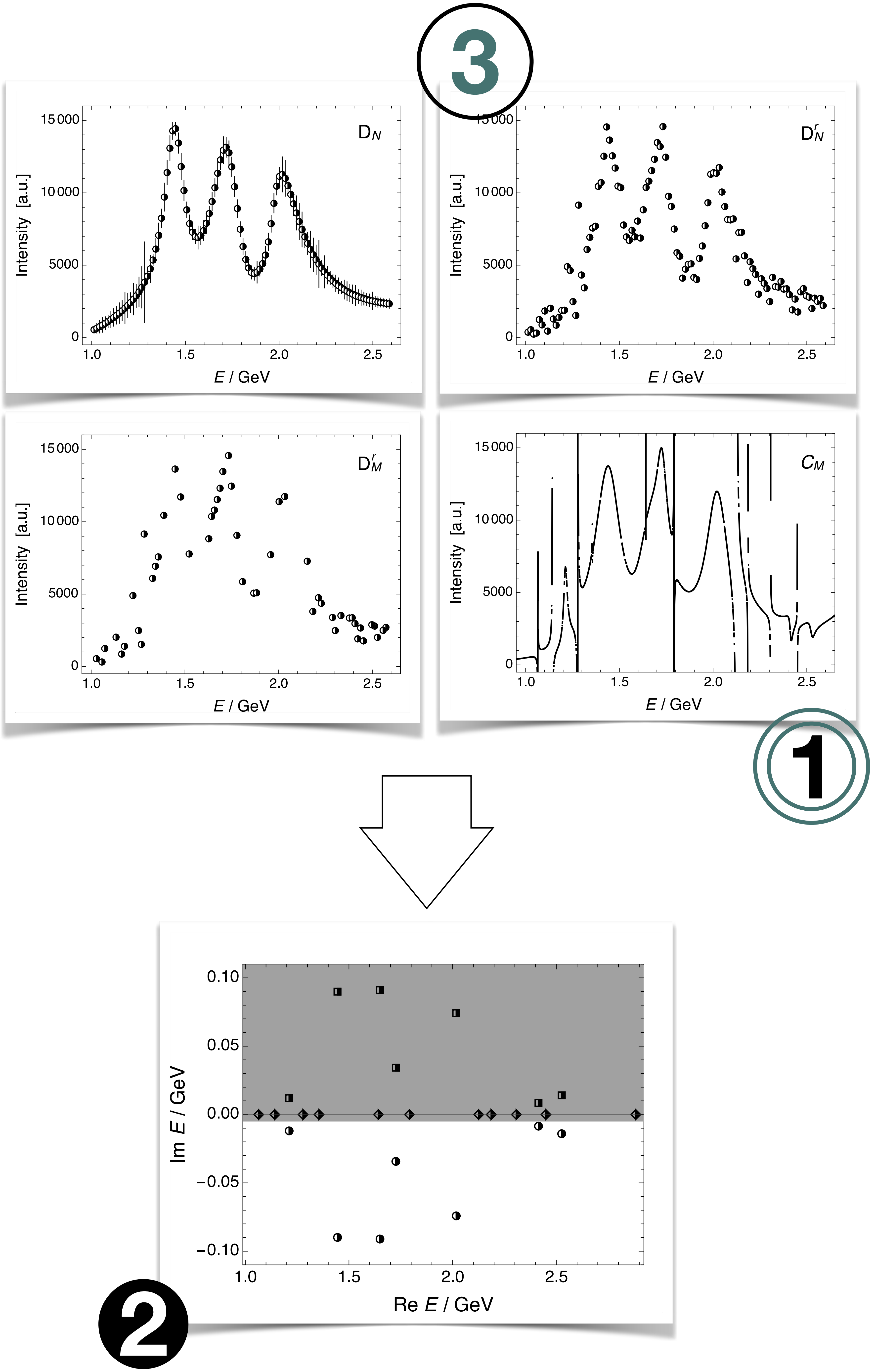

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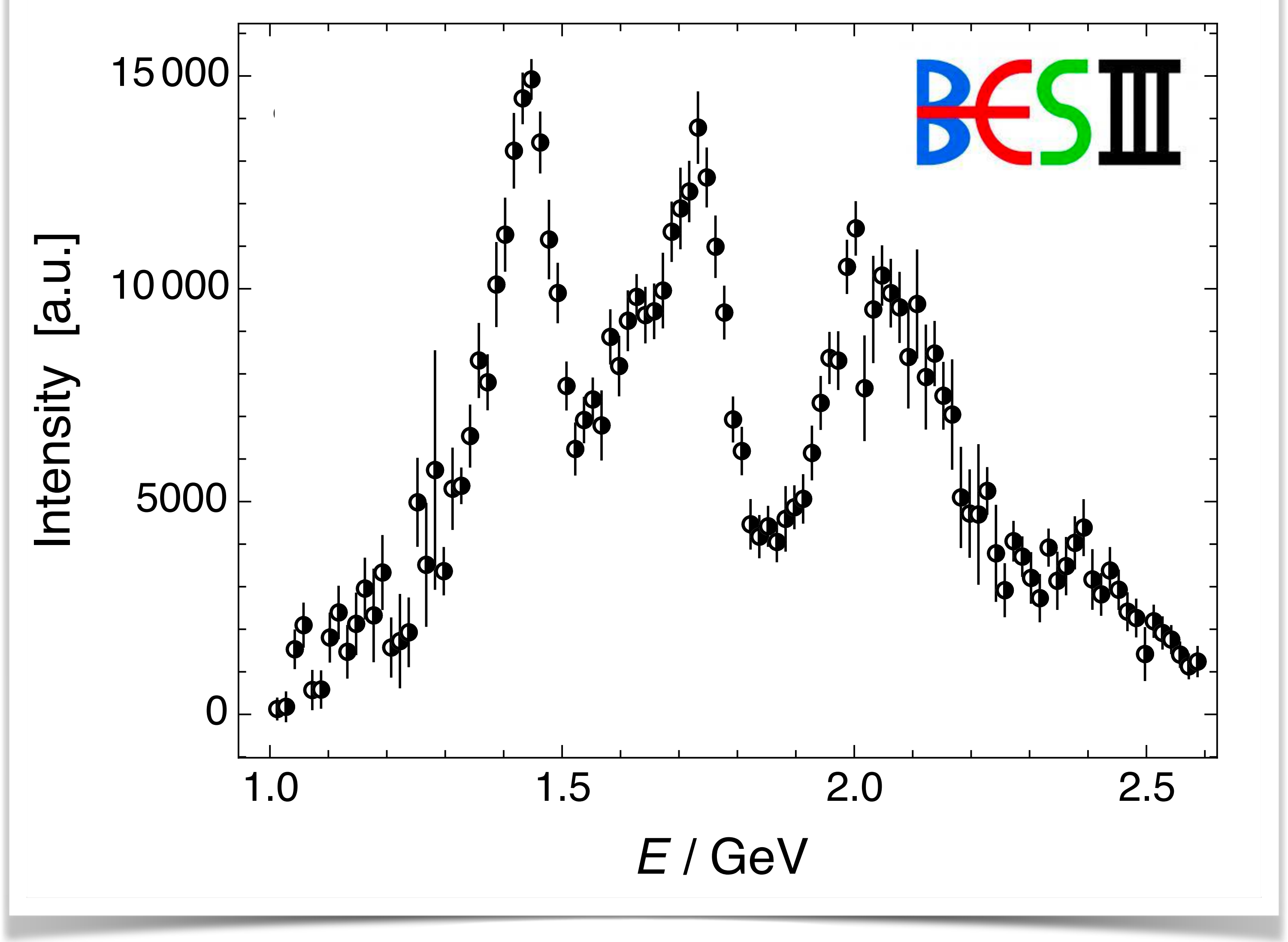

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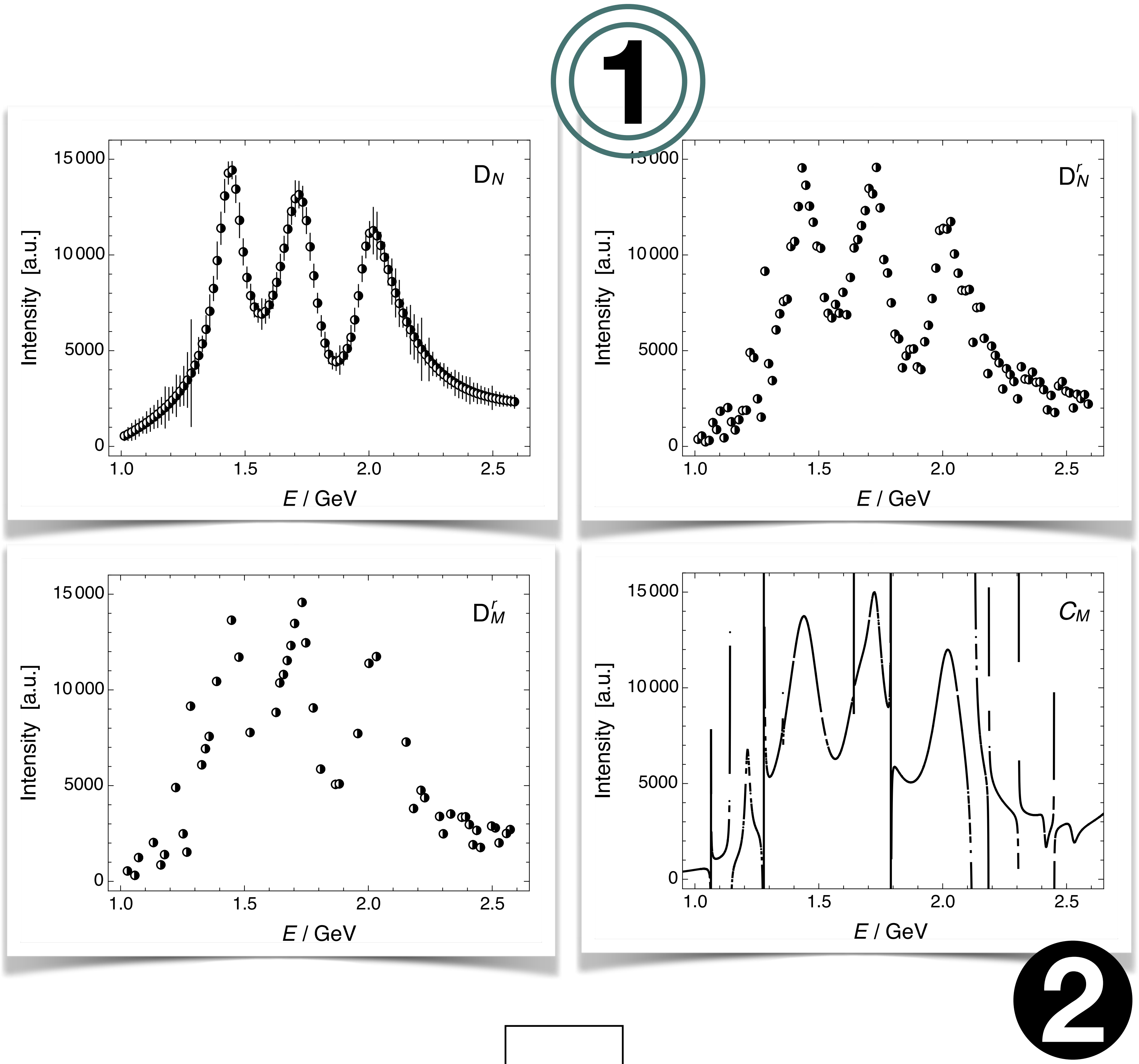

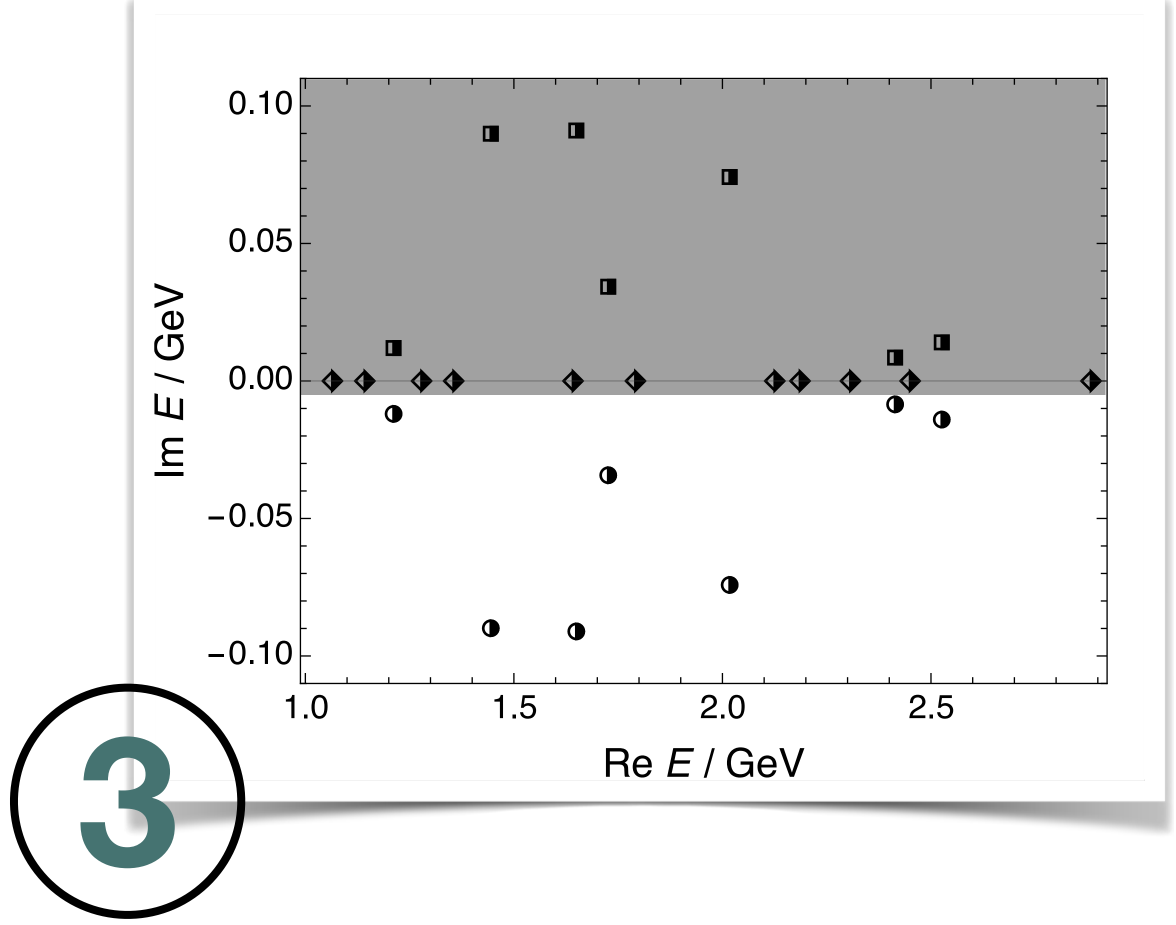

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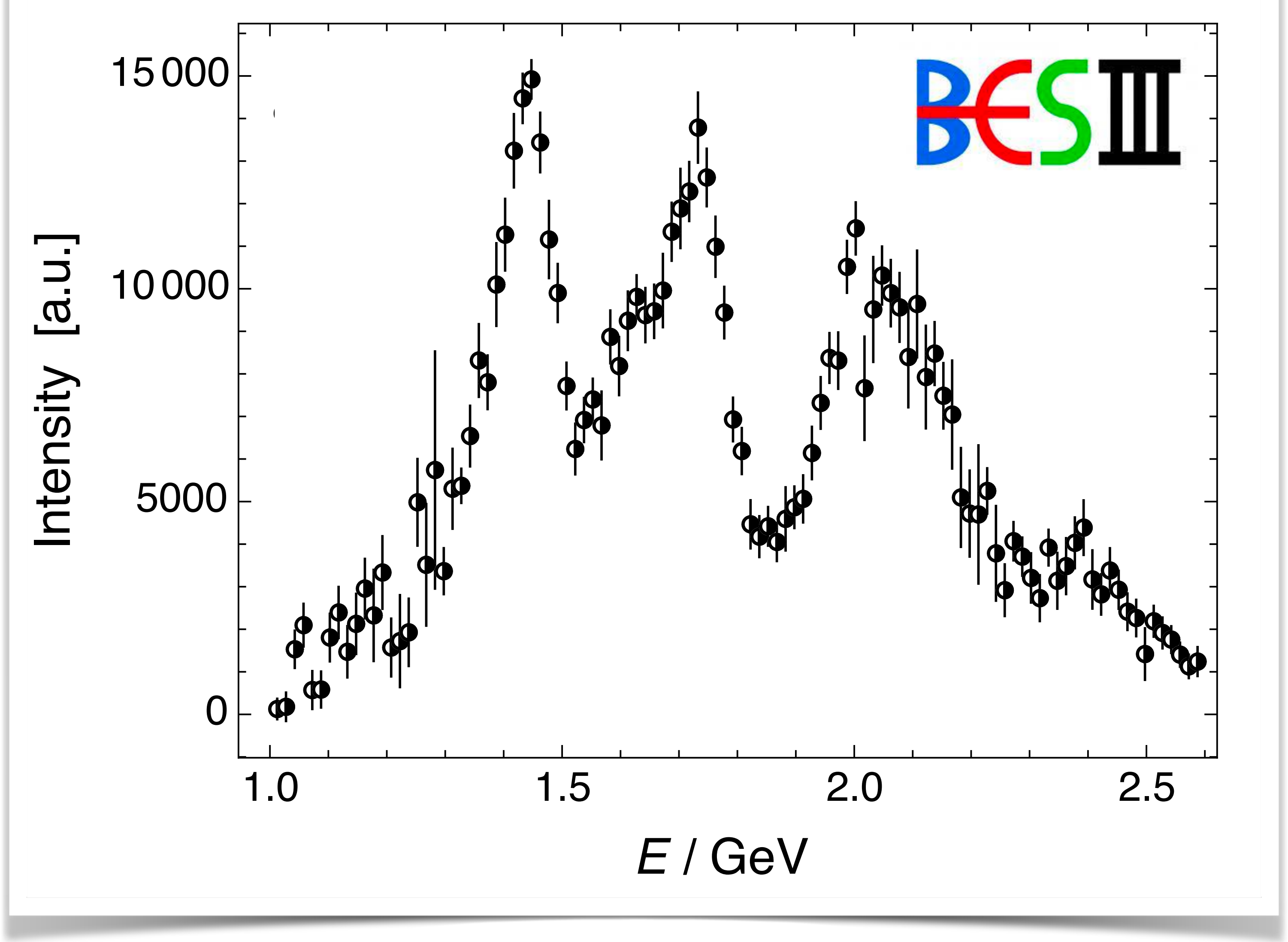

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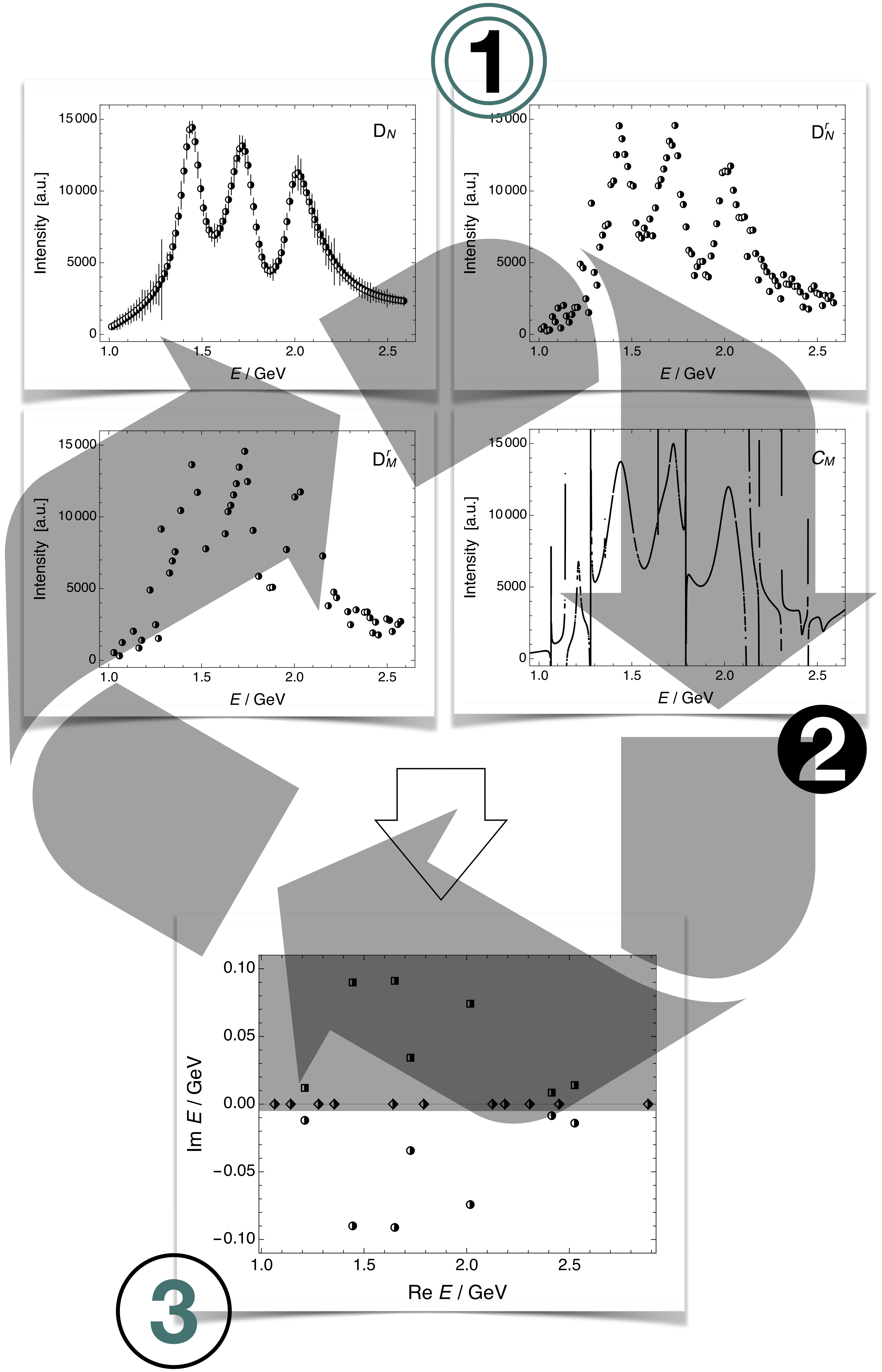

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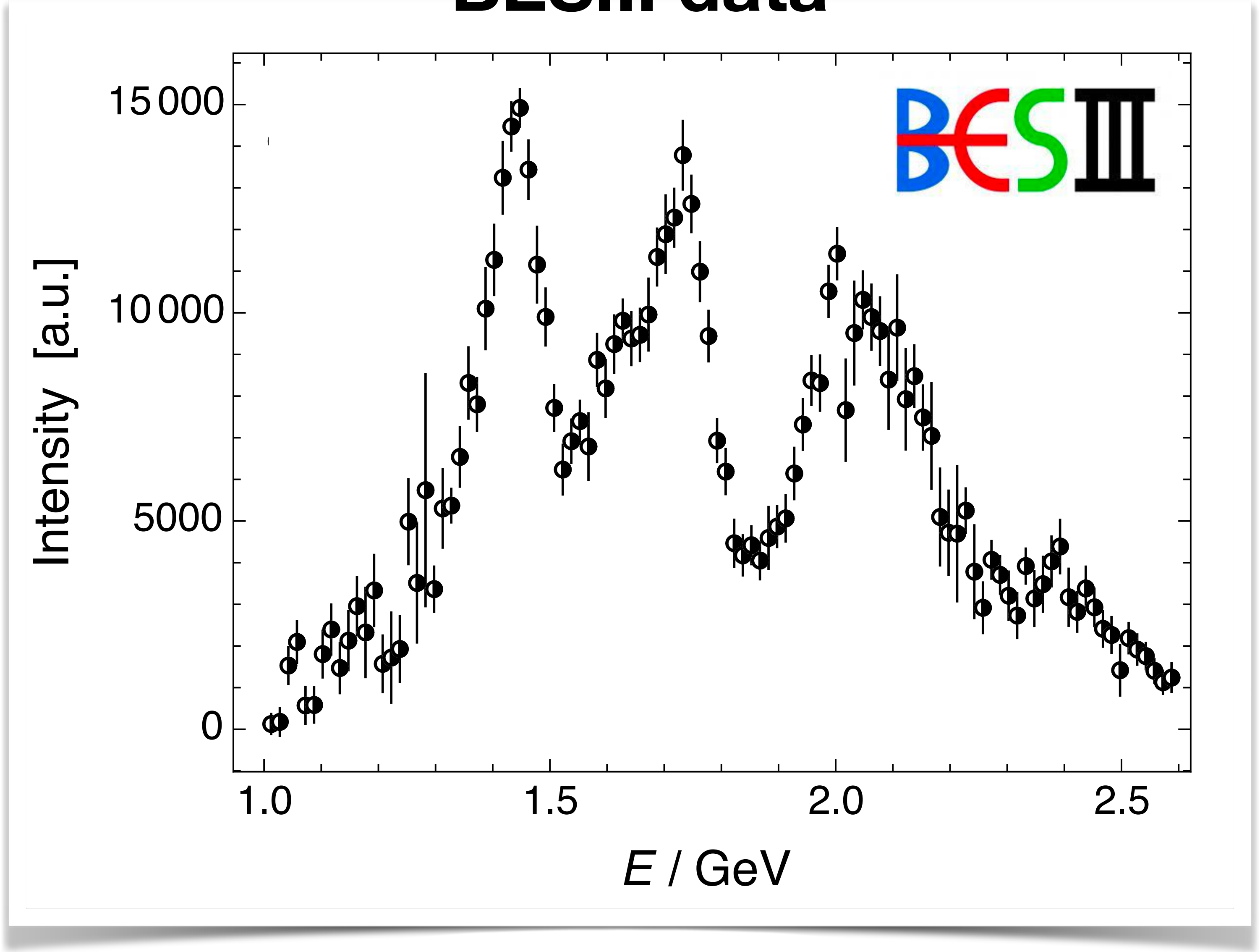

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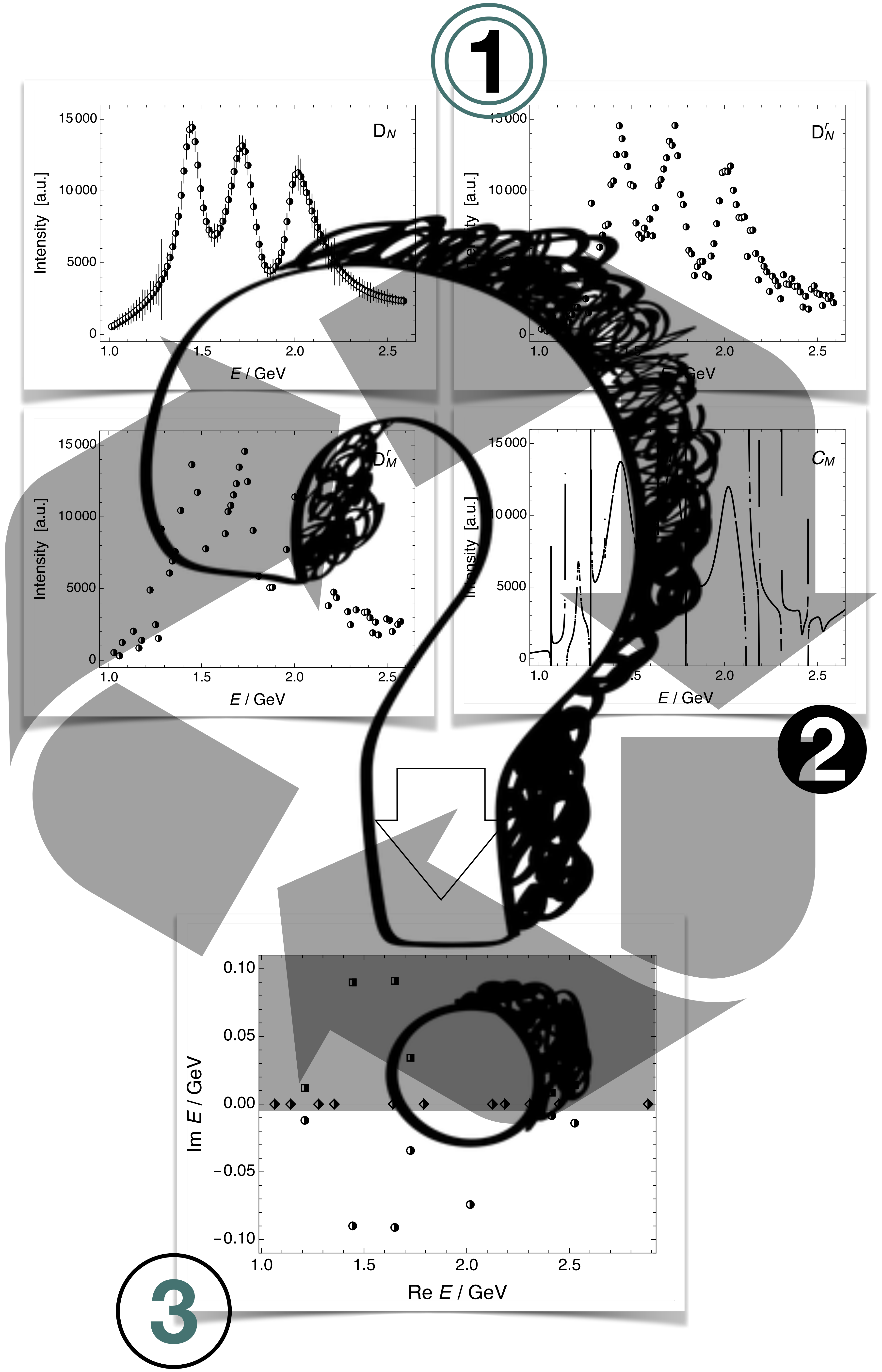

