Some Studies on flipped GUT models

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and arXiv: 2303.10298 (By Fei, To appear in PRD)

Outline

- Flavor structures of quarks and leptons from flipped $SU(5)$ GUT with A_4 modular flavor symmetry
  - Flipped SU(5)
  - Modular flavor symmetry with multiple modular symmetries
  - Flavor structure from orbifold flipped SU(5) GUT with A_4 modular flavor symmetry
- Flipped SU(6) Unification of $SU(3)_c \times SU(3)_L \times U(1)_X$
  - $SU(3)_c \times SU(3)_L \times U(1)_X$ model
  - SU(6) unification
  - Flipped SU(6) unification

FERMIONS

matter constituents
spin = 1/2, 3/2, 5/2, ...

Leptons spin = 1/2		
Flavor	Mass GeV/c^2	Electric charge
ν_e electron neutrino	$<1\times10^{-8}$	0
e electron	0.000511	–1
ν_μ muon neutrino	<0.0002	0
μ muon	0.106	–1
ν_τ tau neutrino	<0.02	0
τ tau	1.7771	–1

Quarks spin = 1/2		
Flavor	Approx. Mass GeV/c^2	Electric charge
u up	0.003	2/3
d down	0.006	–1/3
c charm	1.3	2/3
s strange	0.1	–1/3
t top	175	2/3
b bottom	4.3	–1/3

BOSONS

force carriers
spin = 0, 1, 2, ...

Unified Electroweak spin = 1		
Name	Mass GeV/c^2	Electric charge
γ photon	0	0
W^-	80.4	–1
W^+	80.4	+1
Z^0	91.187	0

Strong (color) spin = 1		
Name	Mass GeV/c^2	Electric charge
g gluon	0	0

H ? Higgs	125	0

PROPERTIES OF THE INTERACTIONS

Property \ Interaction	Gravitational	Weak (Electroweak)	Electromagnetic (Electroweak)	Strong: Fundamental	Strong: Residual
Acts on:	Mass – Energy	Flavor	Electric Charge	Color Charge	See Residual Strong Interaction Note
Particles experiencing:	All	Quarks, Leptons	Electrically charged	Quarks, Gluons	Hadrons
Particles mediating:	Graviton (not yet observed)	W^+ W^- Z^0	γ	Gluons	Mesons
Strength relative to electromag for two u quarks at: 10^{-18} m	10^{-41}	0.8	1	25	Not applicable to quarks
3×10^{-17} m	10^{-41}	10^{-4}	1	60	
for two protons in nucleus	10^{-36}	10^{-7}	1	Not applicable to hadrons	20

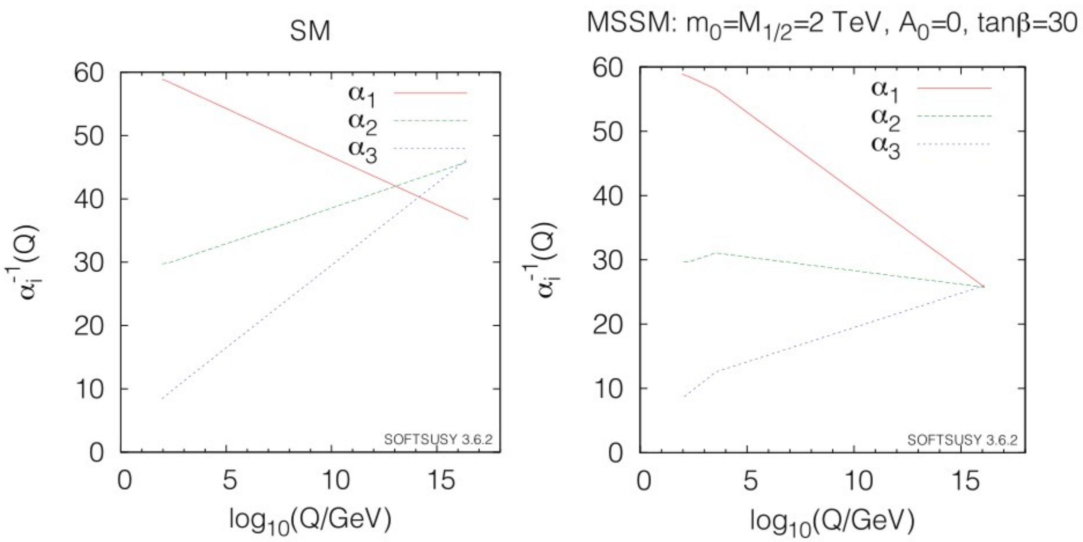
Motivations to GUT

- Why the charges of the proton and positron are exactly the same.
- Why there is apparent disparity between gauge couplings of strong interactions and electroweak interactions at low-energies
- The origin of so many low energy Yukawa couplings. (13 parameters associated with the Yukawa couplings. In addition, Majorana neutrinos introduce 3 more masses and 6 mixing angles and phases.)
- The origin of Baryon and Lepton Number Violation.
- Is proton absolutely stable?
- Why there are distinctions between quarks and leptons?
- Any possible explanations to quark-lepton complementarity?

SU(5) GUT Model

G_{SM} has rank four rank-four group $SU(5)$ is the minimal choice for unification in a simple group

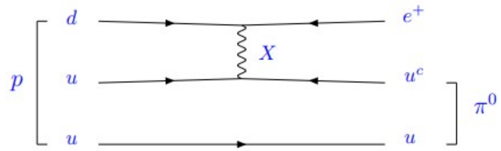
$$\mathbf{10}: \begin{pmatrix} 0 & u_b^c & -u_g^c & u_r & d_r \\ -u_b^c & 0 & u_r^c & u_g & d_g \\ u_g^c & -u_r^c & 0 & u_b & d_b \\ -u_r & -u_g & -u_b & 0 & e^c \\ -d_r & -d_g & -d_b & -e^c & 0 \end{pmatrix} \quad \text{and} \quad \bar{\mathbf{5}}: \begin{pmatrix} d_r^c \\ d_g^c \\ d_b^c \\ e \\ -\nu_e \end{pmatrix}.$$



Proton Decay in the Supersymmetry SU(5)

D type gauge boson exchange

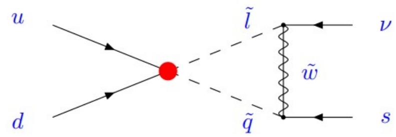
dimension-six operators $\dfrac{1}{M_U^2}\int d^4\theta \Phi^+\Phi\Phi^+\Phi,$



$p \to e^+ \pi^0$ $(n \to e^+ \pi^-)$

F type triplet Higgs exchange

dimension-five operators $\dfrac{1}{M_U}\int d^2\theta \Phi^4.$



$p \to K^+ \bar{\nu}$ $(n \to K^0 \bar{\nu})$

dimension-four operators in $(\mathbf{10}\ \bar{\mathbf{5}}\ \bar{\mathbf{5}})$

eliminated by requiring R parity

Problem of SU(5)

- Doublet-Triplet splitting

in $SU(5)$ one requires (at least) the set $5+\bar{5}+50+\overline{50}+75$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} \bar{3} \\ \text{other} \end{pmatrix} \quad \begin{pmatrix} 3 \\ \text{other} \end{pmatrix} \quad \begin{pmatrix} \bar{3} \\ \bar{2} \end{pmatrix}$$

$$\begin{array}{cccc} \| & \| & & \\ 5_H & \overline{50}_H & 50_H & \bar{5}_H \end{array}$$

the **75** uniquely breaks $SU(5)$ down to $SU(3)\otimes SU(2)\otimes U(1)$

In the $50(\overline{50})$ there is a color $3(\bar{3})$ but no weak $2(\bar{2})$

$$\lambda 5_H \overline{50}_H \langle 75_H \rangle + \lambda' \bar{5}_H 50_H \langle 75_H \rangle$$

give mass to the triplets in $5_H+\bar{5}_H$ but not to the doublets.

- Undesirable mass relations?

$$m_\tau = m_b \quad ☑$$

$$\frac{m_e}{m_\mu} = \frac{m_d}{m_s}, \quad ☒$$

- Neutrino mass
- Minimal version of SUSY SU(5) rule out ?

include a $\{45\}$-dimensional H^P_{qs}

$$\langle H^i_{i5}\rangle = -\tfrac{1}{3}\langle H^4_{45}\rangle$$

$$\begin{array}{c} \\ e \\ \mu \end{array}\!\!\begin{array}{c} \begin{matrix} e & \mu \end{matrix} \\ \begin{pmatrix} 0 & a \\ a & 3b \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ d \\ s \end{array}\!\!\begin{array}{c} \begin{matrix} d & s \end{matrix} \\ \begin{pmatrix} 0 & a \\ a & b \end{pmatrix} \end{array}$$

$$\frac{m_e}{m\mu} = \frac{1}{9}\frac{m_d}{m_s}$$

Doublet-Triplet splitting problem

1. The natural separation of doublet and triplet masses. Here, " natural" mean the absence of large cancellations between a priori unrelated free parameters.

2. Sufficient suppression of triplet-Higgs mediated dimension-five operators that trigger proton decay.

3. Precision gauge coupling unification: any additional gauge representation at the GUT scale will contribute threshold corrections that change unification of gauge couplings.

4. $\mu/B\mu$ problem. SUSY breaking soft mass between the two Higgs doublets $B\mu$ is of the same order as the supersymmetric doublet mass μ^2.

sliding singlet | **missing partner** | **missing VEV** | **pseudo Nambu-Goldstone** | **Orbifold** | **Clockwork**

Sliding Singlet for D-T Splitting

$W = W(\Phi) + \overline{H}_{\bar{5}}\,(\Phi + S)\,H_5$ Φ is an $SU(5)$ adjoint Higgs field, S is a SM singlet

$$\Phi = \mathrm{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right) V_\Phi$$

F-flatness conditions for the F-terms of $\overline{H}_{\bar{5}}$ and H_5

$$(\langle\Phi\rangle + \langle S\rangle)\,\langle H_5\rangle = 0\ ,\ \ \langle\overline{H}_{\bar{5}}\rangle\,(\langle\Phi\rangle + \langle S\rangle) = 0$$

$$\langle S\rangle = -\frac{1}{2}V_\Phi$$

$$\langle\Phi\rangle + \langle S\rangle = \mathrm{diag}\left(-\frac{5}{6}, -\frac{5}{6}, -\frac{5}{6}, 0, 0\right) V_\Phi$$

the color triplets in $\overline{H}_{\bar{5}}$ and H_5 will obtain vector-like mass around V_Φ

the doublets will remain massless

supersymmetry breaking $T_1 = \mathcal{O}(m_g^2 M_{GUT})S + \text{ H.C.}\ ,\ \ T_2 = \mathcal{O}(m_g M_{GUT})F_S + \text{ H.C.}\ ,$

⇩

shift the VEV of S from its supersymmetric minimum $-V_\Phi/2$

$$\delta\langle S\rangle \sim \frac{\mathcal{O}(m_g^2 M_{GUT})}{\mathcal{O}(M_S^2) + (\langle H_d^0\rangle)^2 + (\langle H_u^0\rangle)^2} \sim \mathcal{O}(M_{GUT})$$

⇩

$$V \supset |\overline{H}_{\bar{5}} H_5 + \mathcal{O}(m_g M_{GUT})|^2$$

See Sen's and Barr's solution with SU(6)

Missing VEV mechanism of SO(10)

under $SO(10) \to SU(5)$, $10_{1H} = \bar{5}_{1H} + 5_{1H}$, and $10_{2H} = \bar{5}_{2H} + 5_{2H}$

$$\begin{pmatrix} 3_1 \\ 2_1 \end{pmatrix} = 5_{1H} \quad \begin{pmatrix} \bar{3}_2 \\ \bar{2}_2 \end{pmatrix} = \bar{5}_{2H} \quad \begin{pmatrix} 3_2 \\ 2_2 \end{pmatrix} = 5_{2H} \quad \begin{pmatrix} \bar{3}_1 \\ \bar{2}_1 \end{pmatrix} = \bar{5}_{1H}$$

- Missing VEV mechanism for SO(10)

$$\langle 45_H \rangle = \eta \otimes \mathrm{diag}(a, a, a, 0, 0,).$$

$$W \supset \lambda 10_{1H} 45_H 10_{2H}.$$

This is just what is needed to give mass to the $SU(3)_C$ - triplet higgs(inos)

must introduce <u>two</u> 10's $\quad 10_H 45_H 10_H$ would vanish by the antisymmetry of the 45.

$$W \supset \lambda 10_{1H} 45_H 10_{2H} + \lambda' 10_{2H} 45'_H 10_{3H} + M 10_{3H} 10_{3H} + \sum_{i,j=1}^{3} f_{ij} 16_i 16_j 10_{1H}$$

$$(\bar{2}_1,\ \bar{2}_2,\ \bar{2}_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda' a' \\ 0 & -\lambda' a' & M \end{pmatrix} \begin{pmatrix} 2_1 \\ 2_2 \\ 2_3 \end{pmatrix}$$

need to give 2_{2H} and $\bar{2}_{2H}$ a superheavy Dirac mass

$$(\bar{3}_1,\ \bar{3}_2,\ \bar{3}_3) \begin{pmatrix} 0 & \lambda a & 0 \\ -\lambda a & 0 & 0 \\ 0 & 0 & M \end{pmatrix} \begin{pmatrix} 3_1 \\ 3_2 \\ 3_3 \end{pmatrix}$$

$\langle 45'_H \rangle = \eta \otimes \mathrm{diag}(0, 0, 0, a', a')$ Inverse D-T splitting

an additional 10

Problem with Missing VEV mechanism of SO(10)

An adjoint $(\mathbf{45})$ alone is not sufficient to break SO(10) to the Standard Model:

Requires either spinorial Higgs $(\mathbf{16}+\overline{\mathbf{16}})$ or rank-five antisymmetric tensor $(\mathbf{126}+\overline{\mathbf{126}})$

two pairs of spinor-antispinor $\{C(16)+\bar{C}(\overline{16})\}$ and $\{C'(16)+\bar{C}'(\overline{16})\}$

See Barr & Rabby

$$W = W_A + W_C + W_{ACC'} + (T_1 A T_2 + S T_2^2).$$

$$W_{ACC'} = \overline{C}'\left(\left(\frac{P}{M_P}\right)A + Z\right)C + \overline{C}\left(\left(\frac{\overline{P}}{M_P}\right)A + \overline{Z}\right)C',$$

$$\langle C' \rangle = \langle \bar{C}' \rangle = 0.$$

- The missing VEV pattern for **45** is stable to a high enough accuracy in the presence of all allowed higher dimensional operators.
- There are no undesirable pseudo-Goldstone bosons.
- No flat directions which would lead to VEVs of fields undetermined.
- GUT-scale threshold corrections not spoil the gauge coupling unification.

pseudo-NG boson mechanisms for D-T splitting

Σ in an adjoint 35 representation $H, \bar{H}$ in $6, \bar{6}$ representations of SU(6)

$$\Sigma = 35 = 24 + 6 + \bar{6} + 1$$

$$H = 6 = 5 + 1$$

$$\bar{H} = \bar{6} = \bar{5} + 1.$$

one of the sectors consists of the fields $H, \bar{H}$ and the other of Σ

$\bar{H}\Sigma H$ are not present

$$\langle\Sigma\rangle = V\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&1&&\\&&&&-2&\\&&&&&-2\end{pmatrix}, \quad \langle H\rangle = \langle\bar{H}\rangle = U\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$$

global SU(6) ⇓ SU(4)⊗SU(2)⊗U(1)

global SU(6) ⇓ SU(5)

SU(6) gauge group ➡ SU(3)⊗SU(2)⊗U(1)

