Introduction to Chiral Perturbation Theory

Feng-Kun Guo

Institute of Theoretical Physics, CAS

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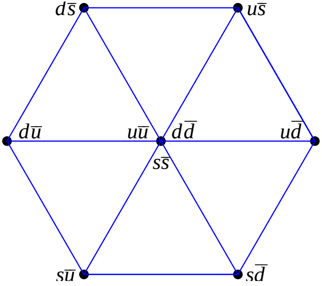
Outline

Light vector and pseudoscalar mesons

Light meson SU(3) $[u, d, s]$ multiplets (octet $+$ singlet):



- Vector mesons

meson	quark content	mass (MeV)
ρ^+/ρ^-	$u\bar{d}\,/\,d\bar{u}$	775
ρ^0	$(u\bar{u} - d\bar{d})/\sqrt{2}$	775
K^{*+}/K^{*-}	$u\bar{s}\,/\,s\bar{u}$	892
$K^{*0}/\bar{K}^{*0}$	$d\bar{s}\,/\,s\bar{d}$	896
ω	$(u\bar{u} + d\bar{d})/\sqrt{2}$	783
ϕ	$s\bar{s}$	1019

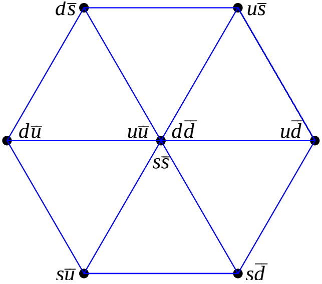
☞ approximate SU(3) symmetry

☞ very good isospin SU(2) symmetry

$$m_{\rho^0} - m_{\rho^\pm} = (-0.7 \pm 0.8)\ \text{MeV}, \quad m_{K^{*0}} - m_{K^{*\pm}} = (6.7 \pm 1.2)\ \text{MeV}$$

Light vector and pseudoscalar mesons

Light meson SU(3) [u, d, s] multiplets (octet + singlet):



- Pseudoscalar mesons

meson	quark content	mass (MeV)
π^+/π^-	$u\bar{d}\,/d\bar{u}$	140
π^0	$(u\bar{u}-d\bar{d})/\sqrt{2}$	135
K^+/K^-	$u\bar{s}\,/s\bar{u}$	494
$K^0/\bar{K}^0$	$d\bar{s}\,/s\bar{d}$	498
η	$\sim(u\bar{u}+d\bar{d}-2s\bar{s})/\sqrt{6}$	548
η'	$\sim(u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{3}$	958

☞ very good isospin SU(2) symmetry

$$m_{\pi^\pm}-m_{\pi^0}=(4.5936\pm0.0005)\ \text{MeV},\quad m_{K^0}-m_{K^\pm}=(3.937\pm0.028)\ \text{MeV}$$

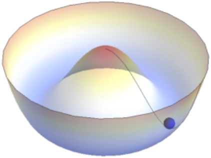
☞ Why are the pions so light? Pseudo-Nambu-Goldstone bosons of the spontaneous breaking of chiral symmetry: $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\to\mathrm{SU}(2)_V$

QCD symmetries

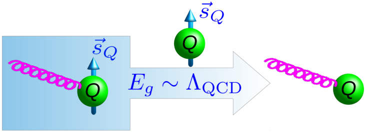
$$\mathcal{L}_{\rm QCD} = \sum_{f=\substack{u,d,s,\\c,b,t}} \bar q_f (i\not{D} - m_f) q_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a} + \frac{g_s^2\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

- Exact: Lorentz-invariance, SU$(3)_c$ gauge, C (and P, T for $\theta = 0$ w/ real m_f)
- Approximate: $\underbrace{\boxed{\overset{\text{u}}{\sim 2\text{ MeV}}}\ \boxed{\overset{\text{d}}{\sim 5\text{ MeV}}}\ \boxed{\overset{\text{s}}{\sim 100\text{ MeV}}}}_{\text{Light quarks}} \ll \Lambda_{\rm QCD} \ll \underbrace{\boxed{\overset{\text{c}}{\sim 1.3\text{ GeV}}}\ \boxed{\overset{\text{b}}{\sim 4.2\text{ GeV}}}\ \boxed{\overset{\text{t}}{\sim 173\text{ GeV}}}}_{\text{Heavy quarks}}$

- ☞ Spontaneously broken chiral symmetry:
  SU$(N_f)_L \times$ SU$(N_f)_R \xrightarrow{\text{SSB}}$ SU$(N_f)_V$



- ☞ Heavy quark spin symmetry (HQSS)
- ☞ Heavy quark flavor symmetry (HQFS)
- ☞ Heavy aiquark-diquark symmetry (HADS)



Chiral symmetry

Chiral symmetry (1)

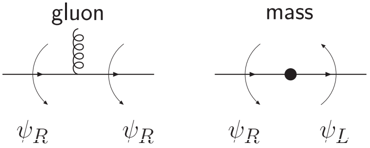
- Masses of the three lightest quarks u, d, s are small
  $\Rightarrow$ approximate chiral symmetry
- Chiral decomposition of fermion fields:

$$\psi = \frac{1}{2}(1-\gamma_5)\psi + \frac{1}{2}(1+\gamma_5)\psi \equiv P_L\psi + P_R\psi \;=\; \psi_L + \psi_R$$

  - ☞ Properties: $(\gamma_5)^2 = 1$, $\{\gamma_5, \gamma_\mu\} = 0$
  - ☞ $P_L{}^2 = P_L, \qquad P_R{}^2 = P_R,$
    $P_L P_R = P_R P_L = 0, \qquad P_L + P_R = 1$

- For massless fermions, left-/right-handed fields do not interact with each other

$$\mathcal{L}[\psi_L, \psi_R] = i\bar{\psi}_L \not{D}\psi_L + i\bar{\psi}_R \not{D}\psi_R - m\big(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\big)$$



Chiral symmetry (2)

- Decompose $\mathcal{L}_{\rm QCD}$ into $\mathcal{L}^0_{\rm QCD}$, the QCD Lagrangian in the 3-flavor chiral limit $m_u = m_d = m_s = 0$, and the light quark mass term:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^0_{\text{QCD}} - \bar{q}\mathcal{M}q\ , \quad q = (u, d, s)^T, \quad \mathcal{M} = \text{diag}(m_u,\, m_d,\, m_s)$$

- $\mathcal{L}^0_{\rm QCD} = i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R + \ldots$

  is invariant under $\text{U}(3)_L \times \text{U}(3)_R$ transformations:

$$\mathcal{L}^0_{\rm QCD}(G_{\mu\nu},\, q',\, D_\mu q') = \mathcal{L}^0_{\rm QCD}(G_{\mu\nu},\, q,\, D_\mu q)$$

$$q' = R\, P_R q + L\, P_L q = R\, q_R + L\, q_L$$

$$R \in \text{U}(3)_R, \quad L \in \text{U}(3)_L$$

- Parity: $q(t, \vec{x}) \xrightarrow{P} \gamma^0 q(t, -\vec{x})$

$$\Rightarrow \qquad q_R(t, \vec{x}) \xrightarrow{P} P_R \gamma^0 q(t, -\vec{x}) = \gamma^0 P_L q(t, -\vec{x}) = \gamma^0 q_L(t, -\vec{x})$$

$$q_L(t, \vec{x}) \xrightarrow{P} \gamma^0 q_R(t, -\vec{x})$$

- $\text{U}(3)_L \times \text{U}(3)_R = \text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_L \times \text{U}(1)_R$

Chiral symmetry (3)

$$q_{L/R} \Rightarrow U_{L/R}\, q_{L/R} = e^{-i\alpha^a_{L/R}T^a} e^{-i\alpha_{L/R}} q_{L/R}.$$

$$q = q_L + q_R = \frac{1-\gamma_5}{2} q + \frac{1+\gamma_5}{2} q$$

$$\longrightarrow \frac{1}{2}\left(e^{-i\alpha_L^aT^a}e^{-i\alpha_L} + e^{-i\alpha_R^aT^a}e^{-i\alpha_R}\right) q + \frac{\gamma_5}{2}\left(-e^{-i\alpha_L^aT^a}e^{-i\alpha_L} + e^{-i\alpha_R^aT^a}e^{-i\alpha_R}\right) q$$

$$= \frac{1}{2}\left(2 - i\alpha_L^aT^a - i\alpha_L - i\alpha_R^aT^a - i\alpha_R\right) q + \frac{\gamma_5}{2}\left(i\alpha_L^aT^a + i\alpha_L - i\alpha_R^aT^a - i\alpha_R\right) q + \dots$$

$$= \Big(1 - i\underbrace{\alpha_V^a}_{\overset{\|}{\frac{1}{2}(\alpha_L^a+\alpha_R^a)}} T^a - i\underbrace{\alpha_V}_{\overset{\|}{\frac{1}{2}(\alpha_L+\alpha_R)}}\Big) q + \Big(-i\underbrace{\alpha_A^a}_{\overset{\|}{\frac{1}{2}(\alpha_R^a-\alpha_L^a)}} T^a - i\underbrace{\alpha_A}_{\overset{\|}{\frac{1}{2}(\alpha_R-\alpha_L)}}\Big)\gamma_5 q + \dots$$

$$= \underbrace{e^{-i\alpha_V^aT^a}}_{SU(N)_V}\ \underbrace{e^{-i\alpha_V}}_{U(1)_V}\ \underbrace{e^{-i\alpha_A^aT^a\gamma_5}}_{\text{axial}}\ \underbrace{e^{-i\alpha_A\gamma_5}}_{U(1)_A}\ q$$