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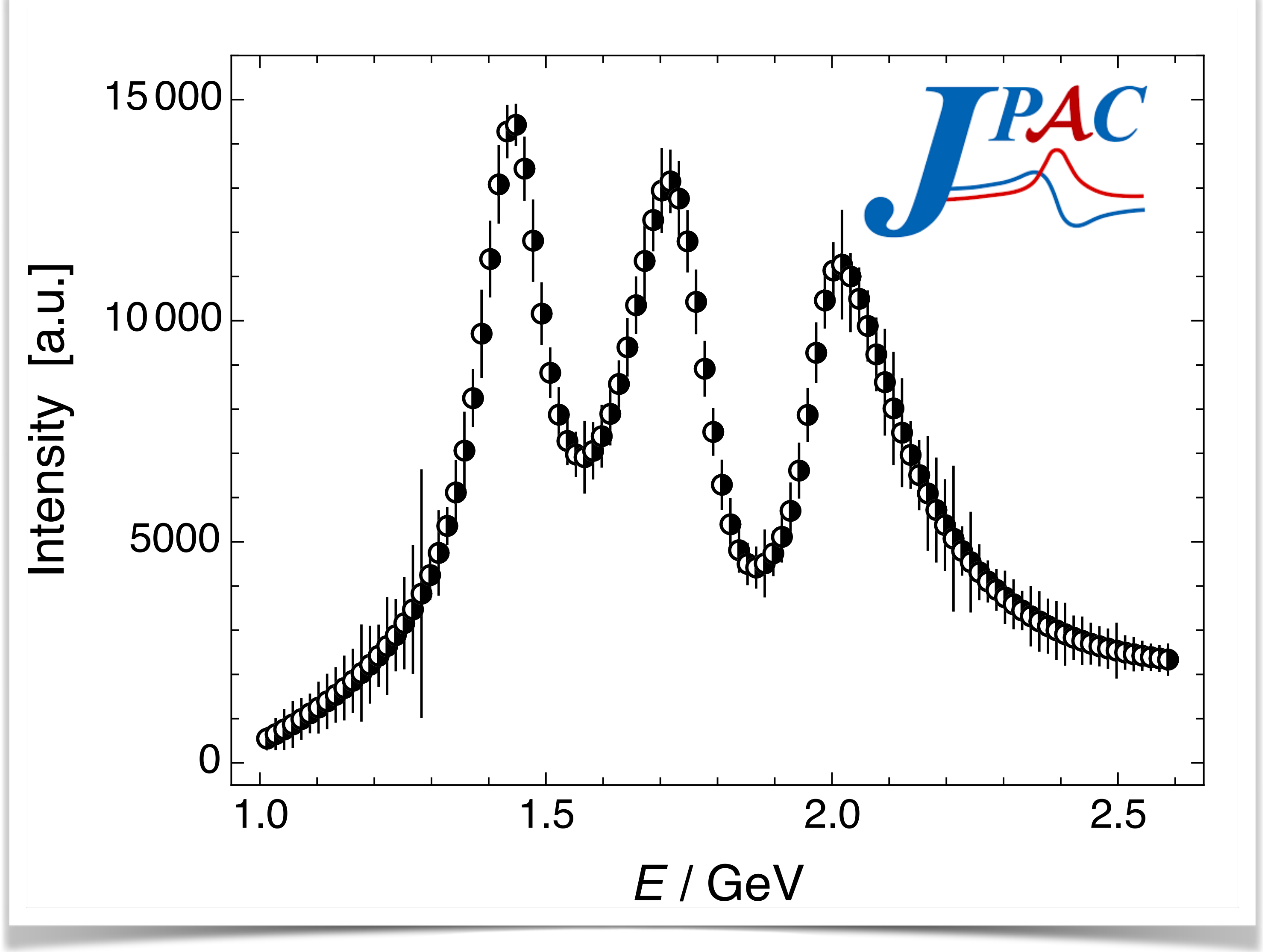

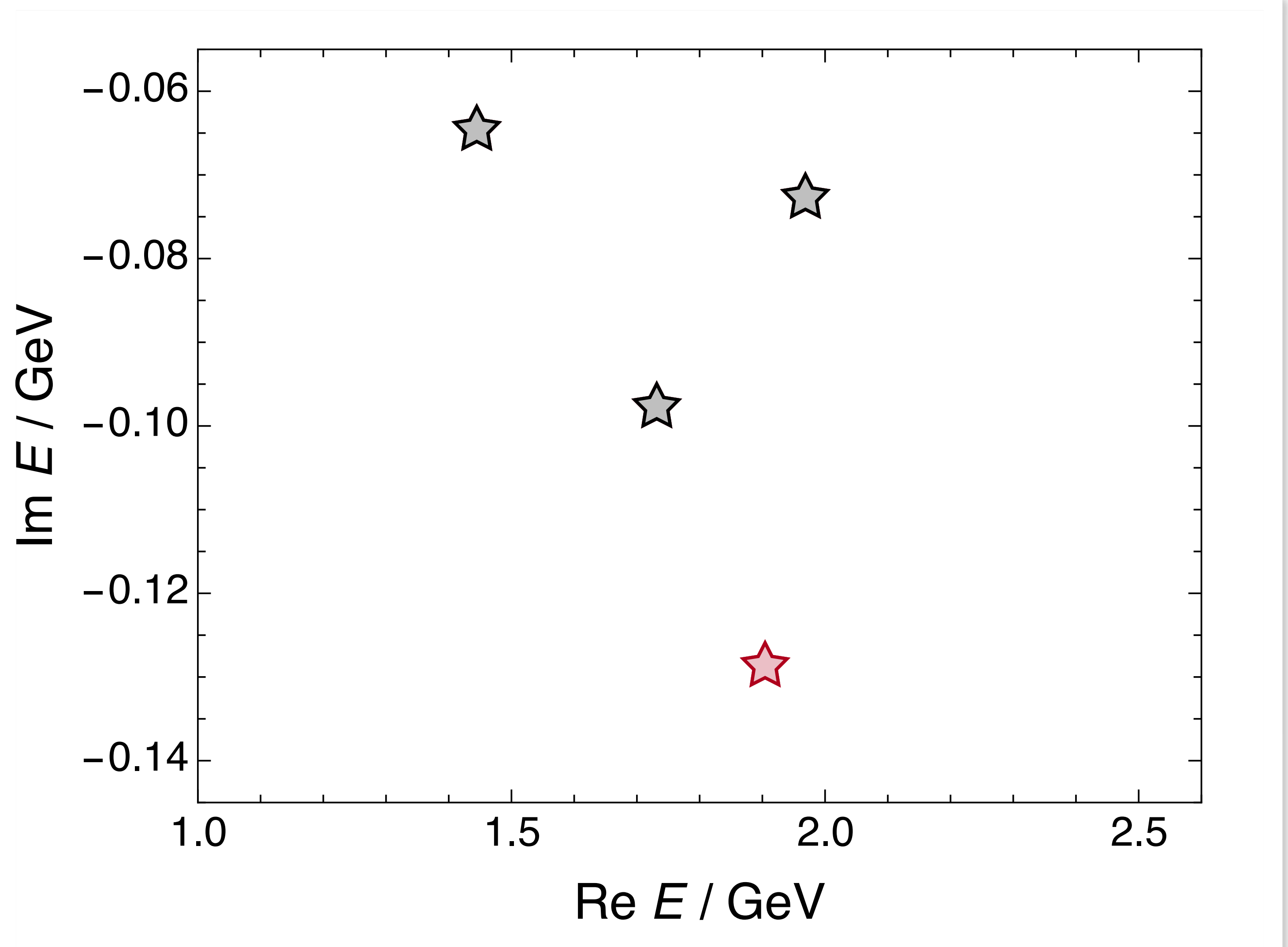

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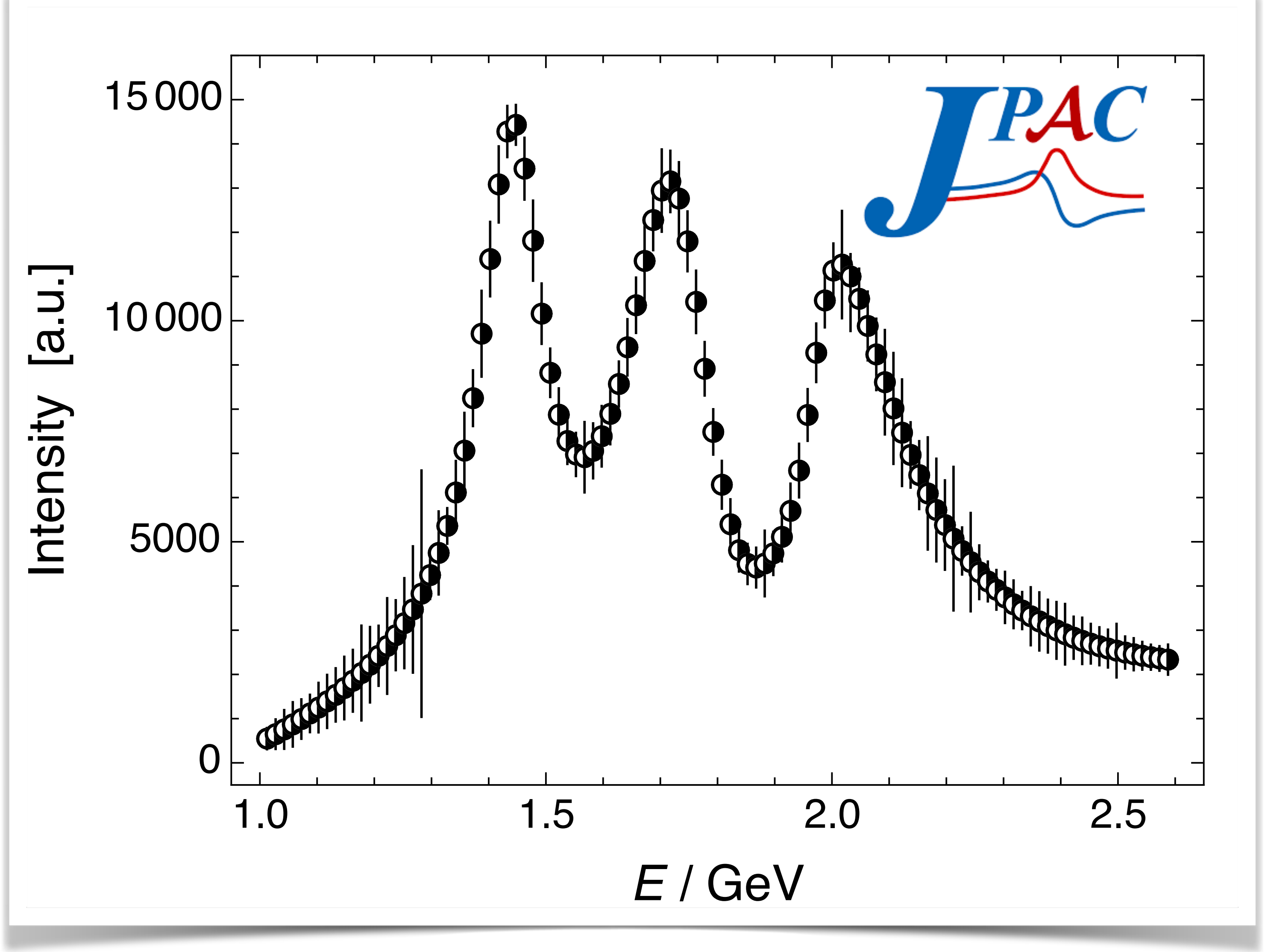

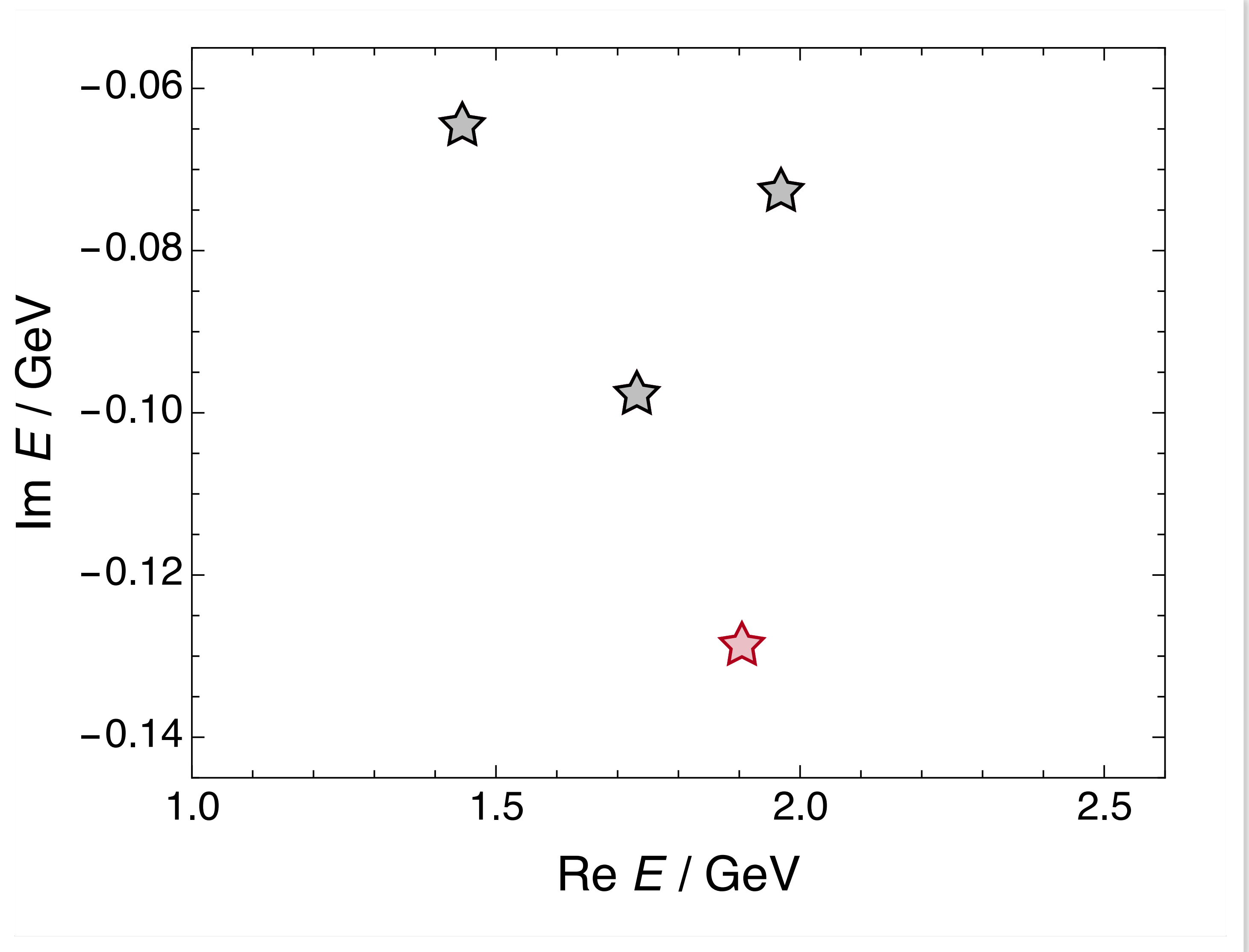

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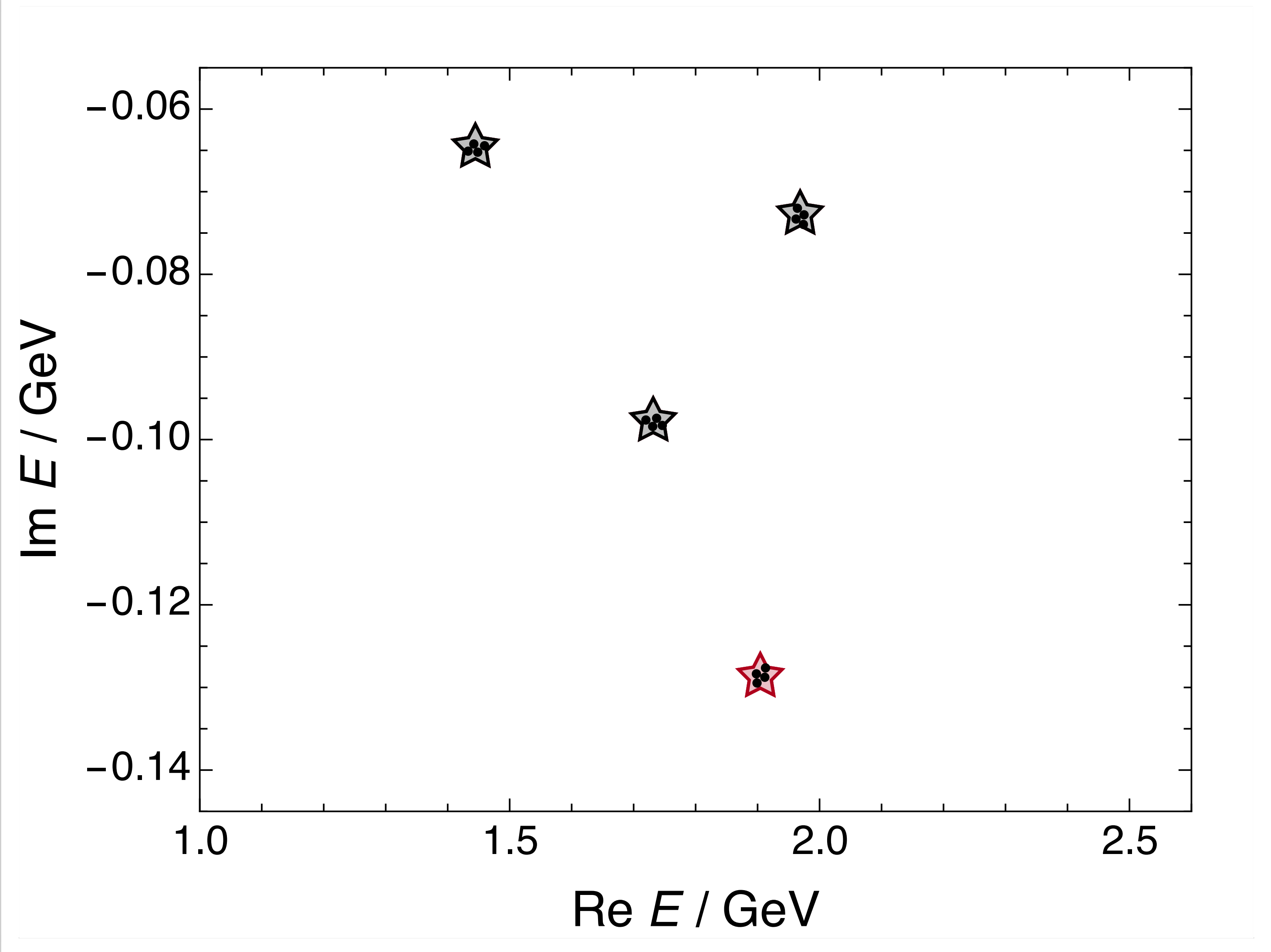

Nearly **exact cancellation** between **poles** and **zeros**

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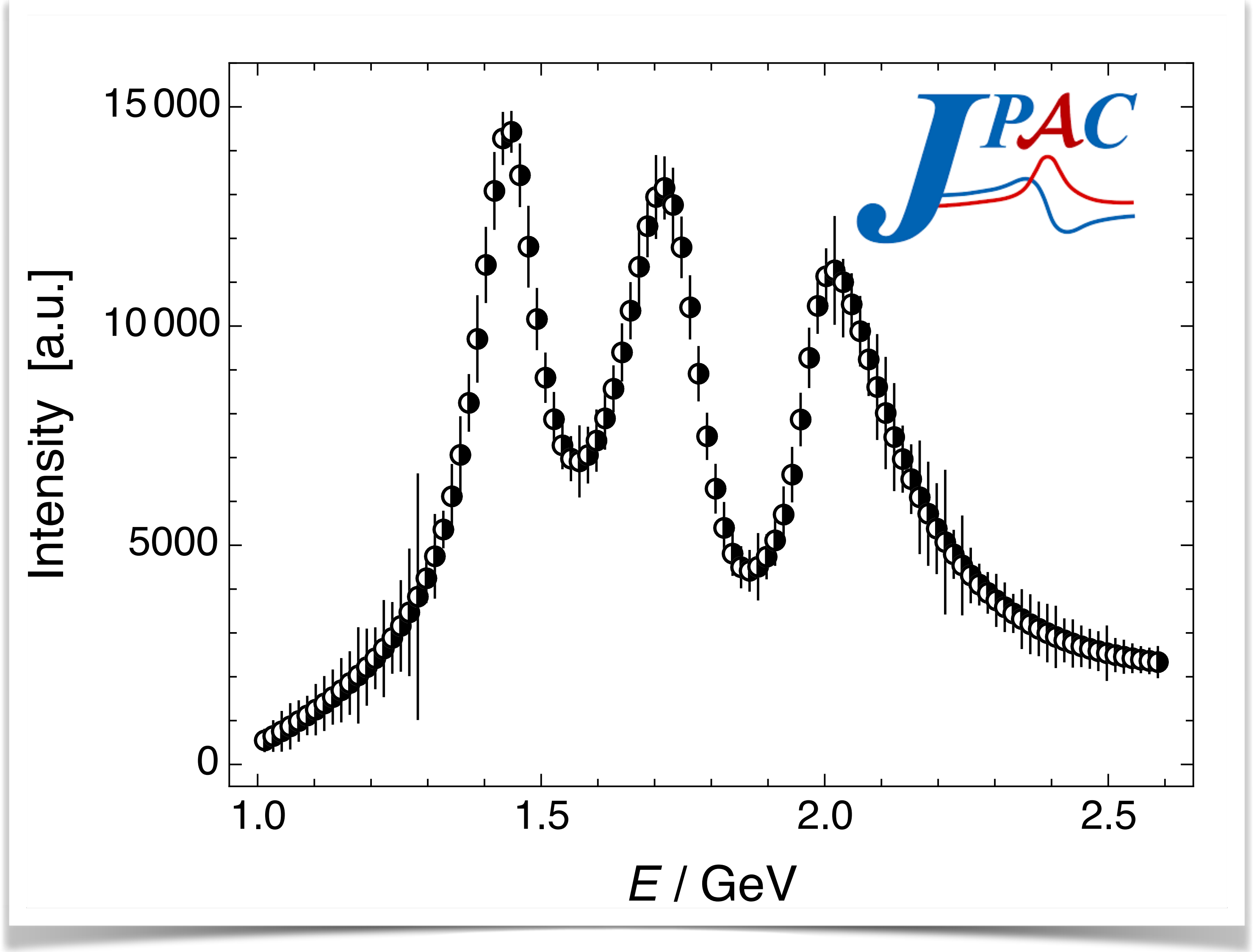

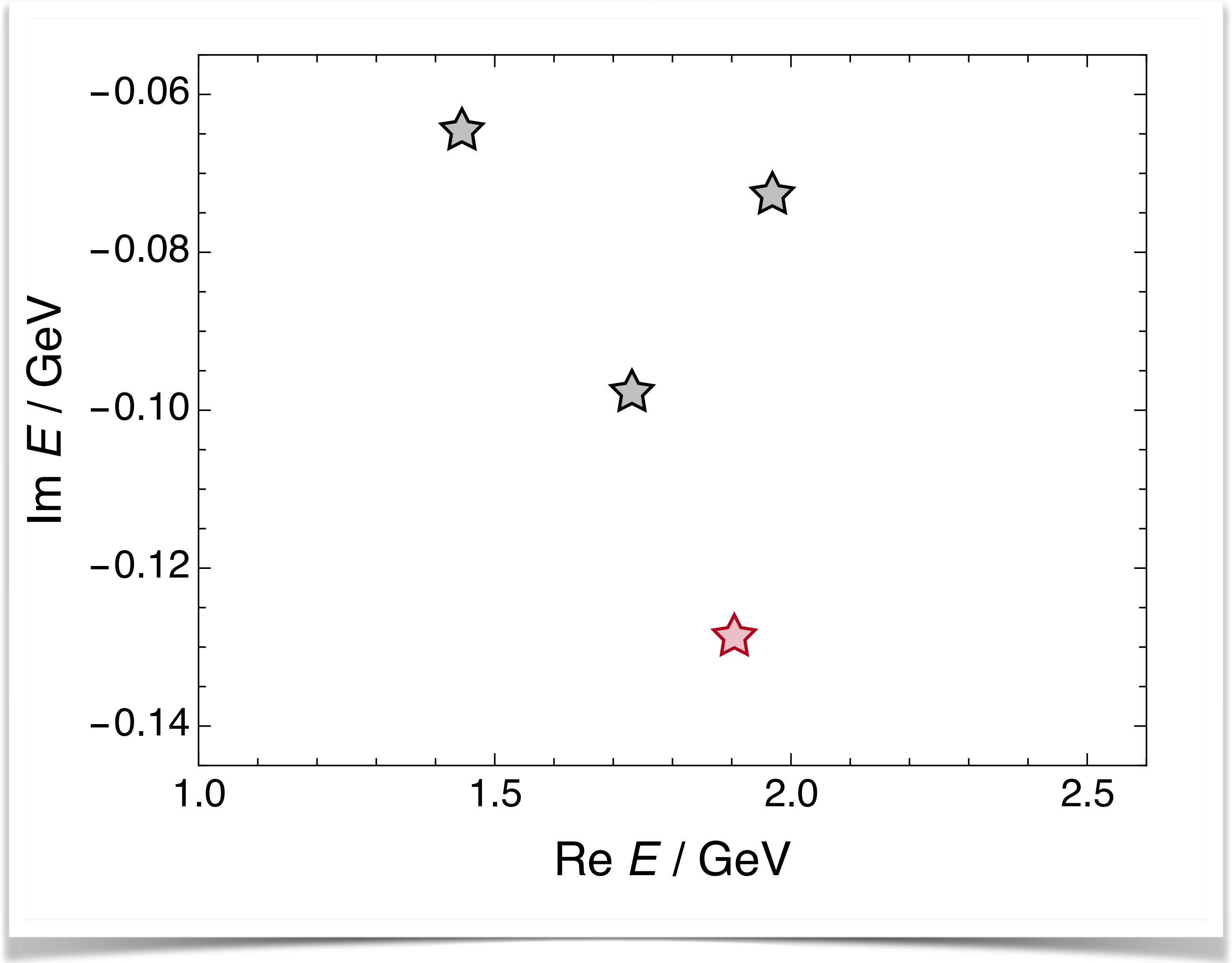

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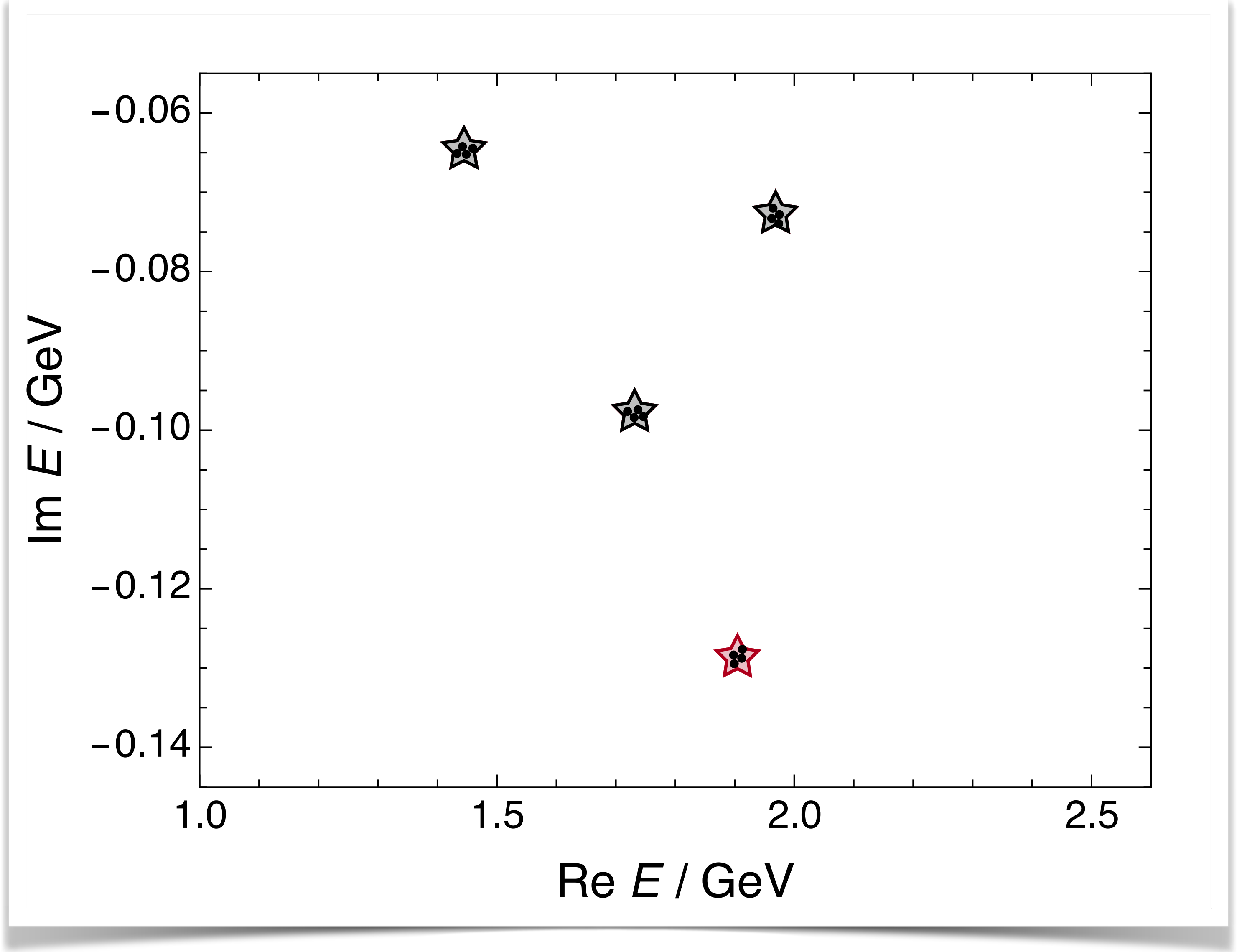