Goldstone bosons

from the breaking SU(6)→SU(4)⊗SU(2)⊗U(1)

$$(\bar{3},2)_{\frac{5}{6}} + (3,2)_{-\frac{5}{6}} + (1,2)_{\frac{1}{2}} + (1,2)_{-\frac{1}{2}}.$$

from the breaking SU(6)→SU(5)

$$(3,1)_{-\frac{1}{3}} + (\bar{3},1)_{\frac{1}{3}} + (1,2)_{\frac{1}{2}} + (1,2)_{-\frac{1}{2}} + (1,1)_0$$

$$(3,1)_{-\frac{1}{3}} + (\bar{3},1)_{\frac{1}{3}} + (3,2)_{-\frac{5}{6}} + (\bar{3},2)_{\frac{5}{6}} + (1,2)_{\frac{1}{2}} + (1,2)_{-\frac{1}{2}} + (1,1)_0.$$

are eaten by the heavy vector bosons

one pair of doublets remains uneaten

$$h_1 = \frac{Uh_\Sigma - 3Vh_H}{\sqrt{9V^2+U^2}}, \quad h_2 = \frac{U\bar{h}_\Sigma - 3V\bar{h}_{\bar{H}}}{\sqrt{9V^2+U^2}},$$

Key Difficulty: suppression of mixing $\bar{H}\Sigma H$

discrete symmetries.

$$\mathrm{SU}(6) \to \mathrm{SU}(5) \to \mathrm{SU}(3)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)$$

$$\langle\Sigma\rangle \sim M_{GUT} \quad \langle H\rangle = \langle\bar{H}\rangle > \langle\Sigma\rangle$$

Clockwork mechanism for D-T Splitting

$$W = \bar{H}(m_1 + m_{24}Y)H + \mathcal{Y}_{ij}\bar{H}A_i\bar{F}_j + \frac{1}{2}\mathcal{Y}'_{ij}HA_iA_j$$

$$Y \equiv \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix}$$

$\mu_D \equiv m_1 + \frac{1}{2}m_{24}$ weak scale

$\mu_T \equiv m_1 - \frac{1}{3}m_{24}$ GUT scale

include N Higgs fields of $\mathbf{5}$ and $\bar{\mathbf{5}}$

$$W = \bar{H}^T(\mathcal{M}_1 + \mathcal{M}_{24}Y)H + \mathcal{Y}_{ij}\bar{H}_1A_i\bar{F}_j + \frac{1}{2}\mathcal{Y}'_{ij}H_NA_iA_j$$

$$\mathcal{M}_1 = \alpha_1 M - \beta_1 K \qquad \mathcal{M}_{24} = \alpha_{24}M - \beta_{24}K$$

$$(M)_{ij} = m_i\delta_{ij} \qquad (K)_{ij} = k_i\delta_{i,j+1}$$

the doublet and triplet mass matrices ($X = D, T$)

$$\mathcal{M}_X \equiv M_X - K_X = \begin{pmatrix} m_{X,1} & & & & \\ -k_{X,1} & m_{X,2} & & & \\ & -k_{X,2} & m_{X,3} & & \\ & & \ddots & \ddots & \\ & & & -k_{X,N-1} & m_{X,N} \end{pmatrix}$$

$$m_{D,i} \equiv \left(\alpha_1 + \frac{1}{2}\alpha_{24}\right)m_i\,, \qquad k_{D,i} \equiv \left(\beta_1 + \frac{1}{2}\beta_{24}\right)k_i$$

$$m_{T,i} \equiv \left(\alpha_1 - \frac{1}{3}\alpha_{24}\right)m_i\,, \qquad k_{T,i} \equiv \left(\beta_1 - \frac{1}{3}\beta_{24}\right)k_i$$

for $m_{D,i} \ll k_{D,j}$ $\qquad \mu_{D,1} \approx \dfrac{\prod_{i=1}^{N} m_{D,i}}{\prod_{i=1}^{N-1} k_{D_i}}$

m_ℓ vanishes

end up with two traditional chiral clockwork chains, with (for $m_{D,i} \ll k_{D,j}$) one $\bar{H}$ zero mode localized near site 0 and one H zero mode localized near site N.

FLIPPED $SU(5)$ UNIFICATION

$$\begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ \nu_e \end{pmatrix}_L \; ; \; \left(\begin{pmatrix} u \\ d \end{pmatrix}_L \; d_L^c \; e_L^c \right) \; ; \; \boxed{\nu_L^c}$$

$\bar{\mathbf{5}}$ $\quad$ $\mathbf{10}$ $\quad$ $\boxed{\mathbf{1}}$

$u_L^c \Leftrightarrow d_L^c$

$\nu_L^c \Leftrightarrow e_L^c$

$$f_{\bar{\mathbf{5}}} = \begin{pmatrix} u_1^c \\ u_2^c \\ u_3^c \\ e \\ \nu_e \end{pmatrix}_L \; ; \; F_{\mathbf{10}} = \left(\begin{pmatrix} u \\ d \end{pmatrix}_L \; d_L^c \; \nu_L^c \right) \; ; \; l_{\mathbf{1}} = e_L^c \, ,$$

Basic GUT tests	$SU(5)$	Flipped $SU(5)$
$\sin^2\theta_W \Rightarrow \alpha_3(M_Z)$	$\times$	$\surd$
Proton decay	$\{p \to \bar{\nu} K^+\}$ $\times$	$\{p \to (e^+/\mu^+)\pi^0\}$
Doublet-triplet splitting	$\times$	$\surd$
Neutrino masses	$\times$	$\surd$
Baryogenesis	$\times$	$\surd$

flipped SU(5) $\quad$ minimal SU(5)

$\Gamma(p \to K^0 e^+)/\Gamma(p \to \pi^0 e^+)$

~ 0.02 vs. ~ 0.003

$(X', Y') = (\mathbf{3}, \mathbf{2}, -1/3)$ $\quad$ $(X, Y) = (\mathbf{3}, \mathbf{2}, 5/3)$

$\Gamma(p \to K^0 \mu^+)/\Gamma(p \to \pi^0 \mu^+)$

~ 0.02 vs. ~ 17

minimal SU(5) $\quad$ flipped SU(5)

$\Gamma(p \to \pi^0 \mu^+)/\Gamma(p \to \pi^0 e^+) \sim 0.008$ $\quad$ ~ 0.1 (NO) light neutrinos

~ 23 (IO) neutrinos.

$\Gamma(p \to \pi^+ \bar{\nu})/\Gamma(p \to \pi^0 e^+) \sim 0.4$ $\quad$ ~ 3.2 NO

~ 95 IO

See Ellis, Oliver et al, JHEP05(2020)021

FLIPPED $SU(5)$

The breaking of the GUT group in 4 dimension while solving doublet-triplet splitting problem without using large GUT representations

$$SU(5)\times U(1) \xrightarrow{\text{scale } M_{32}} SU(3)_c\times SU(2)_L\times U(1)_Y$$

$$\left(\begin{pmatrix} u \\ d \end{pmatrix}_L \quad d_L^c \quad \boxed{\nu_L^c} \right)$$

$$\frac{25}{\alpha_1'} = \frac{1}{\alpha_5} + \frac{24}{\alpha_1}$$

$$H_{\mathbf{10}} = \{Q_H, d_H^c, \nu_H^c\} \quad ; \quad H_{\bar{\mathbf{10}}} = \{Q_{\bar{H}}, d_{\bar{H}}^c, \nu_{\bar{H}}^c\} \ ,$$

$$h_{\mathbf{5}} = \{H_{\mathbf{2}}, H_{\mathbf{3}}\} \quad ; \quad h_{\bar{\mathbf{5}}} = \{H_{\bar{\mathbf{2}}}, H_{\bar{\mathbf{3}}}\}$$

$$W_G = HHh + \bar{H}\bar{H}\bar{h} + F\bar{H}\Phi + \mu h\bar{h} \ ,$$

missing partner

$$HHh \;\rightarrow\; d_H^c\langle\nu_H^c\rangle H_{\mathbf{3}}$$
$$\bar{H}\bar{H}\bar{h} \;\rightarrow\; \bar{d}_H^c\langle\bar{\nu}_H^c\rangle H_{\bar{\mathbf{3}}} \ ,$$

The $SU(5)$ and $U(1)$ gauge couplings continue to evolve above the scale M_{32}, eventually becoming equal at higher scale M_{51} ($\approx M_U$).

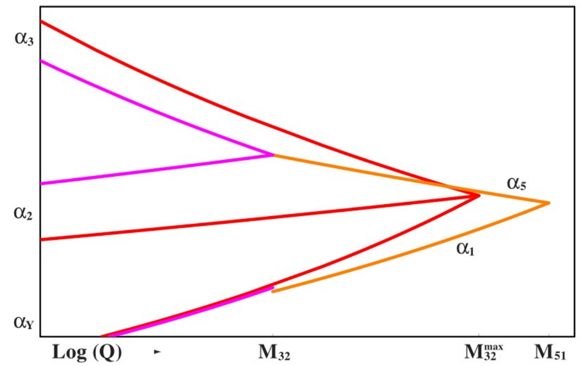
consistency condition $\quad M_{51} \geq M_{32}$

$$\alpha_1' \equiv \alpha_Y(M_{32}) \ \leq \alpha_5(M_{32})$$

when $\alpha_1' = \alpha_5(M_{32})$ $\qquad M_{32}^{\max} = M_{SU(5)}$

$$\alpha_s(M_Z) = \frac{\frac{7}{3}\alpha_{em}}{5\ \sin^2\theta_W - 1 + \frac{11}{2\pi}\alpha_{em}\ \ln\left(\frac{M_{32}^{\max}}{M_{32}}\right)} \ ,$$

$$M_{32} \leq M_{32}^{\max} \ \Longrightarrow \ \alpha_s^{\text{flipped } SU(5)}(M_Z) \leq \alpha_s^{SU(5)}(M_Z)$$



Motivation of Flipped SU(5)

Advantages:√

1. Breaking the GUT group in 4 dimensions while solving doublet-triplet splitting without using large GUT representations--Simple Higgs sector

the breaking of flipped SU(5) by nonzero VESs with $\mathbf{10}/\overline{\mathbf{10}}$ pair.

2. Achieve gauge coupling unification at the string scale 5×10^{17} GeV if extra vector-like particles are added in the weakly coupled heterotic string theory

---see Tianjun Li's work [hep-ph/0610054]

3. Natural from perturbative type II GUT constructions based on intersecting branes

4. Supression of Dim-5 proton decay.

Disadvantages:×

Not genuine GUT

flipped SU(5) with extra fermions

- d=6 proton decay modes can be rotated away.
- Also in ordinary flipped SU(5) with proper mixing angles
- Natural in non-minimal flipped SU(5) with extra fermions:

$$SU(5)\times U(1)_X \to SU(3)_C \times SU(2)_L \times U(1)_Y$$

$$\langle N^c_H\rangle = \langle \overline{N}^c_H\rangle \neq 0$$

$$\mathcal{F}_{(10,1)} = (q, \nu^c, D'^c) \qquad \mathcal{H}_{(10,1)} = (Q_H, N^c_H, D^c_H)$$

$$\overline{f}_{(\overline{5},-3)} = (L', u^c) \qquad \overline{\mathcal{H}}_{(\overline{10},-1)} = (\overline{Q}_H, \overline{N}^c_H, \overline{D}^c_H)$$

$$\ell^c_{(1,5)} = e^c \qquad h_{(5,-2)} = (h_d, \overline{\delta}^c_h)$$

$$\mathcal{G}_{(5,-2)} = (L, \overline{D'}^c) \qquad \overline{h}_{(\overline{5},2)} = (h_u, \delta^c_h)\,.$$

$$\overline{\mathcal{G}}_{(\overline{5},2)} = (\overline{L}', D^c)$$

D-T splitting

$$\mathcal{W}_0 = \mathcal{Y}_u \mathcal{F}\overline{f}\,\overline{h} + \mathcal{Y}_D \mathcal{F}\mathcal{G}\mathcal{H} + \mathcal{Y}_L \overline{f}\,\overline{\mathcal{G}}\mathcal{H} + \mu\mathcal{G}\overline{\mathcal{G}} + \lambda\mathcal{H}\mathcal{H}h + \overline{\lambda}\overline{\mathcal{H}}\overline{\mathcal{H}}\overline{h}$$

together with the following non-renormalisable terms

$$\mathcal{W}_N = \mathcal{Y}_\nu \mathcal{G}N\overline{h} + M_N NN$$

$$\delta\mathcal{W} = \frac{\mathcal{Y}'_d}{M}\mathcal{F}\overline{\mathcal{G}}h\overline{\mathcal{H}} + \frac{\mathcal{Y}'_e}{M}\mathcal{G}\ell^c h\overline{\mathcal{H}}$$

$$\mu\left(L\overline{L'} + \overline{D'}^c D^c\right) + \mathcal{Y}_D\langle N^c_H\rangle D'^c\overline{D'}^c + \mathcal{Y}_L\langle N^c_H\rangle L'\overline{L'}$$

$$= \mu_L \overline{L'}\mathcal{L} + \mu_D\overline{D'}^c\mathcal{D}^c\,,$$

heavy fields

$$\mathcal{Y}_u\,(q\,u^c + \nu^c L')\,h_u + \mathcal{Y}'_d\frac{\langle \overline{N}^c_H\rangle}{M}q\,D^c\,h_d + \mathcal{Y}'_e\frac{\langle \overline{N}^c_H\rangle}{M}Le^c h_d$$

$$\mathcal{L} \equiv \cos\theta_L\,L' + \sin\theta_L\,L \qquad \mathcal{D}^c \equiv \cos\theta_D\,D'^c + \sin\theta_D\,D^c$$

$$\ell \equiv -\sin\theta_L\,L' + \cos\theta_L\,L \qquad d^c \equiv -\sin\theta_D\,D'^c + \cos\theta_D\,D^c$$

light fields masses

$$\mathcal{Y}_u\,(q\,u^c - \sin\theta_L\,\nu^c\ell)\,h_u + \mathcal{Y}'_d\frac{\langle \overline{N}^c_H\rangle}{M}(\cos\theta_D\,q\,d^c)\,h_d + \mathcal{Y}'_e\frac{\langle \overline{N}^c_H\rangle}{M}(\cos\theta_L\,\ell\,e^c)\,h_d$$

$$\mu \ll \mathcal{Y}_L\langle N^c_H\rangle \longrightarrow \theta_L \to 0$$

$$\mu \ll \mathcal{Y}_D\langle N^c_H\rangle \longrightarrow \theta_D \to 0$$

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gauge mediated $D=6$ operator $\quad \mathcal{F}\mathcal{F}^\dagger\overline{f}\,\overline{f}^\dagger \longrightarrow \quad qD'^{c\dagger}L'u^{c\dagger} \to \sin\theta_L\,\sin\theta_D\;q\,d^{c\dagger}\ell\,u^{c\dagger}$

Supressed d=6 proton decay

$SU(6)\times SU(2)_R$ from E_6

arXiv:1405.2274

$$\Psi_{(15,1)} = (\mathcal{F},\mathcal{G}) \qquad \begin{aligned}\Phi_{(15,1)} &= (\mathcal{H},h_1)\\ \overline{\Phi}_{(\overline{15},1)} &= (\overline{\mathcal{H}},\overline{h}_1)\end{aligned} \qquad \psi_{(\overline{6},2)} = (\ell^c,\overline{f},N,\overline{\mathcal{G}}) \qquad \begin{aligned}\phi_{(\overline{6},2)} &= (\ell^c_H,\overline{f}_H,N_H,\overline{h}_2)\\ \overline{\phi}_{(6,2)} &= (\overline{\ell}^c_H,f_H,\overline{N}_H,h_2)\end{aligned}$$

New Fields in addition to non-minimal flipped SU(5):

$\left(\ell^c_H,\overline{f}_H,N_H,\overline{\ell}^c_H,f_H,\overline{N}_H\right)$ and extra matter singlet field (N)

$$W = \mathcal{W}_1+\mathcal{W}_2$$

$$\mathcal{W}_1 = \mathcal{Y}_D\,\Psi\Psi\Phi + \mathcal{Y}_L\,\psi\psi\Phi + \lambda_1\overline{\Phi}^3 + \lambda_2\,\phi^2\Phi$$

$$\mathcal{W}_2 = \frac{\mathcal{Y}}{M}\Psi\psi\overline{\phi}\,\overline{\Phi} + \frac{\lambda'}{M}\Phi^2\overline{\phi}^2\,,$$

$$SU(6)\times SU(2)_R \overset{\langle N_H,\overline{N}_H\rangle}{\longrightarrow} SU(5)\times U(1)_X \overset{\langle N^c_H,\overline{N}^c_H\rangle}{\longrightarrow} SU(3)_C\times SU(2)_L\times U(1)_Y$$

higgsed away $\ell^c_H,\overline{\ell}^c_H,f_H,\overline{f}_H,(N_H-\overline{N}_H)/\sqrt{2}$

leaving $h_2,\overline{h}_2 \quad (N_H+\overline{N}_H)/\sqrt{2}$.