- $\mathrm{U}(3)_L \times \mathrm{U}(3)_R = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$
- $\{e^{-i\alpha_A^a T^a\gamma_5}\}(\alpha_A^a \in \mathbb{R})$ do not form a group (the closure property not satisfied). Baker-Compbell-Hausdorff formula: $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$; $(\gamma_5)^2 = 1$

Chiral symmetry (4)

- Decompose U$(3)_L \times$ U$(3)_R =$ SU$(3)_L \times$ SU$(3)_R \times$ U$(1)_V \times$ U$(1)_A$:

  18 conserved (Noether) currents for massless QCD at the classical level

$$J^{\mu,a}_{L,R} = \bar{q}_{L,R}\gamma^\mu \frac{\lambda^a}{2} q_{L,R}\,, \quad J^{\mu,0}_{L,R} = \bar{q}_{L,R}\gamma^\mu q_{L,R}, \qquad (\lambda_a: \text{ Gell-Mann matrices})$$

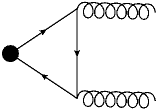
- rewritten in terms of vector ($V = L + R$) and axial vector ($A = R - L$) currents

$$V^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q\,, \qquad A^{\mu,a} = \bar{q}\gamma^\mu\gamma_5 \frac{\lambda^a}{2} q$$
$$\partial_\mu V^{\mu,a} = 0\,, \qquad \partial_\mu A^{\mu,a} = 0$$

- U$(1)_V$: baryon or quark number conservation

$$V^{\mu,0} = \bar{q}\gamma_\mu q, \quad \partial_\mu V^{\mu,0} = 0$$

- U$(1)_A$: explicitly broken by quantum effects, anomaly



$$A^{\mu,0} = \bar{q}\gamma_\mu\gamma_5 q, \quad \partial_\mu A^{\mu,0} = \frac{N_f g_s^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

  $\Rightarrow$ the θ-term, $|\bar{\theta}| \lesssim 10^{-10}$ ($\bar{\theta} \equiv \theta + \arg\det\mathcal{M}$) $\Rightarrow$ strong CP problem

- Is SU$(3)_L \times$ SU$(3)_R$ realized in hadron spectrum?

Wigner–Weyl v.s. Nambu–Goldstone

- Noether's theorem: continuous symmetry ⇒ conserved currents

  Let Q^a be symmetry charges: $Q^a = \int d^3\vec{x}\, J^{a,0}(t,\vec{x}), \quad \partial_\mu J^{a,\mu} = 0$

- Q^a is the symmetry generator: $g = e^{i\alpha^a Q^a}$, H: Hamiltonian, thus

$$gHg^{-1} = H \;\Rightarrow\; [Q^a, H] = 0,$$

$$[Q^a, H]|0\rangle = Q^a \underbrace{H|0\rangle}_{=0} - HQ^a|0\rangle = 0$$

- Wigner–Weyl mode: $Q^a|0\rangle = 0$ or equivalently $g|0\rangle = |0\rangle$

  degeneracy in mass spectrum

- Nambu–Goldstone mode: $g|0\rangle \neq |0\rangle$, spontaneously broken (hidden)

  $Q^a|0\rangle \neq 0$: new states degenerate with vacuum ($HQ^a|0\rangle = 0$), massless Goldstone bosons

  - ☞ spontaneously broken continuous global symmetry ⇒ massless GBs
  - ☞ the same quantum numbers as $Q^a|0\rangle$ ⇒ spinless
  - ☞ #(GBs) = #(broken generators)

Fate of SU(3)$_L\times$SU(3)$_R$ (1)

Vector and axial charges:

$$Q_V^a = Q_L^a + Q_R^a, \quad Q_A^a = Q_R^a - Q_L^a, \quad [Q_{V,A}^a, H_{\text{QCD}}^0] = 0$$

here H_{QCD}^0: QCD Hamiltonian in the chiral limit.

Under parity transformation: $q_R \to \gamma^0 q_L,\ q_L \to \gamma^0 q_R$

$$J_L^{\mu,a} \to J_{R,\mu}^a, \qquad J_R^{\mu,a} \to J_{L,\mu}^a$$

$$\Rightarrow \quad P Q_V^a P^{-1} = Q_V^a, \qquad P Q_A^a P^{-1} = -Q_A^a$$

For an eigenstate of H_{QCD}^0, $|\psi_\alpha\rangle = b_\alpha^\dagger|0\rangle$: $H_{\text{QCD}}^0|\psi_\alpha\rangle = E|\psi_\alpha\rangle$ with $P|\psi_\alpha\rangle = \eta_P|\psi_\alpha\rangle$, then

$$H_{\text{QCD}}^0 Q_A^a|\psi_\alpha\rangle = Q_A^a H_{\text{QCD}}^0|\psi_\alpha\rangle = E\, Q_A^a|\psi_\alpha\rangle,$$

$$P\, Q_A^a|\psi_\alpha\rangle = P\, Q_A^a P^{-1} P|\psi_\alpha\rangle = -\eta_P Q_A^a|\psi_\alpha\rangle$$

Parity doubling: $Q_A^a|\psi_\alpha\rangle = Q_A^a b_\alpha^\dagger|0\rangle = \underbrace{\left[Q_A^a, b_\alpha^\dagger\right]|0\rangle}_{=(t^a)_{\alpha\beta} b_{\beta-}^\dagger|0\rangle} + b_\alpha^\dagger Q_A^a|0\rangle$ has the same mass

but opposite parity

Fate of SU(3)$_L \times$SU(3)$_R$ (2)

- But, in the hadron spectrum:

$$m_{\text{Nucleon},\ P=+} = 939\text{ MeV} \ll m_{N^*(1535),\ P=-} = 1535\text{ MeV},$$

$$m_{\pi,\ P=-} = 139\text{ MeV} \ll m_{a_0(980),\ P=+} = 980\text{ MeV}$$

- No parity doubling in hadron spectrum ⇒

$$Q_A^a|0\rangle \neq 0, \quad \text{or} \quad e^{i\alpha^a Q_A^a}|0\rangle \neq |0\rangle$$

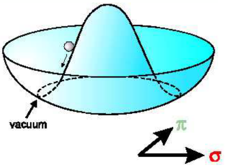
Nambu–Goldstone mode (hidden symmetry):

In QCD, SU(3)$_L \times$SU(3)$_R$ is spontaneously broken down to SU(3)$_V$

- GBs should have $J^P = 0^-$

SSB in linear σ model (1)

- Linear σ model with an $O(4)$ symmetry



$$\mathcal{L}_{\text{LSM}} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi - V(\Phi),$$

$$V(\Phi) = \frac{\lambda}{4}\left(\Phi^T\Phi - v^2\right)^2, \quad \text{with } \Phi^T = (\sigma, \pi_1, \pi_2, \pi_3)$$

- $\sigma = \pi_a = 0$ is not a minimum of $V(\Phi)$ for $v^2 > 0$

  There is a continuum of degenerate vacua: $\Phi^T_{\text{min}}\Phi_{\text{min}} = v^2$

  Choose $\Phi_{\text{min}} = (v, 0, 0, 0)$, vacuum is only invariant under $O(3)$ rotations

  spontaneous symmetry breaking: $O(4) \to O(3)$

- Perturb around $\Phi_{\text{min}} = (v, \vec{0})$, let $\Phi = \Phi_{\text{min}} + (\sigma', \pi_1, \pi_2, \pi_3)^T$, then

$$\mathcal{L}_{\text{LSM}} = \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi_a + \frac{1}{2}\partial_\mu \sigma' \partial^\mu \sigma' - \frac{\lambda}{4}\left(\sigma'^2 + \pi_a\pi_a + 2v\sigma'\right)^2$$

☞ Goldstone bosons (GBs): π_a's are massless

☞ σ' is massive: $m^2_{\sigma'} = 2\lambda v^2$

☞ $m_{\sigma'}$ and 5 interaction terms described by only 2 parameters

SSB in linear σ model (2)

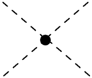
$$\mathcal{L}_{\text{LSM}} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}\partial_\mu\sigma'\partial^\mu\sigma' - \frac{\lambda}{4}\left(\sigma'^{\,2} + \pi_a\pi_a + 2v\sigma'\right)^2$$

Only 2 parameters, important for cancellation!

Examples: tree-level scattering amplitudes

- $\pi_3(p_1)\pi_3(p_2) \to \pi_3(p_3)\pi_3(p_4)$ (individually large terms!)

$i\mathcal{A} =$



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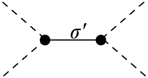

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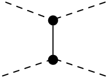

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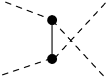


$$= -i6\lambda + \frac{i(-2i\lambda v)^2}{s-(2\lambda v^2)} + \frac{i(-2i\lambda v)^2}{t-(2\lambda v^2)} + \frac{i(-2i\lambda v)^2}{u-(2\lambda v^2)}$$

$$= -i6\lambda + \frac{i}{2\lambda v^2}(2\lambda v)^2\left(3 + \frac{s+t+u}{2\lambda v^2}\right) + \mathcal{O}\left(\frac{p_\pi^4}{m_\sigma^4}\right) = \mathcal{O}\left(\frac{p_\pi^4}{m_\sigma^4}\right)$$

p_π: a generic momentum of GBs

Mandelstam variables: $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, $u=(p_1-p_4)^2$,

$s+t+u=\sum_i p_i^2$

SSB in linear σ model (3)

- $\pi_3\sigma' \to \pi_3\sigma'$

  Exercise: Show that at tree-level

$$\mathcal{A}(\pi_3\sigma' \to \pi_3\sigma') = \mathcal{O}\left(\frac{p_\pi^2}{m_\sigma^2}\right)$$

Lessons from the linear σ model:

☞ SSB happens when there is a degeneracy of the vacuum

☞ nonvanishing VEV of some Hermitian operator, here $\langle\sigma\rangle = v$

☞ SSB $\Rightarrow$ massless GBs (π_a), $\# = \text{dim}(O(4)) - \text{dim}(O(3)) = 3$

☞ GBs decouple at vanishing momenta!