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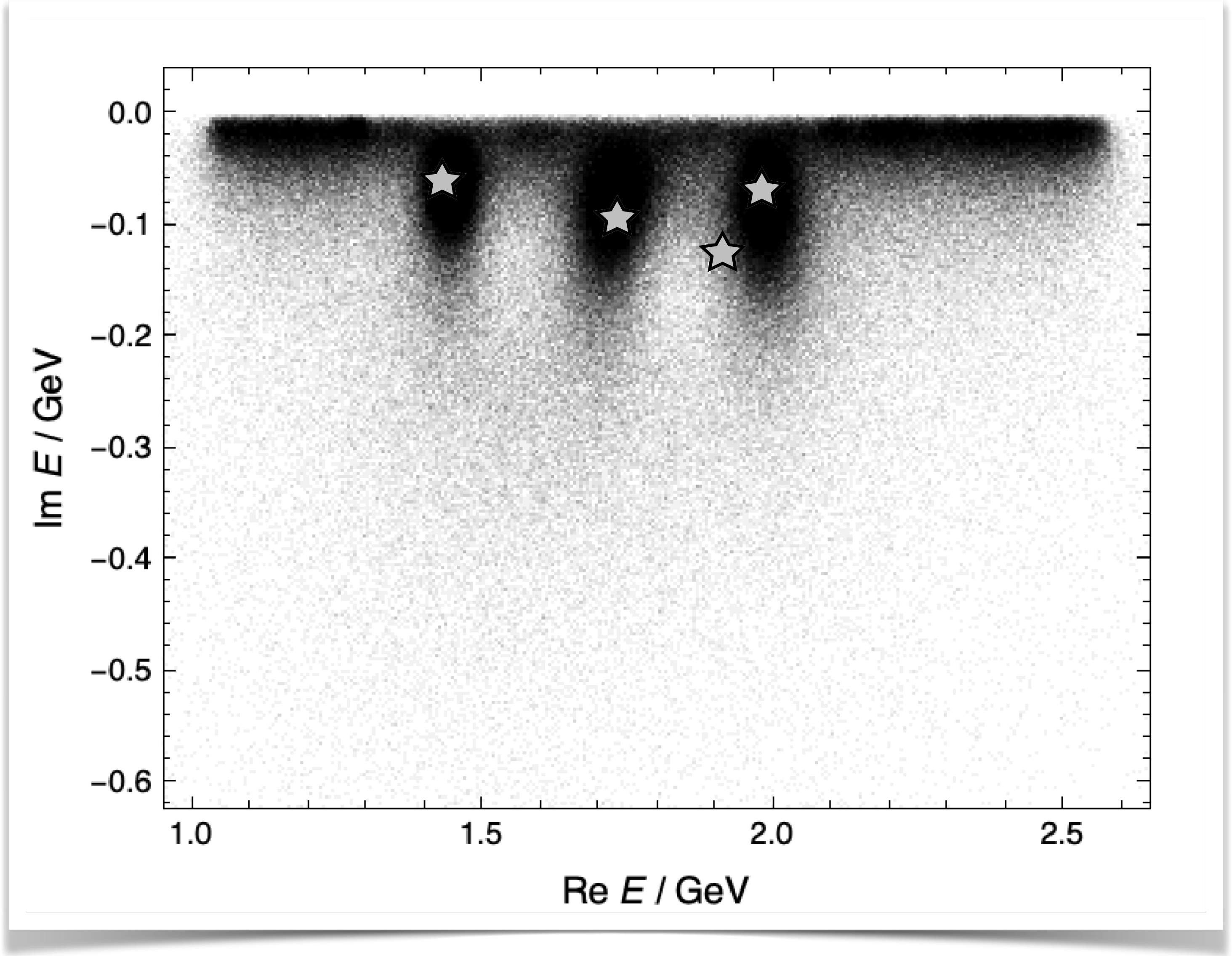
Adding **uncertainties**



Need to separate **noise poles** from **signal poles**

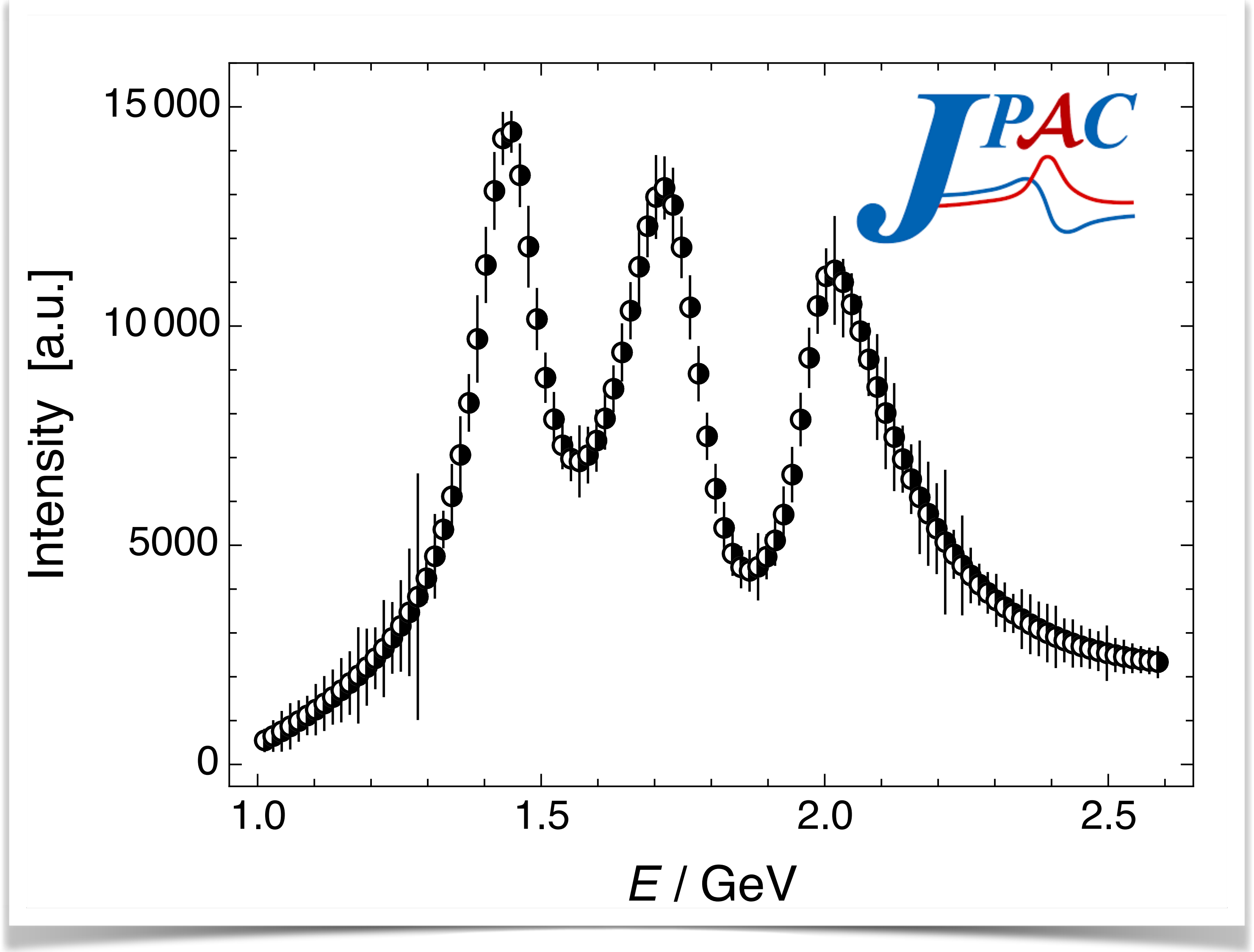
SPM
VALIDATION

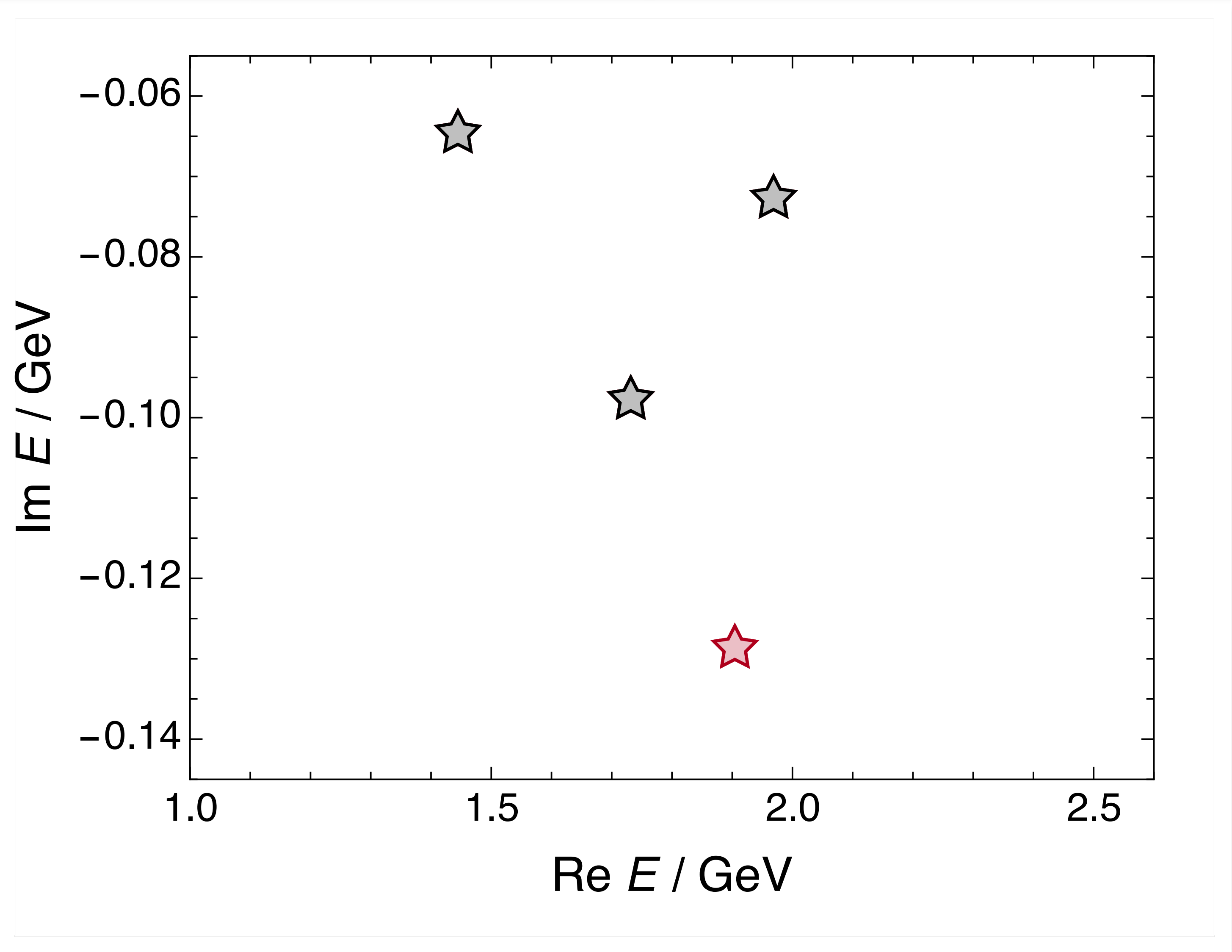
$$\mathsf{D}_N = \{(E_i, I_i = I(E_i)),\ i = 1, \ldots, N\}$$

$$\mathsf{D}_M \subseteq \mathsf{D}_N$$

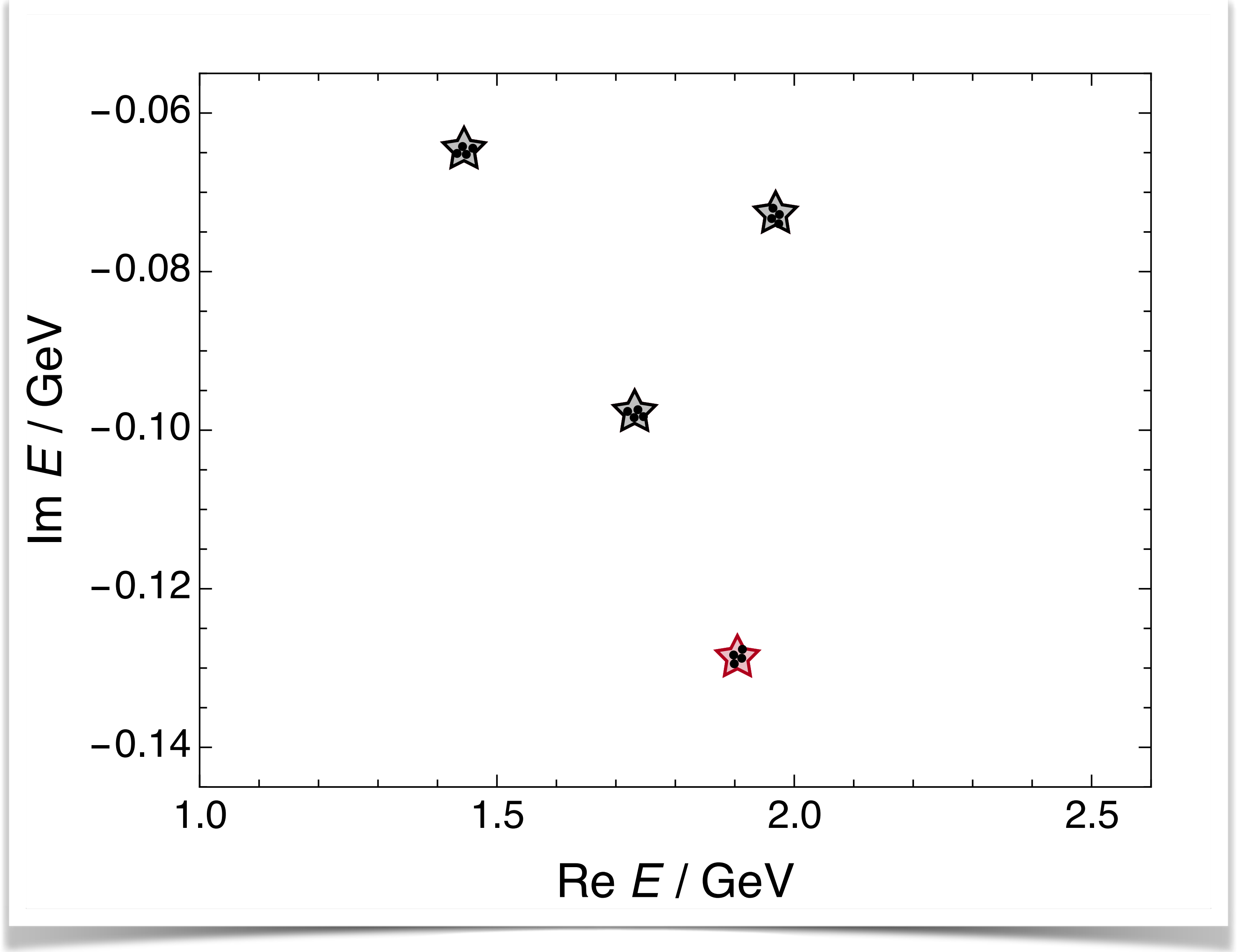
$$C_M(E) = \frac{P(E)}{Q(E)}$$

$$C_M(E_i) = I_i \forall\ E_i \in \mathsf{D}_M$$



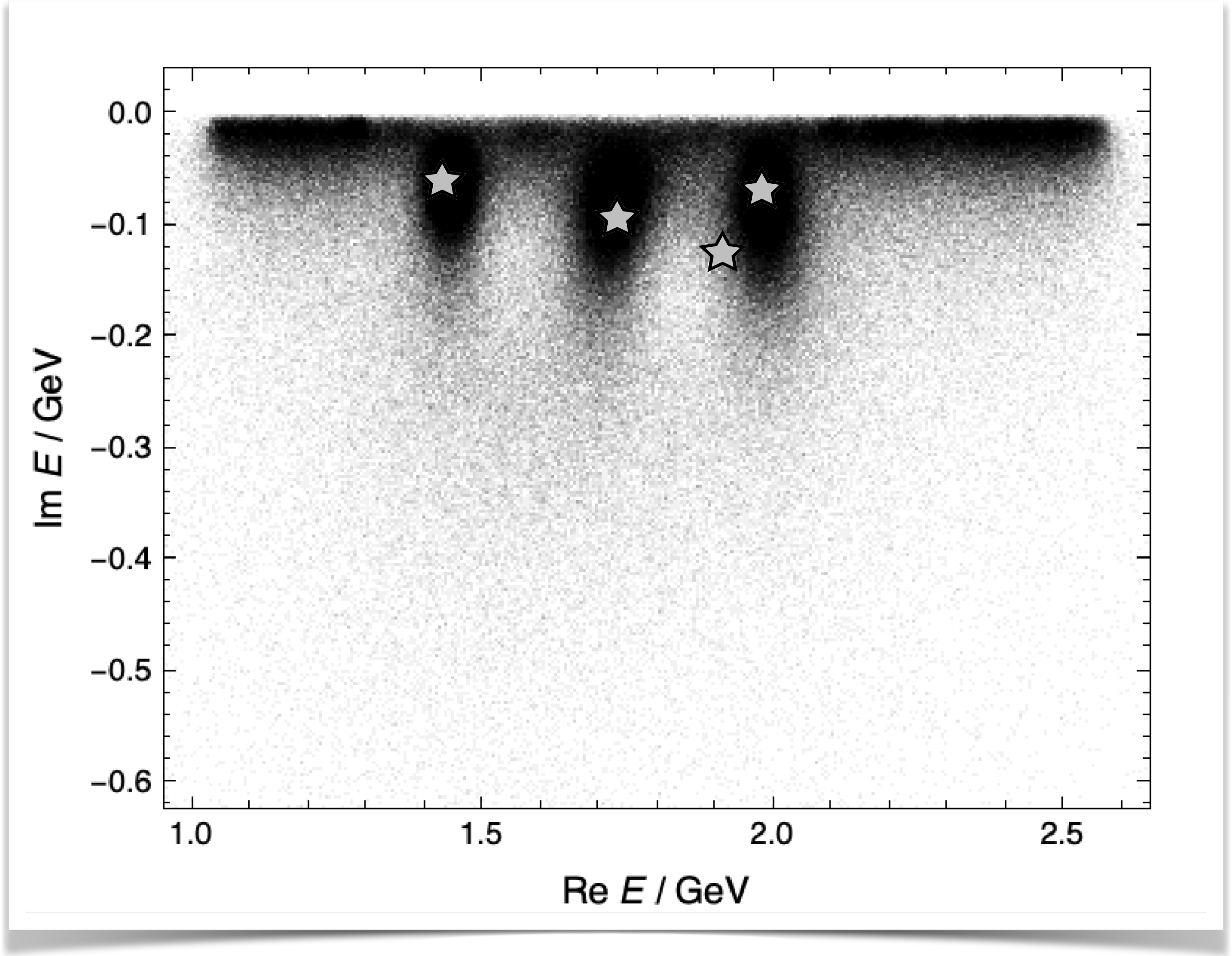


Exact data



Nearly **exact cancellation** between **poles** and **zeros**

Adding **uncertainties**



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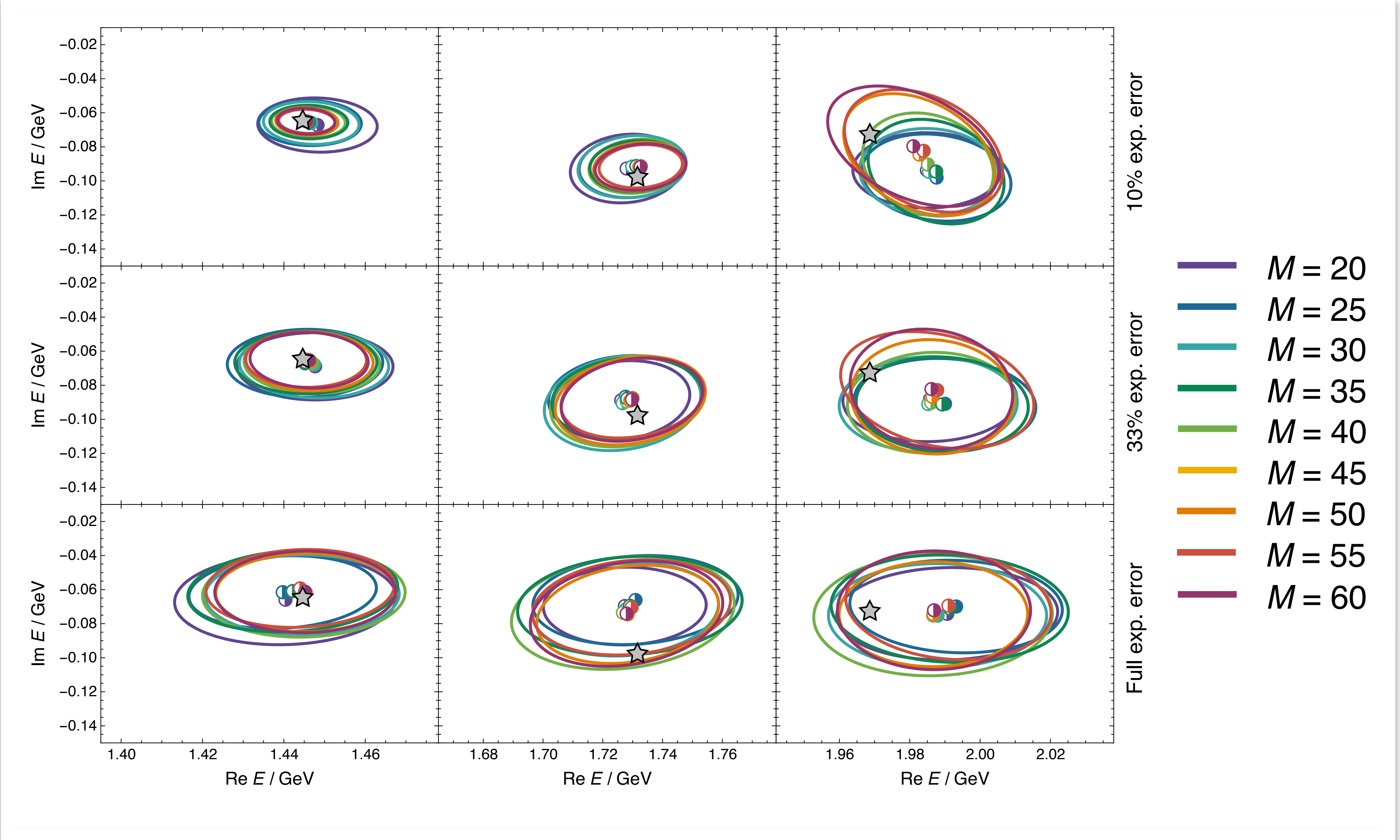
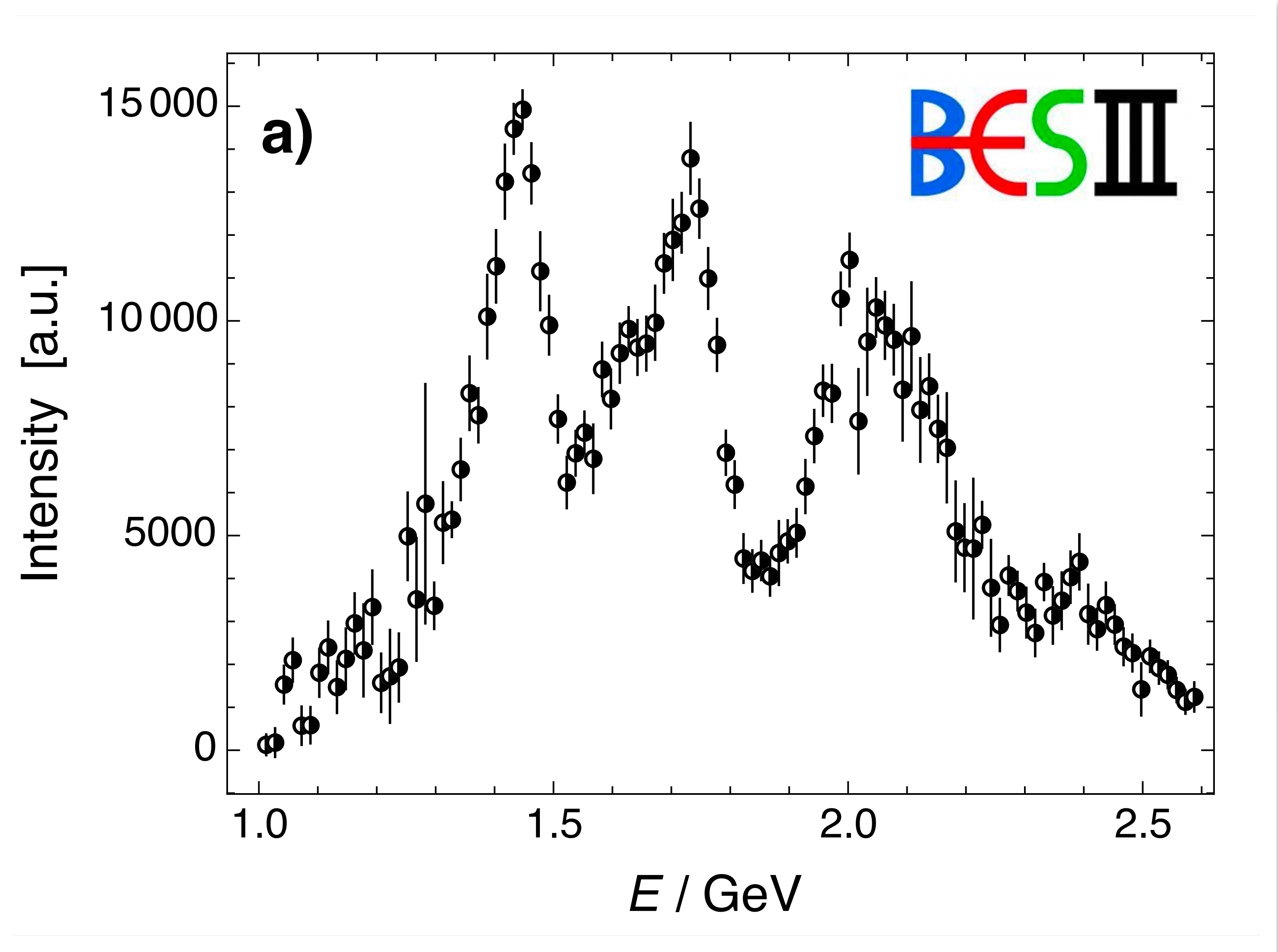
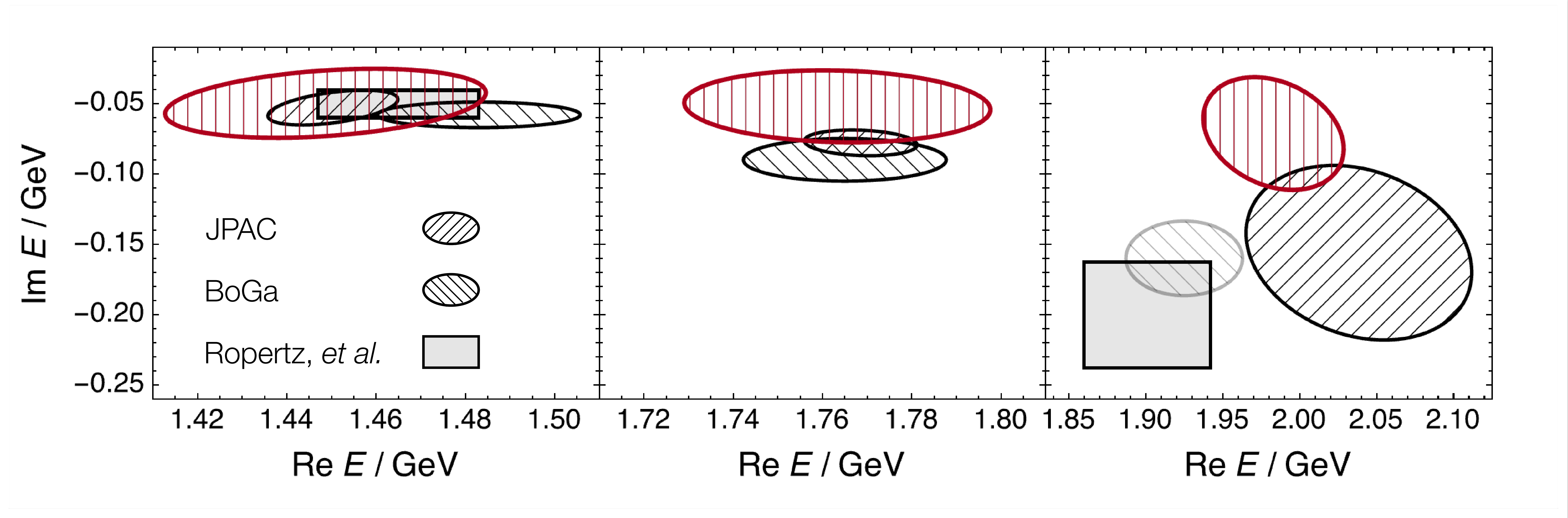