Higgs superpotential

$$\mathcal{W}_H = \lambda_1\overline{\mathcal{H}}^2\overline{h}_1 + \lambda_2\,\langle N_H\rangle\overline{h}_2h_1 + \frac{\lambda'}{M}\langle\overline{N}_H\rangle\mathcal{H}^2h_2$$

$(h_1,\overline{h}_2)$ becomes heavy   irrelevant for fermion masses.

Missing partner mechanism

$$\text{up quarks}\;:\; \frac{\mathcal{Y}}{M}\mathcal{F}\overline{f}\,\overline{N}_H\overline{h}_1 \sim \mathcal{Y}\frac{\langle\overline{N}_H\rangle}{M}\mathcal{F}\overline{f}\,\overline{h}_1$$

$$\text{down quarks}\;:\; \frac{\mathcal{Y}}{M}\mathcal{F}\overline{\mathcal{G}}h_2\overline{\mathcal{H}} \sim \mathcal{Y}\frac{\langle\overline{N}^c_H\rangle}{M}\mathcal{F}\overline{\mathcal{G}}h_2$$

$$\text{charged leptons}\;:\; \frac{\mathcal{Y}}{M}\mathcal{G}\,\ell^ch_2\overline{\mathcal{H}} \sim \mathcal{Y}\frac{\langle\overline{N}^c_H\rangle}{M}\mathcal{G}\ell^ch_2\,.$$

The terms associated with proton suppression mechanism

$$\mathcal{W}_M = \mathcal{Y}_D\,\mathcal{F}\mathcal{G}\mathcal{H} + \mathcal{Y}_L\,\overline{f}\,\overline{\mathcal{G}}\mathcal{H} + \frac{\mathcal{Y}}{M}\mathcal{G}\overline{\mathcal{G}}h_2\overline{h}_1$$

Supressed d=6 proton decay as in extended flipped SU(5)

Modular Symmetry

- Flavor symmetries are interesting approaches to attack the origin of fermion mass hierarchies and mixing angles.
- The modular symmetry is a geometrical symmetry of T^2 and T^2/Z_2, and corresponds to change of their cycle basis.
- Matter modes transform non-trivially under the modular symmetry. That is, the modular symmetry is a flavor symmetry.
- Texture zeros of the fermion mass matrix provide an attractive approach to understand the flavor mixing. Those can be possibly related with modular flavor symmetries.
- The modular flavor symmetry also control higher dimensional operators.Allowed couplings are controlled by stringy symmetries and n-point couplings are written by products of 3-point couplings.

Modular group

See Gui-jun Ding's lecture notes

modular group $\Gamma \equiv \mathrm{SL}(2,Z)$ can act on the upper half plane

$$\gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \text{for } \gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,Z), \quad \mathtt{Im}(\tau) > 0\,.$$

infinite normal subgroups

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,Z) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ N) \right\}$$

inhomogeneous modular group is $\bar{\Gamma} = \Gamma/\{I, -I\}$

The group $\overline{\Gamma}$ is generated by S and T

$$S^2 = \mathbb{1}\ ,\quad (ST)^3 = \mathbb{1}$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S : \tau \to -\frac{1}{\tau} \qquad T : \tau \to \tau + 1$$

$$\overline{\Gamma}(N) = \begin{cases} \Gamma(N) \text{ for } N > 2 \\ \Gamma(N)/\{I, -I\} \text{ for } N = 2 \end{cases}$$

inhomogeneous finite modular group Γ_N $\qquad \Gamma_N \cong \overline{\Gamma}/\overline{\Gamma}(N)$

$\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$

Modular forms of weight k and level N $\qquad$ k an even and non-negative integer

holomorphic functions $f(\tau)$ transforming under the action of $\Gamma(N)$ $\qquad f(\gamma\tau) = (c\tau + d)^k f(\tau)\,, \quad \gamma \in \Gamma(N)$

Modular forms of weight $2k$ and level N form a linear space $\mathcal{M}_{2k}(\Gamma(N))$ finite dimension $d_{2k}(\Gamma(N))$

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)\,, \quad \gamma \in \bar{\Gamma}$$

$\rho(\gamma)_{ij}$ is a unitary matrix under Γ_N

Modular GUT for flipped SU(5)

See Feruglio arXiv:1706.08749

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\; K(\Phi, \bar{\Phi}) + \int d^4x d^2\theta\; w(\Phi) + h.c.$$

the supermultiplets $\varphi^{(I)}$ transform in representation $\rho^{(I)}$ of a quotient group Γ_N

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ \varphi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \varphi^{(I)} \end{cases}$$

γ an element of Γ_N.

The invariance of the action $\mathcal{S}$ requires
$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + f(\Phi) + f(\overline{\Phi}) \end{cases}$$

invariance of the Kahler potential

$$K(\Phi, \bar{\Phi}) = -h \log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

invariance of the superpotential $w(\Phi)$ under the modular group

$$w(\Phi) = \sum_n Y_{I_1 \ldots I_n}(\tau)\; \varphi^{(I_1)} \ldots \varphi^{(I_n)}$$

the functions $Y_{I_1 \ldots I_n}(\tau)$ modular forms of weight $k_Y(n)$ transforming in the representation ρ of Γ_N

$$k_Y(n) = k_{I_1} + \ldots + k_{I_n}$$

The product $\rho \times \rho^{I_1} \times \ldots \times \rho^{I_n}$ contains an invariant singlet

Multiple modular symmetries

See Ye-Ling Zhou's paper 1906.02208

finite modular transformations $\gamma_1, \ldots, \gamma_M$ in $\Gamma^1_{N_1} \times \Gamma^2_{N_2} \times \cdots \times \Gamma^M_{N_M}$

$$\gamma_J : \tau_J \to \gamma_J \tau_J = \frac{a_J \tau_J + b_J}{c_J \tau_J + d_J}\,,$$

$$\begin{aligned}\phi_i(\tau_1, \ldots, \tau_M) &\to \phi_i(\gamma_1\tau_1, \ldots, \gamma_M\tau_M)\\ &= \prod_{J=1,\ldots,M} (c_J\tau_J + d_J)^{-2k_{i,J}} \bigotimes_{J=1,\ldots,M} \rho_{I_{i,J}}(\gamma_J)\phi_i(\tau_1, \tau_2, \ldots, \tau_M)\end{aligned}$$

$$\begin{aligned}Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1, \ldots, \tau_M) &\to Y_{(I_{Y,1},\ldots,I_{Y,M})}(\gamma_1\tau_1, \ldots, \gamma_M\tau_M)\\ &= \prod_{J=1,\ldots,M} (c_J\tau_J + d_J)^{2k_{Y,J}} \bigotimes_{J=1,\ldots,M} \rho_{I_{Y,J}}(\gamma_J) Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1, \ldots, \tau_M)\,.\end{aligned}$$

$$K(\phi_i, \overline{\phi}_i; \tau_1, \ldots, \tau_M, \overline{\tau}_1, \ldots, \overline{\tau}_M) = -\sum_{J=1,\ldots,M} h_J \log(-i\tau_J + i\overline{\tau}_J) + \sum_i \frac{\overline{\phi}_i \phi_i}{\prod_{J=1,\ldots,M} (-i\tau_J + i\overline{\tau}_J)^{2k_{i,J}}}\,,$$

$$W(\phi_i; \tau_1, \ldots, \tau_M) = \sum_n \sum_{\{i_I, \cdots, i_n\}} \left(Y_{(I_{Y,1},\ldots,I_{Y,M})}\phi_{i_1} \cdots \phi_{i_n}\right)_{\mathbf{1}}\,,$$

Modular GUT for flipped SU(5)

A_4 is the group of even permutations of four objects.

regular tetrahedron

The finite modular group $A_4 \cong \Gamma_3$ can generated by $S^2 = (ST)^3 = T^3 = 1$

Four irreducible representations

	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	3 (Real)	3 (Complex)
S	1	1	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$
T	1	ω	ω^2	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

$$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'', \qquad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}, \qquad \mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}',$$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3}_S \oplus \mathbf{3}_A,$$

two triplets $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ and $\psi = (\psi_1, \psi_2, \psi_3)$

	Real basis	Complex basis
$(\varphi\psi)_{1}$	$\varphi_1\psi_1 + \varphi_2\psi_2 + \varphi_3\psi_3$	$\varphi_1\psi_1 + \varphi_2\psi_3 + \varphi_3\psi_2$
$(\varphi\psi)_{1'}$	$\varphi_1\psi_1 + \omega^2\varphi_2\psi_2 + \omega\varphi_3\psi_3$	$\varphi_3\psi_3 + \varphi_1\psi_2 + \varphi_2\psi_1$
$(\varphi\psi)_{1''}$	$\varphi_1\psi_1 + \omega\varphi_2\psi_2 + \omega^2\varphi_3\psi_3$	$\varphi_2\psi_2 + \varphi_3\psi_1 + \varphi_1\psi_3$
$(\varphi\psi)_{3_S}$	$\begin{pmatrix} \varphi_2\psi_3 + \varphi_3\psi_2 \\ \varphi_3\psi_1 + \varphi_1\psi_3 \\ \varphi_1\psi_2 + \varphi_2\psi_1 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 2\varphi_1\psi_1 - \varphi_2\psi_3 - \varphi_3\psi_2 \\ 2\varphi_3\psi_3 - \varphi_1\psi_2 - \varphi_2\psi_1 \\ 2\varphi_2\psi_2 - \varphi_3\psi_1 - \varphi_1\psi_3 \end{pmatrix}$
$(\varphi\psi)_{3_A}$	$\begin{pmatrix} \varphi_2\psi_3 - \varphi_3\psi_2 \\ \varphi_3\psi_1 - \varphi_1\psi_3 \\ \varphi_1\psi_2 - \varphi_2\psi_1 \end{pmatrix}$	$\begin{pmatrix} \varphi_2\psi_3 - \varphi_3\psi_2 \\ \varphi_1\psi_2 - \varphi_2\psi_1 \\ \varphi_3\psi_1 - \varphi_1\psi_3 \end{pmatrix}$

A_4 is the minimal choice which admits triplet representations

Modular GUT for flipped SU(5)

In modular GUT framework, the superfields within a multiplet of GUT gauge group should transform identically with the same modulus field.

It seems that the unification of matter contents will be spoiled if different values of modulus are assigned separately for quarks and leptons.

We propose to reconcile such an inconsistency in the orbifold GUT 5D $\mathcal{M}_4 \times S^1/Z_2$ orbifold.

Compactification on S^1/Z_2 is obtained by identifying the fifth coordinate y under the two operations

$$Z : y \to -y, \qquad T : y \to y + 2\pi R.$$

5D $N = 1$ SUSY (corresponding to 4D $N = 2$ SUSY) to 4D $N = 1$ SUSY by proper boundary conditions

$$S = \int d^5x \frac{1}{kg^2} \mathrm{Tr}\left[\frac{1}{4}\int d^2\theta \, (W^\alpha W_\alpha + \mathrm{h.c.}) + \int d^4\theta \left((\sqrt{2}\partial_5 + \bar{\Sigma})e^{-V}(-\sqrt{2}\partial_5 + \Sigma)e^V + \partial_5 e^{-V}\partial_5 e^V\right)\right]$$
$$+ \int d^5x \left[\int d^4\theta \left(\Phi^c e^V \bar{\Phi}^c + \bar{\Phi}e^{-V}\Phi\right) + \int d^2\theta \left(\Phi^c(\partial_5 - \frac{1}{\sqrt{2}}\Sigma)\Phi + \mathrm{h.c.}\right)\right]$$

vector multiplet

$$V(x^\mu, y) \to V(x^\mu, -y) = PV(x^\mu, y)P^{-1},$$
$$\Sigma(x^\mu, y) \to \Sigma(x^\mu, -y) = -P\Sigma(x^\mu, y)P^{-1}$$

hypermultiplet fundamental representations,

$$\Phi(x^\mu, y) \to \Phi(x^\mu, -y) = \eta_\Phi P\Phi(x^\mu, y),$$
$$\Phi^c(x^\mu, y) \to \Phi^c(x^\mu, -y) = -\eta_\Phi P\Phi^c(x^\mu, y)$$

Modular GUT for flipped SU(5)

$$P_{O(y=0)} = \mathrm{diag}\,(\ +1,\ \ +1,\ \ +1,\ \ +1,\ \ +1)\,,$$
$$P_{O'(y=\pi R)} = \mathrm{diag}\,(\ +1,\ \ +1,\ \ +1,\ \ -1,\ \ -1)\,,$$

and proper brane mass terms

For Yukawa couplings $W \supseteq y^D_{ij} F_i F_j h + y^U_{ij} F_i \bar{f}_j \bar{h} + y^E_{ij} \bar{f}_i E_j h\,,$

For Higgs sector $W \supseteq HHh + \overline{H}\,\overline{H}\bar{h} + X(\overline{H}H - M_H^2)$

introduce additional neutral superfield $S_i(1,0)$ for neutrino sector

inverse seesaw mechanism $\mathcal{W} \supseteq Y^U_{ij} F_i \bar{f}_j \bar{h} + Y^S_{ij} \overline{H} S_i F_j + \frac{M_{SS;ij}}{2} S_i S_j$

$$m_\nu \approx m_D^T M_{SN}^{-1} M_{SS} (M_{SN}^T)^{-1} m_D$$

neutrino mass matrix

$$\mathcal{M} = \begin{pmatrix} 0 & m_D^T & 0 \\ m_D & 0 & M_{SN} \\ 0 & M_{SN} & M_{SS} \end{pmatrix}$$

$$m_D \sim Y_2 v_u \qquad M_{SN} \sim Y_S M_H$$

$$T^a_F(\mathbf{10_1}) = Q^a_L(\mathbf{3},\mathbf{2})^{(+,-)}_{(\mathbf{1},\mathbf{1/6})} \oplus D^{c,a}_L(\bar{\mathbf{3}},1)^{(+,-)}_{(\mathbf{1},\mathbf{-2/3})} \oplus N^{c,a}_L(1,1)^{(+,+)}_{(\mathbf{1},\mathbf{1})}\,,$$
$$T'^{,a}_F(\mathbf{10_1}) = Q^a_L(\mathbf{3},\mathbf{2})^{(+,-)}_{(\mathbf{1},\mathbf{1/6})} \oplus D^{c,a}_L(\bar{\mathbf{3}},1)^{(+,+)}_{(\mathbf{1},\mathbf{-2/3})} \oplus N^{c,a}_L(1,1)^{(+,-)}_{(\mathbf{1},\mathbf{1})}$$
$$T''^{,a}_F(\mathbf{10_1}) = Q^a_L(\mathbf{3},\mathbf{2})^{(+,+)}_{(\mathbf{1},\mathbf{1/6})} \oplus D^{c,a}_L(\bar{\mathbf{3}},1)^{(+,-)}_{(\mathbf{1},\mathbf{-2/3})} \oplus N^{c,a}_L(1,1)^{+,-}_{(\mathbf{1},\mathbf{1})}\,,$$
$$F^a_{\bar{f}}(\bar{\mathbf{5}}_{\mathbf{-3}}) = U^{c,a}_L(\bar{\mathbf{3}},1)^{(+,+)}_{(\mathbf{-3},\mathbf{1/3})} \oplus L^a_L(\mathbf{1},\mathbf{2})^{(+,-)}_{(\mathbf{-3},\mathbf{-1/2})}\,,$$
$$F'^{,a}_{\bar{f}}(\bar{\mathbf{5}}_{\mathbf{-3}}) = U^{c,a}_L(\bar{\mathbf{3}},1)^{(+,-)}_{(\mathbf{-3},\mathbf{1/3})} \oplus L^a_L(\mathbf{1},\mathbf{2})^{(+,+)}_{(\mathbf{-3},\mathbf{-1/2})}\,,$$
$$O^a_E(\mathbf{1_{-5}}) = E^{c,a}_L(\mathbf{1},\mathbf{1})^{(+,+)}_{(\mathbf{-5},\mathbf{0})}\,, \quad O^a_S(\mathbf{1_0}) = S^a(\mathbf{1},\mathbf{1})^{(+,+)}_{(\mathbf{0},\mathbf{0})}\,,$$