Derivative coupling

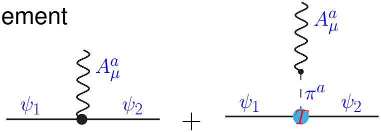
Symmetry implies a derivative coupling for GBs, i.e.,

$$\boxed{\text{GBs do not interact at vanishing momenta}}$$

- Consider GB π^a: $\langle \pi^a | Q^a_A | 0 \rangle = \int d^3x \, \langle \pi^a | A^a_0(x) | 0 \rangle \neq 0$

  Lorentz invariance $\Rightarrow$ $\left\langle \pi^a(q) | A^a_\mu(0) | 0 \right\rangle = -i q_\mu F_\pi$
- Consider the matrix element

$$\langle \psi_1 | A^a_\mu(0) | \psi_2 \rangle =$$



$$= \quad R^a_\mu \quad + \quad F_\pi q^\mu \frac{1}{q^2} T^a$$

Current conservation $\Rightarrow$ $q^\mu A^a_\mu = 0$, thus

$$q^\mu R^a_\mu + F_\pi T^a = 0 \quad \Rightarrow \quad \lim_{q^\mu \to 0} T^a = 0$$

- $\Rightarrow$ GBs couple in a derivative form !!

SSB in QCD

- Hamiltonian invariant under a group $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, vacuum invariant under its vector subgroup $H = \mathrm{SU}(N_f)_V$.

$$Q_V^a|0\rangle = 0, \quad Q_A^a|0\rangle \neq 0$$

- Nonvanishing chiral condensate: $\langle \bar{q}q \rangle = \langle 0 \,|(\bar{q}_L q_R + \bar{q}_R q_L)|\, 0 \rangle \neq 0$
- SSB ⇒ massless pseudoscalar Goldstone bosons

$\#(\mathrm{GBs}) = \dim(G) - \dim(H) = N_f^2 - 1$

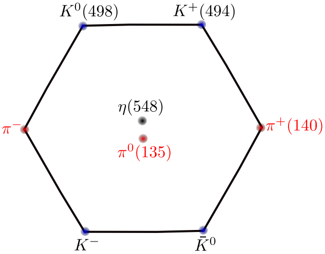
for $N_f = 3$, 8 GBs: $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

for $N_f = 2$, 3 GBs: $\pi^\pm, \pi^0$

Pions get a small mass due to explicit symmetry breaking by tiny $m_{u,d}$ (a few MeV)

$M_\pi \ll M_{\text{other hadron}}$

also, $m_s \gg m_{u,d} \;\; \Rightarrow \;\; M_K \gg M_\pi$

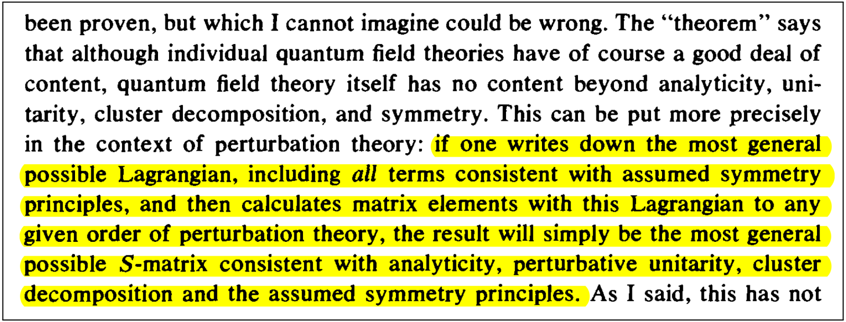


- Mechanism for $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V$ in QCD not well understood

Chiral perturbation theory

Low-energy effective field theory (EFT)

S. Weinberg, “Phenomenological Lagrangians”, Physica **96A** (1979) 327



been proven, but which I cannot imagine could be wrong. The “theorem” says that although individual quantum field theories have of course a good deal of content, quantum field theory itself has no content beyond analyticity, unitarity, cluster decomposition, and symmetry. This can be put more precisely in the context of perturbation theory: if one writes down the most general possible Lagrangian, including *all* terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible *S*-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. As I said, this has not

Translation:

- The most general effective Lagrangian, up to a given order, consistent with the symmetries of the underlying theory

  ⇒ results consistent with the underlying theory (with unitarity satisfied only perturbatively)!
- The degrees of freedom can be different from those of the underlining theory

  ⇒ we can work with hadrons directly for low-energy QCD, really effective!

Low-energy EFT (2)

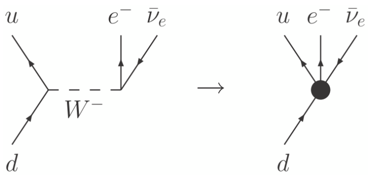
However, the “most general” means

- an infinite number of parameters ⇒ intractable (?)
- nonrenormalizable (in contrast to, e.g., QED and QCD)

Solution: systematic expansion with a power counting

- only a finite number of parameters at a given order, can be determined from
  - experiments
  - lattice calculations for QCD
- renormalize order by order
- existence of a small (dimensionless) quantity, e.g.,
  separation of energy scales, $E \ll \Lambda \Rightarrow$ expansion in powers of (E/Λ)
  Neutron decay ($n \to p e^- \bar{v}_e$): weak interactions for $|q^2| \ll M_W^2$ (decoupling EFT)

$$\frac{e^2}{2\sin^2\theta_W}\frac{1}{M_W^2-q^2}$$
$$=\frac{e^2}{2M_W^2\sin^2\theta_W}\left(1+\frac{q^2}{M_W^2}+\dots\right)$$
$$=\frac{4G_F}{\sqrt{2}}+\mathcal{O}\left(\frac{q^2}{M_W^2}\right)$$



Low-energy EFT (3)

- Pro: model-independent, controlled uncertainty
- Con: number of parameters increases fast when going to higher orders

Things need to be remembered for an EFT:

- separation of energy scales:
  systematic expansion with a power counting
- symmetry constraints from the full theory
  For low-energy QCD, we consider
  - ☞ (approximate) chiral symmetry of light quarks
    ⇒ CHiral Perturbation Theory (non-decoupling EFT)
    full theory ⇒ EFT via spontaneous symmetry breaking (SSB)

    generation of new light degrees of freedom
  - ☞ heavy quark symmetry: spin and flavor

Chirl perturbation theory (CHPT) as low-energy EFT for QCD

CHPT: a low energy EFT for QCD:

- an example for a non-decoupling EFT:

  degrees of freedom are different from those of the underlying theory
- a theory for the Goldstone bosons of SU$(N_f)_L \times$ SU$(N_f)_R \to$ SU$(N_f)_V$
- most general Lagrangian with the same global symmetries as QCD

How do the Goldstone bosons transform under SU$(N_f)_L \times$ SU$(N_f)_R$?

Nonlinear realization of chiral symmetry (1)

Weinberg (1968); Coleman, Wess, Zumino (1969); Callan, Coleman, Wess, Zumino (1969)

To study the transformation properties of the Goldstone bosons (GBs)

- Assume $G \xrightarrow{\text{SSB}} H$, we have GBs

$$\Phi = (\phi_i, \ldots, \phi_n), \quad n = \text{dim}(G) - \text{dim}(H)$$

- Recall for SSB, $g|0\rangle \neq |0\rangle$ produces massless GBs

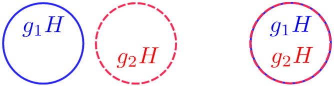
  $g|0\rangle = gh|0\rangle$ since the vacuum is invariant under $h \in H$

  GBs are defined up to the equivalence $g \sim gh$, *i.e.*, GBs live in the coset space G/H

- Left coset of H with respect to $g \in G$: $gH = \{gh|h \in H\}$

  coset space G/H: set of all left cosets $\{gH|g \in G\}$

  - ☞ Two left cosets either completely overlap or completely disjoint



If $\underbrace{g_1h_1}_{\in g_1H} = \underbrace{g_2h_2}_{\in g_2H}$, then $\underbrace{g_1h_1H}_{=g_1H} = \underbrace{g_2h_2H}_{=g_2H}$

  - ☞ $\text{dim}(G/H) = \dim(G) - \dim(H)$

- GBs live in the coset space G/H:

Nonlinear realization of chiral symmetry (2)

- Cosets either completely overlap or completely disjoint

  ⇒ free to choose the set of representative elements / set of coordinates on G/H

  E.g., for $h_{1,2} \in H$, we can choose either gh_1 or gh_2 to represent the coset gH

- Transformation properties of the GBs uniquely determined once the set of rep. elements have been chosen:

  Parameterizing GBs by $u \in G/H$

  transformation under $g \in G$

$$g\, u = u'\, h(g, u)$$

  since any element of the coset $\{u'h(g,u)|h(g,u) \in H\}$ can be used

  ⇒ Nonlinear transformation of GBs

$$\boxed{u \overset{g\in G}{\to} u' = g\, u\, h^{-1}(g, u)}$$

Application to QCD (1)

For QCD, $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \xrightarrow{\text{SSB}} H = \mathrm{SU}(N_f)_V$

- $g = (g_L, g_R) \in \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ for $g_L \in \mathrm{SU}(N_f)_L$, $g_R \in \mathrm{SU}(N_f)_R$

$$g_1 g_2 = (g_{L_1}, g_{R_1})(g_{L_2}, g_{R_2}) = (g_{L_1} g_{L_2}, g_{R_1} g_{R_2})$$