$J/\psi \to \gamma\pi^0\pi^0$

S-WAVE INTENSITY

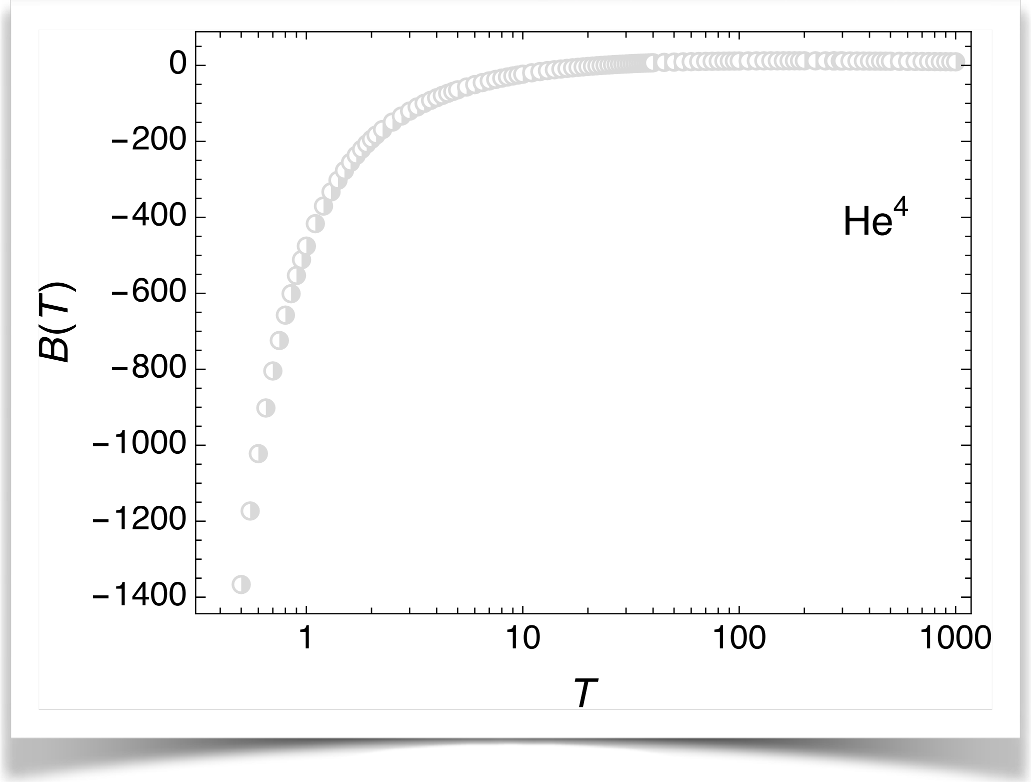


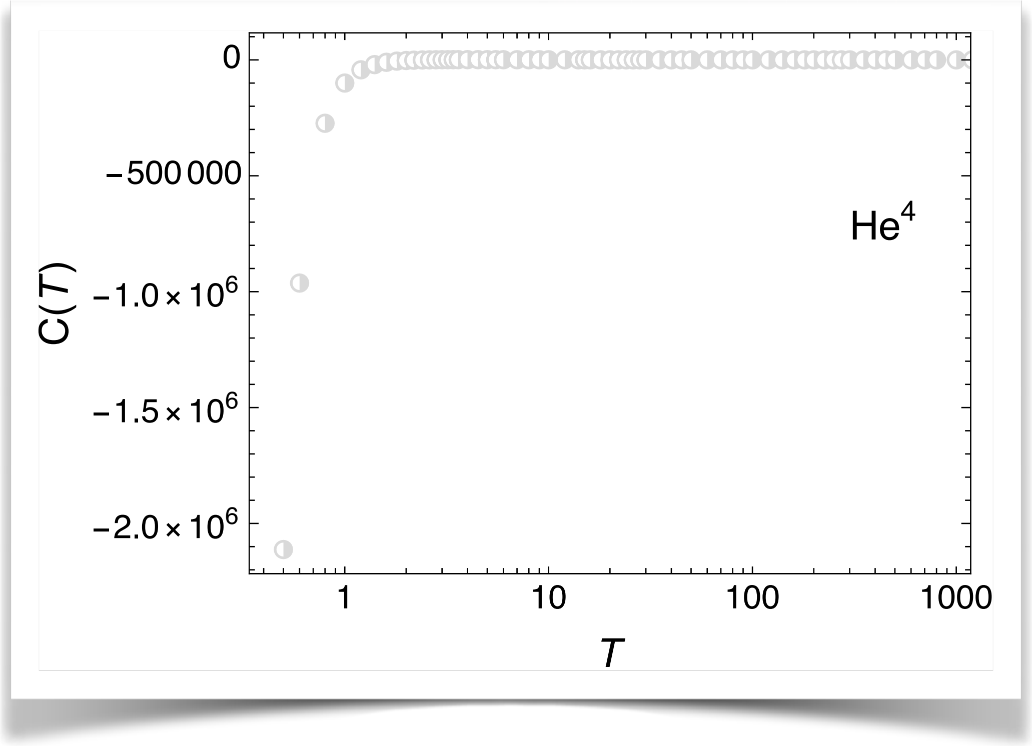


VIRIAL COEFFICIENTS

Describe the deviation from ideal-gas behavior

$$\frac{p}{\rho RT} = 1 + B(T)\rho + C(T)\rho^2 + \cdots$$





PRECISION TEMPERATURE METROLOGY

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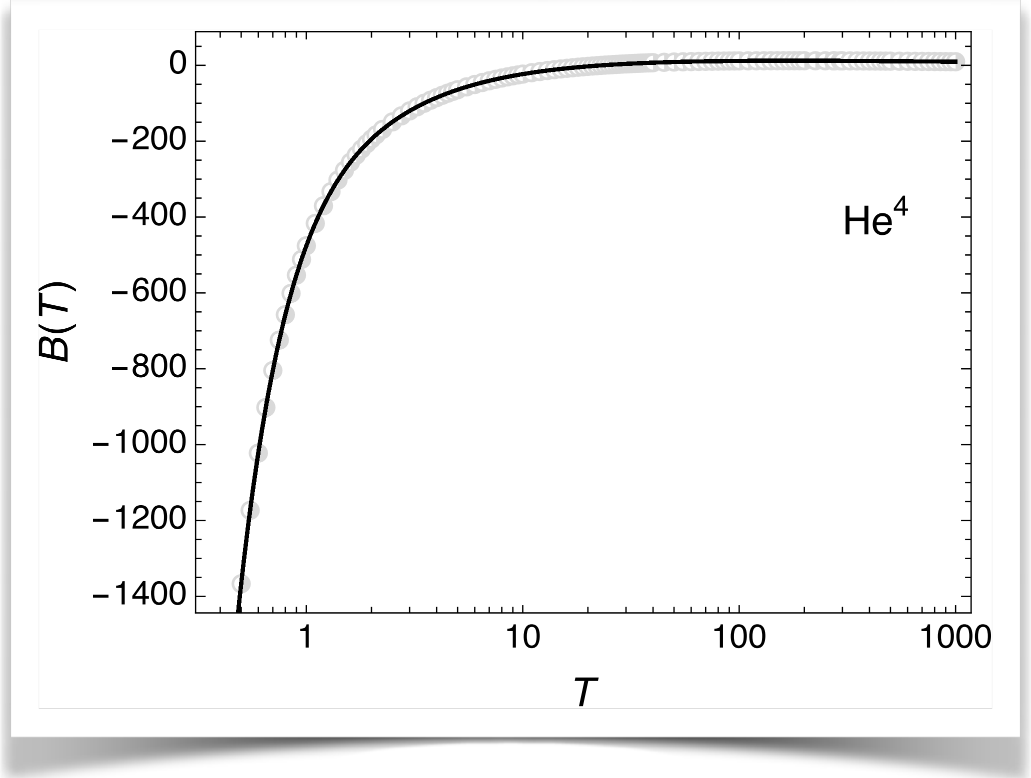

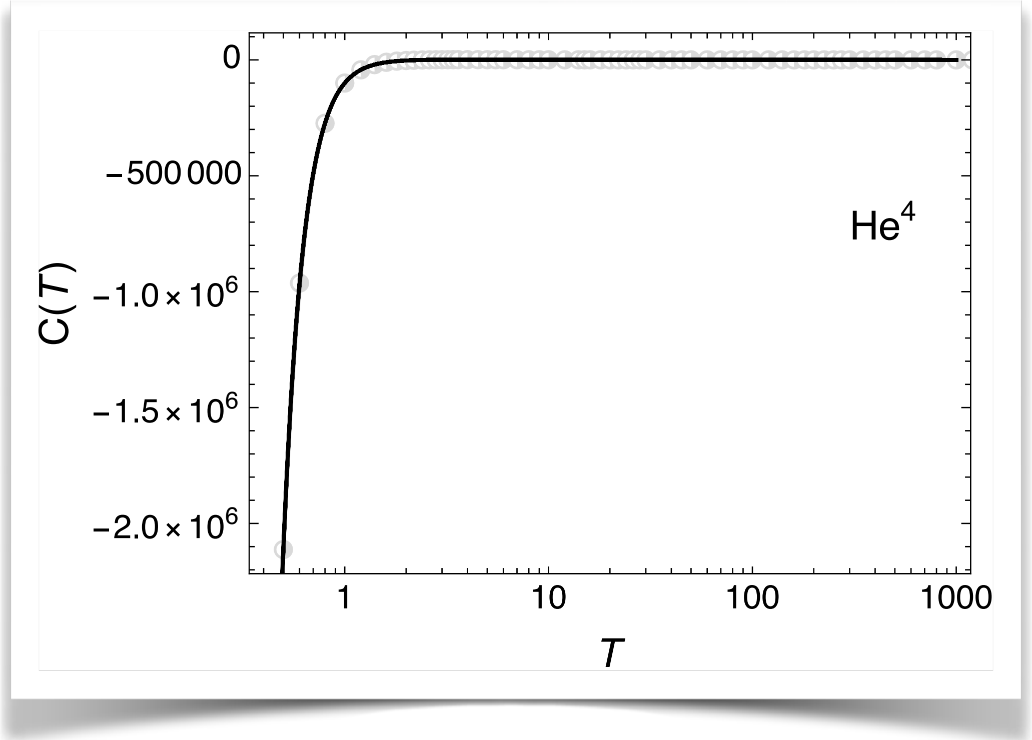

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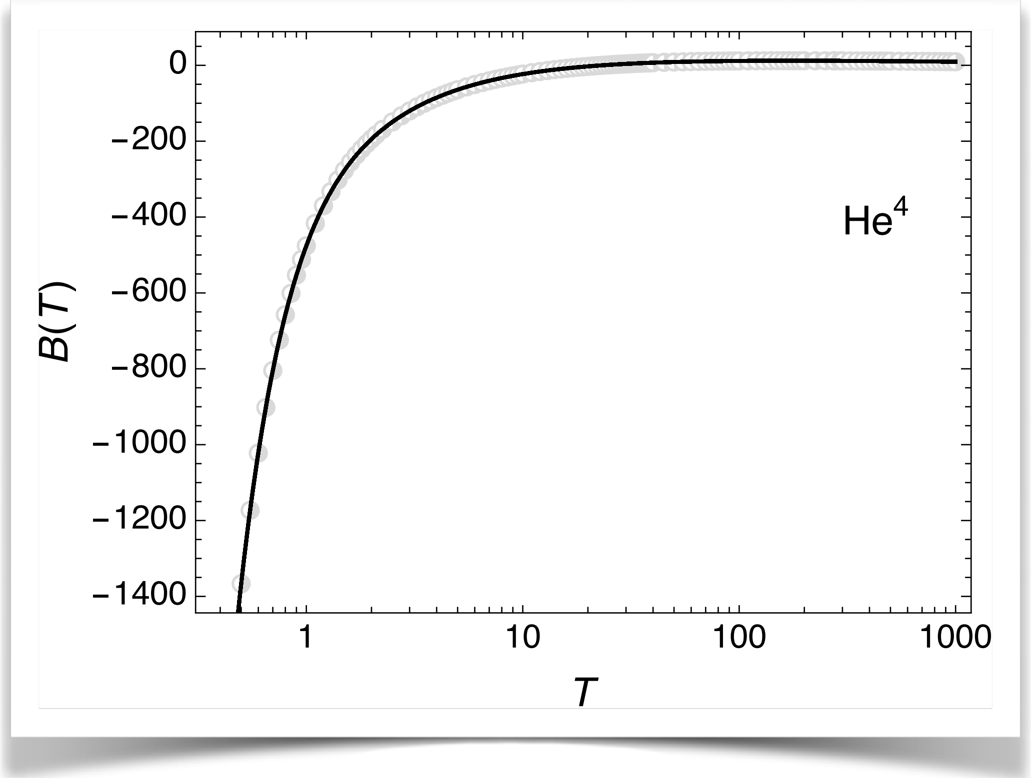

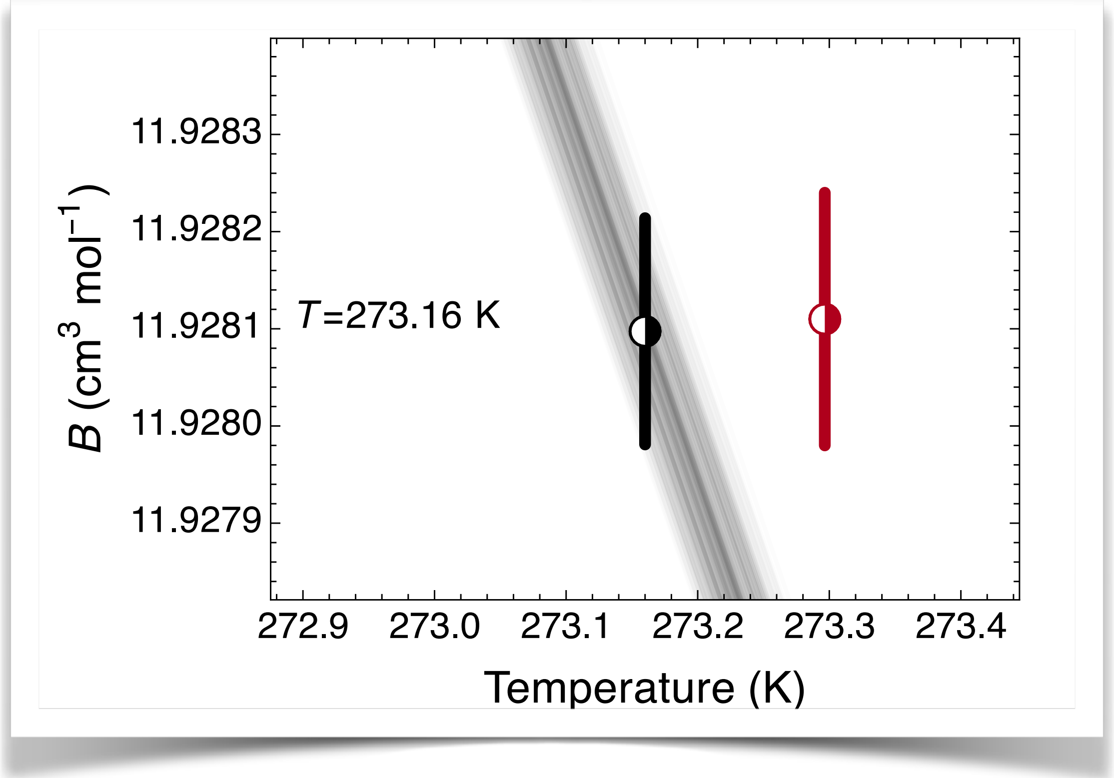

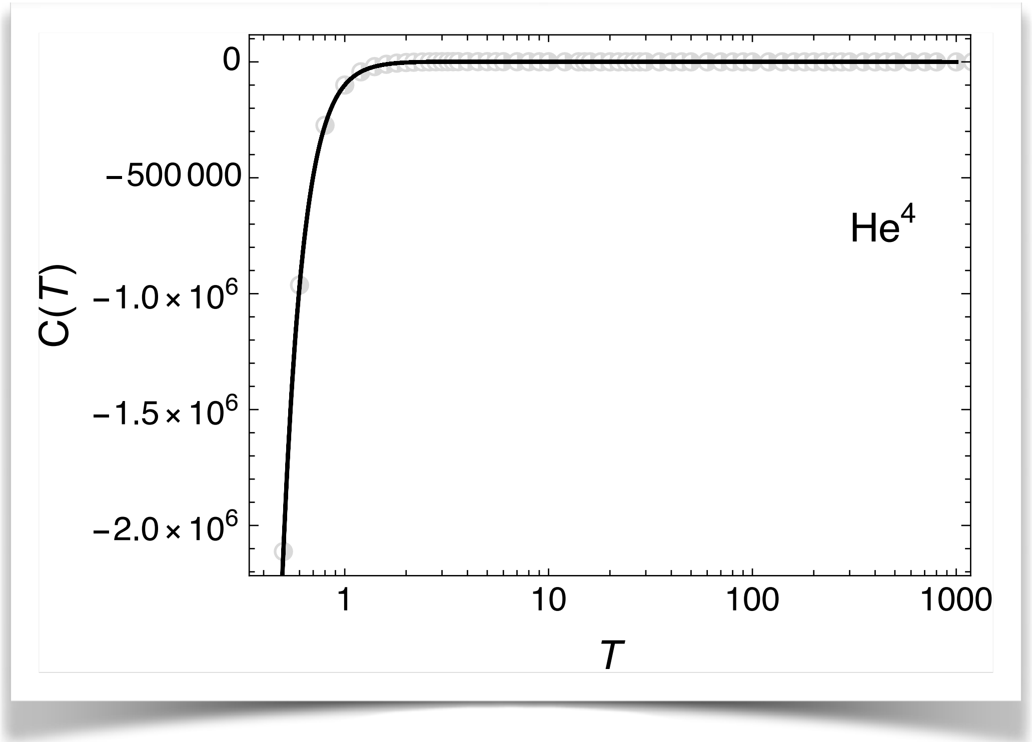

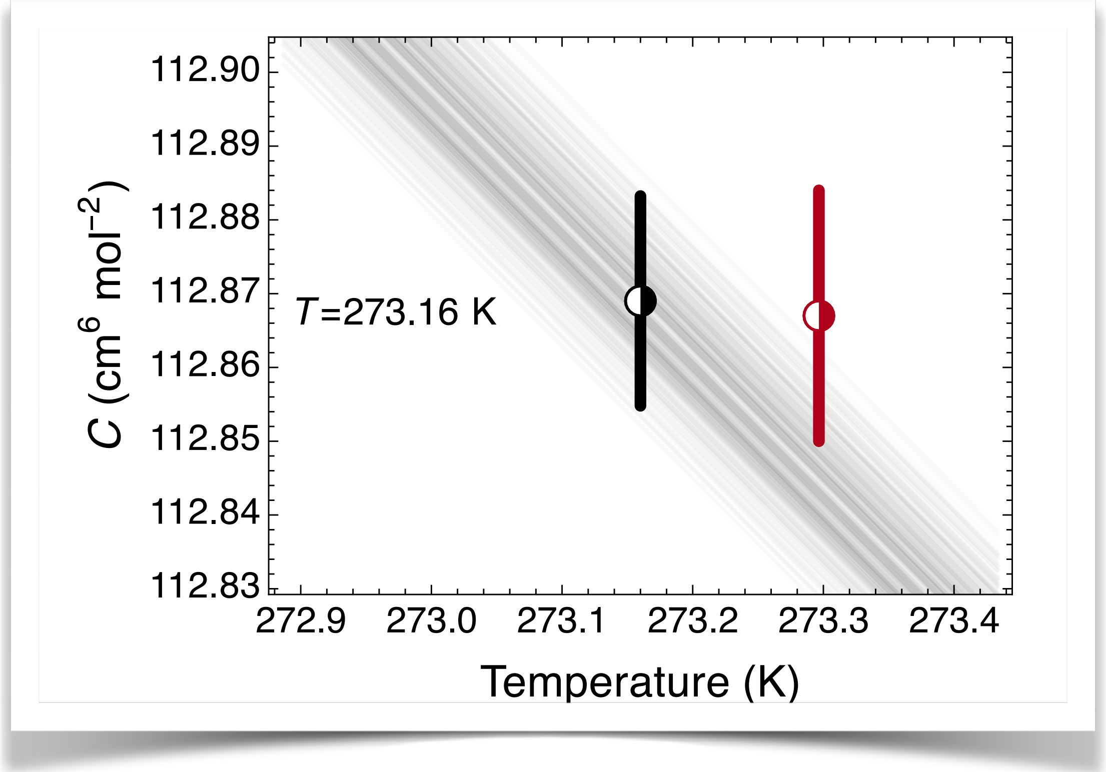

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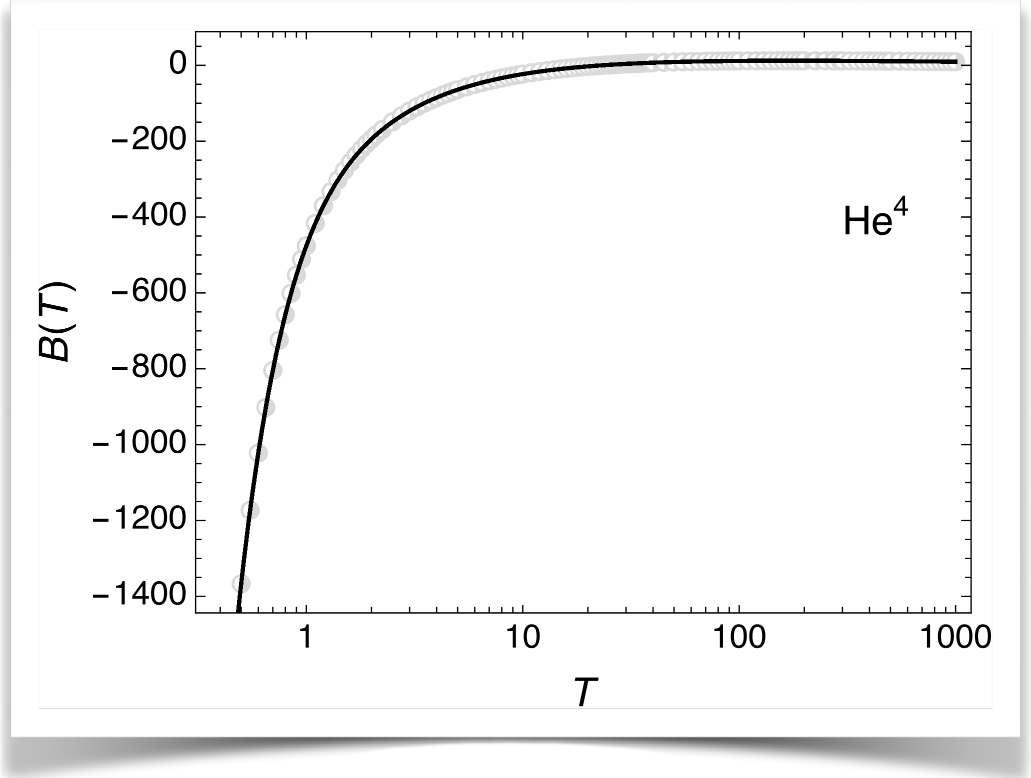

