Revisiting the Leading Order Formula for the $J/\Psi \rightarrow ggg$ Decay

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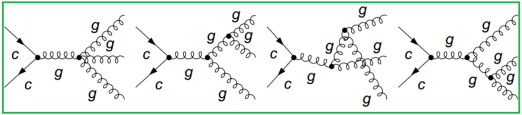
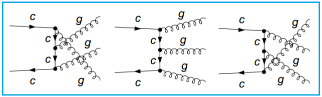
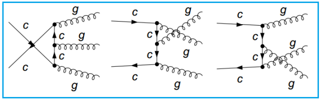
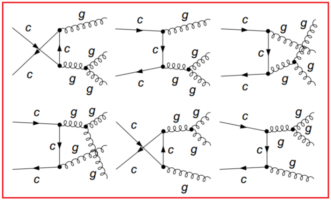
$J/\Psi \to g^a g^b g^c$: diagrams



$\propto \mathrm{Tr}[t^d] = 0$

$M_1 =$  $\propto \mathrm{Tr}[t^a t^c t^b]$

$M_2 =$  $\propto \mathrm{Tr}[t^a t^b t^c]$

$M_3 =$  $\propto f^{abc}$

Colors and helicities of initial $q\bar{q}$

- The initial $q\bar{q}$ is color siglet:

$$c\bar{c}: \ \frac{1}{\sqrt{3}}(|1\bar{1}\rangle + |2\bar{2}\rangle + |3\bar{3}\rangle)_{\text{color}} \tag{1}$$

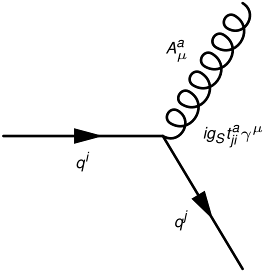
- Spin triplet and singlet (J/Ψ is $|1, \pm 1\rangle$):

$$\text{triplet} \begin{cases} |1,1\rangle = |\frac{1}{2}, \frac{1}{2}\rangle \\ |1,-1\rangle = |-\frac{1}{2}, -\frac{1}{2}\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}, -\frac{1}{2}\rangle + |-\frac{1}{2}, \frac{1}{2}\rangle\right) \end{cases} \tag{2}$$

$$\text{singlet}: |0,0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}, -\frac{1}{2}\rangle - |-\frac{1}{2}, \frac{1}{2}\rangle\right) \tag{3}$$

- J/Ψ is $|1, \pm 1\rangle$.

Coupling between quarks and gluon



$$t^1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad t^2 = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad t^3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$t^4 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad t^5 = \frac{1}{2}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad t^6 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$t^7 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad t^8 = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\mathrm{Tr}[t^a] = 0\,, \qquad \mathrm{Tr}[t^a t^b] = \mathrm{Tr}[t^b t^a]\,, \qquad \mathrm{Tr}[t^a t^b t^c] = \mathrm{Tr}[t^b t^c t^a] \neq \mathrm{Tr}[t^a t^c t^b] \tag{4}$$

$$[t^a, t^b] = if^{abc}t^c, \qquad \{t^a, t^b\} = \frac{1}{3}\delta^{ab} + d^{abc}t^c \tag{5}$$

表 5.1 SU(3) 群结构常数的非零值

f_{123}	f_{147}	f_{156}	f_{246}	f_{257}	f_{345}	f_{367}	f_{458}	f_{678}
1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
d_{118}	d_{146}	d_{157}	d_{228}	d_{247}	d_{256}	d_{338}	d_{344}	d_{355}
$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$
d_{366}	d_{377}	d_{448}	d_{558}	d_{668}	d_{778}	d_{888}		
$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$		

$q\bar{q} \to g^a g^b g^c$: discussion

The colors of final state gluons can be divided into three classes:

- $\mathrm{Tr}[t^a t^b t^c] = 0$. In this case we have

$$\mathrm{Tr}[t^a t^c t^b] = 0 \,, \tag{6}$$
$$f^{abc} = 0 \,. \tag{7}$$

  No diagrams contribute.