- Choice of a representative element inside each left coset is free

$$(g_L, g_R)H = (g_L g_R^\dagger, \mathbb{1}) \underbrace{(g_R, g_R)}_{\in H = \mathrm{SU}(N_f)_V} H = (g_L g_R^\dagger, \mathbb{1})\, H$$

☞ Goldstone bosons can be parameterized by

$$\boxed{U = g_L g_R^\dagger = \exp\left(i \frac{\phi}{F'}\right)}$$

here, $\phi = \sum_{a=1}^{8} \lambda_a \phi_a$ for SU(3), and $\vec{\tau} \cdot \vec{\pi}$ for SU(2)

λ_a: Gell-Mann matrices, $\tau_i (i = 1, 2, 3)$: Pauli matrices

ϕ_a (π_i) : Goldstone boson fields

F' : dimensionful constant (to be determined later)

Application to QCD (2)

- Acting $g = (L,\, R) \in G$ on the coset $(U, \mathbb{1})H$

$$g\,(U, \mathbb{1})H = (L\,U,\, R)H = (L\,U\,R^\dagger\,R,\, R)H = (L\,U\,R^\dagger, \mathbb{1})\,\underbrace{(R,\, R)}_{\in \mathrm{SU}(N_f)_V}\,H$$

$\Rightarrow$ transformation property of U:

$$\boxed{U \xrightarrow{g} L\,U\,R^\dagger}$$

One can also parametrize the GBs such that $U \xrightarrow{g} R\,U\,L^\dagger$. Any one is okay if used consistently.

- For $g \in H = \mathrm{SU}(N_f)_V$, we have $R = L$

$$U \to L\,U\,L^\dagger \quad \Rightarrow \quad \phi \to L\,\phi\,L^\dagger$$

i.e., GB fields transform linearly under $\mathrm{SU}(N_f)_V$

CHPT at leading order

Construction of the effective Lagrangian for GBs (1)

Aim: reproduce low-energy structure of QCD

- Effective Lagrangian invariant under $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ (and C, P)

$$U \overset{G}{\to} L\,U\,R^\dagger, \qquad U \overset{C}{\to} U^T, \qquad U \overset{P}{\to} U^\dagger$$

- What does "low-energy" mean here?

  Goldstone boson fields (contained in U) as the only degrees of freedom

  $\Rightarrow$ energy range restricted to well below 1 GeV

  (separation of energy scales, more see later)

- Low energies: expand in powers of momenta (= number of derivatives)
- Lorentz invariance $\Rightarrow$ only even number of derivatives are allowed

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$$

- $\mathcal{L}^{(0)}$? U is unitary, $UU^\dagger = \mathbb{1}$, hence

$$\langle (UU^\dagger)^n \rangle \ = \ \text{const.}\ , \quad \langle \ldots \rangle \equiv \mathrm{Tr}_{\text{flavor}}[\ldots]$$

  $\Rightarrow$ Leading non-trivial term is $\mathcal{L}^{(2)}$

Construction of the effective Lagrangian for GBs (2)

- Leading term of the effective Lagrangian is $\mathcal{L}^{(2)}$
  just one single term (nonlinear σ model):

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle} \quad \text{with} \quad U = \exp\left(\frac{i\phi}{F'}\right)$$

$$\phi_{\text{SU(2)}} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \quad \phi_{\text{SU(3)}} = \sqrt{2}\begin{pmatrix} \frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\phi_8}{\sqrt{6}}, \end{pmatrix}$$

  i.e. $\phi_{\text{SU(2)}} = \tau^a\pi^a$, $\phi_{\text{SU(3)}} = \lambda^a\phi^a$, with $\tau^a(a = 1, 2, 3)$ and $\lambda^a(a = 1, 2, \ldots, 8)$ the Pauli and Gell-Mann matrices, respectively

- $\mathcal{L}^{(2)}$ is invariant under $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$:

$$\begin{aligned} \langle \partial_\mu U \partial^\mu U^\dagger \rangle &\to \langle \partial_\mu U' \partial^\mu U'^\dagger \rangle \\ &= \langle L\partial_\mu U R^\dagger R \partial^\mu U^\dagger L^\dagger \rangle = \langle \partial_\mu U \partial^\mu U^\dagger \rangle \end{aligned}$$

Note that $\left\langle U\partial^\mu U^\dagger \right\rangle = 0$, therefore $\left\langle U\partial_\mu U^\dagger \right\rangle \left\langle U\partial^\mu U^\dagger \right\rangle$ is not present.

The low-energy constant F

- Expand U in powers of ϕ, $\quad U = 1 + \frac{i\phi}{F'} - \frac{\phi^2}{2F'^2} + \ldots$

  $\Rightarrow$ canonical kinetic terms $\quad \mathcal{L}^{(2)} = \partial_\mu \pi^+ \partial^\mu \pi^- + \partial_\mu K^+ \partial^\mu K^- + \ldots$

  for $\quad \boxed{F' = F}$

- Calculate the Noether currents L_a^μ, R_a^μ from $\mathcal{L}^{(2)}$ $\quad \Rightarrow$

$$\begin{aligned} V_a^\mu &= R_a^\mu + L_a^\mu = i\frac{F^2}{4}\langle \lambda_a \left[\partial^\mu U, U^\dagger\right]\rangle \\ A_a^\mu &= R_a^\mu - L_a^\mu = i\frac{F^2}{4}\langle \lambda_a \left\{\partial^\mu U, U^\dagger\right\}\rangle \end{aligned}$$

- Expand the currents in powers of ϕ, $\quad A_a^\mu = -F\partial^\mu \phi_a + \mathcal{O}\left(\phi^3\right)$

$$\langle 0|A_a^\mu(x)|\phi_b(p)\rangle = ip^\mu F\, e^{-ip\cdot x}\, \delta_{ab}$$

  $\Rightarrow$ F is the pion decay constant in the chiral limit

$$\boxed{F \approx F_\pi}$$

  $F_\pi = 92.2$ MeV measured in the leptonic decay of the pion $\pi^+ \to \ell^+ \nu_\ell$

PCAC

So far, only considered chiral limit $m_u = m_d = m_s = 0$.

For non-zero quark masses,

- the singlet vector current still conserved (baryon number conservation)

$$\partial^\mu V_\mu = 0$$

- vector currents V_μ^a conserved when $m_u = m_d = m_s = m$ (i.e., $\mathcal{M} = m\mathbb{1}$)

$$\partial^\mu V_\mu^a = i\,\bar{q}\left[\mathcal{M}, \frac{\lambda^a}{2}\right] q$$

- A_μ^a not conserved any more: Partially Conserved Axial Current (PCAC)

$$\partial^\mu A_\mu^a = i\,\bar{q}\left\{\mathcal{M}, \frac{\lambda^a}{2}\right\} q$$

Explicit symmetry breaking: quark masses

- In the chiral limit $m_u = m_d = m_s = 0$

  ☞ $\mathcal{L}^{(2)} = \frac{F^2}{4}\left\langle \partial_\mu U \partial^\mu U^\dagger \right\rangle$ contains no terms $\propto M^2\phi^2$

  ☞ theory for massless Goldstone bosons, but pions have nonvanishing masses

- In nature, u, d, s quark masses are small ($m_{u,d,s} \ll \Lambda_{\text{QCD}}$), but non-zero

  ☞ chiral symmetry explicitly broken

$$\mathcal{L}_m = -\bar{q}_R\, m\, q_L - \bar{q}_L\, m\, q_R$$

  ☞ if symmetry breaking is weak ⇒

  a perturbative expansion in the quark masses

- Effective Lagrangian is still an appropriate tool to systematically derive all symmetry relations

CHiral Perturbation Theory (CHPT):

double expansion in low momenta and quark masses

Explicit symmetry breaking: the spurion trick (1)

Spurion in 3 steps: very useful trick for explicit symmetry breaking

1. Introduce a spurion field (e.g. quark mass, electric charge, $\gamma_\mu, \ldots$) with a transformation property so that the symmetry breaking term in the full theory is invariant
2. Write down invariant operators in EFT including the spurion field
3. Set the spurion field to the value which it should take

Explicit symmetry breaking: the spurion trick (2)

- Apply the spurion trick to quark masses:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^0_{\text{QCD}} - \bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M}^\dagger q_L$$

  - ☞ Treat $\mathcal{M}$ as a complex spurion field

$$\boxed{\mathcal{M} \to \mathcal{M}' = L\mathcal{M}R^\dagger}$$

  - ☞ Then construct Lagrangian invariant under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots, \mathcal{M})$$

  - ☞ This procedure guarantees that chiral symmetry is broken in exactly the same way in the effective theory as it is in QCD

$$\mathcal{L}_2 = \frac{F^2}{4}\left[\langle \partial_\mu U \partial^\mu U^\dagger \rangle + 2B\langle \mathcal{M}U^\dagger + \mathcal{M}^\dagger U \rangle\right] \quad \text{with } U \to L\,U R^\dagger$$

- The spurion trick is very useful to construct EFT operators with a given symmetry transformation property

LO chiral Lagrangian with the mass term (1)

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\left[\langle\partial_\mu U \partial^\mu U^\dagger\rangle + 2B\langle\mathcal{M}U^\dagger + \mathcal{M}^\dagger U\rangle\right]}$$

- At LO [SU(3)]:

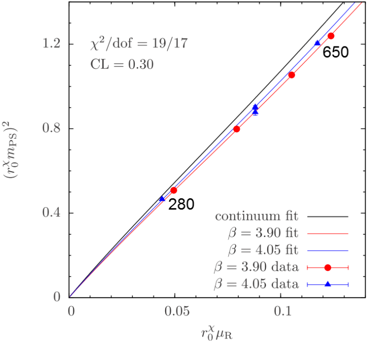
$$\boxed{\begin{aligned} M_{\pi^\pm}^2 &= B(m_u + m_d) \\ M_{K^\pm}^2 &= B(m_u + m_s) \\ M_{K^0}^2 &= B(m_d + m_s) \end{aligned}}$$

- Gell-Mann–Oakes–Renner (GMOR) relation:

$$M_{\text{GB}}^2 \propto m_q$$

☞ CHPT can be used to extrapolate lattice results from large to the physical values of $m_{u,d}$ (or equivalently pion masses)

GMOR relation on lattice:



ETM Col., JHEP08(2010)097

- Unified power counting for derivative and quark mass expansions:

$$m_q = \mathcal{O}\left(p^2\right)$$

LO chiral Lagrangian with the mass term (2)

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\left[\langle \partial_\mu U \partial^\mu U^\dagger \rangle + 2B\langle \mathcal{M}U^\dagger + \mathcal{M}^\dagger U \rangle\right]}$$

- Flavor-neutral states ϕ_3, ϕ_8 are mixed:

$$\frac{B}{2}\begin{pmatrix} \phi_3 \\ \phi_8 \end{pmatrix}^T \begin{pmatrix} m_u + m_d & \frac{1}{\sqrt{3}}(m_u - m_d) \\ \frac{1}{\sqrt{3}}(m_u - m_d) & \frac{1}{3}(m_u + m_d + 4m_s) \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_8 \end{pmatrix}$$

- Diagonalize with

$$\begin{pmatrix} \pi^0 \\ \eta \end{pmatrix} = \begin{pmatrix} \cos \epsilon_{\pi^0\eta} & \sin \epsilon_{\pi^0\eta} \\ -\sin \epsilon_{\pi^0\eta} & \cos \epsilon_{\pi^0\eta} \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_8 \end{pmatrix}$$

- Exercise: derive the $\pi^0\eta$ mixing angle:

$$\epsilon_{\pi^0\eta} = \frac{1}{2}\arctan\left(\frac{\sqrt{3}(m_d - m_u)}{2m_s - m_u - m_d}\right) \simeq \frac{\sqrt{3}}{2}\frac{m_d - m_u}{2m_s - m_u - m_d}$$

LO chiral Lagrangian with the mass term (3)

- Mass eigenvalues:

$$\boxed{\begin{aligned} M_{\pi^0}^2 &= B(m_u + m_d) - \mathcal{O}\left((m_u - m_d)^2\right) \\ M_\eta^2 &= \frac{B}{3}(m_u + m_d + 4m_s) + \mathcal{O}\left((m_u - m_d)^2\right) \end{aligned}}$$

- At LO w/o electromagnetic effects,

$$M_{\pi^\pm}^2 = M_{\pi^0}^2\,,$$

in the isospin limit $m_u = m_d$:

$$M_{K^\pm}^2 = M_{K^0}^2 \qquad \text{(of course!)}$$

- Gell-Mann–Okubo (GMO) mass formula for pseudoscalars:

$$4M_K^2 = 3M_\eta^2 + M_\pi^2$$

$\Rightarrow$ a LO relation, fulfilled in nature at 7% accuracy

LO chiral Lagrangian with the mass term (4)

Exercise: One possible solution of the strong CP problem is the Peccei-Quinn (PQ) mechanism which introduces a global U(1) symmetry, called the PQ symmetry. Axion is the pseudo-Goldstone boson of the spontaneous breaking of this symmetry. Its properties can be studied in CHPT by changing the quark mass matrix $\mathcal{M}$ to $\mathcal{M}e^{i\,Xa/f_a}$ with a the axion field, f_a the axion decay constant, and X satisfying $\langle X\rangle = 1$. Consider the LO mass term of the SU(2) version of CHPT with axion,

$$\mathcal{L}_a^{(2)} = \frac{F^2}{2}B\langle\mathcal{M}e^{i\,Xa/f_a}U^\dagger + \text{h.c.}\rangle,$$

where h.c. represents the Hermitian conjugated term.

1) show that there will be no a-π^0 mixing if we choose $X = \mathcal{M}^{-1}/\langle\mathcal{M}^{-1}\rangle$;

2) show that the axion mass squared is given by

$$m_a^2 = \frac{F^2M_\pi^2}{f_a^2}\frac{m_u m_d}{(m_u+m_d)^2}.$$

Quark mass ratios

- Unknown parameter B prevents quark mass determination
- Quark mass ratios:

$$\frac{m_u}{m_d} = \frac{M_{K^+}^2 - M_{K^0}^2 + M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \approx 0.67$$

$$\frac{m_s}{m_d} = \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \approx 22$$

- Sizable m_u/m_d, but why no large isospin violation in nature?

☞ $(m_d - m_u)/m_s$ (see $\pi^0\eta$ mixing angle) is small

☞ $(m_d - m_u)/\Lambda_{\text{QCD}}$ is small

- Pion mass difference due to $m_d - m_u$:

$$M_{\pi^0}^2 = M_{\pi^+}^2 \left\{ 1 - \frac{(m_d - m_u)^2}{8\hat{m}(m_s - \hat{m})} + \ldots \right\}$$

this leads to

$$M_{\pi^+} - M_{\pi^0} \approx 0.2 \text{ MeV}$$

vs. $$\left(M_{\pi^+} - M_{\pi^0}\right)_{\text{exp}} \approx 4.6 \text{ MeV}$$

Electromagnetic effects

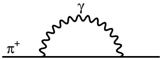
- Two sources of isospin symmetry breaking:

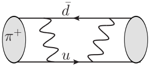
  ☞ $m_u \neq m_d$

  ☞ electromagnetic effects, $Q_u(=\frac{2}{3}) \neq Q_d(=-\frac{1}{3})$ (in units of e)

- Coupling of $\mathcal{L}^{(2)}$ to external vector (v_μ) / axial vector (a_μ) currents via covariant derivative straightforward:

$$\partial_\mu U \longrightarrow D_\mu U = \partial_\mu U - i[v_\mu, U] + i\{a_\mu, U\}$$

$\left\langle D_\mu U D^\mu U^\dagger \right\rangle$ contains 

, but misses, e.g., 

- Including photons only through the minimal substitution does not generate the most general e.m. effects

  ⇒ has to include chirally invariant local operators of $\mathcal{O}\left(e^2\right)$ for virtual photons

Virtual photons (1)

- Using the spurion trick for quark electric charge matrix $Q = e$ diag $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$ as for the quark mass term:

$$\mathcal{L}_{\text{em}} = -\bar{q} Q \not{A} q \longrightarrow -\bar{q}_L Q_L \not{A} q_L - \bar{q}_R Q_R \not{A} q_R$$

☞ Pretend $Q_{L,R}$ transform as:

$$Q_L \longmapsto L Q_L L^\dagger \,, \quad Q_R \longmapsto R Q_R R^\dagger$$

☞ Construct terms invariant under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$

☞ Set $Q_L = Q_R = Q$ in the end

- Power counting: $Q_{L,R} = \mathcal{O}(p)$
- Only one term at $\mathcal{O}\left(e^2\right) = \mathcal{O}\left(p^2\right)$

$$\boxed{\mathcal{L}_{\text{em}}^{(2)} = C \left\langle Q_L U Q_R U^\dagger \right\rangle}$$

Virtual photons (2)

- Contribution to the meson masses:

$$M_{\pi^\pm}^2 = B(m_u + m_d) + \frac{2Ce^2}{F^2}, \quad M_{K^\pm}^2 = B(m_u + m_s) + \frac{2Ce^2}{F^2}$$

  no contributions to neutral meson masses or $\pi^0\eta$ mixing

- Dashen's theorem:

$$\boxed{\left(M_{\pi^+}^2 - M_{\pi^0}^2\right)_{\text{em}} = \left(M_{K^+}^2 - M_{K^0}^2\right)_{\text{em}}}$$

  in the chiral limit

- Constant C fixed: $C = \frac{F_\pi^2}{2e^2}\left(M_{\pi^+}^2 - M_{\pi^0}^2\right)$
- Improved quark mass ratio

$$\frac{m_u}{m_d} = \frac{M_{K^+}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 0.56 \quad \text{instead of } 0.67$$

- The π^0-η mixing angle can be constructed:

$$\epsilon_{\pi^0\eta} \simeq \frac{\sqrt{3}}{2}\frac{m_d - m_u}{2m_s - m_u - m_d} \simeq \frac{\sqrt{3}}{2}\frac{M_{K^0}^2 - M_{K^\pm}^2 - M_{\pi^0}^2 + M_{\pi^\pm}^2}{M_{K^0}^2 + M_{K^\pm}^2 - M_{\pi^0}^2 - M_{\pi^\pm}^2} \simeq 0.0099$$

Representation invariance

- Freedom to choose coordinates on coset space G/H
- Haag's theorem on field redefinition: Haag (1958); Coleman, Wess, Zumino (1969)

  If fields ϕ and χ are related nonlinearly by a local function as

  $$\phi = \chi\, F[\chi] \quad \text{with} \quad F[0] = 1,$$

  then the same physical observables (on-shell S-matrices) can be obtained using either field ϕ with Lagrangian $\mathcal{L}[\phi]$ or χ with $\mathcal{L}[\chi F[\chi]]$.