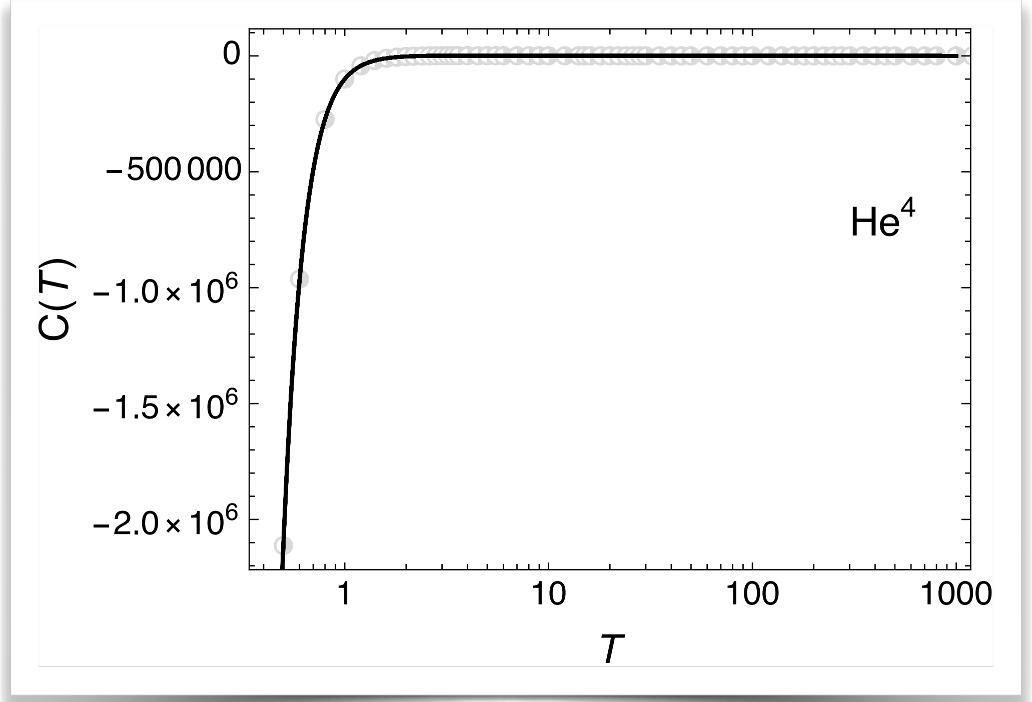

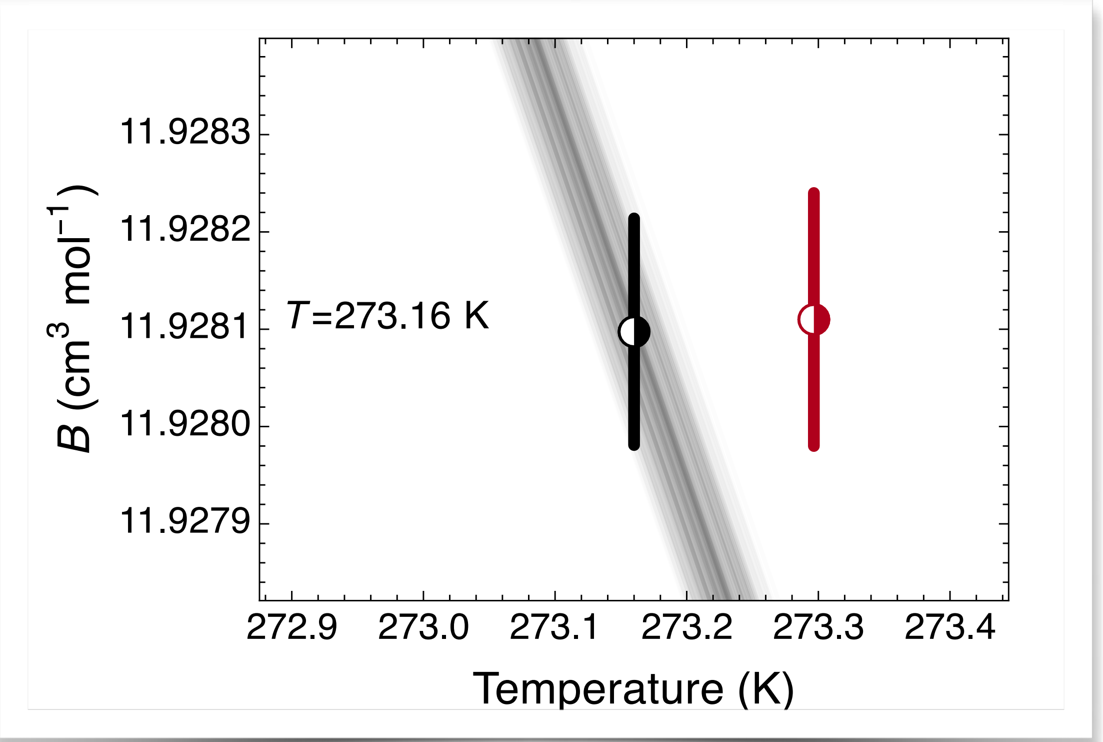

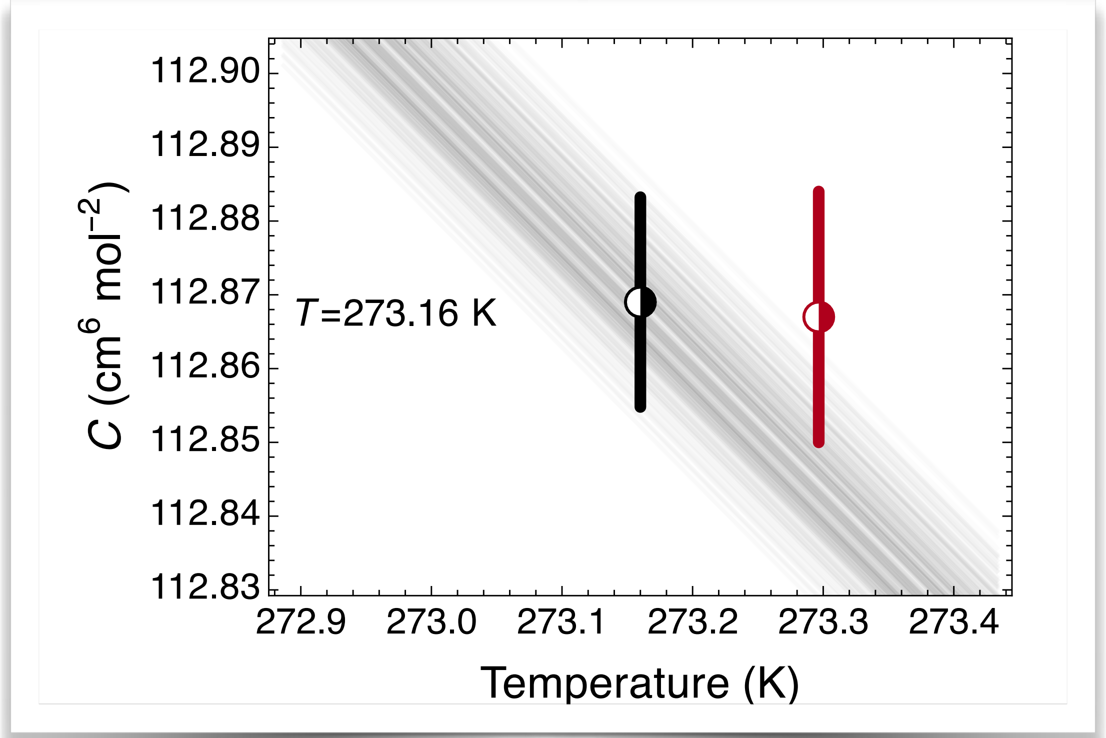

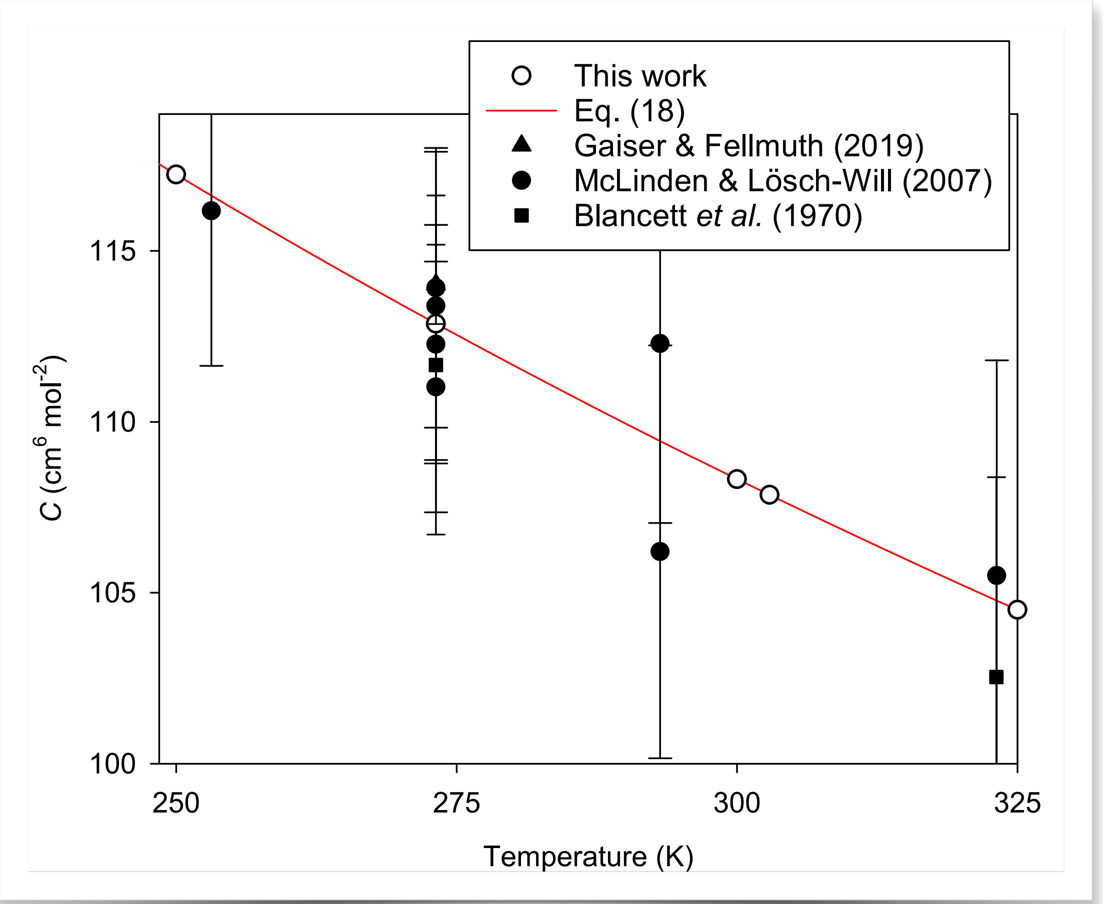

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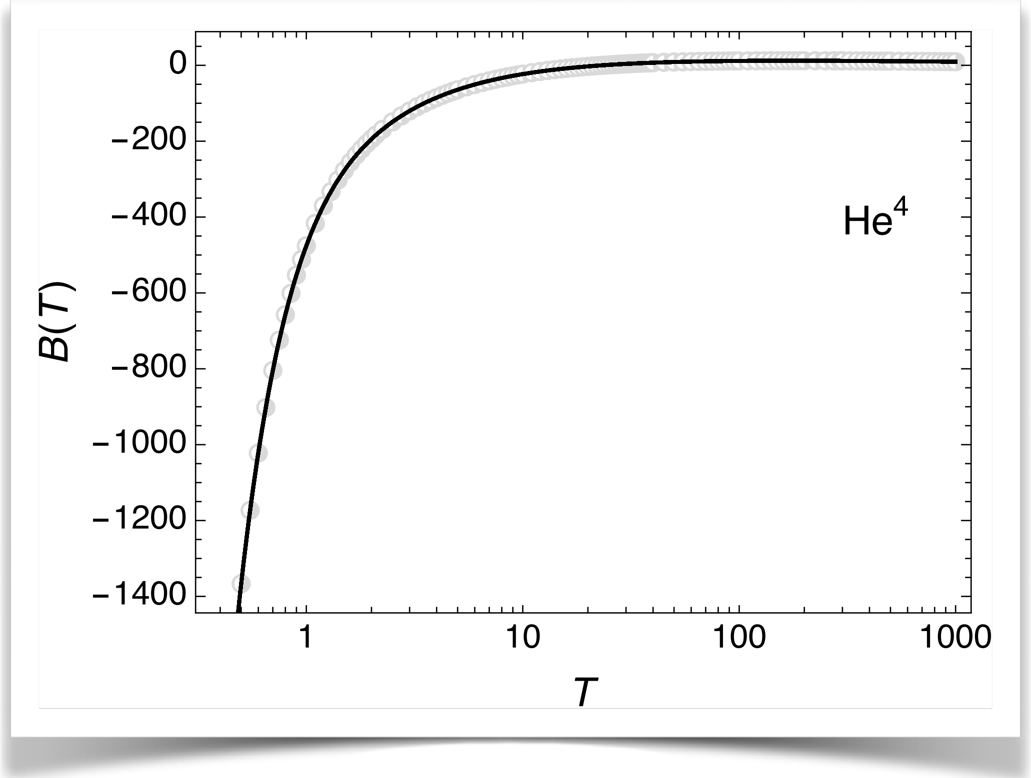
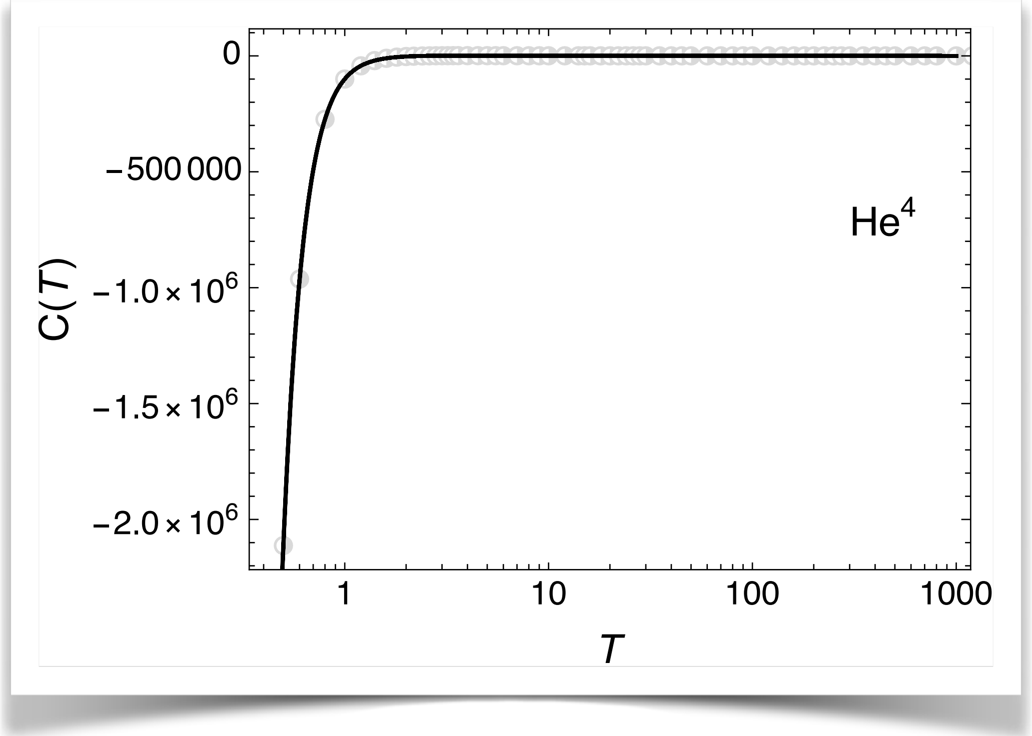
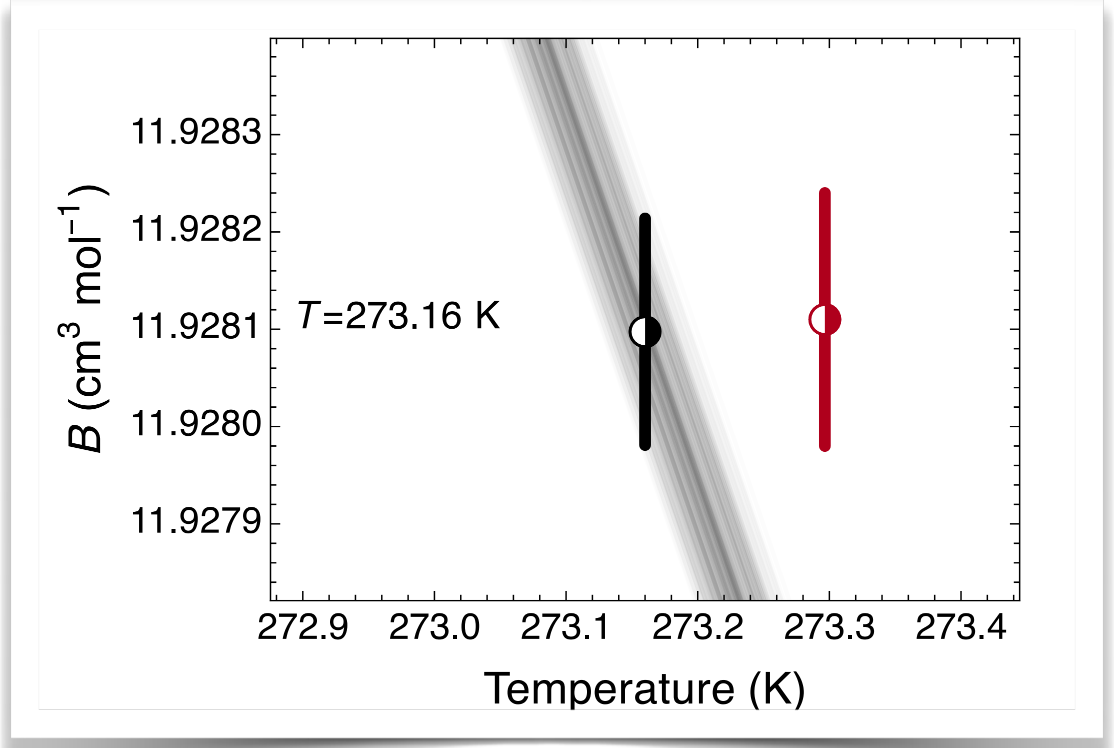
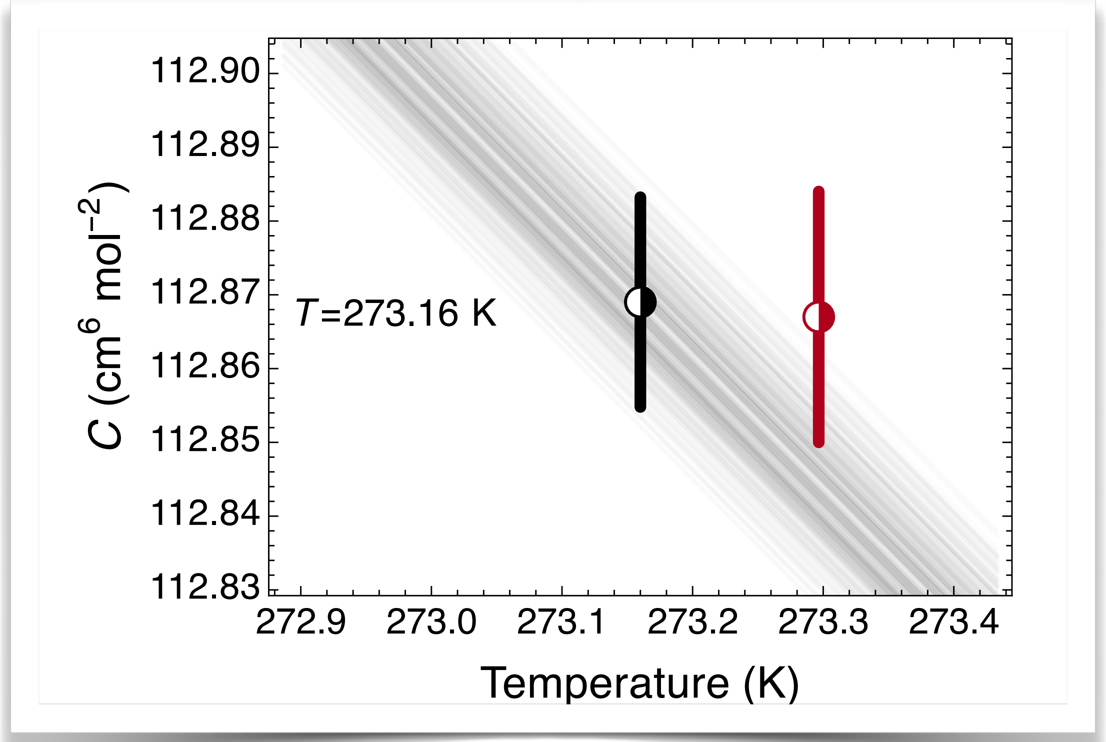
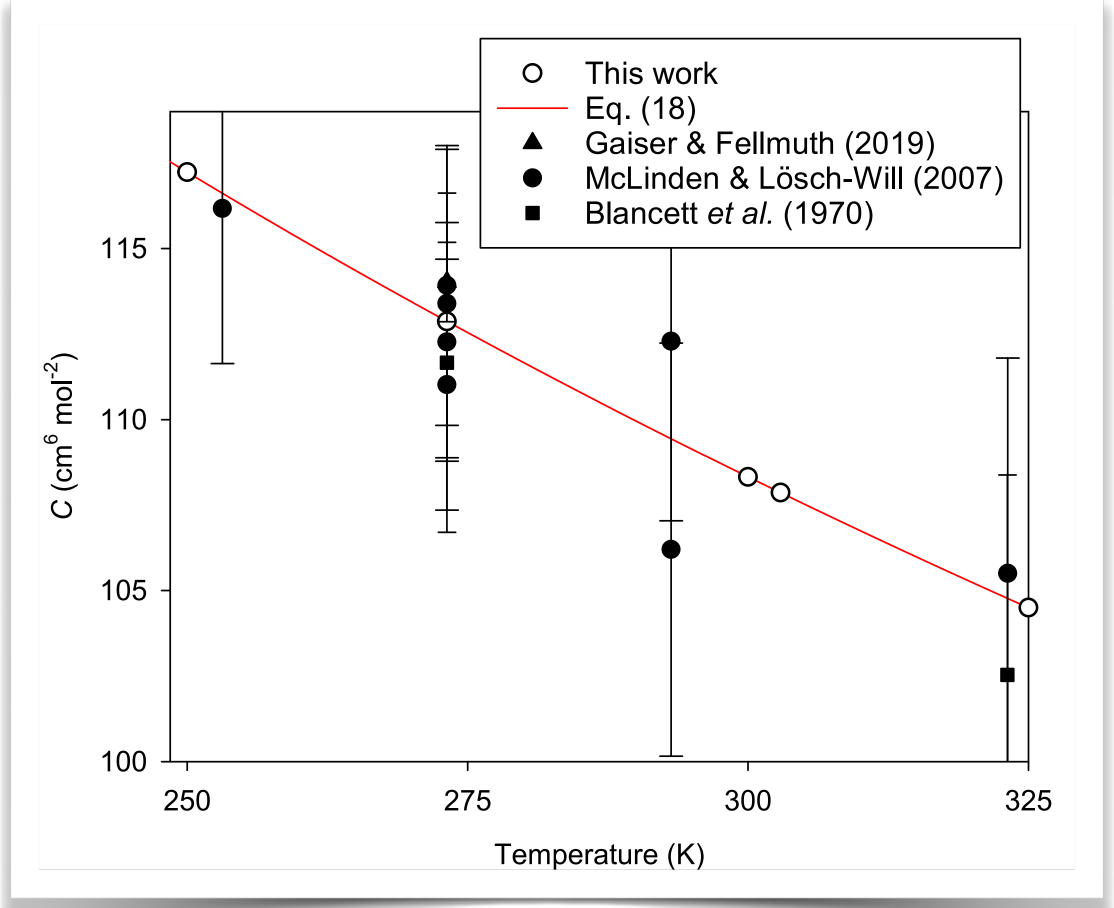






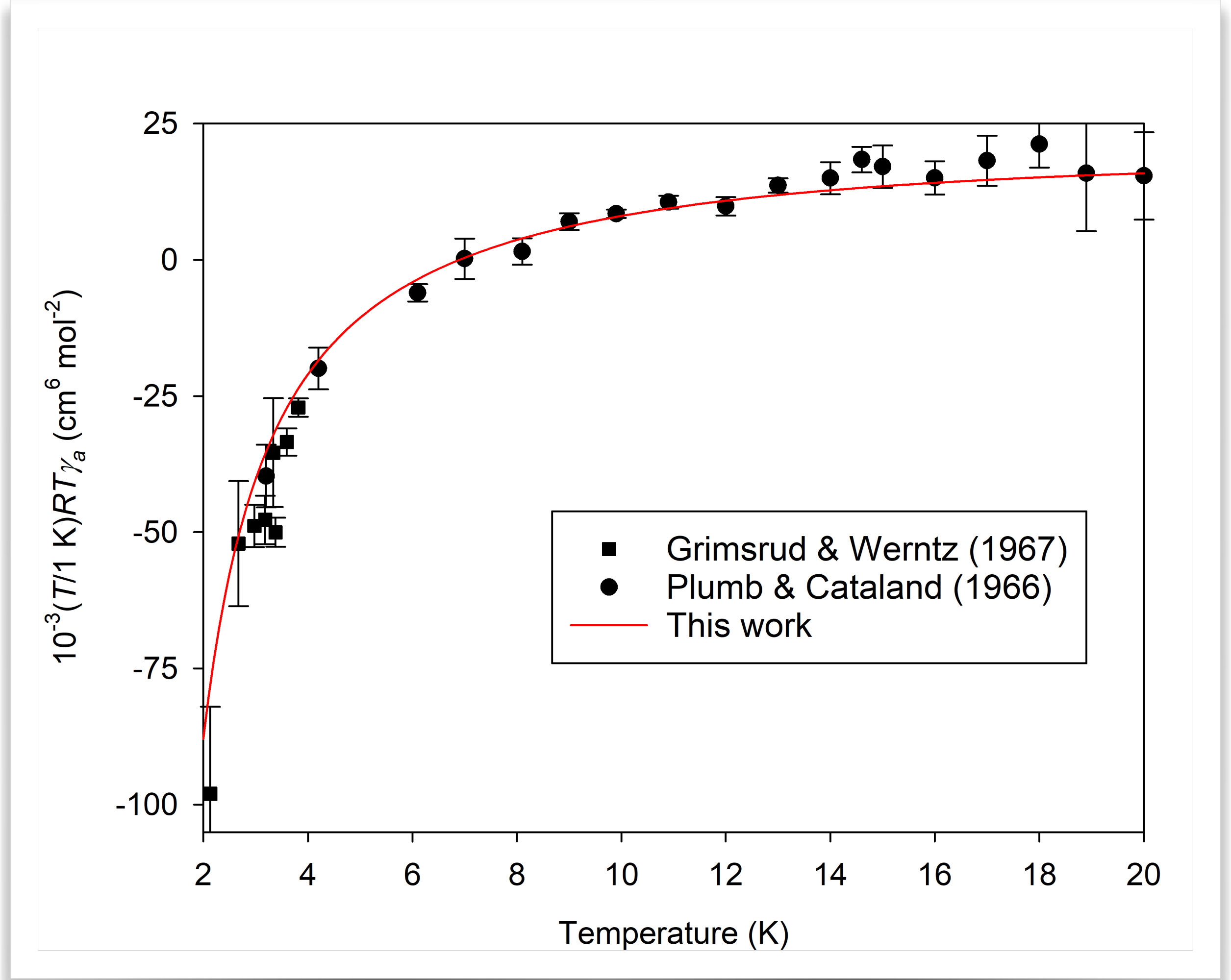


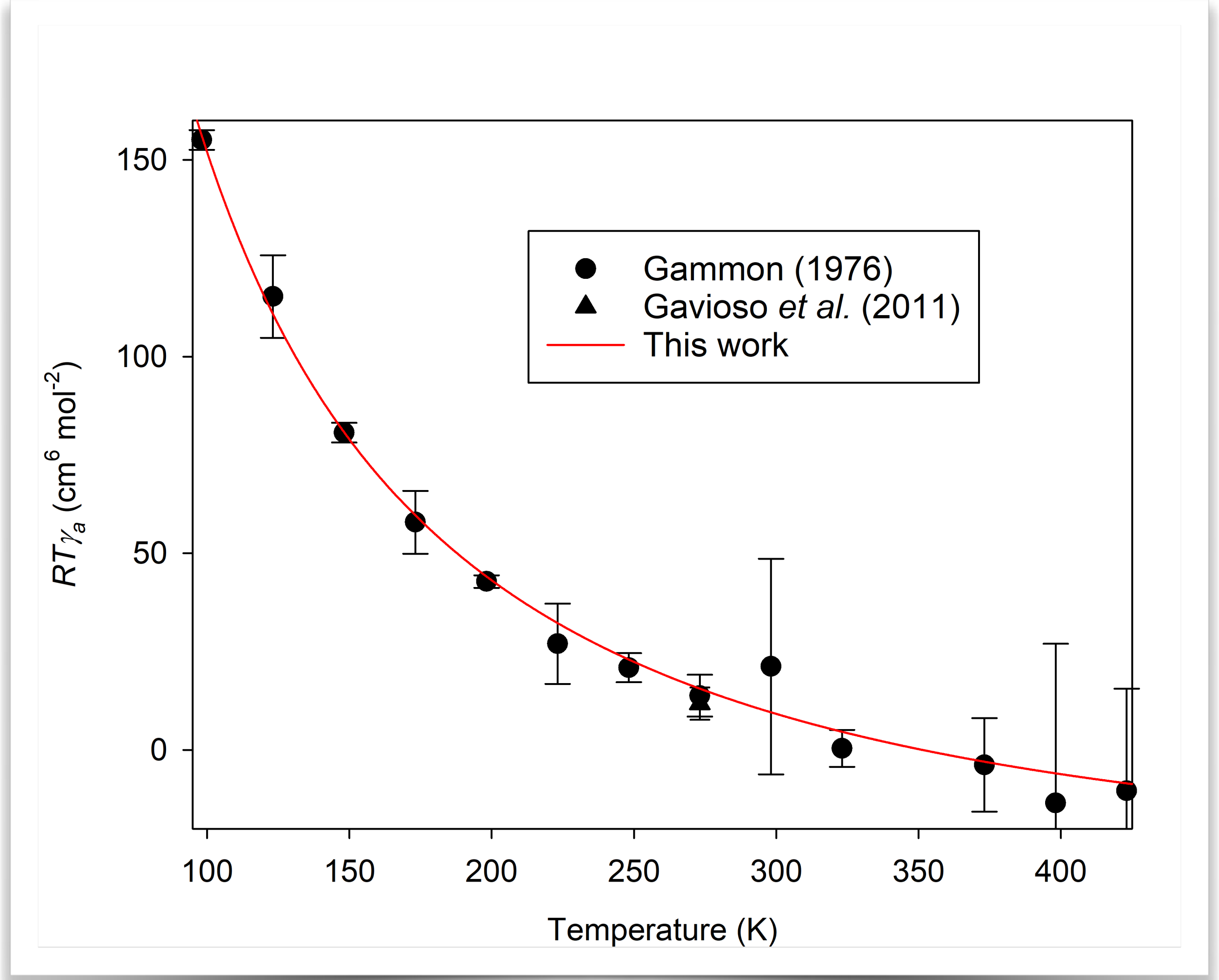


$$\beta_a(T) = 2B + 2(\gamma_0 - 1)T\frac{\mathrm{d}B}{\mathrm{d}T} + \frac{(\gamma_0-1)^2}{\gamma_0}T^2\frac{\mathrm{d}^2B}{\mathrm{d}T^2}$$

$$Q = B + (2\gamma_0 - 1)T\frac{\mathrm{d}B}{\mathrm{d}T} + (\gamma_0 - 1)T^2\frac{\mathrm{d}^2B}{\mathrm{d}T^2}$$

$$RT\gamma_a = \frac{\gamma_0 - 1}{\gamma_0}Q^2 - \beta_a(T)B(T) + \frac{2\gamma_0+1}{\gamma_0}C + \frac{\gamma_0^2-1}{\gamma_0}T\frac{\mathrm{d}C}{\mathrm{d}T} + \frac{(\gamma_0-1)^2}{2\gamma_0}T^2\frac{\mathrm{d}^2C}{\mathrm{d}T^2}$$