- $\mathrm{Tr}[t^a t^b t^c] \neq 0$ and $\mathrm{Tr}[t^a t^b t^c]$ is real. In this case we have

$$\mathrm{Tr}[t^a t^b t^c] = \mathrm{Tr}[t^a t^c t^b] \,, \tag{8}$$
$$f^{abc} = 0 \,. \tag{9}$$

  Only diagrams in the blue boxes contribute.

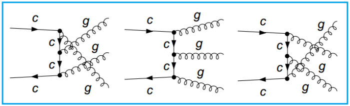
- $\mathrm{Tr}[t^a t^b t^c] \neq 0$ and $\mathrm{Tr}[t^a t^b t^c]$ is pure imaginary. In this case we have

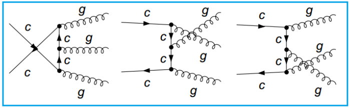
$$\mathrm{Tr}[t^a t^b t^c] = -\mathrm{Tr}[t^a t^c t^b] \,, \tag{10}$$
$$f^{abc} = -4i \times \mathrm{Tr}[t^a t^b t^c] \,. \tag{11}$$

  Both diagrams in the blue and red boxes contribute.

Class II: $\mathrm{Tr}[t^a t^b t^c] \neq 0$ and $\mathrm{Tr}[t^a t^b t^c]$ is real

$$\mathrm{Tr}[t^a t^b t^c] = \mathrm{Tr}[t^a t^c t^b]\,, \qquad f^{abc} = 0$$

$M_1 =$  $\propto \mathrm{Tr}[t^a t^c t^b]$

$M_2 =$  $\propto \mathrm{Tr}[t^a t^b t^c]$

$$M(g^a g^b g^c) = (M_1 + M_2) \propto \mathrm{Tr}[t^a t^b t^c] \times M(\gamma\gamma\gamma) \tag{12}$$

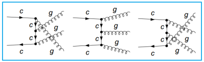
Numerical results in this case:

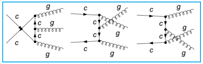
$$M(\text{triplet} \to g^a g^b g^c) \neq 0 \tag{13}$$

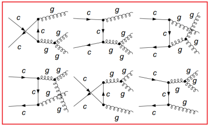
$$M(\text{singlet} \to g^a g^b g^c) = 0 \tag{14}$$

Class III: $\mathrm{Tr}[t^a t^b t^c] \neq 0$ and $\mathrm{Tr}[t^a t^b t^c]$ is pure imaginary

$$\mathrm{Tr}[t^a t^b t^c] = -\mathrm{Tr}[t^a t^c t^b]\,, \qquad f^{abc} = -4i \times \mathrm{Tr}[t^a t^b t^c]\,.$$

$M_1 =$  $\propto \mathrm{Tr}[t^a t^c t^b]$

$M_2 =$  $\propto \mathrm{Tr}[t^a t^b t^c]$

$M_3 =$  $\propto f^{abc}$

$$M(g^a g^b g^c) = M_1 + M_2 + M_3 \tag{15}$$

Numerical results in this case:

$$M(\text{triplet} \to g^a g^b g^c) = 0 \tag{16}$$

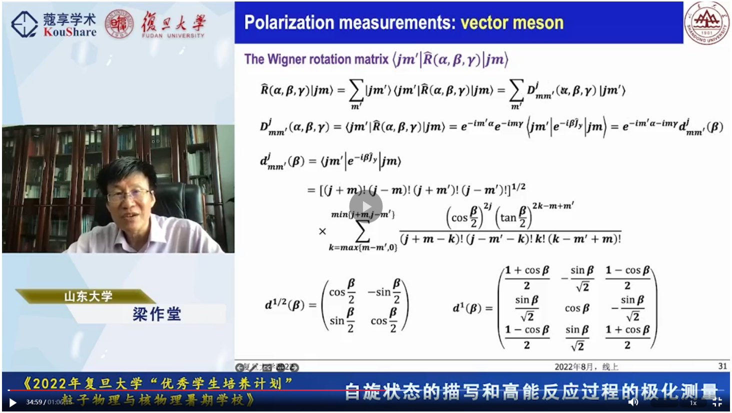
$$M(\text{singlet} \to g^a g^b g^c) \neq 0 \tag{17}$$

$J/\Psi \to g^a g^b g^c$

- Since J/Ψ is $|1, \pm 1\rangle$, we only need to consider the processes $J/\Psi \to g^a g^b g^c$ satisfying that $\mathrm{Tr}[t^a t^b t^c]$ are real and diagrams in blue boxes.