$\pi\pi$ scattering (1)

In the chiral limit, $\mathcal{L}^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle$

Amplitude for $\pi^+(p_1)\pi^-(p_2) \to \pi^0(p_3)\pi^0(p_4)$: $\mathcal{A}(s,t,u)$

- Exponential representation:

$$U = \exp\frac{i\phi}{F}$$

with $\phi = \vec{\tau}\cdot\vec{\pi}$, $\pi^0 = \pi_3$, $\pi^\pm = \frac{1}{\sqrt{2}}(\pi_1 \mp i\pi_2)$ for SU(2).

$$\mathcal{A}(s,t,u) = \frac{s}{F^2}$$

☞ parameter-free prediction

☞ in accordance with Goldstone theorem: vanishes at zero-momentum

- Square-root representation:

$$U = \frac{1}{F}\left(\sqrt{F^2 - \vec{\pi}'^{\,2}} + i\vec{\tau}\cdot\vec{\pi}'\right)$$

calculating with $\vec{\pi}'$ as the pion fields gives the same scattering amplitude.

$\pi\pi$ scattering (2)

Exercise: Calculate the amplitude and show that the two representations are related by the following field redefinition:

$$\vec{\pi}' = \vec{\pi}\frac{F}{|\vec{\pi}|}\sin\left(\frac{|\vec{\pi}|}{F}\right) = \vec{\pi} + \text{nonlinear terms}, \quad \text{here } |\vec{\pi}| \equiv \sqrt{\vec{\pi}^2}$$

Hint: The following relations of Pauli matrices might be useful:

$$[\tau^a, \tau^b] = 2i\epsilon^{abc}\tau^c, \quad \{\tau^a, \tau^b\} = 2\delta^{ab}\mathbb{1},$$

$$\vec{\tau}\cdot\vec{A}\vec{\tau}\cdot\vec{B} = \vec{A}\cdot\vec{B}\mathbb{1} + i\vec{\tau}\cdot(\vec{A}\times\vec{B}), \quad \exp(i\vec{\tau}\cdot\vec{\pi}) = \cos(|\vec{\pi}|)\mathbb{1} + i\frac{\vec{\tau}\cdot\vec{\pi}}{|\vec{\pi}|}\sin(|\vec{\pi}|),$$

$$\mathrm{Tr}(\tau^a) = 0, \quad \mathrm{Tr}(\tau^a\tau^b) = 2\delta^{ab}, \quad \mathrm{Tr}(\tau^a\tau^b\tau^c) = 2i\epsilon^{abc},$$

$$\mathrm{Tr}(\tau^a\tau^b\tau^c\tau^d) = 2(\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd})$$

- The amplitude with the quark mass term included reads

$$\boxed{A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}}$$

Weinberg (1966)

$\pi\pi$ scattering (3)

- Pions: isospin $I = 1$. From particle basis to isospin basis ($|I, I_3\rangle$), choosing phase convention:

  $|\pi^+\rangle = -|\pi; I = 1, I_3 = 1\rangle$, $|\pi^0\rangle = |\pi; 1, 0\rangle$, $|\pi^-\rangle = |\pi; 1, -1\rangle$, then

$$|\pi^+\pi^-\rangle = -\left(\frac{1}{\sqrt{6}}|\pi\pi; 2, 0\rangle + \frac{1}{\sqrt{2}}|\pi\pi; 1, 0\rangle + \frac{1}{\sqrt{3}}|\pi\pi; 0, 0\rangle\right),$$

$$|\pi^0\pi^0\rangle = \sqrt{\frac{2}{3}}|\pi\pi; 2, 0\rangle - \frac{1}{\sqrt{3}}|\pi\pi; 0, 0\rangle,$$

$$|\pi^+\pi^0\rangle = -\frac{1}{\sqrt{2}}\left(|\pi\pi; 2, 1\rangle + |\pi\pi; 1, 1\rangle\right),$$

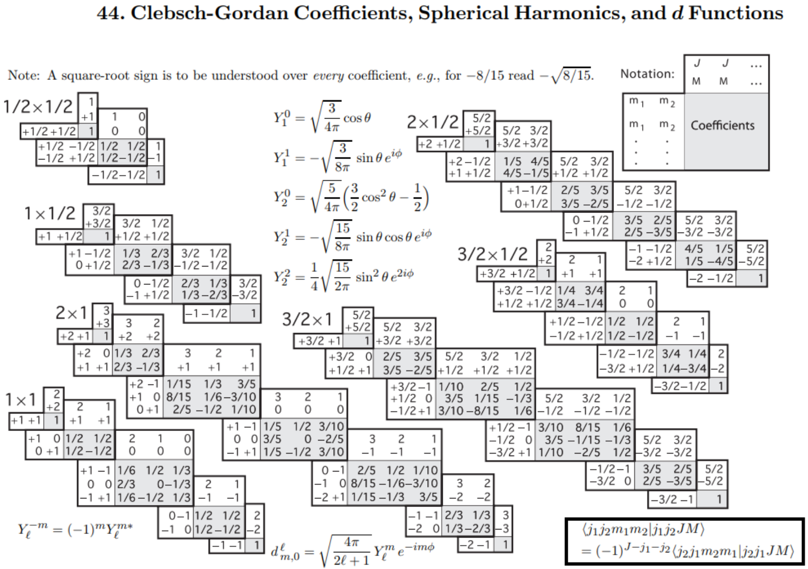
$$|\pi^0\pi^+\rangle = -\frac{1}{\sqrt{2}}\left(|\pi\pi; 2, 1\rangle - |\pi\pi; 1, 1\rangle\right).$$

- Crossing symmetry (recall $s = (p_1 + p_2)^2, t = (p_1 - p_3)^2, u = (p_1 - p_4)^2$):
  Let $T(s, t, u)$ denotes the amplitude for $A(p_1)B(p_2) \to C(p_3)D(p_4)$, then

$$T(t, s, u): \qquad A(p_1)\bar{C}(-p_3) \to \bar{B}(-p_2)D(p_4)$$

$$T(u, t, s): \qquad A(p_1)\bar{D}(-p_4) \to C(p_3)\bar{B}(-p_2)$$

CG coefficients



44. Clebsch-Gordan Coefficients, Spherical Harmonics, and d Functions

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	…
	M	M	…
m_1 m_2	Coefficients		
m_1 m_2			
⋮ ⋮			

1/2×1/2

	1
	+1
+1/2 +1/2	1

	1	0
	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	1
	−1
−1/2 −1/2	1

1×1/2

	3/2
	+3/2
+1 +1/2	1

	3/2	1/2
	+1/2	+1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

	3/2	1/2
	−1/2	−1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

	3/2
	−3/2
−1 −1/2	1

2×1

	3
	+3
+2 +1	1

	3	2
	+2	+2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

	3	2	1
	+1	+1	+1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

	3	2	1
	0	0	0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

	3	2	1
	−1	−1	−1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

	3	2
	−2	−2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

	3
	−3
−2 −1	1

1×1

	2
	+2
+1 +1	1

	2	1
	+1	+1
+1 0	1/2	1/2
0 +1	1/2	−1/2

	2	1	0
	0	0	0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

	2	1
	−1	−1
0 −1	1/2	1/2
−1 0	1/2	−1/2

	2
	−2
−1 −1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

3/2×1

	5/2
	+5/2
+3/2 +1	1

	5/2	3/2
	+3/2	+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

	5/2	3/2	1/2
	+1/2	+1/2	+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

	5/2	3/2	1/2
	−1/2	−1/2	−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

	5/2	3/2
	−3/2	−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

	5/2
	−5/2
−3/2 −1	1

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

2×1/2

	5/2
	+5/2
+2 +1/2	1

	5/2	3/2
	+3/2	+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2	3/2
	+1/2	+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2	3/2
	−1/2	−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2	3/2
	−3/2	−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2
	−5/2
−2 −1/2	1

3/2×1/2

	2
	+2
+3/2 +1/2	1

	2	1
	+1	+1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

	2	1
	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	2	1
	−1	−1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

	2
	−2
−3/2 −1/2	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

http://pdg.lbl.gov/2018/reviews/rpp2018-rev-clebsch-gordan-coefs.pdf

$\pi\pi$ scattering (4)

- Isospin symmetry: isospin is conserved (for strong interaction with $m_u = m_d$)

$$\begin{aligned}
\mathcal{A}(s,t,u) &= T_{\pi^+\pi^-\to\pi^0\pi^0}(s,t,u) = \frac{1}{3}\left(T^{I=0}(s,t,u) - T^{I=2}(s,t,u)\right), \\
\mathcal{A}(t,s,u) &= T_{\pi^+\pi^0\to\pi^+\pi^0}(s,t,u) = \frac{1}{2}\left(T^{I=1}(s,t,u) + T^{I=2}(s,t,u)\right), \\
\mathcal{A}(u,t,s) &= T_{\pi^+\pi^0\to\pi^0\pi^+}(s,t,u) = \frac{1}{2}\left(T^{I=2}(s,t,u) - T^{I=1}(s,t,u)\right).
\end{aligned}$$

- Crossing symmetry + isospin symmetry ⇒ isospin amplitudes

$$\begin{aligned}
T^{I=0}(s,t,u) &= 3\mathcal{A}(s,t,u) + \mathcal{A}(t,s,u) + \mathcal{A}(u,t,s) \\
T^{I=1}(s,t,u) &= \mathcal{A}(t,s,u) - \mathcal{A}(u,t,s) \\
T^{I=2}(s,t,u) &= \mathcal{A}(t,s,u) + \mathcal{A}(u,t,s)
\end{aligned}$$

$\pi\pi$ scattering (5)

- S-wave scattering lengths (unit: M_π^{-1}): $a_0^I = \frac{1}{32\pi} T^I(s = 4M_\pi^2, t = u = 0)$

$$\boxed{a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16, \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045}$$

- How do the LO predictions compare with data?