THE PROTON RADIUS PUZZLE

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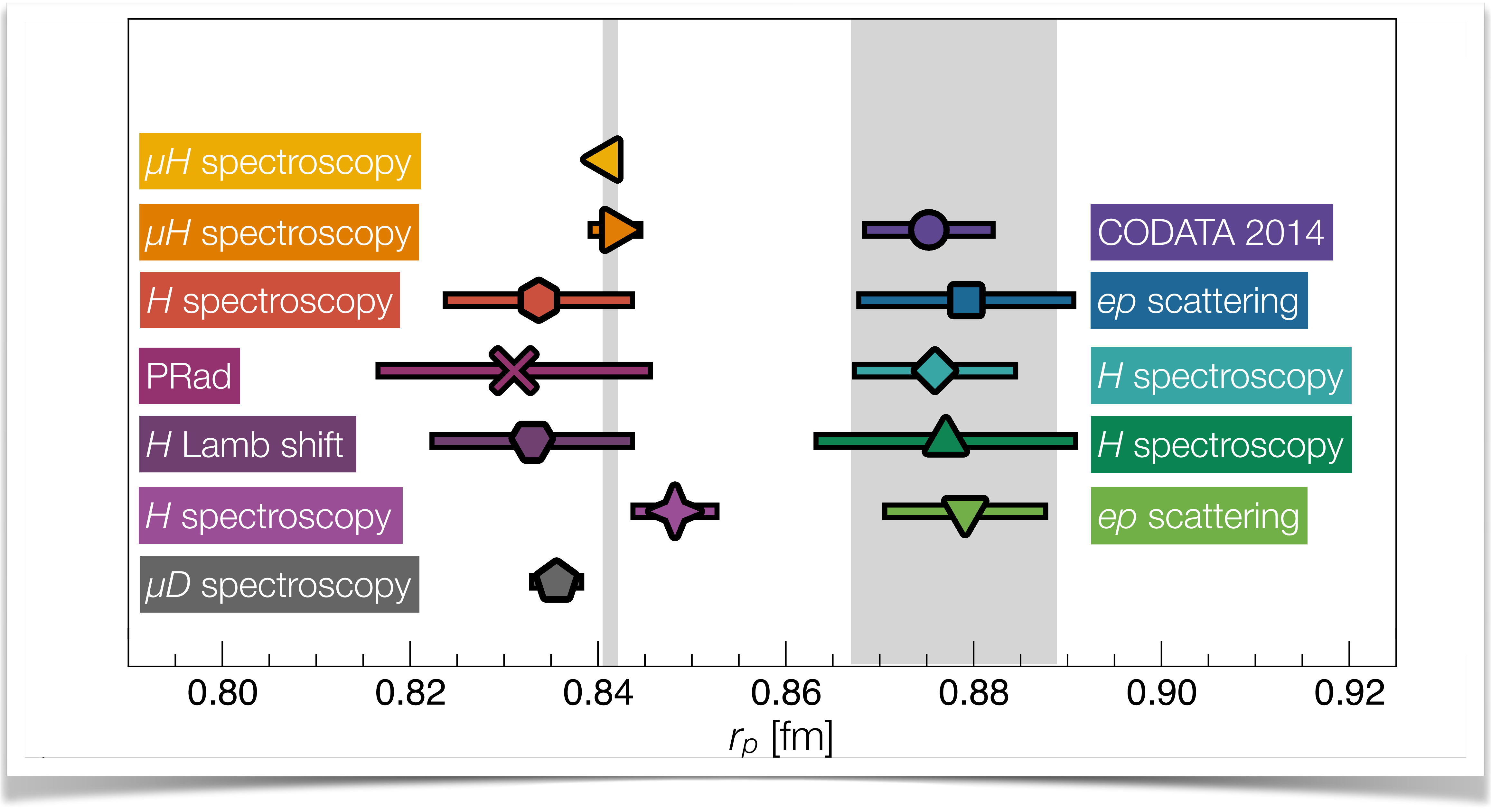

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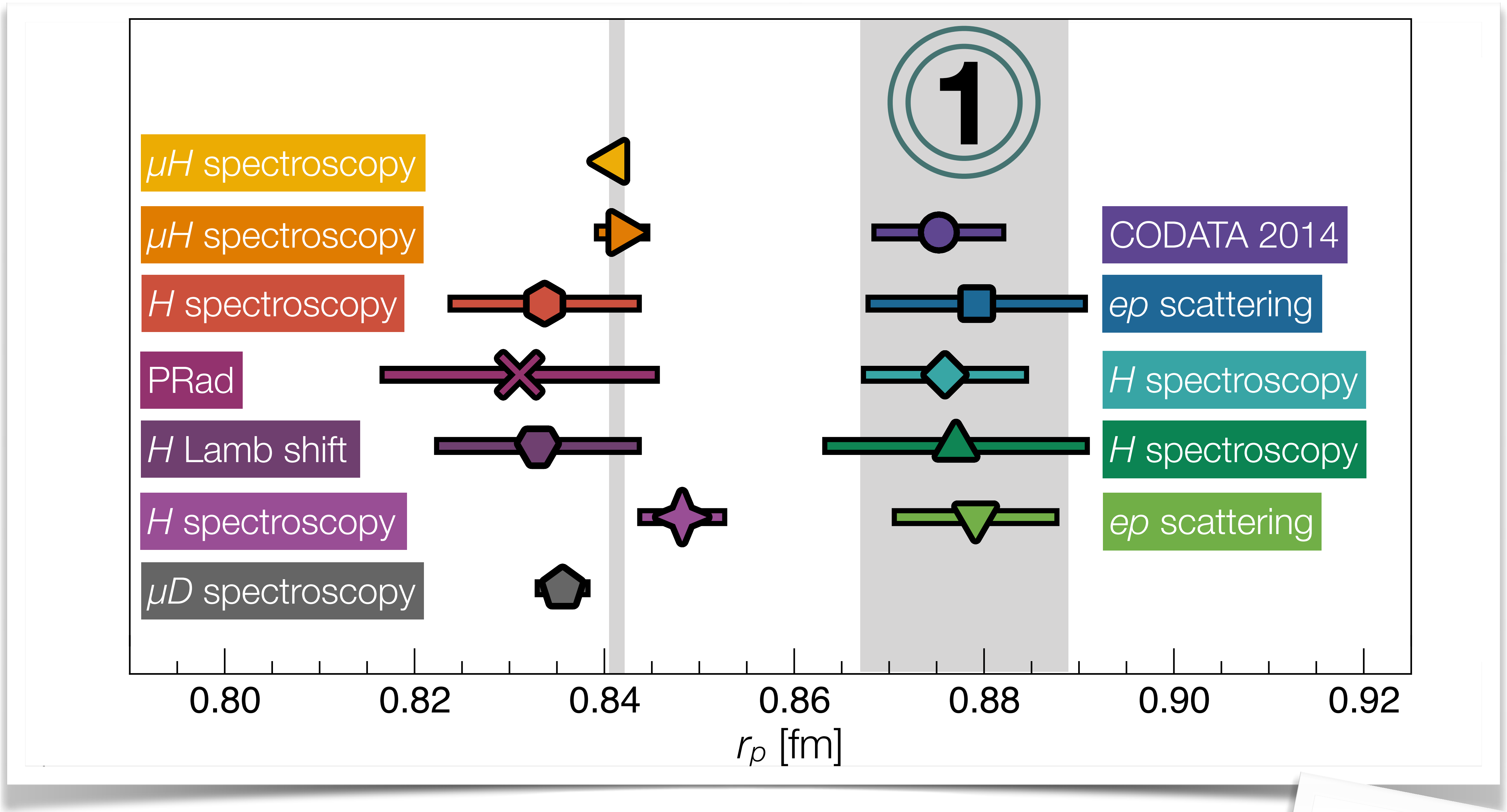

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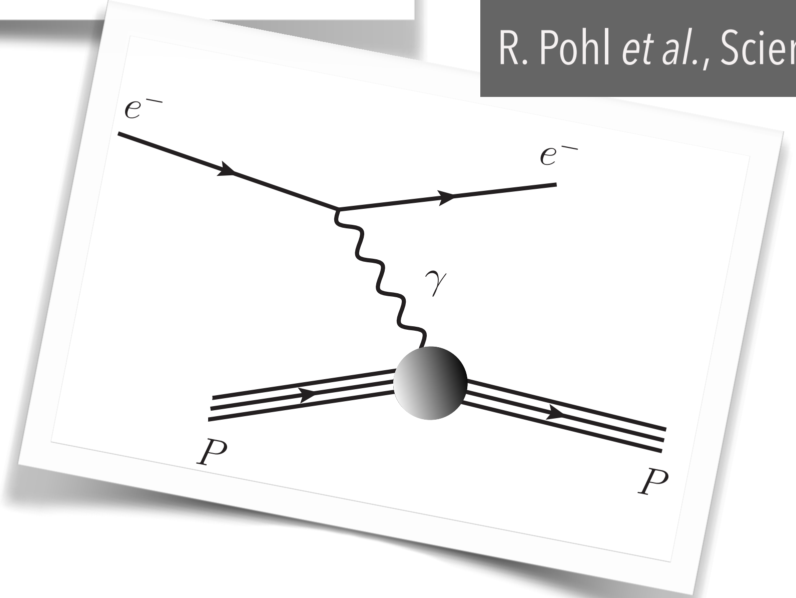

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point-like probe, QED only

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electric and magnetic form factor encode the shape of the proton

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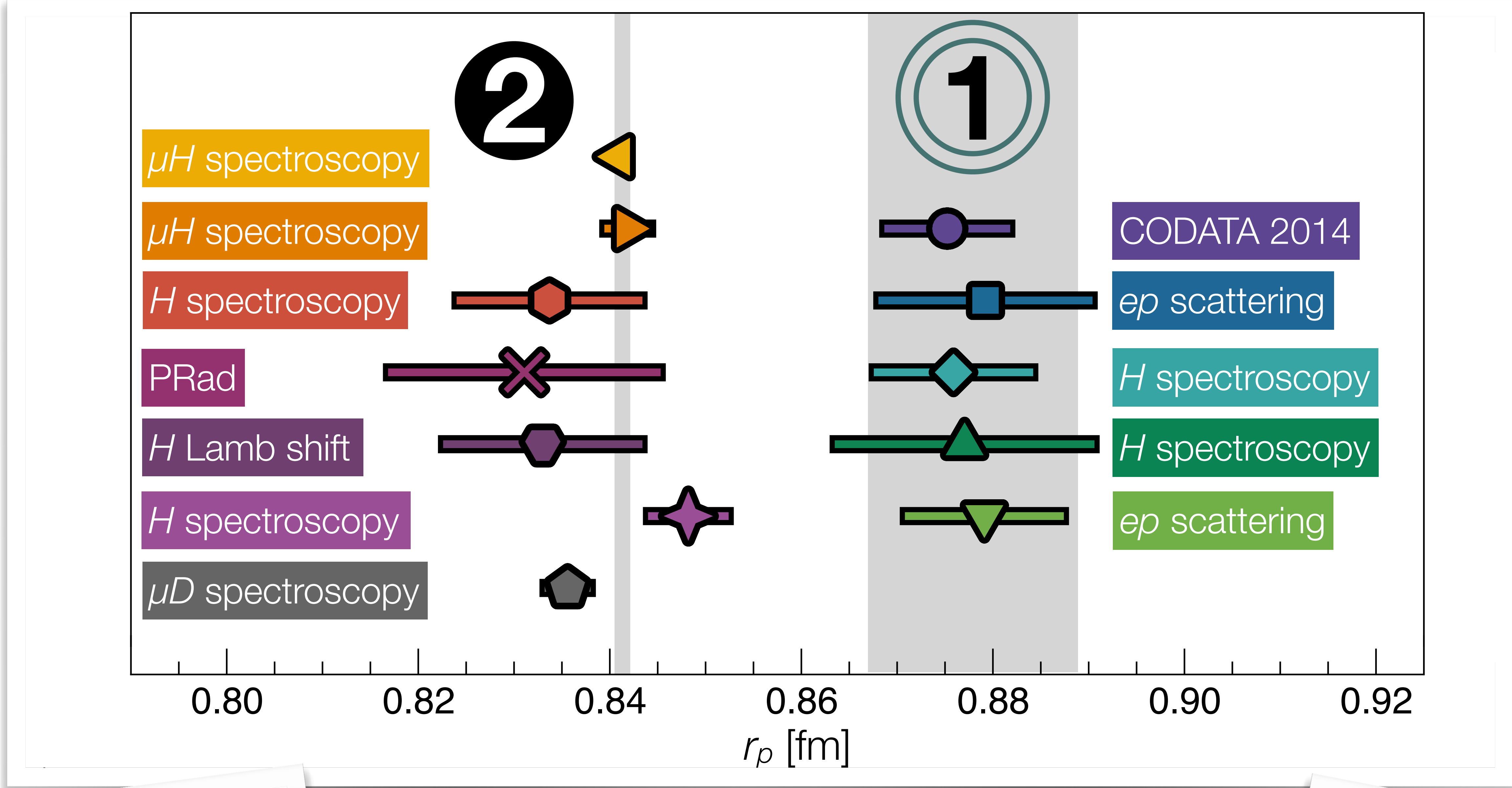

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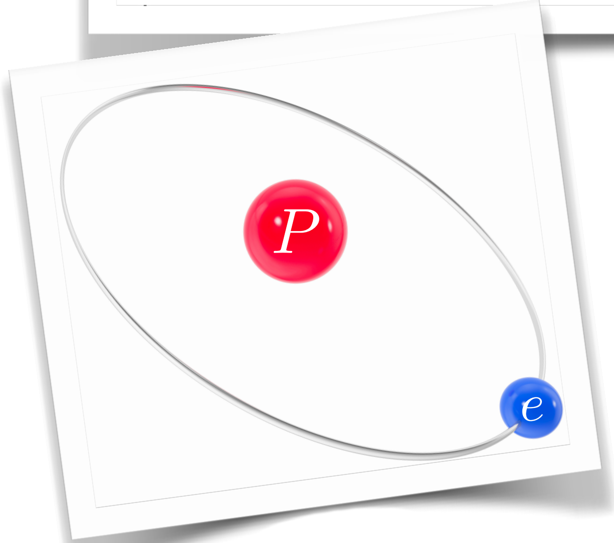
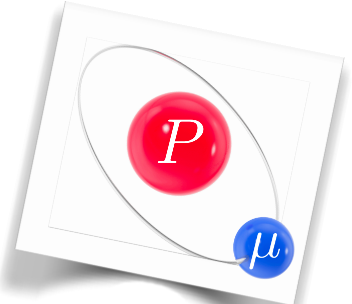
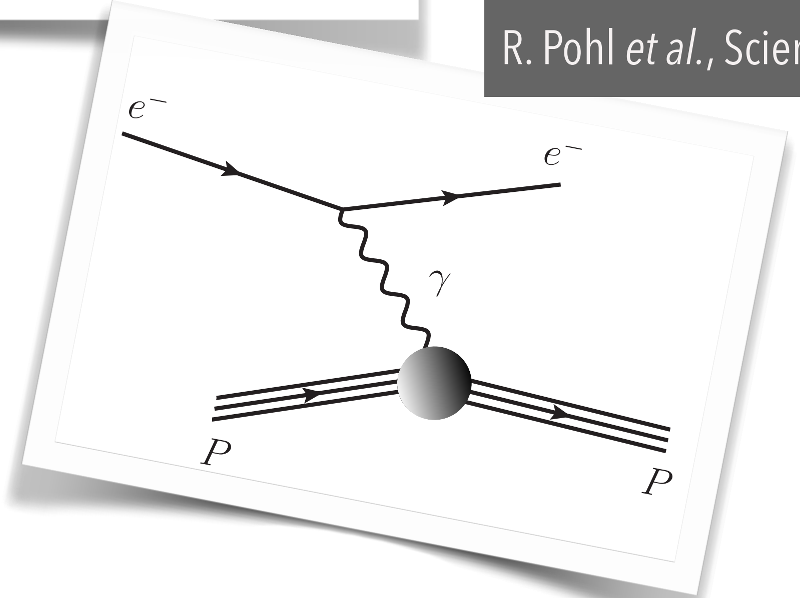
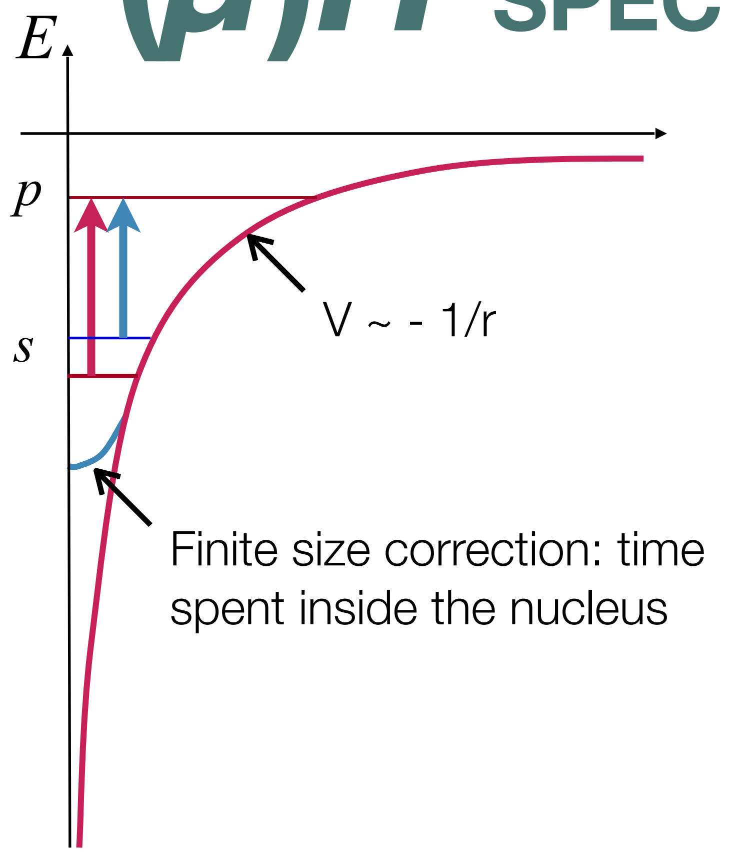






(μ)H SPECTROSCOPY ❷



probability of lepton inside proton

$\sim (r_p \alpha)^3 m^3$ $m \simeq m_e$

muon ~ 200 heavier than electron
~10^7 more sensitive to r_p

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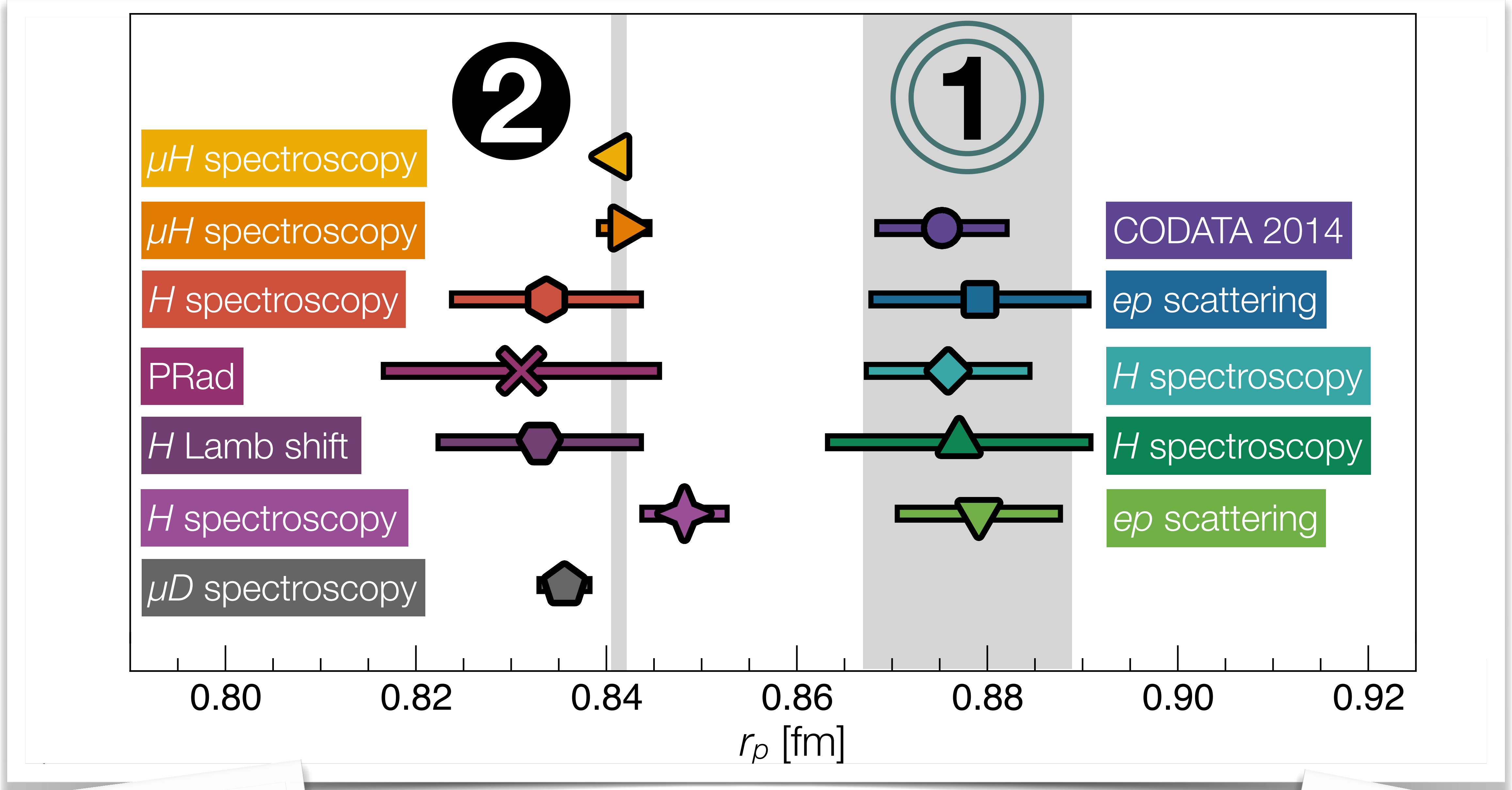

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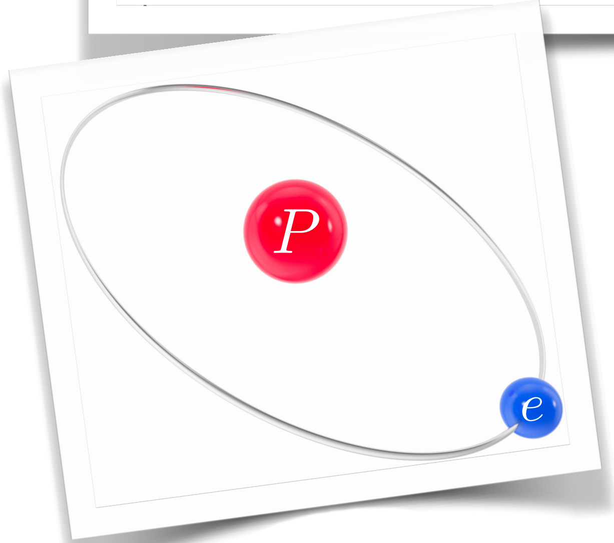

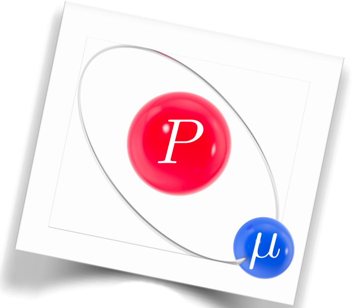

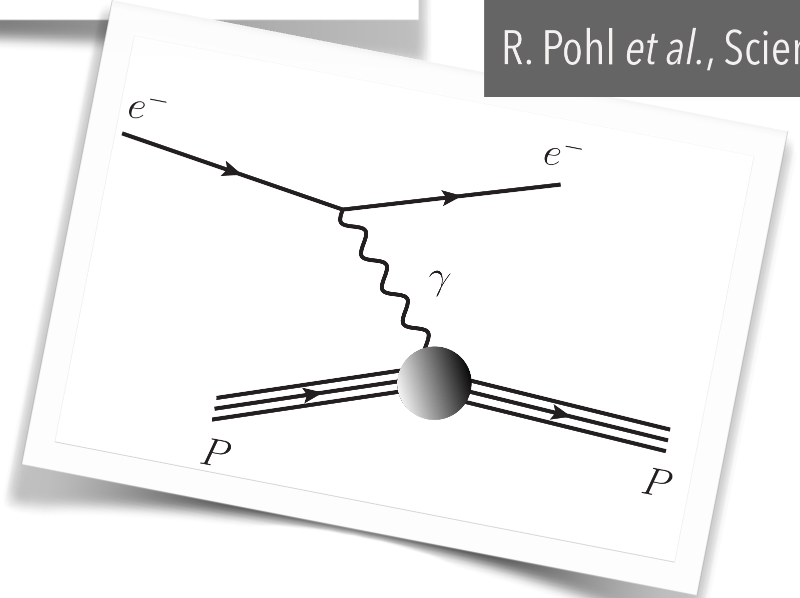

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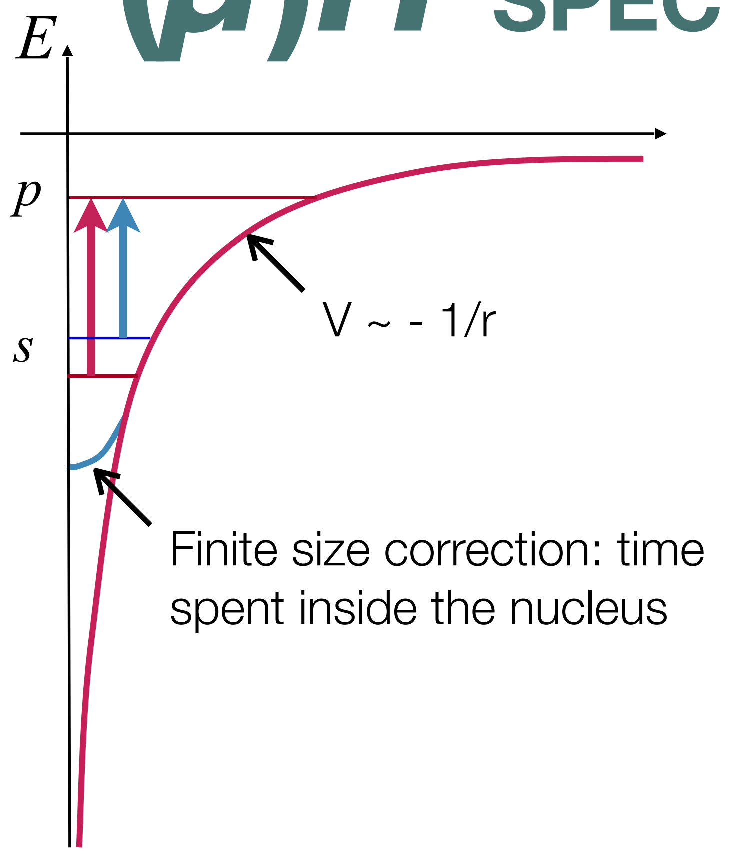