$J/\Psi \to g^a g^b g^c$

- Since c and $\bar{c}$ in J/Ψ are all still ($\vec{p}_{c/\bar{c}}$=0) in the calculations, each of the three states ($|1,\pm1\rangle$, $|1,0\rangle$) can be obtained by rotations of the other two. So, decay widths of these three states are same though the angular distributions of final states are different.



$J/\Psi \to g^a g^b g^c$

- In this case, we have

$$\int d\Pi_f \, |M(|1,\pm 1\rangle \to g^a g^b g^c)|^2 = \int d\Pi_f \, |M(|1,0\rangle \to g^a g^b g^c)|^2 \tag{18}$$

$$= \int d\Pi_f \, |M(|1/2,1/2\rangle \to g^a g^b g^c)|^2 = \int d\Pi_f \, |M(|-1/2,-1/2\rangle \to g^a g^b g^c)|^2 \tag{19}$$

$$=2 \int d\Pi_f \, |M(|1/2,-1/2\rangle \to g^a g^b g^c)|^2 = 2 \int d\Pi_f \, |M(|-1/2,1/2\rangle \to g^a g^b g^c)|^2 \tag{20}$$

$$=\frac{1}{3} \int d\Pi_f \sum_{\lambda_{c/\bar{c}}=\pm 1/2} |M(|\lambda_c, \lambda_{\bar{c}}\rangle \to g^a g^b g^c)|^2 \tag{21}$$

where we have used

$$|1/2,1/2\rangle = |1,1\rangle \tag{22}$$

$$|-1/2,-1/2\rangle = |1,-1\rangle \tag{23}$$

$$|1/2,-1/2\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,0\rangle) \tag{24}$$

$$|-1/2,1/2\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle - |0,0\rangle) \tag{25}$$

$$M(|0,0\rangle \to g^a g^b g^c) = 0 \tag{26}$$

$J/\Psi \to ggg$

$$\Gamma^{\rm LO}_{ggg} = \frac{1}{3}\int d\Pi_f \sum_{\mathrm{Tr}[t^a t^b t^c]\text{ are real}} \sum_{\lambda_{c/\bar{c}}=\pm 1/2} |M(|\lambda_c,\lambda_{\bar{c}}\rangle \to g^a g^b g^c)|^2 \frac{|\Psi_{\rm NR}(0)|^2}{M^2_{J/\Psi}} \tag{27}$$

$$= \frac{1}{3}\int d\Pi_f \sum_{\lambda_{c/\bar{c}}=\pm 1/2} |M(|\lambda_c,\lambda_{\bar{c}}\rangle \to \gamma\gamma\gamma)|^2 \frac{|\Psi_{\rm NR}(0)|^2}{M^2_{J/\Psi}}$$

$$\times \frac{\alpha_S^3}{\alpha_{\rm em}^3} \times \frac{1}{Q_c^6} \times \sum_{\mathrm{Tr}[t^a t^b t^c]\text{ are real}} \left(\frac{\mathrm{Tr}[t^a t^b t^c]}{\sqrt{3}}\right)^2 \tag{28}$$

$$= \frac{5}{18}\,\frac{128}{9}\,\alpha_S^3 \frac{|\Psi_{\rm NR}(0)|^2}{M^2_{J/\Psi}} \int_0^1 dx \int_{1-x}^1 dy$$