$$h(\mathbf{5_{-2}}) = H_T(\mathbf{3},\mathbf{1})^{(+,-)}_{(-2,-1/3)} \oplus H_D(\mathbf{1},\mathbf{2})^{(+,+)}_{(-2,1/2)}\,,$$
$$\bar{h}(\bar{\mathbf{5}}_{\mathbf{2}}) = H'_T(\mathbf{3},\mathbf{1})^{(+,-)}_{(2,1/3)} \oplus H_{\mathrm{U}}(\mathbf{1},\mathbf{2})^{(+,+)}_{(2,-1/2)}\,,$$
$$H(\mathbf{10}_1) = H_{TQ}(\mathbf{3},\mathbf{2})^{(+,-)}_{(\mathbf{1},\mathbf{1/6})} \oplus H_{TD}(\bar{\mathbf{3}},1)^{(+,-)}_{(\mathbf{1},\mathbf{-2/3})} \oplus H_N(1,1)^{(+,+)}_{(\mathbf{1},\mathbf{1})}\,,$$
$$\overline{H}(\overline{\mathbf{10}}_{-1}) = \overline{H}_{TQ}(\bar{\mathbf{3}},\bar{\mathbf{2}})^{(+,-)}_{(\mathbf{-1},\mathbf{-1/6})} \oplus \overline{H}_{TD}(\bar{\mathbf{3}},1)^{(+,-)}_{(\mathbf{1},\mathbf{-2/3})} \oplus \overline{H}_N(1,1)^{(+,+)}_{(\mathbf{-1},\mathbf{-1}}$$
$$O_X(\mathbf{1}_0) = X(\mathbf{1},\mathbf{1})^{(+,+)}_{(\mathbf{0},\mathbf{0})}\,.$$

Further breaking[1] of $\mathrm{U}(1)_X \times \mathrm{U}(1)_{Y'}$ into $\mathrm{U}(1)_Y$ triggered via proper Higgs field

$(\mathcal{Z},\mathcal{T})$	KK modes	4D masses
$(+,+)$	$\cos[ny/R]$	n/R
$(+,-)$	$\cos[(n+1/2)y/R]$	$(n+1/2)/R$
$(-,+)$	$\sin[(n+1)y/R]$	$(n+1)/R$
$(-,-)$	$\sin[(n+1/2)y/R]$	$(n+1/2)/R$

three families	$A_4^Q \times A_4^L$
$T_F^{\prime,a}(\mathbf{10_1}), T_F^{\prime\prime,a}(\mathbf{10_1})$	$(3,1)$
$T_F^a(\mathbf{10_1})$	$(1,3)$
$F_{\bar f}^a(\overline{\mathbf{5}}_{\mathbf{-3}})$	$(3,1)$
$F_{\bar f}^{\prime,a}(\overline{\mathbf{5}}_{\mathbf{-3}})$,$O_E^a(\mathbf{1_{-5}})$	$(1,3)$
$O_S^a(\mathbf{1_0})$	$(1,3)$

The fittings of the SM plus neutrino flavor structure are sometimes not good enough with a single modulus field, which on the other hand prefer multiple values of modulus VEVs for various sectors.

zero modes for the quarks and leptons transform as $(3,1)$ and $(1,3)$ under $A_4^Q \times A_4^L$, respectively.

The superpotential in the SU(5) preserving $O(y=0)$ brane can be written as

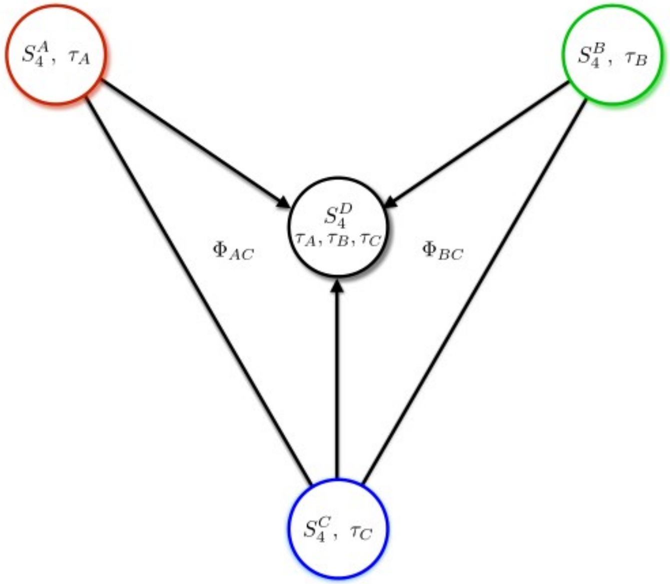
$$\mathcal{L} \supseteq \delta(y) \int d^2\theta \left[Y_{ab;23}^D T_F^{a;2} T_F^{b;3} h + Y_{ab;12}^N T_F^{a;1} F_{\bar f}^{b;2} \bar h + Y_{ab;31}^U T_F^{a;3} F_{\bar f}^{b;1} \bar h + Y_{ab;1}^E F_{\bar f}^{a;1} E^b h \right.$$
$$\left. + \; Y_{ab;1}^S \overline{H} O_S^a T_F^{b;1} + \frac{M_{SS;ab}}{2} O_S^a O_S^b + O_X(\overline{H}H - M_H^2) \right] \, .$$

to break the multiple $A_4^Q \times A_4^L$ by BCs to diagonal A_4^D

Break the Multiple Modular Symmetries via Higgs Fields

See Ye-Ling Zhou's paper 1906.02208

Illustration of the breaking of $S_4^A \times S_4^B \times S_4^C \to S_4^D$



Vacuum alignments

for the bi-triplet scalars Φ_{AC} and Φ_{BC}

Fields	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
χ_{AC}	**3**	**1**	**3**	0	0	0
χ_{BC}	**1**	**3**	**3**	0	0	0
χ_A	**3**	**1**	**1**	0	0	0
χ_B	**1**	**3**	**1**	0	0	0

$$\begin{aligned} w_d &= \Phi_{AC}\Phi_{AC}\chi_{AC} + \mathrm{M}_A\Phi_{AC}\chi_{AC} + \Phi_{AC}\Phi_{AC}\chi_A\,, \\ &+ \Phi_{BC}\Phi_{BC}\chi_{BC} + \mathrm{M}_B\Phi_{BC}\chi_{BC} + \Phi_{BC}\Phi_{BC}\chi_B\,, \end{aligned}$$

$$\sum_{j,k=1,2,3;}\ \sum_{\beta,\gamma=1,2,3} |\epsilon_{ijk}||\epsilon_{\alpha\beta\gamma}|(\tilde{\Phi}_{AC})_{j\beta}(\tilde{\Phi}_{AC})_{kc} + \mathrm{M}_A(\tilde{\Phi}_{AC})_{i\alpha} = 0$$

24 solutions $\quad \langle\Phi_{AC}\rangle = \rho_{\mathbf{3}}(\gamma)P_{23}v_{AC}$

$$S_4^A \times S_4^C \to S_4^D$$

$$\sum_{j,k=1,2,3;}\ \sum_{\alpha=1,2,3} |\epsilon_{ijk}|(\tilde{\Phi}_{AC})_{j\alpha}(\tilde{\Phi}_{AC})_{k\alpha} = 0$$

$\langle\breve{\Phi}_{AC}\rangle_{i\alpha} = (\breve{P}_{23})_{i\alpha}v_{AC}$ corresponding to $\langle\breve{\tilde{\Phi}}_{AC}\rangle_{i\alpha} \propto \delta_{i\alpha}$

Reduce the Multiple Modular Symmetries via Boundary Conditions

It is possible to break the multiple $A_4^Q \times A_4^L$ by BCs to diagonal A_4^D, which is then identified to be the (single) modular A_4 symmetry in the low energy effective theory.

choices with $\Phi_{i\alpha}^{(++)}$ for fixed $'i'$ (or $'\alpha'$) corresponds to the breaking of $A_4^Q \times A_4^L$ to A_4^Q (or A_4^L), respectively.

We can introduce bi-triplets to reduce the modular symmetries into the diagonal one.

assign the following BCs for the bi-triplet $(\mathbf{3},\mathbf{3})$ fields $\Phi_{i\alpha}$ of $A_4^Q \times A_4^L$,

i, α the indices for A_4^Q and A_4^L, respectively.

$A_4^Q \times A_4^L$ to the diagonal A_4^D

$\gamma^Q \in A_4^Q$ and $\gamma^L \in A_4^L$ being associated to the $\gamma^D \in A_4^D$ by $\gamma^Q = \gamma^L = \gamma^D$.

$$\mathbf{3}\otimes\mathbf{3} = \mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}''\oplus\mathbf{3_s}\oplus\mathbf{3_a}, \qquad \Phi_\kappa^{\mathbf{r}} = \sum_{i,\alpha} C_{i\alpha;\kappa}^{\mathbf{r}}\Phi_{i\alpha}\,,$$

$$\Phi = (\Phi^{\mathbf{1}})^{(++)} \oplus (\Phi^{\mathbf{1}'})^{(+-)} \oplus (\Phi^{\mathbf{1}''})^{(+-)} \oplus (\Phi_\kappa^{\mathbf{3_s}})^{(+-)} \oplus (\Phi_\kappa^{\mathbf{3_a}})^{(+-)}\,.$$

$$\Phi^{\mathbf{1}} = \frac{1}{\sqrt{3}}\left(\Phi_{11} + \Phi_{23} + \Phi_{32}\right)\,,$$ only the zero modes of the singlet (that is, $\Phi^{\mathbf{1}}$) survives.

any combination of the representations can be allowed to act as the BCs that break the $A_4^Q \times A_4^L$ to A_4^D

Classification according to the choice of representation and modulaı weights

Up-type quark sector

- $\rho_{\bar{f}} = \mathbf{3},\ \rho_F = \mathbf{3}$.

$$k_{\bar{f}} + k_F = 0; \quad (y_U)_{ij} = \beta_1 S^0_{\mathbf{1}}(\tau)\,,$$

$$k_{\bar{f}} + k_F = 2; \quad (y_U)_{ij} = \beta_1 S^{(2)}_{\mathbf{3}}(\tau) + \beta_2 A^{(2)}_{\mathbf{3}}(\tau)\,,$$

$$k_{\bar{f}} + k_F = 4; \quad (y_U)_{ij} = \beta_1\, S^{(4)}_{\mathbf{3}} + \beta_2 A^{(4)}_{\mathbf{3}} + \beta_3 S^{(4)}_{\mathbf{1}} + \beta_4 S^{(4)}_{\mathbf{1'}}\,,$$

$$k_{\bar{f}} + k_F = 6; \quad (y_U)_{ij} = \beta_1\, S^{(6)}_{\mathbf{3}I} + \beta_2 A^{(6)}_{\mathbf{3}I} + \beta_3 S^{(6)}_{\mathbf{3}II} + \beta_4 A^{(6)}_{\mathbf{3}II} + \beta_5 S^{(6)}_{\mathbf{1}}\,,$$

$$k_{\bar{f}} + k_F = 8;$$

$$(y_U)_{ij} = \beta_1\, S^{(8)}_{\mathbf{3}I} + \beta_2 A^{(8)}_{\mathbf{3}I} + \beta_3 S^{(8)}_{\mathbf{3}II} + \beta_4 A^{(8)}_{\mathbf{3}II} + \beta_5 S^{(8)}_{\mathbf{1}} + \beta_6 S^{(8)}_{\mathbf{1'}} + \beta_7 S^{(8)}_{\mathbf{1''}}\,,$$

$$S^{(k)}_1(\tau) = Y^{(k)}_{\mathbf{1}}(\tau)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S^{(k)}_{\mathbf{1'}} = Y^{(k)}_{\mathbf{1'}}(\tau)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A^{(k)}_{\mathbf{3}}(\tau) = \begin{pmatrix} 0 & Y^{(k)}_{\mathbf{3},3}(\tau) & -\,Y^{(k)}_{\mathbf{3},2}(\tau) \\ -Y^{(k)}_{\mathbf{3},3}(\tau) & 0 & Y^{(k)}_{\mathbf{3},1}(\tau) \\ Y^{(k)}_{\mathbf{3},2}(\tau) & -Y^{(k)}_{\mathbf{3},1}(\tau) & 0 \end{pmatrix}$$

$$S^{(k)}_{\mathbf{1''}} = Y^{(k)}_{\mathbf{1''}}(\tau)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S^{(k)}_{\mathbf{3}}(\tau) = \begin{pmatrix} 2Y^{(k)}_{\mathbf{3},1}(\tau) & -Y^{(k)}_{\mathbf{3},3}(\tau) & -Y^{(k)}_{\mathbf{3},2}(\tau) \\ -Y^{(k)}_{\mathbf{3},3}(\tau) & 2Y^{(k)}_{\mathbf{3},2}(\tau) & -Y^{(k)}_{\mathbf{3},1}(\tau) \\ -Y^{(k)}_{\mathbf{3},2}(\tau) & -Y^{(k)}_{\mathbf{3},1}(\tau) & 2Y^{(k)}_{\mathbf{3},3}(\tau) \end{pmatrix}$$

Classification according to the choice of representation and modulaı weights

Up-type quark sector

- $\rho_{\bar{f}} = \mathbf{3},\ \rho_F = \mathbf{1}, \mathbf{1}', \mathbf{1}''.$
- $\rho_{\bar{f}} = \mathbf{1}, \mathbf{1}', \mathbf{1}'',\ \rho_F = \mathbf{3}.$
- $\rho_{\bar{f}} = \mathbf{1}, \mathbf{1}', \mathbf{1}'',\ \rho_F = \mathbf{1}, \mathbf{1}', \mathbf{1}''.$

Down-type quark sector

- $\rho_F = \mathbf{3}.$
- $\rho_F = \mathbf{1}, \mathbf{1}', \mathbf{1}''.$

Charged lepton sector

- $\rho_{\bar{f}} = \mathbf{3}, \rho_E = \mathbf{3}.$
- $\rho_{\bar{f}} = \mathbf{3}, \rho_E = \mathbf{1}, \mathbf{1}', \mathbf{1}''.$
- $\rho_{\bar{f}} = \mathbf{1}, \mathbf{1}', \mathbf{1}'', \rho_E = \mathbf{3}.$
- $\rho_{\bar{f}} = \mathbf{1}, \mathbf{1}', \mathbf{1}'', \rho_E = \mathbf{1}, \mathbf{1}', \mathbf{1}''.$

Neutrino sector

- $\rho_F = \mathbf{3}, \rho_S = \mathbf{3}.$
- $\rho_F = \mathbf{3}, \rho_S = \mathbf{1}, \mathbf{1}', \mathbf{1}''.$
- $\rho_F = \mathbf{1}, \mathbf{1}', \mathbf{1}'', \rho_S = \mathbf{3}.$
- $\rho_F = \mathbf{1}, \mathbf{1}', \mathbf{1}'', \rho_S = \mathbf{1}, \mathbf{1}', \mathbf{1}'$