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IMPRECISE DATA

direct interpolation does not work
requires **smoothing** with **roughness penalty**:
seek $g \in \mathbb{S}$ minimising

$$\mathsf{P}(g, \lambda) = \lambda \sum_{i=1}^{\ell} [y_i - g(x_i)]^2 + (1 - \lambda) \int_a^b \mathrm{d}x \, [g''(x)]^2$$

smoothing par. | data fidelity | roughness penalty

THEOREM: g is the *natural spline* interpolant of nodes $\{x_i\}$

optimal smoothing parameter determined via generalised cross validation

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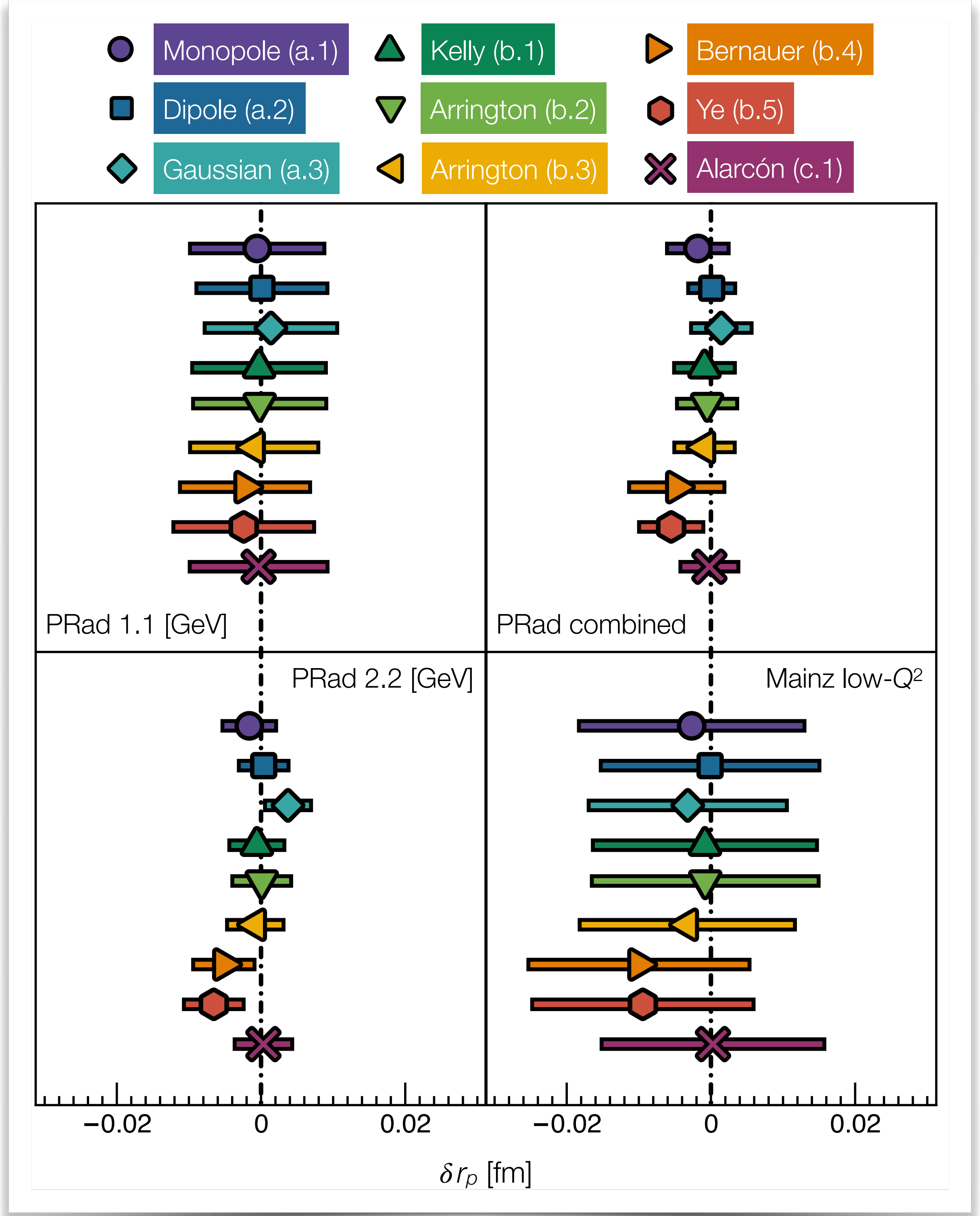
ROBUSTNESS



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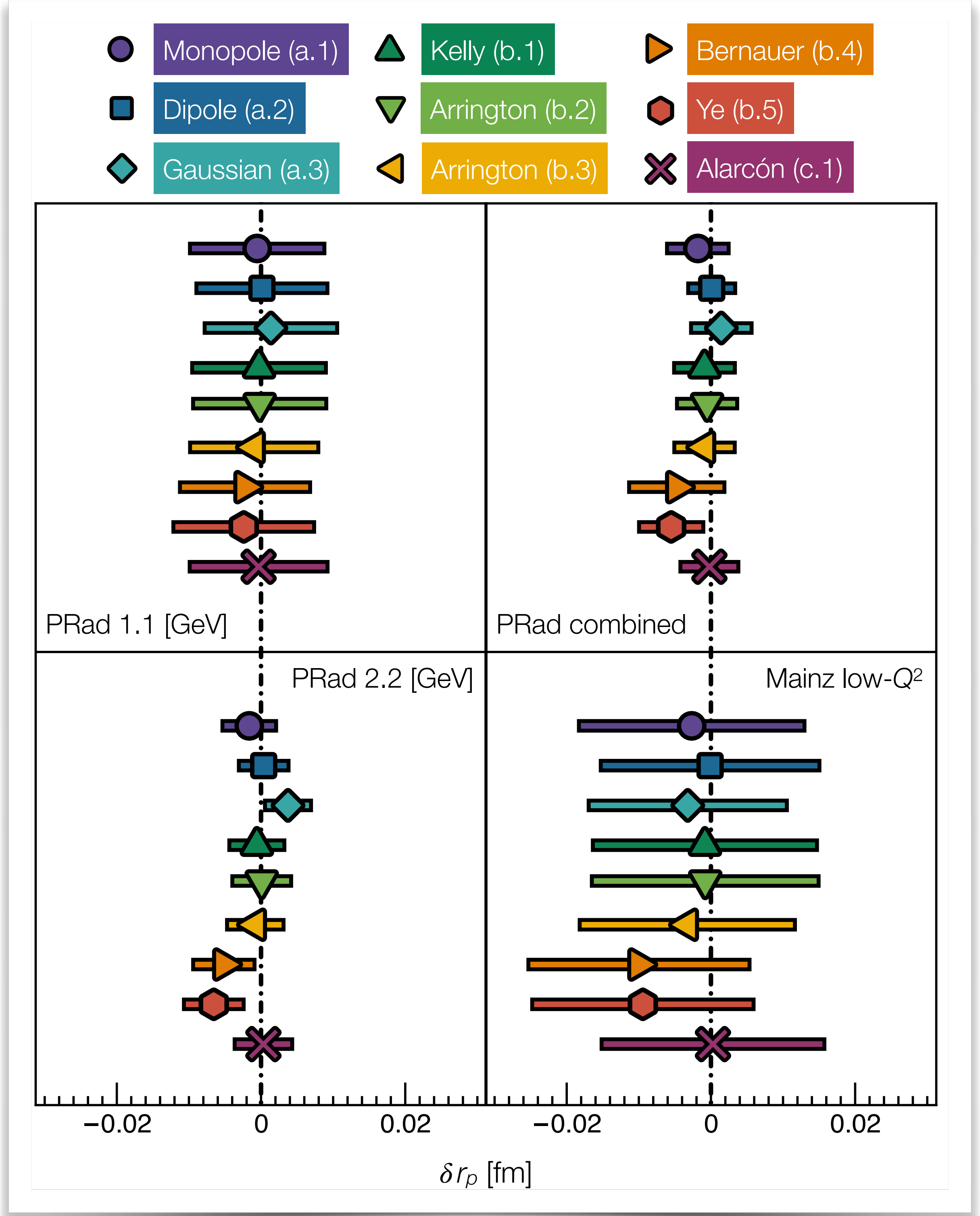

① PRad DATA

lowest yet achieved momentum transferred

$$2.1 \times 10^{-4} \leq Q^2/[\mathrm{GeV}^2] \leq 6 \times 10^{-2}$$

two datasets at different energy beams 1.1 and 2.2 [GeV]

$$r_p^{1.1} = 0.842 \pm 0.008_{\mathrm{stat}}\ [\mathrm{fm}]$$
$$r_p^{2.2} = 0.824 \pm 0.003_{\mathrm{stat}}\ [\mathrm{fm}]$$

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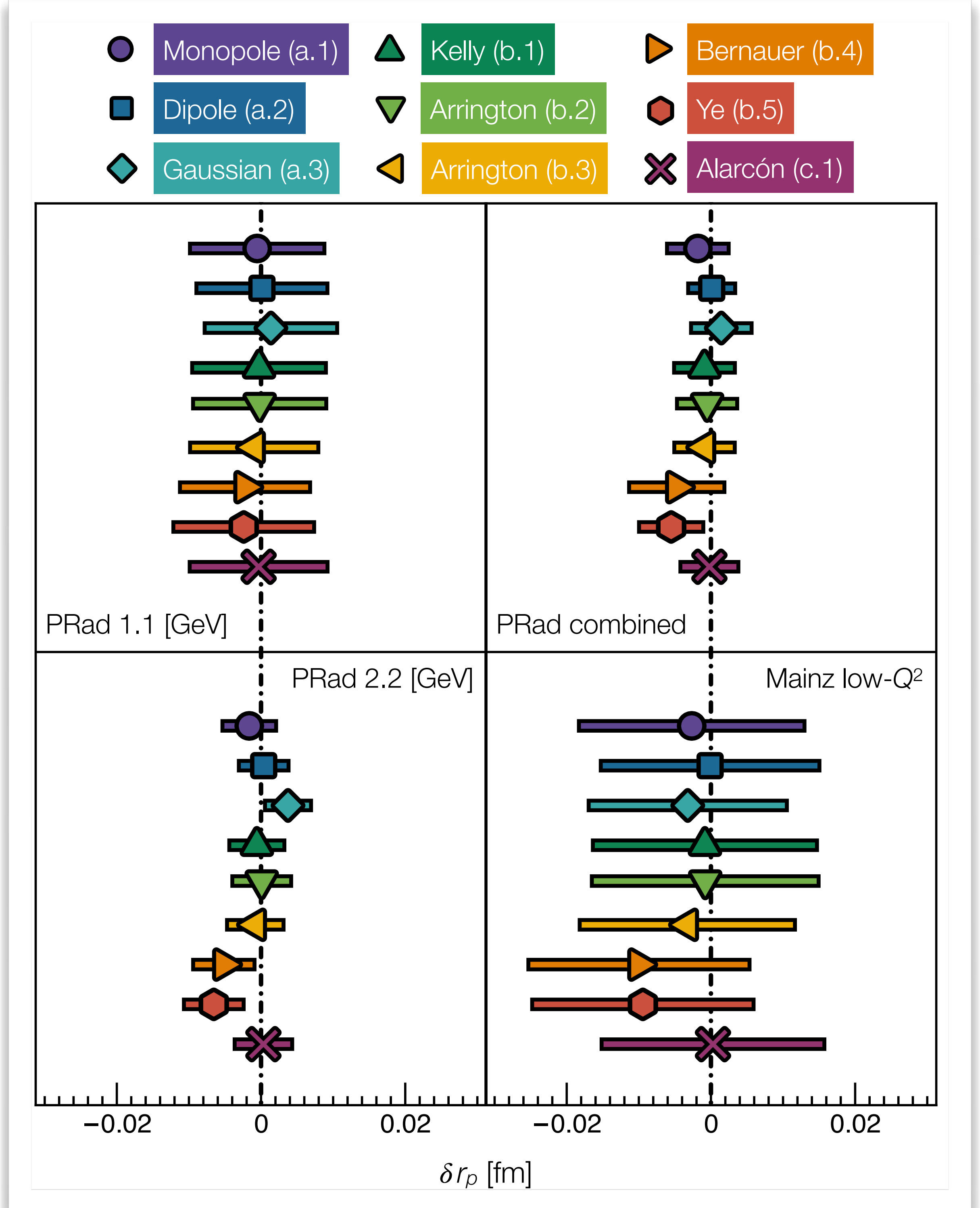
ROBUSTNESS



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② A1 DATA

extends toward low-Q^2

$$3.8 \times 10^{-3} \leq Q^2/[\mathrm{GeV}^2] \leq 1$$

use first 40 low-Q^2 data

$$r_p^{\mathrm{A1-low}Q^2} = 0.856 \pm 0.014_{\mathrm{stat}}\ [\mathrm{fm}]$$

all data yield

$$r_p^{\mathrm{A1}} = 0.857 \pm 0.021_{\mathrm{stat}}\ [\mathrm{fm}]$$

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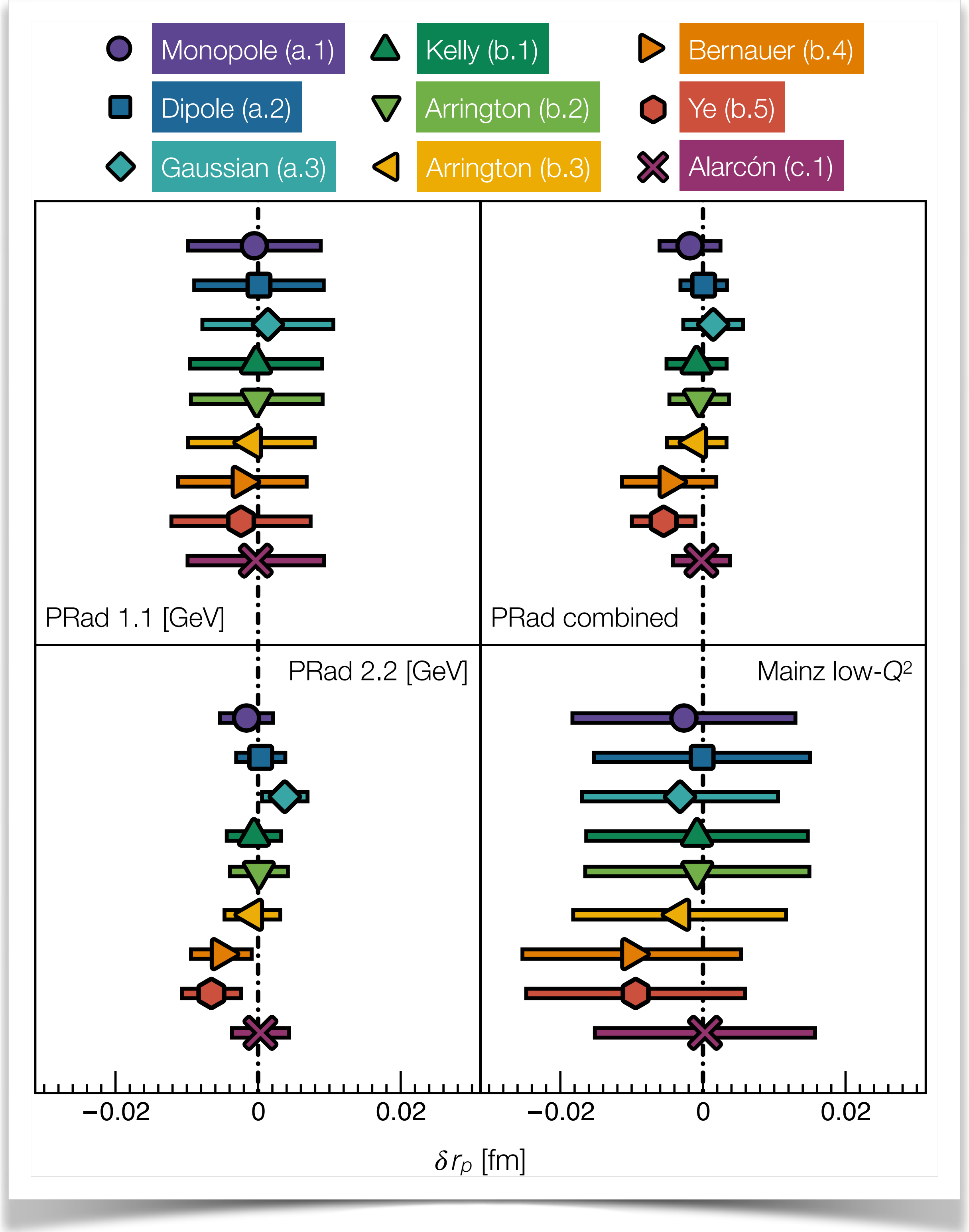
optimal smoothing parameter determined via generalised cross validation

ROBUSTNESS



① PRad DATA

lowest yet achieved momentum transferred

$$2.1 \times 10^{-4} \leq Q^2/[\mathrm{GeV}^2] \leq 6 \times 10^{-2}$$

two datasets at different energy beams 1.1 and 2.2 [GeV]

$$r_p^{1.1} = 0.842 \pm 0.008_{\mathrm{stat}}\ [\mathrm{fm}]$$
$$r_p^{2.2} = 0.824 \pm 0.003_{\mathrm{stat}}\ [\mathrm{fm}]$$

$$r_p^{\mathrm{PRad}} = 0.838 \pm 0.005_{\mathrm{stat}}\ [\mathrm{fm}]$$

② A1 DATA

extends toward low-Q^2

$$3.8 \times 10^{-3} \leq Q^2/[\mathrm{GeV}^2] \leq 1$$

use first 40 low-Q^2 data

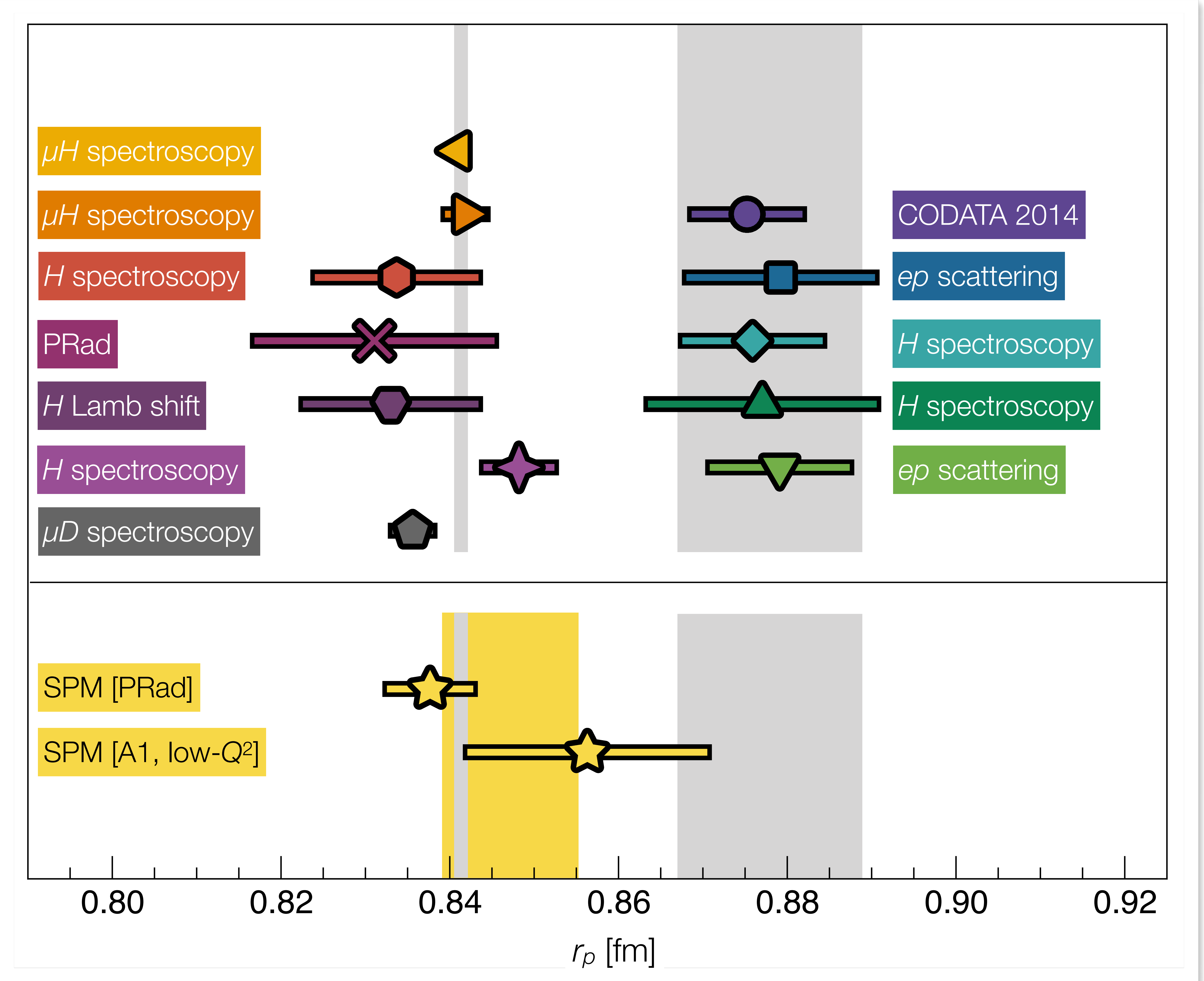
$$r_p^{\mathrm{A1-low}Q^2} = 0.856 \pm 0.014_{\mathrm{stat}}\ [\mathrm{fm}]$$

all data yield

$$r_p^{\mathrm{A1}} = 0.857 \pm 0.021_{\mathrm{stat}}\ [\mathrm{fm}]$$

③ Combined:

$$r_p^{\mathrm{SPM}} = 0.847 \pm 0.008_{\mathrm{stat}}\ [\mathrm{fm}]$$



PROTON RADIUS PUZZLE?

IMPRECISE DATA

direct interpolation does not work
requires **smoothing** with **roughness penalty**:
seek $g \in \mathbb{S}$ minimising

$$\mathsf{P}(g,\lambda) = \lambda \sum_{i=1}^{\ell} [y_i - g(x_i)]^2 + (1-\lambda) \int_a^b \mathrm{d}x\, [g''(x)]^2$$

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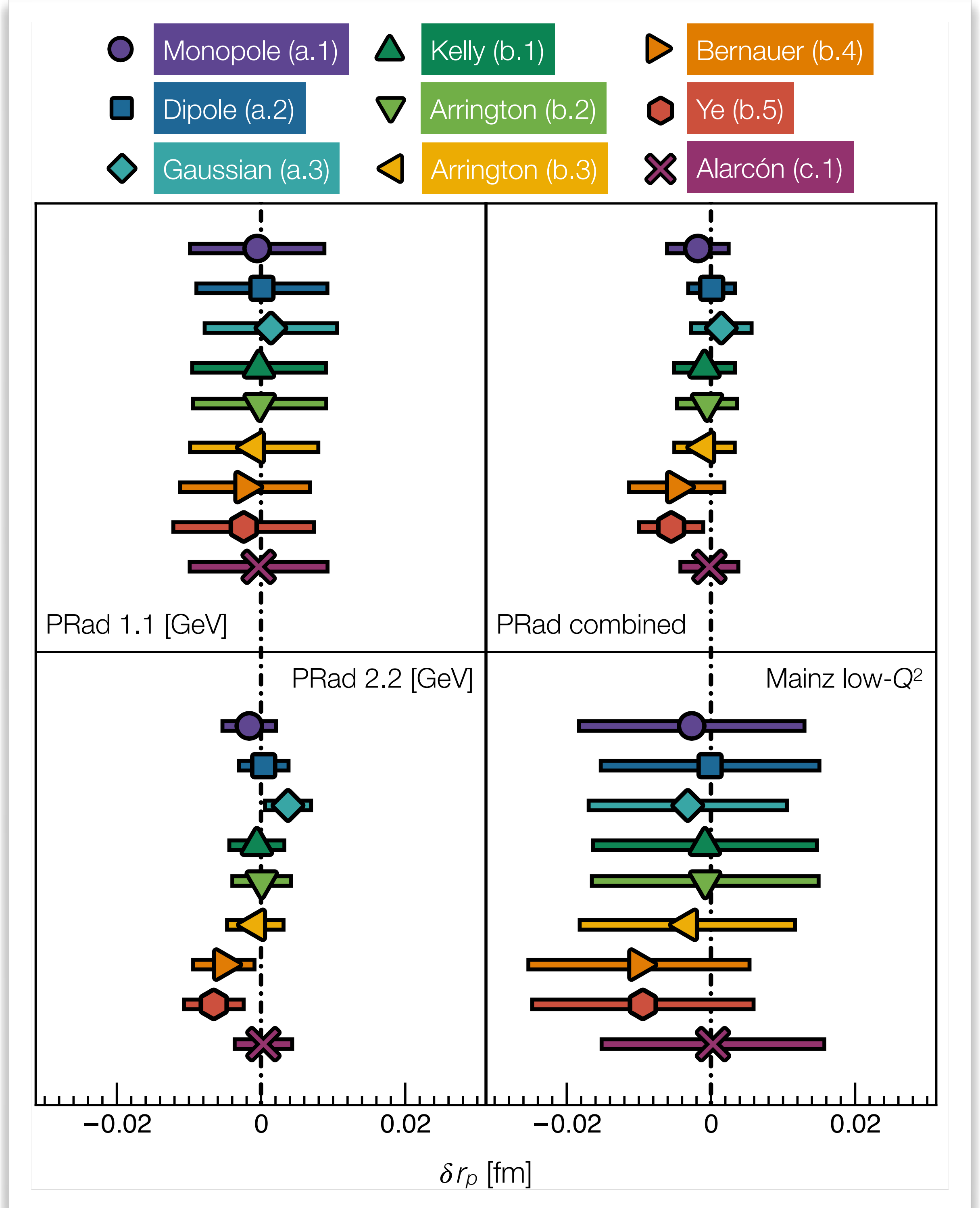

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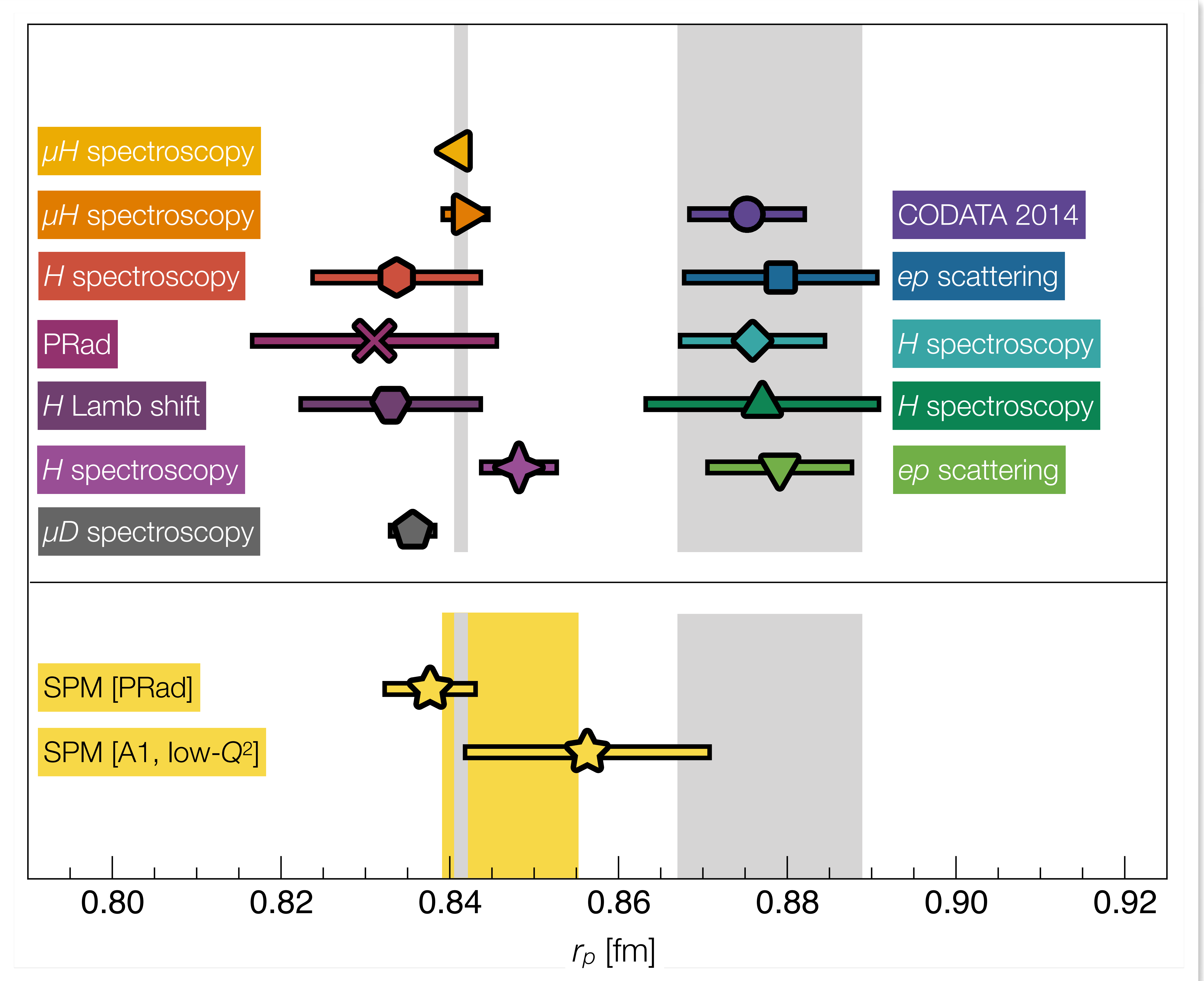