☞ Precise measurements from NA48/2 (K_{e4} decays & $K^\pm \to \pi^0\pi^0\pi^\pm$)

B.Bloch-Devaux, Kaon09

$$\begin{aligned} a_0^0 &= 0.2210 \pm 0.0047_{\text{stat}} \pm 0.0015_{\text{syst}} \\ a_0^2 &= -0.0429 \pm 0.0044_{\text{stat}} \pm 0.0016_{\text{syst}} \\ a_0^0 - a_0^2 &= 0.2639 \pm 0.0020_{\text{stat}} \pm 0.0004_{\text{syst}} \end{aligned}$$

☞ from DIRAC (life time of $\pi^+\pi^-$ atom)

B.Adeva et al., PLB704(2011)24

$$\left|a_0^0 - a_0^2\right| = 0.2533^{+0.0080}_{-0.0078}\Big|_{\text{stat}}{}^{+0.0078}_{-0.0073}\Big|_{\text{syst}}$$

CHPT with matter fields

Including matter fields

- So far, EFT for (pseudo-)Goldstone bosons
- Matter fields (fields which are not Goldstone bosons) can be included as well, e.g.
  - ☞ baryon CHPT:

    nucleons [SU(2)] / baryon ground state octet [SU(3)]
  - ☞ SU(2) kaon CHPT:

    kaons treated as matter fields rather than GBs
  - ☞ heavy-hadron CHPT:

    heavy-flavor (charm, bottom) mesons and baryons

    . . .
- Feature:

  a new mass scale: mass of the matter field, non-vanishing in the chiral limit

$$m_N\Big|_{m_q=0} = \mathcal{O}\left(m_N\Big|_{m_q=m_q^{\text{phys.}}}\right)$$

  will cause a problem in power counting
- At low-energies, 3-momenta remain small $\sim M_\pi$, derivative expansion is feasible

Transformation of matter fields

- Proceed as before
  - ☞ need to know how matter fields transform under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$
  - ☞ construct effective Lagrangians according to increasing number of momenta
- Transformation properties of matter fields:
  - ☞ well-defined transformation rule under the unbroken $\mathrm{SU}(N_f)_V$
  - ☞ not necessarily form representations of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$:
    transformation of matter fields under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$
    not uniquely defined,
    related by field redefinition (again: representation invariance)

Two examples:

❶ Baryon CHPT

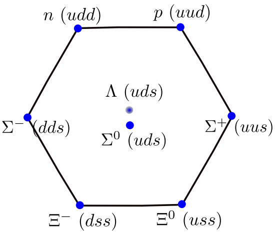
❷ Heavy meson CHPT (see backup slides)

Baryon CHPT at LO

Baryon CHPT: transformation of baryon fields (1)

Consider the SU(3) case, the baryon ground state octet:

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}$$



transform under the global unbroken $H = \mathrm{SU}(3)_V$ as

$$B \xrightarrow{V\in H} V\,B\,V^\dagger$$

- Representation invariance:
  free to choose how B transforms under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as long as it reduces to the above under $\mathrm{SU}(3)_V$
- Example:
  describe the baryons by B_1 or B_2, under $g = (L, R) \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$

$$B_1 \xrightarrow{g} L\,B_1\,L^\dagger, \quad B_2 \xrightarrow{g} L\,B_2\,R^\dagger$$

  both transform as an octet under $(V, V) \in \mathrm{SU}(3)_V$

Baryon CHPT: transformation of baryon fields (2)

- Example:

  describe the baryons by B_1 or B_2, under $g = (L, R) \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$

  $$B_1 \xrightarrow{g} L\, B_1\, L^\dagger, \quad B_2 \xrightarrow{g} L\, B_2\, R^\dagger$$

  both transform as an octet under $(V, V) \in \mathrm{SU}(3)_V$

- Related to each other through field redefinition:

  $$B_2 = B_1 U = B_1 + \frac{i}{F} B_1 \phi + \ldots, \qquad U = \exp\left(\frac{i}{F}\phi\right) \xrightarrow{g} L\, U\, R^\dagger$$

- But $B_{1,2}$ are inconvenient: not parity (P) “invariant"

  $L(L^\dagger)$ needs to be replaced by $R(R^\dagger)$ under parity $\Rightarrow$

  $$B_1(t, \vec{x}) \xrightarrow{P} U^\dagger \gamma^0 B_1(t, -\vec{x}) U \xrightarrow{g} R\, U^\dagger \gamma^0 B_1(t, -\vec{x}) U\, R^\dagger$$

Baryon CHPT: transformation of baryon fields (3)

- An elegant/convenient way:

  introduce $$u = \exp\left(\frac{i\phi}{2F}\right) \quad \text{or} \quad u^2 = U$$

  it transforms under $g \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as (recall $U \xrightarrow{g} L\,U\,R^\dagger$)

  $$u \xrightarrow{g} L\,u\,h^\dagger(L,R,\phi) = h(L,R,\phi)\,u\,R^\dagger$$

  $h(L,R,\phi)$: space-time dependent nonlinear function, called compensator field

  For for SU$(3)_V$ transformations ($L = R$), reduces to $h(L,R,\phi) = L = R$

- We can construct $B = u^\dagger B_1 u$, it transforms as

  $$B \xrightarrow{g} h\,B\,h^\dagger$$

  h is invariant under parity, i.e. $h(t,\vec{x}) \xrightarrow{P} h(t,-\vec{x}) \qquad \Rightarrow B \xrightarrow{P} \gamma^0 B$

- For the SU(2) case, proton and neutron form an isospin doublet $N = (p,n)^T$, construct N such that

  $$N \xrightarrow{g} h\,N$$

Building blocks of the meson–baryon Lagrangian

- Useful to introduce combinations of u whose transformations only involve h:

$$\Gamma^\mu = \frac{1}{2}\left[u^\dagger(\partial^\mu - i\,l^\mu)u + u(\partial^\mu - i\,r^\mu)u^\dagger\right]$$
$$u^\mu = i\left[u^\dagger(\partial^\mu - i\,l^\mu)u - u(\partial^\mu - i\,r^\mu)u^\dagger\right]$$

Γ^μ: chiral connection, vector; u^μ: chiral vielbein, axial vector

$$\Gamma^\mu \xrightarrow{g} h\,\Gamma^\mu h^\dagger + h\,\partial^\mu h^\dagger, \qquad u^\mu \xrightarrow{g} h\,u^\mu h^\dagger$$

- Introduce a covariant derivative:

$$\mathcal{D}^\mu B = \partial^\mu B + [\Gamma^\mu, B]\,, \qquad \mathcal{D}^\mu N = (\partial^\mu + \Gamma^\mu)N$$

which transform as the baryon fields under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$

$$\mathcal{D}^\mu B \xrightarrow{g} h\,\mathcal{D}^\mu B\,h^\dagger, \qquad \mathcal{D}^\mu N \xrightarrow{g} h\,\mathcal{D}^\mu N$$

- Include the quark mass term $\chi = 2B(s + i\,p) = 2B\mathcal{M} + \ldots$ by introducing

$$\chi_+ = u^\dagger\chi u^\dagger + u\chi^\dagger u, \qquad \chi_+ \to h\,\chi_+ h^\dagger$$

All fields transform in terms of h, convenient to construct the effective Lagrangians

The leading-order meson–baryon chiral Lagrangian

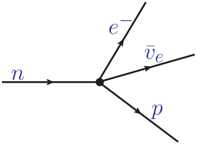
$$\mathcal{L}_{\pi N}^{(1)} = \bar{N}\left(i\gamma_\mu \mathcal{D}^\mu - m + \frac{g_A}{2}\gamma_\mu\gamma_5 u^\mu\right) N$$

$$\mathcal{L}_{\phi B}^{(1)} = \left\langle \bar{B}\left(i\gamma_\mu \mathcal{D}^\mu - m\right) B\right\rangle + \frac{D}{2}\left\langle \bar{B}\gamma_\mu\gamma_5\{u^\mu, B\}\right\rangle + \frac{F}{2}\left\langle \bar{B}\gamma_\mu\gamma_5[u^\mu, B]\right\rangle$$

Remark:

- in meson CHPT, Lagrangian has even powers of momenta (Lorentz invariance)
- here, due to Dirac structures, odd powers possible

New parameters:



- m: nucleon (baryon) mass in the chiral limit
- g_A: expand $u_\mu = l_\mu - r_\mu + \mathcal{O}(\phi) \Rightarrow$ axial vector coupling known from neutron beta decay, $g_A = 1.27$
- D/F: two axial vector couplings in SU(3), can be determined from semileptonic hyperon decays ($D \approx 0.804, F \approx 0.463$), SU(2) constraint

$$D + F = g_A$$

Intermediate summary

- Lagrangians need to respect the global symmetries of QCD, explicit symmetry breaking can be included using the spurion technique
- The LO mesonic chiral Lagrangian invariant under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$

  $$\mathcal{L}^{(2)} = \frac{F^2}{4}\left[\langle\partial_\mu U \partial^\mu U^\dagger\rangle + 2B\langle\mathcal{M}U^\dagger + \mathcal{M}^\dagger U\rangle\right] \quad \text{with} \;\; U = e^{i\phi/F}$$

  GMOR relation: $M_\pi^2 \propto m_q$
- Physical observables do not change under nonlinear field redefinition satisfying:

  $$\phi = \chi\, F[\chi] \quad \text{with} \;\; F[0] = 1$$

- Under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, matter fields do not have a unique transformation law, convenient to introduce the compensator field h, and use a field transforming in the same way under $\mathrm{SU}(N_f)_V$, e.g.,

  $$\text{if} \;\; X \overset{V\in H}{\longrightarrow} V\,X\,V^\dagger, \quad \text{then choose} \;\; X \overset{g\in G}{\longrightarrow} h\,X h^\dagger$$