Form of the mass matrix for representations

$\boldsymbol{\rho_{\bar{f}} = 3,\ \rho_F = 3,\ \rho_E = 3,\ \rho_S = 3}$

$$W \supseteq \left[\alpha_1 Y_{\mathbf{3}}^{(4)} (FF)_{\mathbf{3}S} h + \alpha_2 Y_{\mathbf{1}}^{(4)} (FF)_{\mathbf{1}} h + \alpha_3 Y_{\mathbf{1}'}^{(4)} (FF)_{\mathbf{1}''} h\right]$$
$$+ \left[\beta_1 Y_{\mathbf{3}}^{(2)} (F\bar{f})_{\mathbf{3}S} \bar{h} + \beta_2 Y_{\mathbf{3}}^{(2)} (F\bar{f})_{\mathbf{3}A} \bar{h}\right]$$
$$+ \gamma_1 (\bar{f}E)_{\mathbf{1}} h + \left[\lambda_1 Y_{\mathbf{3}}^{(2)} (SF)_{\mathbf{3}S} \overline{H} + \lambda_2 Y_{\mathbf{3}}^{(2)} (SF)_{\mathbf{3}A} \overline{H}\right] + \frac{\kappa_1}{2} \Lambda_2 (SS)_{\mathbf{1}}$$

- $(k_{\bar{f}}, k_F, k_E, k_S) = (0, 2, 0, 0).$

$$\begin{aligned}
\mathcal{M}_U / v_u \equiv (y_U)_{ij} &= \beta_1 S_{\mathbf{3}}^{(2)}(\tau) + \beta_2 A_{\mathbf{3}}^{(2)}(\tau)\,, \\
\mathcal{M}_D / v_d \equiv (y_D)_{ij} &= \alpha_1 S_{\mathbf{3}}^{(4)} + \alpha_2 S_{\mathbf{1}}^{(4)} + \alpha_3 S_{\mathbf{1}'}^{(4)}\,, \\
\mathcal{M}_E / v_d \equiv (y_E)_{ij} &= \gamma_1 S_{\mathbf{1}}^{0}(\tau)\,, \\
\mathcal{M}_N^{Dirac} / v_u \equiv (y_N)_{ij}^{Dirac} &= (y_U)^T\,, \\
\mathcal{M}_{ij}^{SN} &= \lambda_1 \Lambda_1 S_{\mathbf{3}}^{(2)}(\tau) + \lambda_2 \Lambda_1 A_{\mathbf{3}}^{(2)}(\tau)\,, \\
\mathcal{M}_{ij}^{SS} &= \kappa_1 \Lambda_2 S_{\mathbf{1}}^{0}(\tau)\,,
\end{aligned}$$

Classification according to the choice of representation and modular

$\rho_{\bar{f}} = 3,\ \rho_F = 3,\ \rho_E = 3,\ \rho_S = 3$

- $(k_{\bar{f}}, k_F, k_E, k_S) = (2, 2, 0, 0)$. 5 real and 7 complex free parameters.
- $(k_{\bar{f}}, k_F, k_E, k_S) = (4, 2, 0, 0)$. 5 real and 10 complex free parameters.
- $(k_{\bar{f}}, k_F, k_E, k_S) = (4, 2, 2, 0)$. 5 real and 11 complex free parameters.
- $(k_{\bar{f}}, k_F, k_E, k_S) = (4, 2, 2, 2)$. 5 real and 15 complex free parameters.
- $(k_{\bar{f}}, k_F, k_E, k_S) = (4, 4, 2, 2)$. 5 real and 20 complex free parameters.
- $(k_{\bar{f}}, k_F, k_E, k_S) = (4, 4, 4, 2)$. 5 real and 22 complex free parameters.
- $(k_{\bar{f}}, k_F, k_E, k_S) = (4, 4, 4, 4)$. 5 real and 26 complex free parameters.

$\rho_{\bar{f}_i} = 1, 1', 1'',\ \rho_F = 3,\ \rho_E = 3, \rho_S = 3$

Discard the scenarios with too many free parameters.

$\rho_{\bar{f}} = 3,\ \rho_F = 1, 1', 1'',\ \rho_E = 3, \rho_S = 3$

$\rho_{\bar{f}} = 3,\ \rho_F = 3,\ \rho_E = 1, 1', 1'', \rho_S = 3$

$\rho_{\bar{f}} = 3,\ \rho_F = 3,\ \rho_E = 3, \rho_S = 1, 1', 1''$

Numerical fitting

To keep the predictive power we concentrate on the scenarios in which at lea the $\bar{f}, F, E, S$ superfields transforms as the triplet of A_4 modular group.

The GUT-scale flavor structures of quarks and leptons predicted by our models need to be evolved to the EW scale with the renormalization group equation (RGE) be the implement of the χ^2 fit to the experimental data of SM and neutrino

we use two-loop RGE to evolve our predicted

observable	Value2
$y_u/10^{-6}$	6.644
$y_c/10^{-3}$	3.445
y_t	0.868
$y_d/10^{-5}$	1.323
$y_s/10^{-4}$	1.841
$y_b/10^{-2}$	1.395
θ^q_{12}	0.22737
$\theta^q_{13}/10^{-3}$	3.716
$\theta^q_{23}/10^{-2}$	4.296
δ^q_{CP}	1.194
χ^2_q	46.122

$y_e/10^{-6}$	2.848
$y_\mu/10^{-4}$	6.104
$y_\tau/10^{-2}$	1.022
$\Delta m^2_{21}/10^{-5}/\mathrm{eV}^{22}$	7.419
$\Delta m^2_{31}/10^{-3}/\mathrm{eV}^2$	2.516
$\sin^2\theta^l_{12}$	0.457
$\sin^2\theta^l_{13}/10^{-2}$	2.304
$\sin^2\theta^l_{23}$	0.806
χ^2_l	236.259

To obtain the best-fit parameters in our fitting, we scan randomly the allowed parameter regions to find good seeds for further MCMC scanning. In practice, we try to find the best-fit points for the quark sector first, and then perform the numerical fitting for the lepton sector with the best-fit value of τ in the quark sector. However, it is sometimes difficult to obtain good fittings with a common τ for both sectors. Multiple τ values (for quark and lepton sectors, respectively) are then used in the fitting if the single modulus scenarios do not work well.

$\rho_{\bar{f}} = \rho_F = \rho_E = \rho_S = \mathbf{3}$ and $k_{\bar{f}} = k_F = 4, k_E = k_S = 2.$

observable	Value2
$y_u/10^{-6}$	6.644
$y_c/10^{-3}$	3.445
y_t	0.868
$y_d/10^{-5}$	1.323
$y_s/10^{-4}$	1.841
$y_b/10^{-2}$	1.395
θ_{12}^q	0.22737
$\theta_{13}^q/10^{-3}$	3.716
$\theta_{23}^q/10^{-2}$	4.296
δ_{CP}^q	1.194
χ_q^2	46.122

$y_e/10^{-6}$	2.848
$y_\mu/10^{-4}$	6.104
$y_\tau/10^{-2}$	1.022
$\Delta m_{21}^2/10^{-5}/\mathrm{eV}^{22}$	7.419
$\Delta m_{31}^2/10^{-3}/\mathrm{eV}^2$	2.516
$\sin^2\theta_{12}^l$	0.457
$\sin^2\theta_{13}^l/10^{-2}$	2.304
$\sin^2\theta_{23}^l$	0.806
χ_l^2	236.259

Parameter	Value1
$\beta_1/10^{-2}$	5.292
$\beta_2/10^{-4}$	$1588.315 - 6.410i$
$\beta_3/10^{-2}$	$-3.704 + 10.670i$
$\beta_4/10^{-3}$	$-5.817 + 2.049i$
$\beta_5/10^{-3}$	$-1058.838 - 1.620i$
$\beta_6/10^{-2}$	$2.055 + 9.933i$
$\beta_7/10^{-1}$	$1.710 - 1.460i$
$\alpha_1/10^{-4}$	3.774
$\alpha_2/10^{-3}$	$-2.821 - 273.122i$
$\alpha_3/10^{-5}$	$-70.375 + 1.104i$
$\alpha_4/10^{-4}$	$-3.691 + 2707.307i$
$\alpha_5/10^{-4}$	$-411.085 - 1.680i$
τ	$1.198 + 2.830i$

$y_e/10^{-6}$	2.847
$y_\mu/10^{-4}$	6.109
$y_\tau/10^{-2}$	1.022
$\Delta m_{21}^2/10^{-5}/\mathrm{eV}^2$	7.405
$\Delta m_{31}^2/10^{-3}/\mathrm{eV}^2$	2.511
$\sin^2\theta_{12}^l$	0.384
$\sin^2\theta_{13}^l/10^{-2}$	2.260
$\sin^2\theta_{23}^l$	0.642
χ_l^2	48.806

low energy predictions of the best-fit point

$$\chi_{q,l}^2 \equiv \sum_{i_{q,l}} \chi_{i_{q,l}}^2 ,$$

$\gamma_1/10^{-2}$	1.203
$\gamma_2/10^{-4}$	$3.536 - 9.922i$
$\gamma_3/10^{-3}$	$1.713 - 3.086i$
$\gamma_4/10^{-4}$	$-6.732 + 105.230i$
$\gamma_5/10^{-4}$	$123.915 - 9.913i$
$\Lambda_1/(10^9\ \mathrm{GeV})$	1.207
$\Lambda_2/(10^2\ \mathrm{GeV})$	2.006
$\lambda_1/10^{-3}$	8.678
$\lambda_2/10^{-3}$	$3.249 - 81.540i$
$\lambda_3/10^{-4}$	$9.383 - 154.099i$
$\lambda_4/10^{-2}$	$8.095 + 18.176i$
$\lambda_5/10^{-2}$	$1.193 - 8.156i$
$\kappa_1/10^{-2}$	-4.752
$\kappa_2/10^{-2}$	$-2.809 + 2.805i$
$\kappa_3/10^{-2}$	$-1.245 - 1.635i$

$\gamma_1/10^{-2}$	1.204
$\gamma_2/10^{-4}$	$3.522 - 9.722i$
$\gamma_3/10^{-3}$	$1.710 - 3.078i$
$\gamma_4/10^{-4}$	$-6.723 + 105.127i$
$\gamma_5/10^{-4}$	$123.760 - 9.751i$
$\Lambda_1/(10^9\ \mathrm{GeV})$	1.298
$\Lambda_2/(10^2\ \mathrm{GeV})$	1.833
$\lambda_1/10^{-3}$	8.674
$\lambda_2/10^{-3}$	$3.246 - 81.548i$
$\lambda_3/10^{-4}$	$9.385 - 154.042i$
$\lambda_4/10^{-2}$	$8.095 + 18.164i$
$\lambda_5/10^{-2}$	$1.194 - 8.155i$
$\kappa_1/10^{-2}$	-4.753
$\kappa_2/10^{-2}$	$-2.808 + 2.804i$
$\kappa_3/10^{-2}$	$-1.244 - 1.635i$
τ_l	$1.180 + 2.711i$

Changing in the lepton sector will feed back into the quark sector by only affecting slightly their RGE evolutions. The best fit point for the quark sector is almost insensitive to the changes in the lepton sector So, we keep fixed the best fit point for the quark sector while best fit the lepton sector with multiple τ.

Numerical fitting

	the same τ	multiple
	$\tau/10^{-2}$	$\tau/10^{-2}$
II: $\rho_{\bar{f}} \in \{\mathbf{1}, \mathbf{1}', \mathbf{1}''\}$, $\rho_F = \mathbf{3}$, $\rho_E = \mathbf{3}$, $\rho_S = \mathbf{3}$.	$3.310 + 359.899i$ $\chi_q^2 = 16.408$	$3.310 + 359.899i$ $1.744 + 13.565i$
– **IX**: $\rho_{\bar{f}_{1,2,3}} = (\mathbf{1}', \mathbf{1}'', \mathbf{1}'')$ with modular weights $k_{\bar{f}_{1,2,3}} = (2, 0, 2)$; $k_F = 4$ and $k_E = k_S = 2$.	$\chi_l^2 = 486.036$	$\chi_l^2 = 30.215$
– **X**: $\rho_{\bar{f}_{1,2,3}} = (\mathbf{1}, \mathbf{1}', \mathbf{1}'')$ with modular weights $k_{\bar{f}_{1,2,3}} = (4, 2, 4)$; $k_F = 4$ and $k_E = k_S = 2$.	$1.198 + 2.835i$ $\chi_q^2 = 69.216$ $\chi_l^2 = 208.261$	$1.198 + 2.835i$ $1.326 + 1.317i$ $\chi_l^2 = 0.878.$
III: $\rho_{\bar{f}} = \mathbf{3}$, $\rho_F \in \{\mathbf{1}, \mathbf{1}', \mathbf{1}''\}$, $\rho_E = \mathbf{3}$, $\rho_S = \mathbf{3}$.	$\tau/10^{-3}$ $3.040 + 719.434i$	$\tau/10^{-3}$ $3.040 + 719.434$
– **IX′**: $\rho_{F_{1,2,3}} = (\mathbf{1}', \mathbf{1}'', \mathbf{1}'')$ with modular weights $k_{F_{1,2,3}} = (2, 4, 2)$; $k_{\bar{f}} = k_E = k_S = 2$.	$\chi_q^2 = 1.221$ $\chi_l^2 = 0.358$	
– **X′**: $\rho_{F_{1,2,3}} = (\mathbf{1}, \mathbf{1}', \mathbf{1}'')$ with modular weights $k_{F_{1,2,3}} = (0, 2, 4)$; $k_{\bar{f}} = k_E = k_S = 2$.	$1.017 + 1.913i$ $\chi_q^2 = 50.511$ $\chi_l^2 = 232.993$	$1.017 + 1.913i$ $5.435 + 9.0761i$ $\chi_l^2 = 58.908$

Conclusions

- Explain the flavor structures of the Standard Model plus neutrinos in the framework of flipped SU(5) GUT with A_4 modular flavor symmetry.
- Reduce the multiple modular symmetries to a single modular symmetry in the low energy effective theory with proper boundary conditions.
- Classify all possible scenarios in this scheme according to the assignments of the modular A_4 representations for matter superfields.
- Predictions of many scenarios can fit nicely to the experimental data both with one modulus value and multiple modulus values.

$SU(3)_c \times SU(3)_L \times U(1)_X$ **Model**

$$Q = T_3 + \beta T_8 + X$$

electroweak mixing angle $\sin^2\theta_W(M_Z) = 0.23113 \lesssim 1/4$

appears to obey, at an energy scale μ, an $SU(3)$

$\sin^2\theta_W(\mu) = 1/4$ S. Weinberg,

$SU(3)_C \otimes SU(3)_L \otimes U(1)_X$

$\langle\chi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix}$

$\downarrow$

$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$

$\langle\rho\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \\ 0 \end{pmatrix}, \langle\eta\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix}$

$\downarrow$

$SU(3)_C \otimes U(1)_Q$

two versions of 3-3-1 gauge models.

the third component of leptonic triplet

$\beta = \sqrt{3}$ → charged anti-lepton

$$f^a = \begin{pmatrix} {\nu_L}^a \\ {e_L}^a \\ e_R^{c\,a} \end{pmatrix}_L \sim (1, 3, 0)\,,$$

scalar sector

$\chi \sim (\mathbf{1},\mathbf{3},\text{-}1)$, $\rho \sim (\mathbf{1},\mathbf{3},1)$.

$\eta \sim (\mathbf{1},\mathbf{3},\ 0)$ and $S \sim (\mathbf{1},\mathbf{6},0)$.