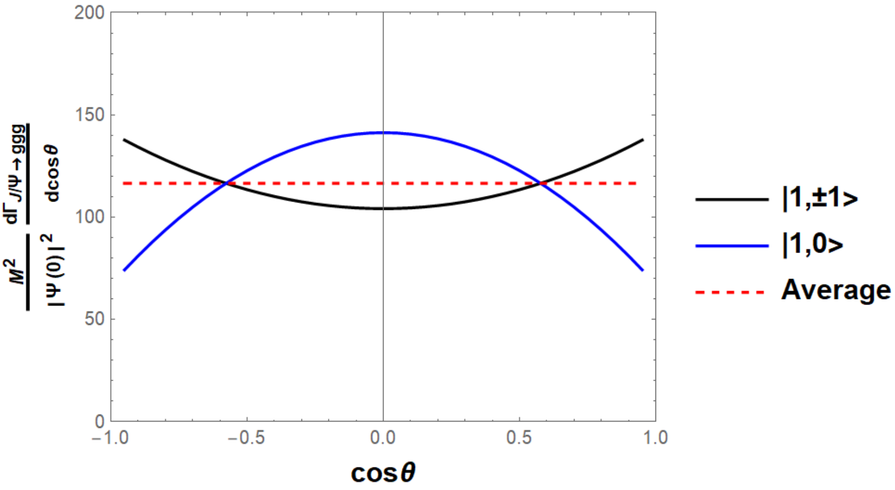
$$\times \frac{x^4 + 2x^3(-2+y) + (-1+y)^2\left(2-2y+y^2\right) + x^2\left(7-9y+3y^2\right) + x\left(-6+13y-9y^2+2y^3\right)}{x^2 y^2 (2-x-y)^2} \tag{29}$$

$$= \frac{5}{18}\,\frac{128}{9}\,\alpha_S^3 \frac{|\Psi_{\rm NR}(0)|^2}{M^2_{J/\Psi}} \frac{\pi^2-9}{2} \tag{30}$$

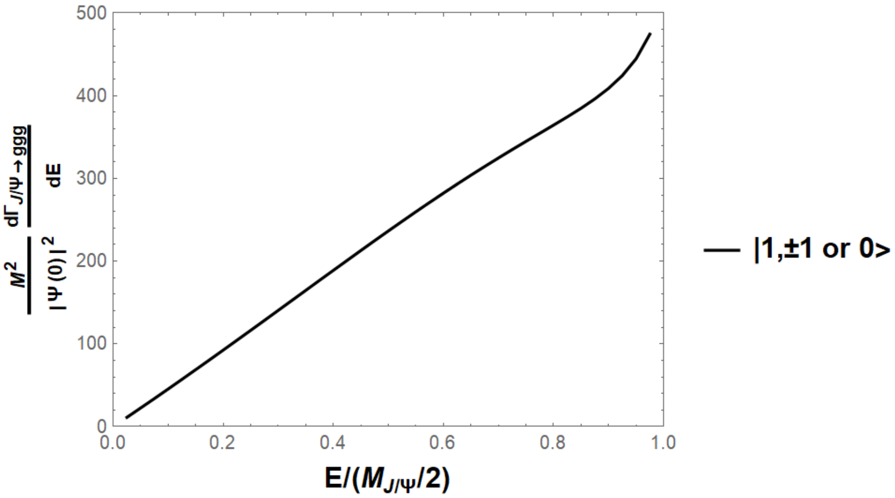
$$= \frac{160}{81}(\pi^2-9)\alpha_S^3 \frac{|\Psi_{\rm NR}(0)|^2}{M^2_{J/\Psi}}\,, \tag{31}$$

which is in agreement with the results from references.