PION NA7 DATA

only data amenable
to an SPM extraction

two measurements (’84, ’86) of the negative pion em form factor

$$0.014 \leq Q^2/[\mathrm{GeV}^2] \leq 0.26$$

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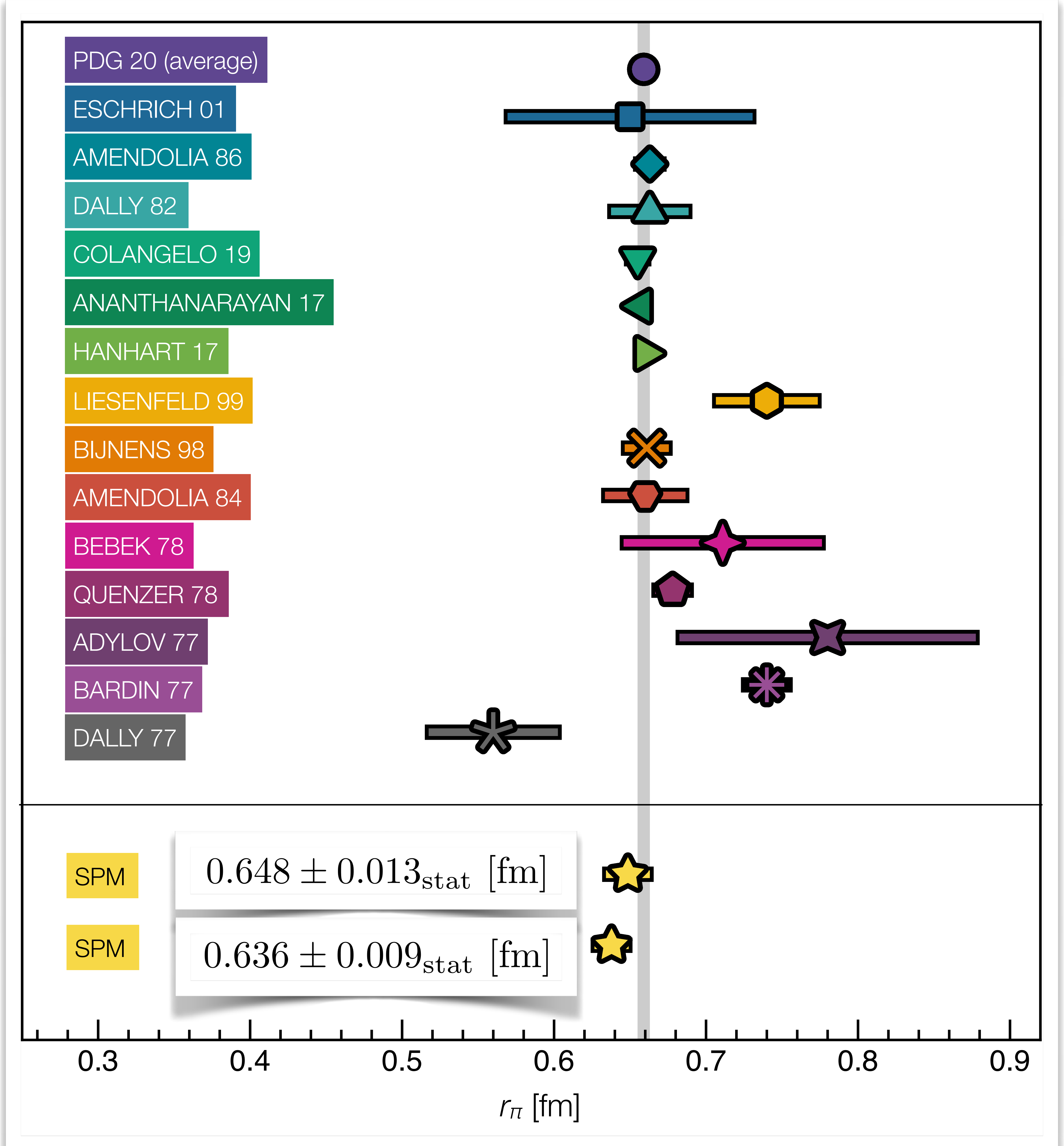

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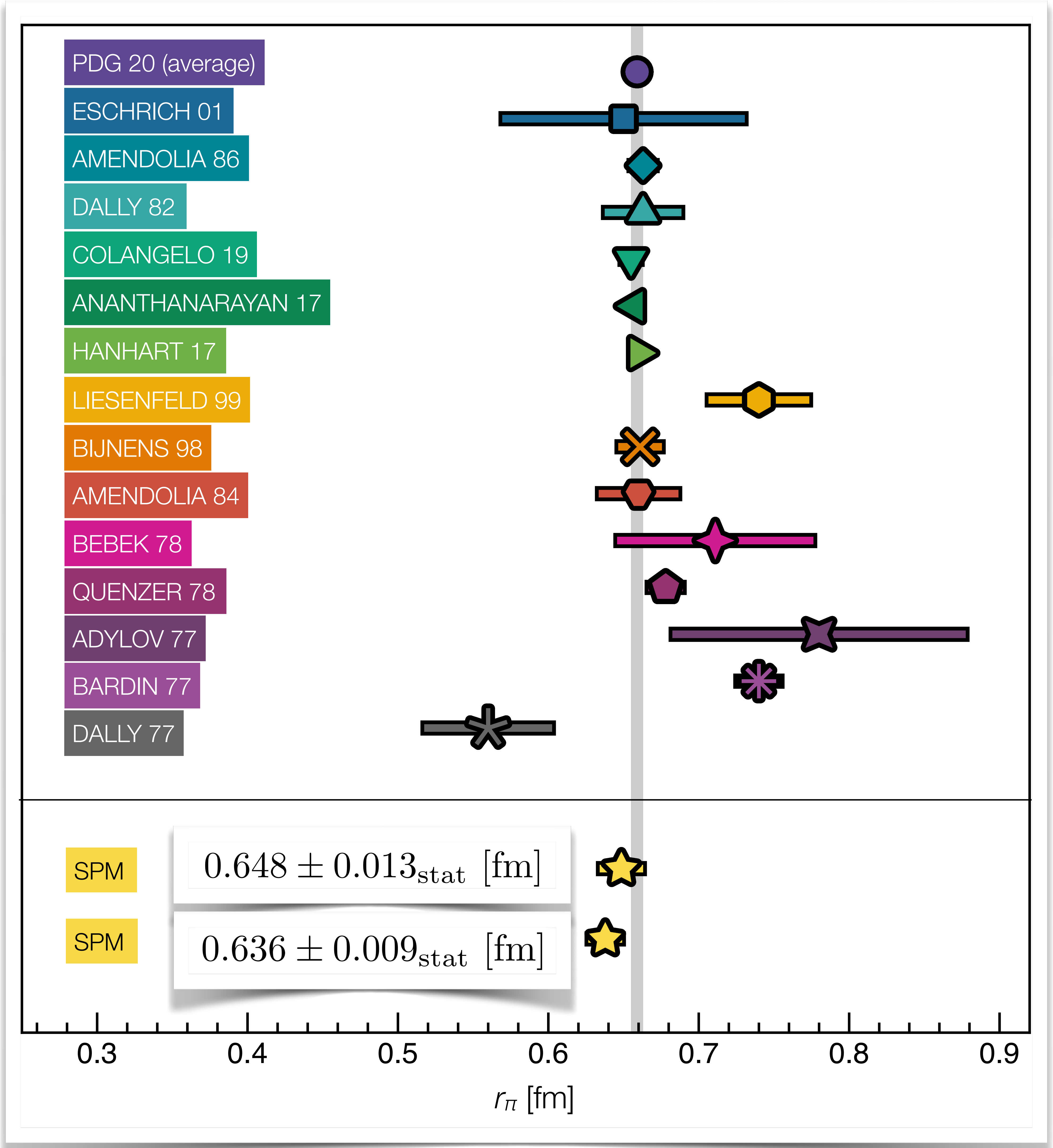

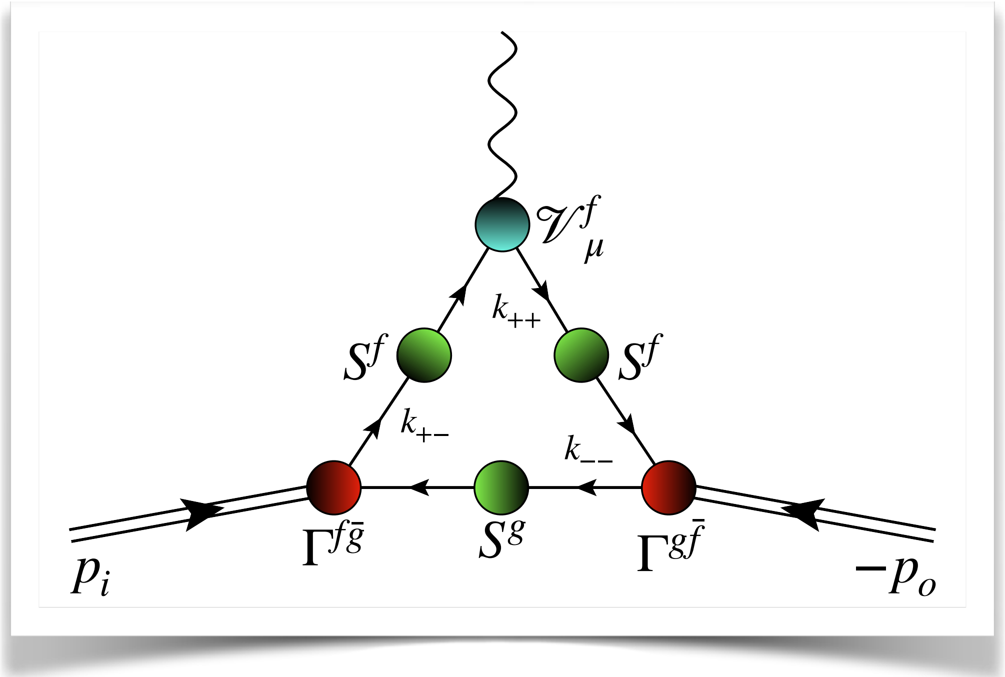

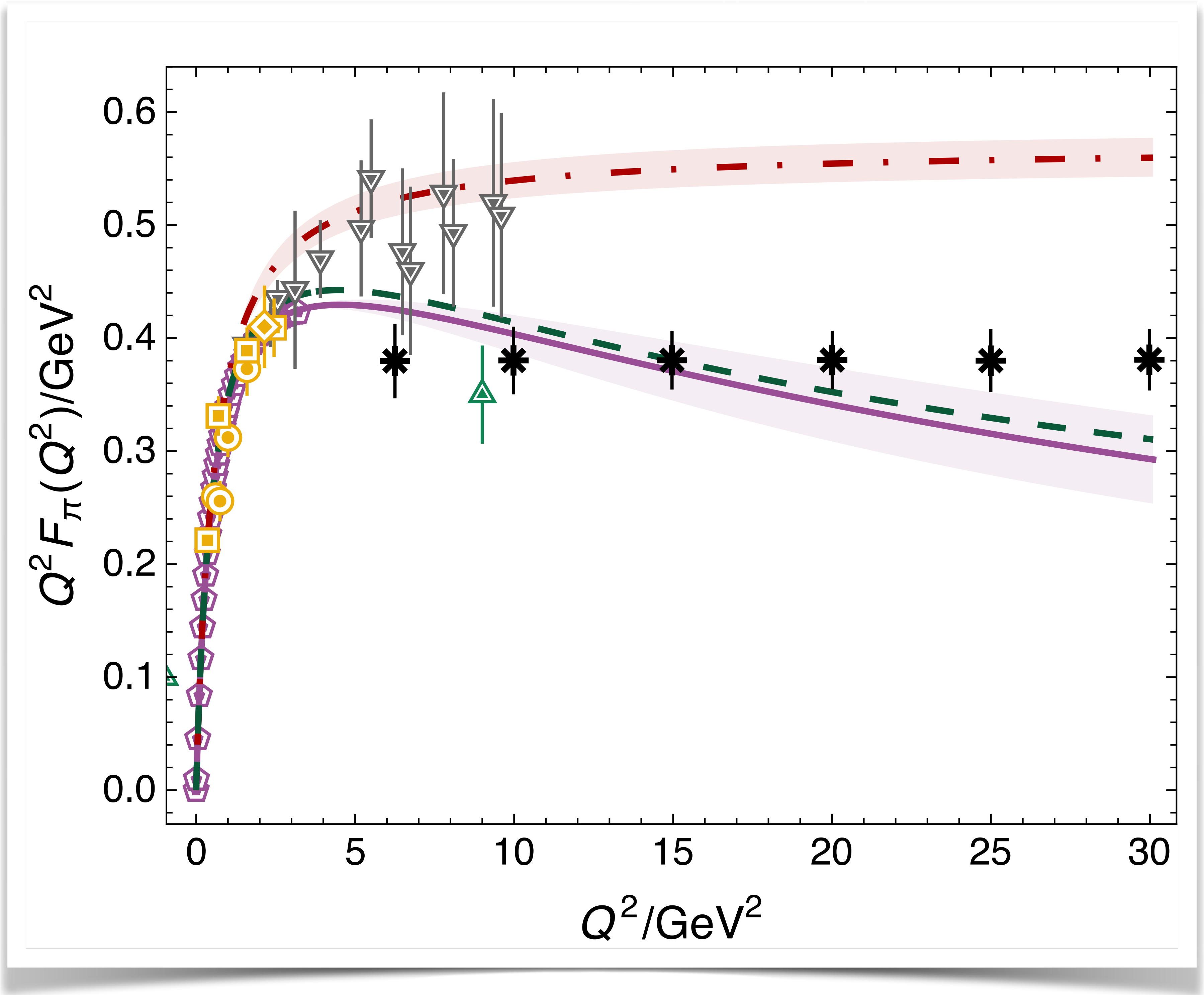

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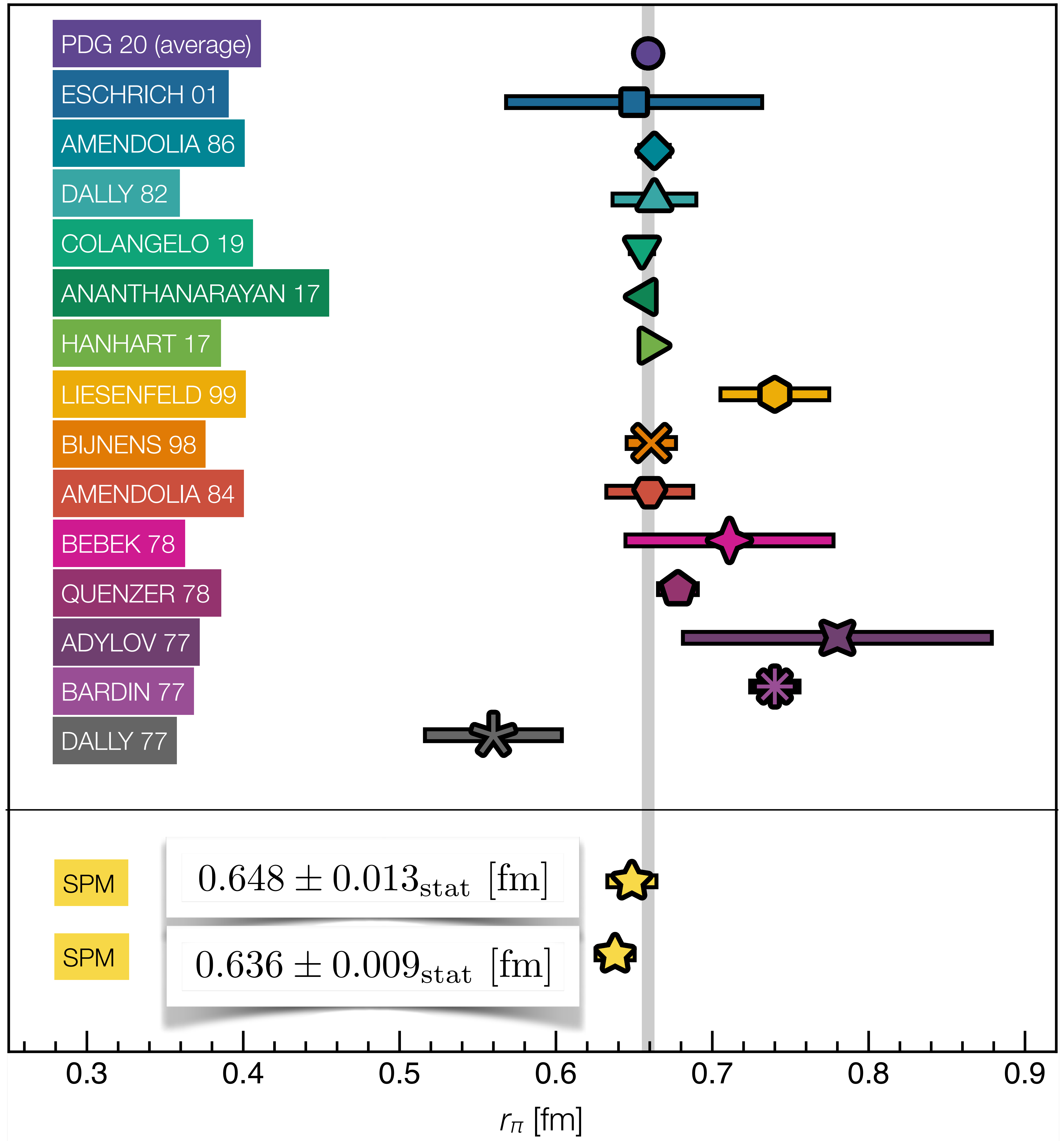

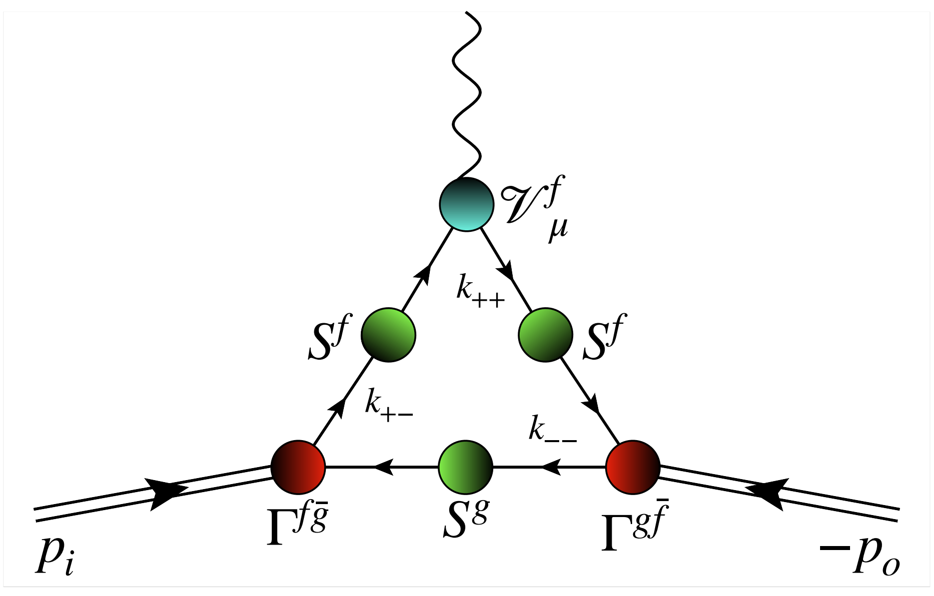

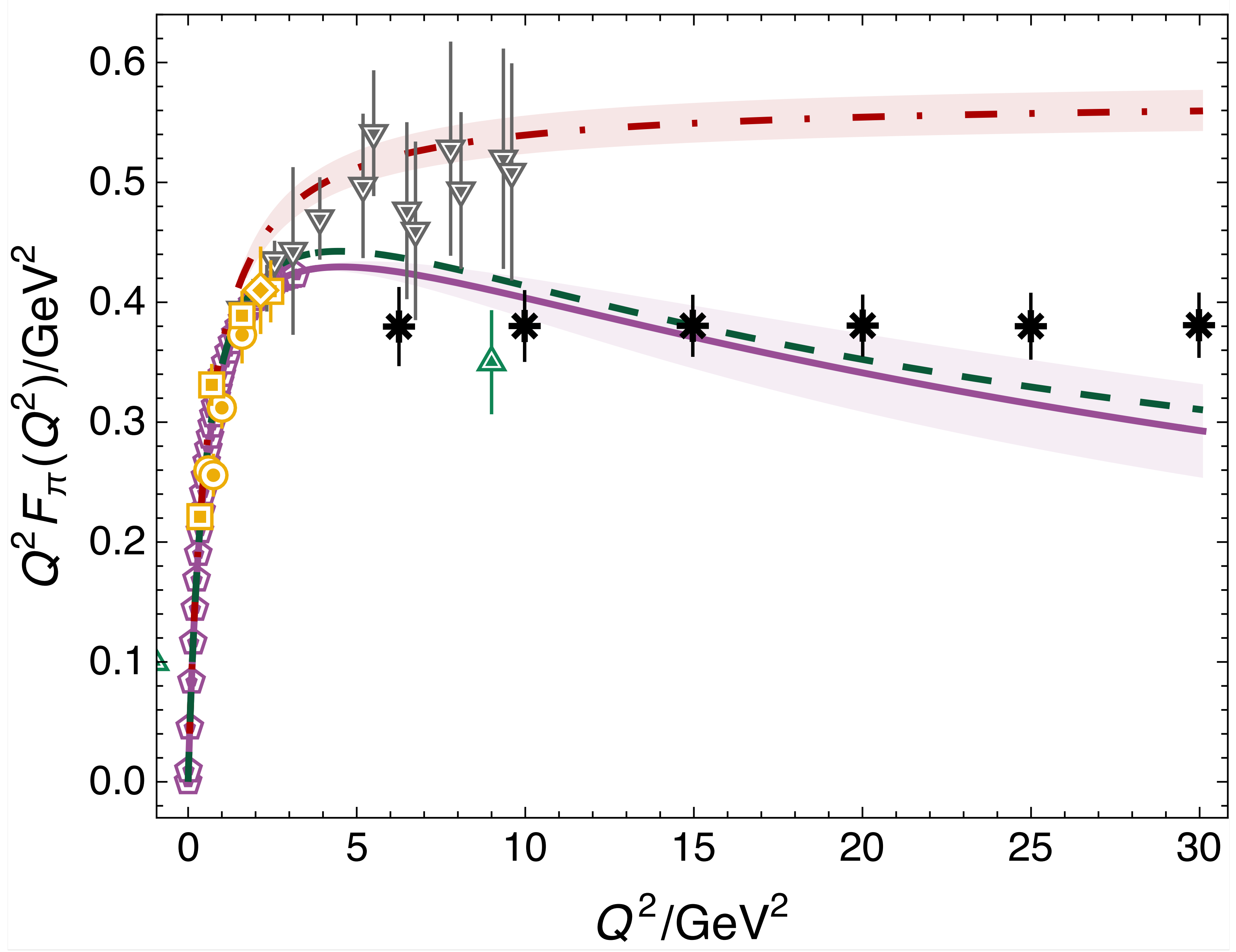

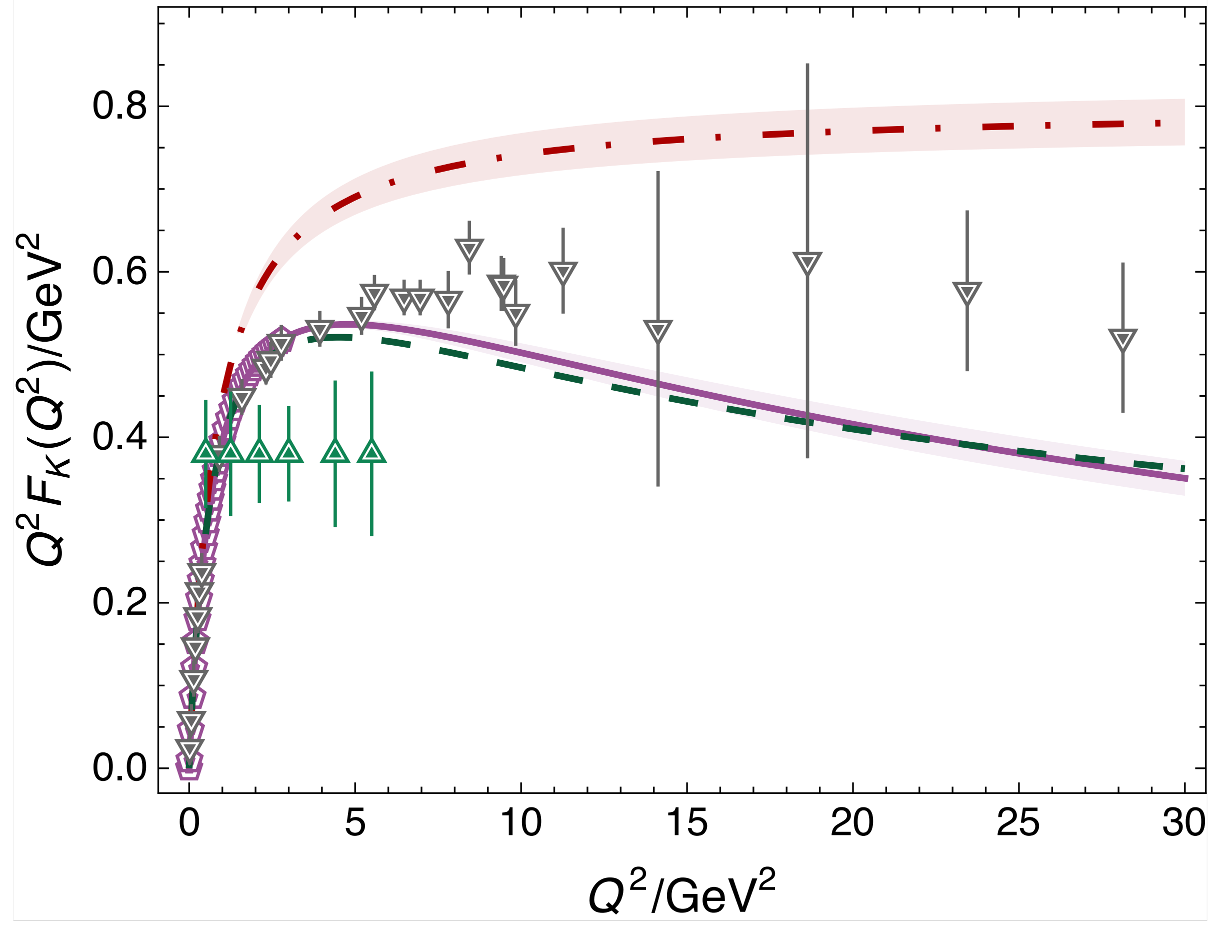

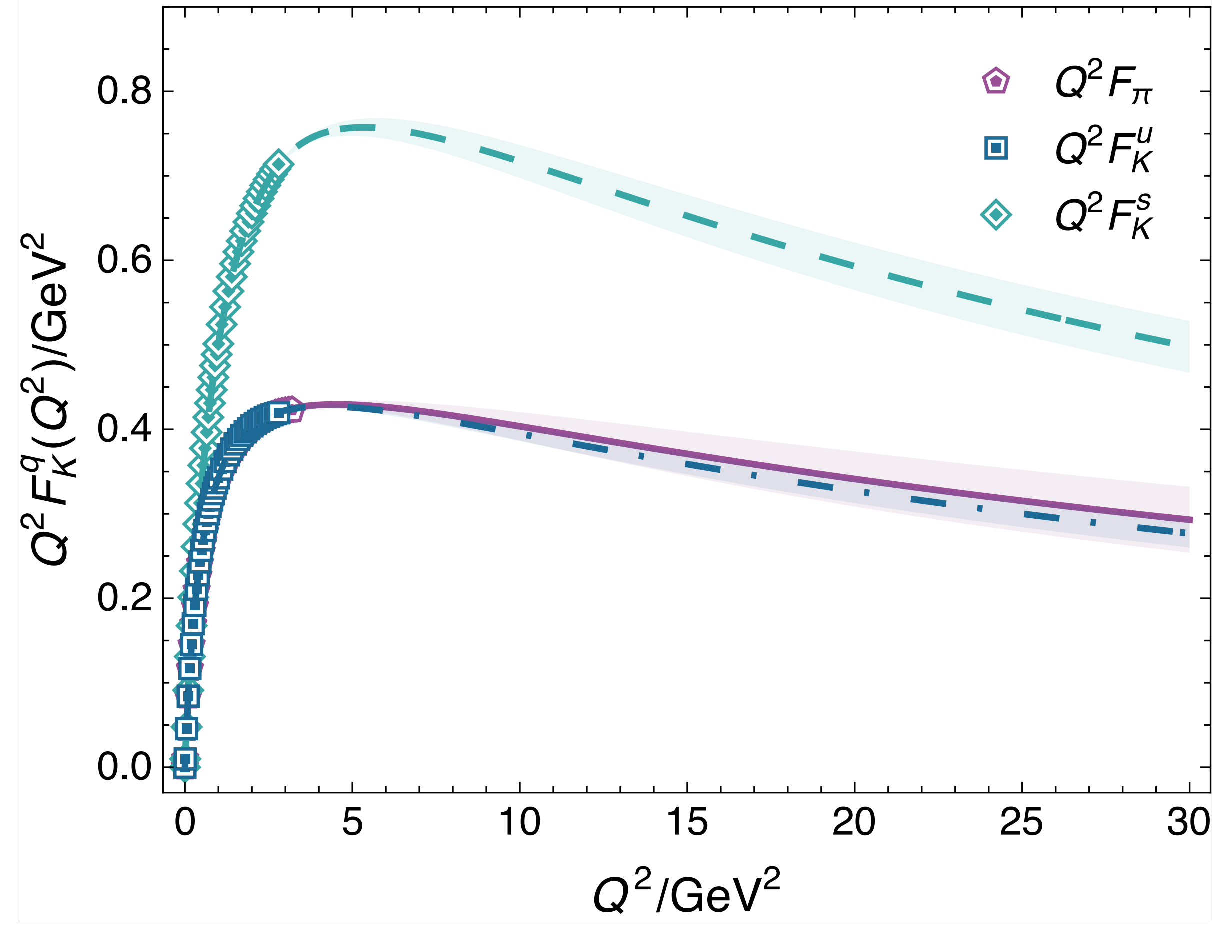

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