$G\bar{f}_L^C S^* f_L$.

texture of m_ν is the same as the m_l

effective dimension five operator

$$\frac{h}{\Lambda}(\bar{f}_L^C\eta^*)(\eta^\dagger f_L), \Longrightarrow m_\nu = 10h\text{GeV}. \ \times$$

Landau pole

about 4-5 TeV

$\beta = \frac{1}{\sqrt{3}}$ → right-handed neutrinos

$$L_a = \begin{pmatrix} \nu_{aL} \\ e_{aL} \\ (\nu_{aR})^c \end{pmatrix} \sim (1\,,\,3\,,\,-1/3)\,,\quad e_{aR} \sim (1, 1, -1),$$

scalar sector

$\eta \sim (\mathbf{1}\,,\,\mathbf{3}\,,\,-1/3)$ $\chi \sim (\mathbf{1}\,,\,\mathbf{3}\,,\,-1/3)$

$\rho \sim (\mathbf{1}\,,\,\mathbf{3}\,,\,2/3)$

$$\mathcal{L}_{M_L} = \frac{f_{ab}}{\Lambda}\left(\overline{L_a^C}\eta^*\right)\left(\eta^\dagger L_b\right) + \text{H.c.}$$

$$\mathcal{L}_{M_R} = \frac{h_{ab}}{\Lambda}\left(\overline{L_a^C}\chi^*\right)\left(\chi^\dagger L_b\right) + \text{H.c.}$$

$SU(3)_c \times SU(3)_L \times U(1)_X$ **Model**

$$\beta = \sqrt{3}$$

$$f^a = \begin{pmatrix} \nu_L{}^a \\ e_L{}^a \\ e_R^{c\,a} \end{pmatrix}_L \sim (1,3,0)$$

$$Q_{1L} = \begin{pmatrix} u_1 \\ d_1 \\ J_1 \end{pmatrix}_L \sim (3,3,2/3),$$

$$u_{1R} \sim (3,1,2/3),\ d_{1R} \sim (3,1,-1/3),\ J_{1R} \sim (3,1,5/3)$$

$$Q_{2L} = \begin{pmatrix} d_2 \\ u_2 \\ J_2 \end{pmatrix}_L \sim (3,3^*,-1/3),$$

$$u_{2R} \sim (3,1,2/3),\ d_{2R} \sim (3,1,-1/3),\ J_{2R} \sim (3,1,-4/3)$$

$$Q_{3L} = \begin{pmatrix} d_3 \\ u_3 \\ J_3 \end{pmatrix}_L \sim (3,3^*,-1/3),$$

$$u_{3R} \sim (3,1,2/3),\ d_{3R} \sim (3,1,-1/3),\ J_{3R} \sim (3,1,-4/3)$$

$$\beta = \frac{1}{\sqrt{3}}$$

$$Q_{iL} = \begin{pmatrix} u_{iL} \\ d_{iL} \\ D_{iL} \end{pmatrix} \equiv [3,3,0], \quad Q_{3L} = \begin{pmatrix} b_L \\ t_L \\ T_L \end{pmatrix} \equiv [3,3^*,1/3],$$

$$u_{iR} \equiv [3,1,2/3], \quad d_{iR} \equiv [3,1,-1/3], \quad D_{iR} \equiv [3,1,-1/3]$$

$$b_R \equiv [3,1,-1/3], \quad t_R \equiv [3,1,2/3], \quad T_R \equiv [3,1,2/3],$$

$$\psi_{aL} = \begin{pmatrix} e_{aL} \\ \nu_{aL} \\ N_{aL} \end{pmatrix} \equiv [1,3^*,-1/3], \quad e_{aR} \equiv [1,1,-1].$$

the anomaly cancellation occurs for the three generations together and not generation by generation.

SEQUENTIAL $\mathrm{SU}(3)_c \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ **MODEL**

see arXiv:1608.05334 by Sarkar,Valle et al

$$Q_{aL} = \begin{pmatrix} u_{aL} \\ d_{aL} \\ D_{aL} \end{pmatrix} \equiv [3,3,0],\ u_{aR} \equiv [3,1,2/3],\ d_{aR} \equiv [3,1,-1/3],\ D_{aR} \equiv [3,1,-1/3],$$

each family is anomaly free

$$\psi_{aL} = \begin{pmatrix} e^-_{aL} \\ \nu_{aL} \\ N^1_{aL} \end{pmatrix} \equiv [1,3^*,-1/3],\ \xi_{aL} = \begin{pmatrix} E^-_{aL} \\ N^2_{aL} \\ N^3_{aL} \end{pmatrix} \equiv [1,3^*,-1/3],\ \chi_{aL} = \begin{pmatrix} N^4_{aL} \\ E^+_{aL} \\ e^+_{aL} \end{pmatrix} \equiv [1,3^*,2/3].$$

$$\mathcal{L}_{\text{quarks}} = y_{u_a}\overline{Q_{aL}}u_{aR}\phi_0^* + y^i_{d_a}\overline{Q_{aL}}d_{aR}\phi_i^* + y^i_{D_a}\overline{Q_{aL}}D_{aR}\phi_i^* + \text{h.c.}$$

$$\mathcal{L}_{\text{leptons}} = \epsilon_{\alpha\beta\gamma}\left[\psi^T_{\alpha L}C^{-1}\left(y_1\xi_{\beta L}\phi_{0\gamma} + y^i_2\chi_{\beta L}\phi_{i\gamma}\right) + \xi^T_{\alpha L}C^{-1}y^i_3\chi_{\beta L}\phi_{i\gamma}\right] + \text{h.c.}$$

$$\phi_0 \equiv [1,3^*,2/3] \quad \text{and} \quad \phi_{1,2} \equiv [1,3^*,-1/3],$$

Neutrino sector

$$\langle\phi_0\rangle = \begin{pmatrix} k_0 \\ 0 \\ 0 \end{pmatrix}, \quad \langle\phi_1\rangle = \begin{pmatrix} 0 \\ k_1 \\ n_1 \end{pmatrix}, \quad \langle\phi_2\rangle = \begin{pmatrix} 0 \\ k_2 \\ n_2 \end{pmatrix}.$$

$k_{0,1,2} \sim m_W$ $\qquad$ $n_{1,2} \sim M_{331}$

$$m_{u_a} = y_{u_a}k_0,$$

$$m^a_{dD} = \begin{pmatrix} y^1_{d_a}k_1 + y^2_{d_a}k_2 & y^1_{D_a}k_1 + y^2_{D_a}k_2 \\ y^1_{d_a}n_1 + y^2_{d_a}n_2 & y^1_{D_a}n_1 + y^2_{D_a}n_2 \end{pmatrix}$$

$$m_{eE} = \begin{pmatrix} -\left(y^1_2k_1 + y^2_2k_2\right) & \left(y^1_2n_1 + y^2_2n_2\right) \\ -\left(y^1_3k_1 + y^2_3k_2\right) & \left(y^1_3n_1 + y^2_3n_2\right) \end{pmatrix}$$

SU(6) Grand Unification of the sequential $\mathrm{SU(3)_c \otimes SU(3)_L \otimes U(1)_X}$ Model

$$\bar{6} \to (1,\ \bar{3},\ \frac{-1}{2\sqrt{3}}) \bigoplus (\bar{3},\ 1,\ \frac{1}{2\sqrt{3}})$$
$$\hookrightarrow f_i = (e_{Li}, -\nu_{Li},\ N_i) \bigoplus d^c_{Ri}\ ,$$
$$\bar{6}' \to (1,\ \bar{3},\ \frac{-1}{2\sqrt{3}}) \bigoplus (\bar{3},\ 1,\ \frac{1}{2\sqrt{3}})$$
$$\hookrightarrow f'_i = (e'_{Li}, -\nu'_{Li},\ N'_i) \bigoplus D^c_{Ri}\ ,$$
$$15 \to (3,\ 3,\ 0) \bigoplus (1,\ \bar{3},\ \frac{1}{\sqrt{3}}) \bigoplus (\bar{3},\ 1,\ \frac{-1}{\sqrt{3}})$$
$$\hookrightarrow F_i = (u_{Li},\ d_{Li},\ D_{Li}) \bigoplus Xf^c_i = (\nu'^c_{Ri},\ e'^c_{Ri},\ e^c_{Ri}) \bigoplus u^c_{Ri}$$
$$\mathcal{L} \supseteq y^A_{ab}\mathbf{15}_a\ \overline{\mathbf{6}}_b\ \overline{\mathbf{6}}_{H;i} + y^B_{ab}\mathbf{15}_a\ \mathbf{15}_b\ \mathbf{15}_H$$

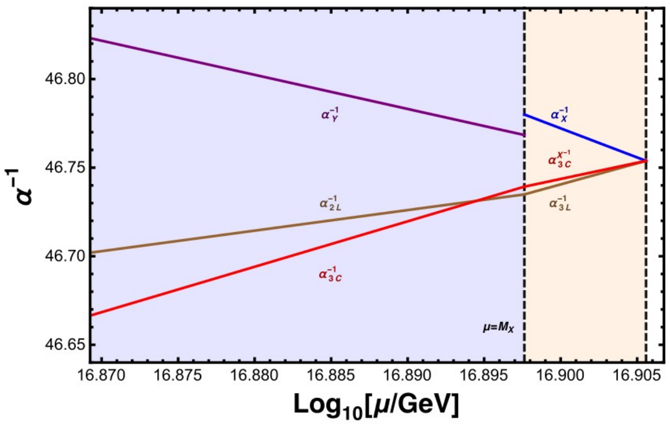
two $\bar{6}$ antifundamental representations and one 15 antisymmetric representation of the fermions are anomaly free.

$$\mathcal{A}[6] = 1,\ \ \mathcal{A}[15] = 2,$$

scalar multiplets coming from two $\bar{6}$ and one 15 representations

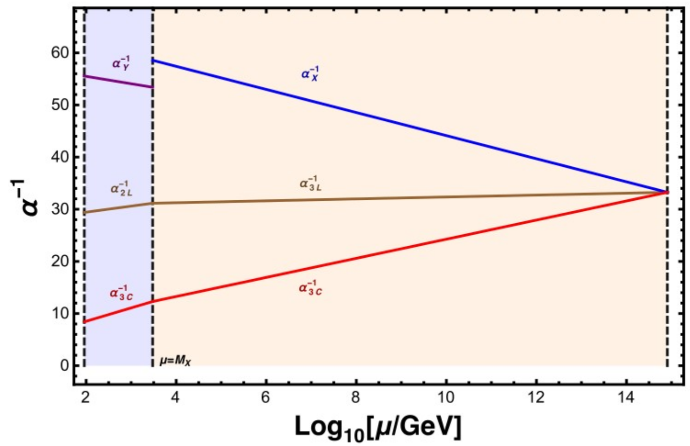
$$\overline{\mathbf{6}}_H = (\bar{3}, 1, \frac{1}{2\sqrt{3}}) \oplus (1, \bar{3}, -\frac{1}{2\sqrt{3}})\ ,$$
$$\mathbf{15}_H = (\bar{3}, 1, -\frac{1}{\sqrt{3}}) \oplus (3, 3, 0) \oplus (1, \bar{3}, \frac{1}{\sqrt{3}})$$

minimal SVS Model



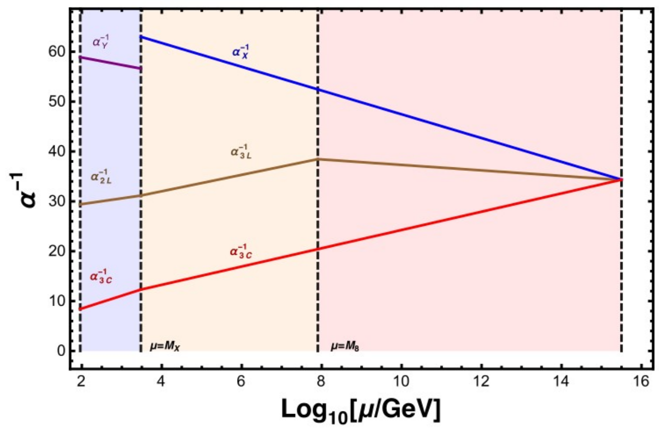
very high scale of M_X

SVS Model with fermionic octets



lower limit of the order of $M_X \gtrsim 10^5$ GeV

sequential 331 with fermionic octets



octet mass scale detached M_X

SU(6) Grand Unification of the sequential $\mathrm{SU(3)_c \otimes SU(3)_L \otimes U(1)_X}$ **Model**

$$\sin^2\theta_w(M_Z) = \frac{3}{8} + \frac{5}{8}\alpha_{\rm em}(M_Z)\left[\frac{4}{5}\left\{\frac{b_{3L}}{2\pi}\ln\left(\frac{M_8}{M_X}\right) + \frac{b_{3L}^8}{2\pi}\ln\left(\frac{M_U}{M_8}\right)\right\} + \frac{(b_{2L}-b_Y)}{2\pi}\ln\left(\frac{M_X}{M_Z}\right) - \frac{4}{5}\frac{b_X}{2\pi}\ln\left(\frac{M_U}{M_X}\right)\right],$$

sequential 331 model $M_X = 3000$ GeV octet mass scale $M_8 = 8\times 10^7$ GeV

unification scale $M_U = 10^{15.5}$ GeV $\sin^2\theta_w(M_Z) \simeq 0.231$

$d = 6$ contributions for proton decay

$$\Gamma^{-1}(p\to e^+\pi^0) \sim 10^{36}\ \text{yrs}\left(\frac{\alpha_{\rm GUT}^{-1}}{35}\right)^2\left(\frac{M_U}{10^{16}\ \text{GeV}}\right)^4.$$

$M_U = 10^{15.5}$ GeV and $\alpha_{\rm GUT}^{-1} \sim 35$ in the SVS and sequential 331 models.

lifetime of the proton decay mode $p \to e^+\pi^0 \sim 10^{34}$ yrs

$SU(3)_c \times SU(3)_L \times U(1)_X$ unification into $SU(6) \times U(1)_K$ model

arXiv: 2303.01298

$$\bar{\mathbf{6}}_{-\frac{1}{2}} \supseteq [U_L^c, L_L],\ \mathbf{15}_0 \supseteq [Q_L, D_L^c, N_L^c],\ \bar{\mathbf{6}}_{\frac{1}{2}} \supseteq [(XD)_L^c, E_L^c]\ ,$$

$$\bar{\mathbf{6}}_{-\frac{1}{2}} = U_L^c(\bar{3}, 1, \frac{1}{2\sqrt{3}})_{-\frac{1}{2}} \oplus (L_L, N_L^s)(1, \bar{3}, -\frac{1}{2\sqrt{3}})_{-\frac{1}{2}}\ ,$$

$$\mathbf{15}_0 = D_L^c(\bar{3}, 1, -\frac{1}{\sqrt{3}})_0 \oplus (Q_L, (XD)_L)(3, 3, 0)_0 \oplus ((XL)_L, N_L^c)(1, \bar{3}, \frac{1}{\sqrt{3}})_0$$

$$\bar{\mathbf{6}}_{\frac{1}{2}} = (XD)_L^C(\bar{3}, 1, \frac{1}{2\sqrt{3}})_{\frac{1}{2}} \oplus ((XL)_L^c, E_L^c)(1, \bar{3}, -\frac{1}{2\sqrt{3}})_{\frac{1}{2}}\ ,$$

the fitting of $(XD)_L^c$ and D_L^C can be exchanged.

subsequent studies.

$$Q_X = -\frac{\sqrt{3}}{3} Q_P + Q_K$$

the anomaly is canceled for $SU(6)$ with two $\bar{\mathbf{6}}$ representation and one antisymmetric $\mathbf{15}$ representation for each generation.