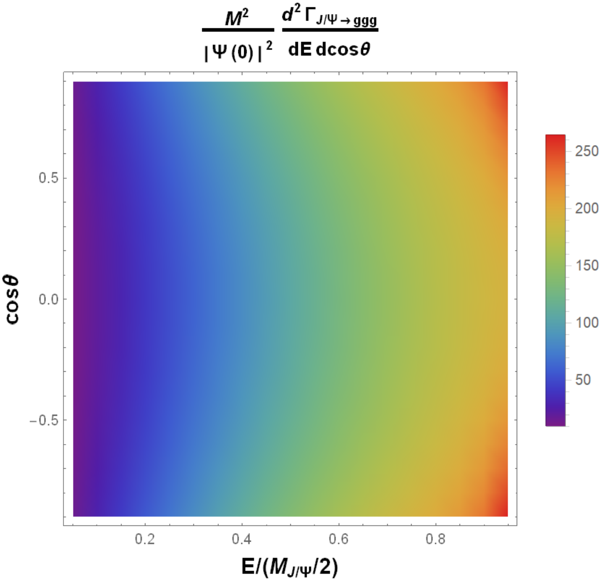
$J/\Psi \to ggg$



$J/\Psi \to ggg$


500
400
300
200
100
0
0.0
0.2
0.4
0.6
0.8
1.0
E/(M_J/Ψ/2)
M^2/|Ψ(0)|^2 dΓ_J/Ψ→ggg/dE
|1,±1 or 0>

$J/\Psi \to ggg$

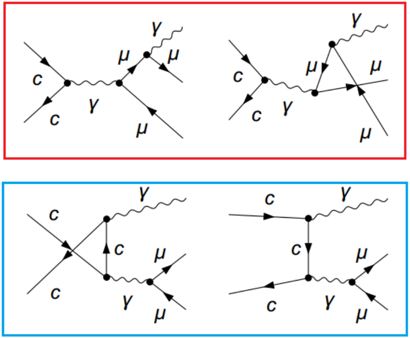


$\frac{M^2}{|\Psi(0)|^2} \frac{d^2\Gamma_{J/\Psi \to ggg}}{dE\, d\cos\theta}$
0.5
0.0
−0.5
$\cos\theta$
0.2
0.4
0.6
0.8
$E/(M_{J/\Psi}/2)$
250
200
150
100
50

$J/\Psi \to \gamma\mu^+\mu^-$

Mt =

Ms =



Numerical results in this case:

$$M_s(\text{triplet} \to \gamma\mu^+\mu^-) = 0 \quad (32)$$

$$M_t(\text{singlet} \to \gamma\mu^+\mu^-) = 0 \quad (33)$$

$J/\Psi \to \gamma\mu^+\mu^-$

$$\int d\Pi_f\, |M(|1,\pm1\rangle \to \gamma\mu^+\mu^-)|^2 = \int d\Pi_f\, |M(|1,0\rangle \to \gamma\mu^+\mu^-)|^2 \tag{34}$$

$$= \int d\Pi_f\, |M(|1/2,1/2\rangle \to \gamma\mu^+\mu^-)|^2 = \int d\Pi_f\, |M(|-1/2,-1/2\rangle \to \gamma\mu^+\mu^-)|^2 \tag{35}$$

$$= 2\int d\Pi_f\, |M(|1/2,-1/2\rangle \to \gamma\mu^+\mu^-)|^2 = 2\int d\Pi_f\, |M(|-1/2,1/2\rangle \to \gamma\mu^+\mu^-)|^2 \tag{36}$$

$$= \frac{1}{3}\int d\Pi_f \sum_{\lambda_{c/\bar{c}}=\pm 1/2} |M(|\lambda_c,\lambda_{\bar{c}}\rangle \to \gamma\mu^+\mu^-)|^2 \tag{37}$$

What is the C parity of gluon?

- For photon:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{38}$$

$$\mathcal{L} \ni -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{39}$$

$$\hat{C} A^\mu \hat{C}^{-1} = -A^\mu \tag{40}$$

- For gluon: $\hat{C} A^{a\mu} \hat{C}^{-1} \neq A^{a\mu}$ due to $J/\Psi \to ggg$

$$F^{a\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g_S f^{abc} A^{b\mu} A^{c\nu} \tag{41}$$

$$\mathcal{L} \ni -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} \tag{42}$$

$$\Longrightarrow \hat{C} A^{a\mu} \hat{C}^{-1} \neq -A^{a\mu} \tag{43}$$

So, gluons are neither eigenstates of $\hat{C}$ transformation nor eigenstates of $\hat{T}$ transformation. But they are eigenstates of $\hat{P}$ ($\hat{C}\hat{T}$) transformation.