$U(1)_K$ anomaly

$$6(\frac{1}{2} - \frac{1}{2}) = 0\ ,\quad 6\left[\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^3\right] = 0$$

To break the flipped SU(6) into $SU(3)_c \times SU(3)_L \times U(1)_X$, introduce $\mathbf{20}_{\frac{1}{2}}$

$$\mathbf{20}_{\frac{1}{2};H} = (1, 1, -\frac{3}{2\sqrt{3}})_{\frac{1}{2};H} \oplus (1, 1, \frac{3}{2\sqrt{3}})_{\frac{1}{2};H} \oplus (3, \bar{3}, -\frac{1}{2\sqrt{3}})_{\frac{1}{2};H} \oplus (\bar{3}, 3, \frac{1}{2\sqrt{3}})_{\frac{1}{2};H}$$

$$Q_X : (1, 1, \frac{3}{2\sqrt{3}})_{\frac{1}{2};H} = 0\ ,\quad Q_X : (1, 1, -\frac{3}{2\sqrt{3}})_{\frac{1}{2};H} = -1$$

The $N_H(1, 1, \frac{3}{2\sqrt{3}})_{\frac{1}{2};H}$ component of $\mathbf{20}_{\frac{1}{2};H}$ can acquire a vacuum expectation value (VEV) $\langle N_H \rangle = M_X$ to break the flipped SU(6) into $SU(3)_c \times SU(3)_L \times U(1)_X$.

$SU(3)_c \times SU(3)_L \times U(1)_X$ unification into $SU(6) \times U(1)_K$ model

arXiv: 2303.01298

Tiny neutrino masses can be generated by seesaw mechanism

Majorana mass terms for RH-neutrinos N_L^c.

the proper choice is $\overline{\mathbf{105}}_s$,

in terms of $SU(3)_c \times SU(3)_L \times U(1)_X$

$$\overline{\mathbf{105}}^s = (\mathbf{1}, \mathbf{6}, -\frac{\mathbf{2}}{\sqrt{\mathbf{3}}}) \oplus (\mathbf{6}, \mathbf{1}, \frac{\mathbf{2}}{\sqrt{\mathbf{3}}}) \oplus (\mathbf{8}, \bar{\mathbf{3}}, \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}) \oplus (\bar{\mathbf{3}}, \mathbf{8}, -\frac{\mathbf{1}}{\sqrt{\mathbf{3}}}) \oplus (\mathbf{6}, \mathbf{6}, \mathbf{0}) \oplus (\mathbf{3}, \mathbf{3}, \mathbf{0}) \ ,$$

$$\mathcal{L} \supseteq Y_{ab}^m \mathbf{15}_0^a \mathbf{15}_0^b (\overline{\mathbf{105}})_{0;H}^s \supseteq (1, \bar{3}, \frac{1}{\sqrt{3}})_0^a \otimes (1, \bar{3}, \frac{1}{\sqrt{3}})_0^b \otimes (1, 6, -\frac{2}{\sqrt{3}})_{0;H} \ ,$$

the $(1, 6, -\frac{2}{\sqrt{3}})_{0;H}$ component of $\overline{\mathbf{105}}_s$ $\quad m_S \sim \frac{\mathcal{M}_U \mathcal{M}_E}{Y_{XD} M_{331}} \sim 1$ eV ~~eak~~ the $SU(3)_c \times SU(3)_L \times U(1)_X$

develops VEV along the $N_{\tilde{H}}(1, 1, 0)$ direction (in terms of SM quantum number)

$$m_D \simeq \mathcal{M}_{\nu;D} \sim \mathcal{O}(1) \text{ GeV} \qquad M_\nu \simeq \frac{M_{\nu;D} M_{\nu;D}^T}{Y^m M_S} \sim 10^{-2}\text{eV}$$

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & \mathcal{M}_{\nu;D}^T \\ \mathcal{M}_{\nu;D} & Y^m M_S \end{pmatrix}$$

the 331 breaking scale M_{331} is constrained to lie at about 10^{11} GeV

$SU(3)_c \times SU(3)_L \times U(1)_X$ unification into $SU(6) \times U(1)_K$ model

arXiv: 2303.01298

To break the residue $SU(3)_c \times SU(3)_L \times U(1)_X$ gauge symmetry into SM

$$\overline{\mathbf{6}}_{\frac{1}{2};H} = (\bar{3}, 1, \frac{1}{2\sqrt{3}})_{\frac{1}{2};H} \oplus (1, \bar{3}, -\frac{1}{2\sqrt{3}})_{\frac{1}{2};H}, \longrightarrow H_u \in H_1(1, \bar{3}, \tfrac{2}{3})$$

$$\overline{\mathbf{6}}_{-\frac{1}{2};H} = (\bar{3}, 1, \frac{1}{2\sqrt{3}})_{-\frac{1}{2};H} \oplus (1, \bar{3}, -\frac{1}{2\sqrt{3}})_{-\frac{1}{2};H}, \longrightarrow H_2(1, \bar{3}, -\tfrac{1}{3})$$

$$\mathbf{15}_{0;H} = (\bar{3}, 1, -\frac{1}{\sqrt{3}})_{0;H} \oplus (3, 3, 0)_{0;H} \oplus (1, \bar{3}, \frac{1}{\sqrt{3}})_{0;H} \longrightarrow H_d \in H_3(1, \bar{3}, -\tfrac{1}{3})$$

$$\langle H_1 \rangle = \sqrt{2} \begin{pmatrix} v_u \\ 0 \\ 0 \end{pmatrix}, \quad \langle H_2 \rangle = \sqrt{2} \begin{pmatrix} 0 \\ 0 \\ M_{331} \end{pmatrix}, \quad \langle H_3 \rangle = \sqrt{2} \begin{pmatrix} v_d \\ 0 \\ 0 \end{pmatrix}$$

$$\mathcal{L} \supseteq -\sum_{a,b=1}^{3} Y_{U;ab} \overline{\mathbf{6}}^a_{-\frac{1}{2}} \mathbf{15}^b_0 \overline{\mathbf{6}}_{\frac{1}{2};H} - \sum_{a,b=1}^{3} Y_{E;ab} \overline{\mathbf{6}}^a_{-\frac{1}{2}} \overline{\mathbf{6}}^b_{\frac{1}{2}} \mathbf{15}_{0,H} - \sum_{a,b=1}^{3} Y_{D,N;ab} \mathbf{15}^a_0 \mathbf{15}^b_0 \mathbf{15}_{0;H}$$
$$- \sum_{a,b=1}^{3} Y_{XD;ab} \overline{\mathbf{6}}^a_{\frac{1}{2}} \mathbf{15}^b_0 \overline{\mathbf{6}}_{-\frac{1}{2};H},$$

$$\overline{\mathbf{6}}^a_{\frac{1}{2}} \mathbf{15}^b_0 \overline{\mathbf{6}}_{-\frac{1}{2};H} \left\{ \begin{array}{l} \left[(1, \bar{3}, -\frac{1}{2\sqrt{3}})_{\frac{1}{2}} \otimes (1, \bar{3}, \frac{1}{\sqrt{3}})_0 \otimes (1, \bar{3}, -\frac{1}{2\sqrt{3}})_{-\frac{1}{2};H}\right] \supseteq [(XL)_L \otimes (XL)^c_L \otimes N_H(1,1,0)] \,, \\ \left[(3,3,0)_0 \otimes (\bar{3}, 1, \frac{1}{2\sqrt{3}})_{\frac{1}{2}} \otimes (1, \bar{3}, -\frac{1}{2\sqrt{3}})_{-\frac{1}{2};H}\right] \supseteq [(XD)_L \otimes (XD)^c_L \otimes N_H(1,1,0)] \end{array} \right.$$

which will generate Dirac mass $Y_{XD} M_{331}$ for vector-like heavy extra leptons $(XL)_L, (XL)^c_L$ and vector-like heavy quarks $(XD)_L, (XD)^c_L$.

$SU(3)_c \times SU(3)_L \times U(1)_X$ unification into $SU(6) \times U(1)_K$ model

arXiv: 2303.01298

The N_L^s new sterile neutrino component within $\bar{\mathbf{6}}_{-\frac{1}{2}}$ can also obtain masses after EWSB,

$$\left[(1,\bar{3},-\frac{1}{2\sqrt{3}})_{-\frac{1}{2}} \otimes (1,\bar{3},\frac{1}{\sqrt{3}})_0 \otimes (1,\bar{3},-\frac{1}{2\sqrt{3}})_{\frac{1}{2};H}\right] \supseteq [N_L^S \otimes (XL)_L \otimes H_u]$$

$$\left[(1,\bar{3},-\frac{1}{2\sqrt{3}})_{-\frac{1}{2}} \otimes (1,\bar{3},-\frac{1}{2\sqrt{3}})_{\frac{1}{2}} \otimes (1,\bar{3},\frac{1}{\sqrt{3}})_{0;H}\right] \supseteq [N_L^S \otimes (XL)_L^c \otimes H_d]$$

the mass matrix for the new sterile neutrinos can be given by

$$M_S' \equiv \begin{pmatrix} 0 & \mathcal{M}_U^T & \mathcal{M}_E^T \\ \mathcal{M}_U & 0 & Y_{XD}M_{331} \\ \mathcal{M}_E & Y_{XD}M_{331} & 0 \end{pmatrix}, \qquad m_S \sim \frac{\mathcal{M}_U\mathcal{M}_E}{Y_{XD}M_{331}} \sim 1 \text{ eV}$$

Problematic !

introduce new Higgs fields $\mathbf{21}_{1;H}$ representation to push heavy such new sterile neutrino: $-y_{S;ab}\overline{\mathbf{6}}^a_{-\frac{1}{2}}\overline{\mathbf{6}}^b_{-\frac{1}{2}}\mathbf{21}_{1;H}$

$$\mathbf{21}_{1;H} = (6,1,-\frac{1}{\sqrt{3}})_{1;H} \oplus (3,3,0)_{1;H} \oplus (1,6,\frac{1}{\sqrt{3}})_{1;H}$$

M_S' all lie at the M_{331} scale

the $(1,\bar{6},\frac{1}{\sqrt{3}})_{1;H}$ component of $\mathbf{21}_{1;H}$ develops a VEV $\langle\mathbf{21}_{1;H}\rangle = M_{S'} \sim M_{331}$ along the $N_{H''}(1,1,0)$ direction

---

if the $(1,\bar{3},1)$ direction (in terms of SM quantum number) of $\mathbf{21}_{1;H}$ develops a small triplet VEV

ordinary SM LH neutrino a mixed type I+II seesaw mechanism

$SU(3)_c \times SU(3)_L \times U(1)_X$ unification into $SU(6) \times U(1)_K$ model

arXiv: 2303.01298

$$SU(6) \times U(1)_K \xrightarrow{M_X} SU(3)_c \times SU(3)_L \times U(1)_X \xrightarrow{M_{331}} SM,$$

$$\frac{1}{g_X^2} = \frac{1}{3}\frac{1}{g_P^2} + \frac{1}{g_K^2}, \qquad \frac{1}{g_Y^2} = \frac{1}{3}\frac{1}{g_{3L}^2} + \frac{1}{g_X^2},$$

colored Higgs fields can acquire masses of order M_X while the uncolored ones can still be as light as M_{331} scale.

(D-T) like splitting can be realized by the missing partner mechanism

missing partner mechanism the SUSY flipped SU(6) model.

$$W \supseteq \epsilon^{ijklmn}\lambda \left(\mathbf{20}_{\frac{1}{2};\mathbf{H}}\right)_{ijk} \left(\mathbf{15}'_{\mathbf{0};\mathbf{H}}\right)_{lm} \left(\mathbf{6}_{-\frac{1}{2};\mathbf{H}}\right)_{n} + \epsilon^{ijklmn}\lambda' \left(\overline{\mathbf{20}}_{-\frac{1}{2};\mathbf{H}}\right)_{ijk} \left(\overline{\mathbf{15}}'_{\mathbf{0};\mathbf{H}}\right)_{lm} \left(\bar{\mathbf{6}}_{\frac{1}{2};\mathbf{H}}\right)_{n}$$

$$+ M_{\mathbf{15}'}\overline{\mathbf{15}}'_{\mathbf{0};\mathbf{H}}\mathbf{15}'_{\mathbf{0};\mathbf{H}} + X\left(\mathbf{20}_{\frac{1}{2};\mathbf{H}}\overline{\mathbf{20}}_{-\frac{1}{2};\mathbf{H}} - M_X^2\right),$$

$\langle \mathbf{20}_{\frac{1}{2};\mathbf{H}}\rangle = \langle \overline{\mathbf{20}}_{-\frac{1}{2};\mathbf{H}}\rangle = M_X$ break the flipped $SU(6)$

The $(\bar{3}, 1, \frac{1}{2\sqrt{3}})_{\frac{1}{2};H}$ component within $\bar{\mathbf{6}}_{\frac{1}{2};\mathbf{H}}$ pairs to the $(3, 1, \frac{1}{\sqrt{3}})_{0;H'}$ components within $\overline{\mathbf{15}}'_{\mathbf{0};\mathbf{H}}$ while the $(1, \bar{3}, -\frac{1}{2\sqrt{3}})_{\frac{1}{2};H}$ component cannot find any partner. Therefore, the colored component within $\bar{\mathbf{6}}_{\frac{1}{2};\mathbf{H}}$ is heavy while the uncolored one is much lighter (at or below the M_{331} scale). Missing partner mechanism of similar settings can also push heavy the colored components within $\bar{\mathbf{6}}_{-\frac{1}{2};\mathbf{H}}$ and $\mathbf{15}_{\mathbf{0};\mathbf{H}}$.

Boundary conditions can also be used to split the colored and uncolored Higgs

S^1/Z_2 orbifold $\mathcal{Z}: y \to -y, \quad \mathcal{T}: y \to y + 2\pi R.$

choose the boundary condition with $P = diag(-1,-1,-1,\ 1,\ 1,\ 1)$ (under $\mathcal{Z}$ reflection) and $P' = diag(\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$ (under the reflection $\mathcal{Z}' = \mathcal{Z}\mathcal{T}$) so as that the Higgs satisfy

$$\overline{\mathbf{6}}_{\frac{1}{2};H} = (\bar{3}, 1, \frac{1}{2\sqrt{3}})^{(-,+)}_{\frac{1}{2};H} \oplus (1, \bar{3}, -\frac{1}{2\sqrt{3}})^{(+,+)}_{\frac{1}{2};H},$$

$$\overline{\mathbf{6}}_{-\frac{1}{2};H} = (\bar{3}, 1, \frac{1}{2\sqrt{3}})^{(-,+)}_{-\frac{1}{2};H} \oplus (1, \bar{3}, -\frac{1}{2\sqrt{3}})^{(+,+)}_{-\frac{1}{2};H},$$

$$\mathbf{15}_{0;H} = (\bar{3}, 1, -\frac{1}{\sqrt{3}})^{(-,+)}_{0;H} \oplus (3,3,0)^{(-,+)}_{0;H} \oplus (1, \bar{3}, \frac{1}{\sqrt{3}})^{(+,+)}_{0;H},$$

$$\mathbf{21}_{1;H} = (6, 1, -\frac{1}{\sqrt{3}})^{(-,+)}_{1;H} \oplus (3,3,0)^{(-,+)}_{1;H} \oplus (1, 6, \frac{1}{\sqrt{3}})^{(+,+)}_{1;H},$$

$SU(3)_c \times SU(3)_L \times U(1)_X$ unification into $SU(6) \times U(1)_K$ model

arXiv: 2303.01298

After the breaking of *331* gauge group into $SU(3)_c \times SU(2)_L \times U(1)_Y$ at about the M_{331}

Case I: 2HDM

$$(b_3, b_2, b_Y) = (-7, -3, 7)$$

Case II: 2HDM plus an $SU(2)_L$ triplet

$$(b_3, b_2, b_Y) = (-7, -\frac{7}{3}, 8)$$

---

assume successful splitting among the colored/uncolored Higgs fields

the Higgs sector for the *331* model

$$H_1(1, \bar{3}, \frac{2}{3}) \ni H_u, \quad H_2(1, \bar{3}, -\frac{1}{3}) \ni H_d, \quad H_3(1, \bar{3}, -\frac{1}{3}), \quad H_S^i(1, 6, \frac{2}{3}) \ (i = 1, 2).$$

$$(b_3^c, b_3^L, b_1^X) = (-5, -\frac{17}{6}, \frac{94}{9}).$$

Upon the M_X scale,

the matter contents for three generations $\overline{\mathbf{6}}_{-\frac{1}{2};i}, \overline{\mathbf{6}}_{\frac{1}{2};i}, \mathbf{15}_{0;i}$ and the Higgs fields $\mathbf{20}_{-\frac{1}{2};H}, \mathbf{21}_{1;H}, \overline{\mathbf{105}}_{0;H}, \overline{\mathbf{6}}_{\frac{1}{2};H}$, $\mathbf{15}_{0,H}$ and $\overline{\mathbf{6}}_{-\frac{1}{2};H}$.

$$(b_6, b_K) = (-\frac{38}{3}, \frac{47}{3})$$

after normalization into $SO(12)$,

$$(b_6, b_{K'}) = (-\frac{38}{3}, \frac{47}{9})$$

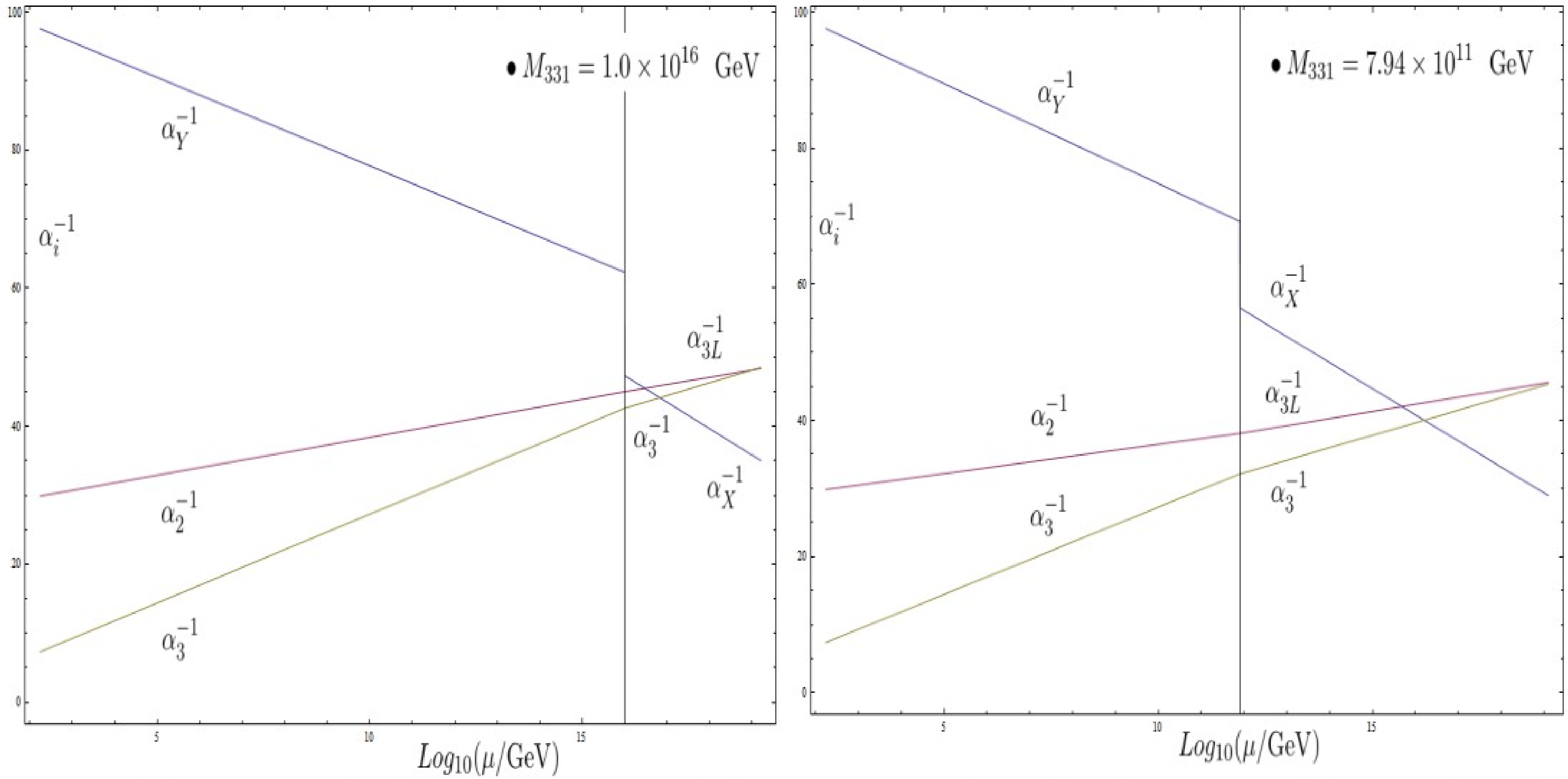
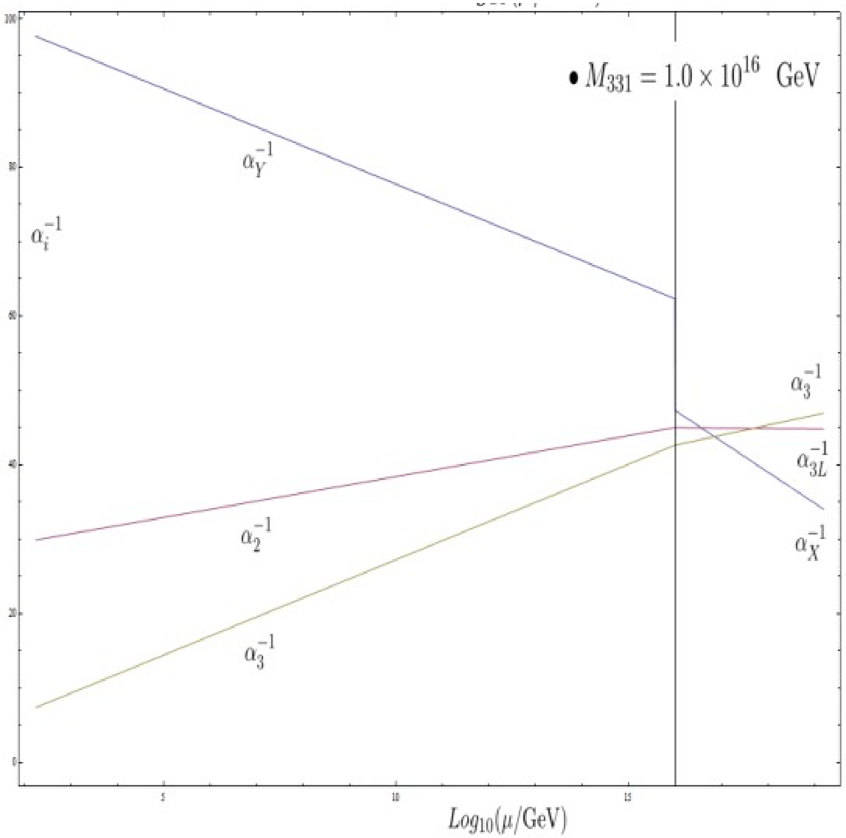
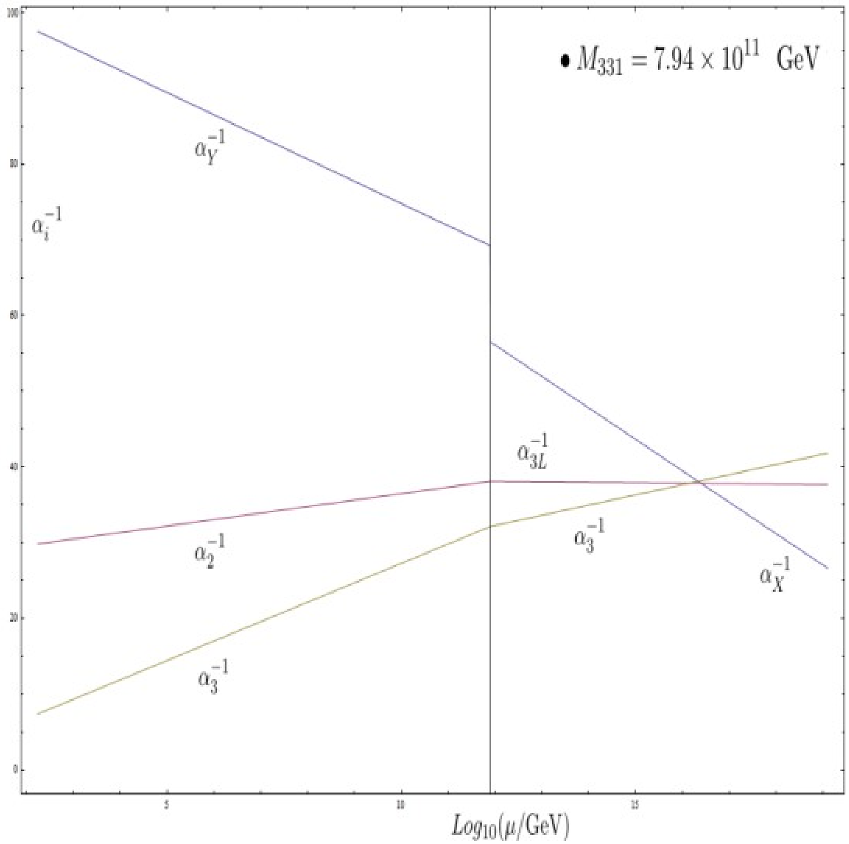


Figure 1. The RGE evolutions of the gauge couplings for the *331* models are shown for scenarios with 2HDM below M_{331} (case I, left panels) and 2HDM plus $SU(2)_L$ triplet Higgs below M_{331} (case II, right panels), respectively. With the *331* symmetry breaking scale $M_{331} = 10^{16}$ GeV (left panels) and $M_{331} = 7.94 \times 10^{11}$ GeV (right panels), the $SU(3)_L$ and $SU(3)_c$ gauge couplings can be unified into the flipped $SU(6)$ GUT model at the scale $M_X = 10^{19.03}$ GeV (left panel) and $M_X = 10^{19.13}$ GeV (right panel), respectively. The upper (and lower) panels correspond to the cases without (and with) the surviving M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs field, respectively.





The requirement that $M_X \gtrsim M_{331}$ set bounds for M_{331} scale to lie below $10^{17.6}$ GeV for case I and below $10^{15.3}$ GeV for case II.

Requiring the unification scales to lie below the Planck scale constrains $M_{331} \gtrsim 10^{15.9}$ GeV for case I and $M_{331} \gtrsim 10^{11.8}$ GeV for case II.

The upper bounds $10^{17.6}$ GeV for case I (and $10^{15.3}$ GeV for case II) on M_{331} scale also apply for case with the surviving M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs field.

Table 1. The flipped SU(6) GUT scales and corresponding $\alpha_6^{-1}(M_X)$ values for some benchmark points in case I/case II with and without light $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$ Higgs, respectively. The '\' in the table denotes that either the unification scale M_X lies upon the Planck scale (cannot unify) or below the M_{331} scale, or the bound on M_{331} from neutrino mass generations is not satisfied (for case I).

$(Log_{10}(M_X/\text{GeV}), \alpha_6^{-1}(M_X))$ \ M_{331} GeV	3.16×10^3	1.0×10^{11}	1.0×10^{13}	3.16×10^{16}
Case I without $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$	\	\	\	$(18.6, 47.8)$
Case II without $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$	\	\	$(18.2, 44.4)$	\
Case I with light $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$	\	$(17.9, 39.1)$	$(17.8, 41.4)$	$(17.7, 45.5)$
Case II with light $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$	$(18.0, 30.0)$	$(16.3, 36.9)$	$(15.9, 38.8)$	\

Proton decay triggered by flipped SU(6) GUT

dimension-6 operators

$$\mathcal{L} \supseteq -\sqrt{2} g_6 \Big[(\epsilon_{\alpha\beta} V_l (U_L^{c;A})^\dagger \gamma^\mu X_{\mu;A}^\beta L_L^\alpha) + \epsilon_{ABC} V_{CKM} (Q_L^A)^\dagger \gamma^\mu X_{\mu;B} D_L^{c;C}$$
$$+ \epsilon_{\alpha\beta} V_N^\dagger (N_L^\alpha)^\dagger \gamma^\mu X_{\mu;B}^\beta D_L^{c;B} \Big] ,$$
$$\mathcal{L} \supseteq \frac{g_6^2}{M_X^2} V_{CKM;11}^* (U_l)_{i1} \left[\epsilon_{ABC} (u_R^A d_R^B)(l_L^i u_L^C) \right]$$
$$\Gamma(p \to \pi^0 l_i^+) = \frac{m_p}{32\pi} \left(1 - \frac{m_\pi^2}{m_p^2} \right)^2 \left| \mathcal{A}(p \to \pi^0 \bar{l}_i^+) \right|^2 ,$$
$$= \frac{g_6^4}{32\pi M_X^4} m_p |V_{ud}^2| |(U_l)_{i1}|^2 \left(1 - \frac{m_\pi^2}{m_p^2} \right)^2 \mathcal{A}_L^2 \mathcal{A}_{S1}^2 \left(\langle \pi^0 | (ud)_R u_L | p \rangle_{l_i} \right)^2$$

$$\mathcal{A}_{S1} \approx \left[\frac{\alpha_3(M_{331})}{\alpha_3(M_X)} \right]^{\frac{2}{5}} \left[\frac{\alpha_3(M_Z)}{\alpha_3(M_{331})} \right]^{\frac{2}{7}} \left[\frac{\alpha_2(M_{331})}{\alpha_2(M_X)} \right]^{\frac{34}{27}} \left[\frac{\alpha_2(M_Z)}{\alpha_2(M_{331})} \right]^{\frac{3}{4}}$$
$$\left[\frac{\alpha_Y(M_{331})}{\alpha_Y(M_X)} \right]^{-\frac{33}{362}} \left[\frac{\alpha_Y(M_Z)}{\alpha_Y(M_{331})} \right]^{-\frac{11}{84}} ,$$

for case I without light $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$ Higgs

$$\mathcal{A}_{S1} \approx \left[\frac{\alpha_3(M_{331})}{\alpha_3(M_X)}\right]^{\frac{6}{11}} \left[\frac{\alpha_3(M_Z)}{\alpha_3(M_{331})}\right]^{\frac{2}{7}} \left[\frac{\alpha_2(M_{331})}{\alpha_2(M_X)}\right]^{-\frac{27}{2}} \left[\frac{\alpha_2(M_Z)}{\alpha_2(M_{331})}\right]^{\frac{27}{28}} \left[\frac{\alpha_Y(M_{331})}{\alpha_Y(M_X)}\right]^{-\frac{33}{410}} \left[\frac{\alpha_Y(M_Z)}{\alpha_Y(M_{331})}\right]^{-\frac{11}{96}}, \quad \text{for case II with light } \tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}} \text{ Higgs.}$$

Table 2. The partial proton decay lifetime of the $(p \to e^+\pi^0)$ mode in flipped SU(6) GUT. We calculate such partial life time for some benchmark points in case I/case II with and without M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$ Higgs, respectively.

$\tau((p \to e^+\pi^0)_{\text{flipped}})/years \setminus M_{331}$ GeV	3.16×10^3	1.0×10^{13}	3.16×10^{16}
Case I without $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$	\	\	6.92×10^{46}
Case II without $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$	\	1.04×10^{45}	\
Case I with $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$	\	3.88×10^{43}	1.71×10^{43}
Case II with $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$	6.72×10^{43}	8.21×10^{35}	\

Conclusions

- We can unify the sequential $SU(3)_c \times SU(3)_L \times U(1)_X$ model (with $\beta = 1/\sqrt{3}$) into a flipped SU(6) GUT model. Anomaly cancelation can easily be satisfied.

- Neutrino masses generation and successful gauge coupling unification can set lower/upper bounds on the *331* breaking scale.

- Certain parameter region with $M_{331} \approx 10^{15}$ GeV of case II (for case with M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs field) will predict a partial proton lifetime of order 10^{34} years for $p \to e^+\pi^0$ mode, which can be tested soon by future DUNE, JUNO and Hyper-Kamiokande experiments.