CHPT at NLO

Unitarity of S-matrix

- Probability conservation ⇒ unitarity of the S-matrix

$$S\,S^\dagger = S^\dagger S = \mathbb{1}$$

T-matrix: $S = \mathbb{1} + i\,T \Rightarrow \qquad T - T^\dagger = i\,T\,T^\dagger$

thus, unitarity dictates a relation for the partial-wave scattering amplitude t_ℓ:

$$\text{Im}\,t_\ell(s) = \sigma(s)|t_\ell(s)|^2$$

here $\sigma(s)$: two-body phase space factor

- From the LO CHPT, the $\pi\pi$ scattering amplitude $A(s,t,u) = \dfrac{s - M_\pi^2}{F^2}$,

  no imaginary part! ⇒ unitarity is broken

Going to higher orders

- Perturbative unitarity: imaginary part given by loops

$$\text{Im}\, t_\ell^{(2)}(s) = 0, \quad \text{Im}\, t_\ell^{(4)}(s) = \sigma(s)|t_\ell^{(2)}(s)|^2, \; \ldots$$

- Symmetries do not forbid higher order terms in effective Lagrangians: more derivatives, more insertion of quark masses
- More derivatives $\Rightarrow$ non-renormalizable

> . . . , if we include in the Lagrangian *all* of the infinite number of interactions allowed by symmetries, then there will be a counterterm available to cancel every ultraviolet divergence. In this sense, . . . , non-renormalizable theories are just as renormalizable as renormalizable theories, as long as we include all possible terms in the Lagrangian.
>
> S.Weinberg, *The Quantum Theory of Fields*, Vol.1

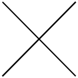
- Q: How should we deal with the infinite number of terms?

  A: Power counting is needed: organize the infinite number of terms according to power of the expansion parameter, finite number of terms up to a given order

Necessity of higher order Lagrangian from renormalization

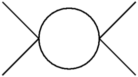
Consider the example of $\pi\pi$ scattering

- At LO, $\mathcal{O}\left(p^2\right)$: two derivatives or one quark mass insertion



$$= \mathcal{O}\left(p^2\right)$$

- One-loop with two $\mathcal{O}\left(p^2\right)$ vertices:



$$= \int d^4q \frac{q_1 q_2 q_3 q_4}{(q^2 - M_\pi^2)[(p-q)^2 - M_\pi^2]}$$

$$= \mathcal{O}\left(p^{4+4-4}\right) = \mathcal{O}\left(p^4\right)$$

- The loop is divergent, divergence absorbed by the counterterms in the $\mathcal{O}\left(p^4\right)$ Lagrangian

  How can we construct the higher order Lagrangian?

Building blocks

- Powerful to include external fields

  ⇒ incorporate electroweak interactions, quark masses:

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= \mathcal{L}^0_{\text{QCD}} + \bar{q}\gamma^\mu \left(v_\mu + \gamma_5 a_\mu\right) q - \bar{q}\left(s + i\gamma_5 p\right) q \\ &= \mathcal{L}^0_{\text{QCD}} + \bar{q}_L\gamma^\mu l_\mu q_L + \bar{q}_R\gamma^\mu r_\mu q_R - \bar{q}_L(s+ip)q_R - \bar{q}_R(s-ip)q_L\end{aligned}$$

$l_\mu = v_\mu - a_\mu, \quad r_\mu = v_\mu + a_\mu$ left-/right-handed sources

$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu], \quad F_L^{\mu\nu} = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu]$

$\chi = 2B(s+ip) = 2B\mathcal{M} + \ldots$ scalar/pseudoscalar sources

- Transformation laws for the building blocks:

$$U \to L\,U\,R^\dagger, \qquad \chi \to L\,\chi\,R^\dagger,$$
$$F_L^{\mu\nu} \to L\,F_L^{\mu\nu}L^\dagger, \qquad F_R^{\mu\nu} \to R\,F_R^{\mu\nu}R^\dagger$$

- Another set of building blocks: u_μ, $\chi_\pm$ and $f_\pm^{\mu\nu}$. $\quad f_\pm^{\mu\nu} = uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u$

  They all transform as $O \to h\,O\,h^\dagger$

Construction of higher order chiral Lagrangian

- We need a counting scheme (more see later):

U	$\sim$	$\mathcal{O}\left(p^0\right)$	
small momentum / derivative	$\sim$	$\mathcal{O}\left(p\right)$	
light quark masses χ	$\sim$	$\mathcal{O}\left(p^2\right)$	$\Leftarrow M_{\text{GB}}^2 \sim m_q$
external fields $l_\mu,\, r_\mu$	$\sim$	$\mathcal{O}\left(p\right)$	$\Leftarrow D_\mu U = \partial_\mu U - i l_\mu U + i U r_\mu$

- At a given order, write down the most general Lagrangian allowed by symmetries (for $\theta = 0$):

  chiral symmetry, P, C and T

  most general $\Rightarrow$ be able to absorb all divergences at the same order

SU(3) chiral Lagrangian at NLO

- SU(3) chiral Lagrangian at $\mathcal{O}\left(p^4\right)$ Gasser, Leutwyler (1985)

$$\begin{aligned}\mathcal{L}^{(4)} = & L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\ & + L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\ & + L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + \chi U^\dagger)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\ & + L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\ & - i\, L_9\langle F_L^{\mu\nu} D_\mu U D_\nu U^\dagger + F_R^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10}\langle U^\dagger F_L^{\mu\nu} U F_{R\mu\nu}\rangle \\ & + 2 \text{ contact terms}\end{aligned}$$

- $L_{1,\ldots,10}$: low-energy constants (LECs)
- NLO SU(2) chiral Lagrangian contains 7 terms: $\ell_{1,\ldots,7}$ Gasser, Leutwyler (1984)

Low-energy constants

- One loop diagrams with vertices from $\mathcal{L}^{(2)}$ are of $\mathcal{O}\left(p^4\right)$, divergences should be absorbed by counterterms in $\mathcal{L}^{(4)}$, can be derived using background field method with heat kernel technique, for detailed derivations, see the attached file *chpt_heat_kernel_renormalization.pdf*
- Low-energy constants generally contain two parts:

$$L_i = L_i^r(\mu) + \Gamma_i \times \text{divergence}$$

  renormalized LECs $L_i^r(\mu)$ are finite, scale-dependent
- Scale dependence of LECs cancel the one from loop integrals

  $\Rightarrow$ physical observables are scale-independent !
- L_i^r's are independent of light quark masses by construction, parameterize the short-distance physics

  Values not fixed by chiral symmetry:

  ☞ extracted using experimental data

  ☞ estimated with models such as resonance saturation Ecker et al (1989)

  ☞ using lattice simulations Flavour Lattice Averaging Group (FLAG) (2019)

Chiral Lagrangian at higher orders

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots$$

- Number of terms allowed by the symmetries increases very fast
- How many LECs are contained in these for SU(N), $N = (2, 3)$?

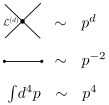
$\mathcal{L}^{(2)}$	contains	(2, 2)	constants	(F, B)
$\mathcal{L}^{(4)}$	contains	(7, 10)	constants	Gasser, Leutwyler (1984, 1985)
$\mathcal{L}^{(6)}$	contains	(53, 90)	constants	Bijnens, Colangelo, Ecker (1999)

- Why different for SU(2) / SU(3)?
  same most general SU(N) Lagrangian, but
  matrix-trace relations [Cayley–Hamilton] render some of the structures redundant.
  Example: Cayley–Hamilton relation for 2×2 matrices A, B,

$$\{A, B\} = A\,\langle B\rangle + B\,\langle A\rangle + \langle A\,B\rangle - \langle A\rangle\,\langle B\rangle$$

Weinberg's power counting

Weinberg (1979)



$\sim p^d$

$\sim p^{-2}$

$\int d^4p \sim p^4$

- Consider an arbitrary Feynman diagram with L loops, I internal lines, V_d vertices of order d:

$$\mathcal{A} \propto \int (d^4p)^L \frac{1}{(p^2)^I} \prod_d (p^d)^{V_d}$$

- The chiral dimension of $\mathcal{A}$

$$D = 4L - 2I + \sum_d dV_d$$

- Use topological identity for L to eliminate I, $L = I - \sum_d V_d + 1$

$$\boxed{D = \sum_d V_d(d-2) + 2L + 2}$$

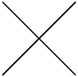
  - ☞ Lowest order is $\mathcal{O}\left(p^2\right)$, i.e. $d \geq 2$ $\Rightarrow$ rhs is a sum of non-negative numbers
  - ☞ For a given order D, there is only a finite number of combinations of L and V_d
  - ☞ Each loop is suppressed by two orders in the momentum expansion

Power counting – example: $\pi\pi$ scattering (1)

$$\boxed{D = \sum_d V_d(d-2) + 2L + 2}$$

- $D = 2$

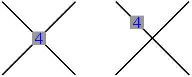
  ☞ $L = 0$, lowest order tree-level diagram only



- $D = 4$

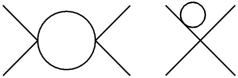
  ☞ $L = 0$, tree-level diagram with one insertion from $\mathcal{L}^{(4)}$

  $V_4 = 1, V_{d>4} = 0$



  ☞ $L = 1$, one-loop diagram with only $d = 2$ vertices

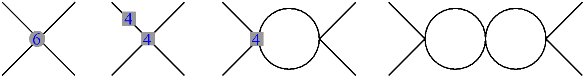
  $V_{d>2} = 0$



Power counting – example: $\pi\pi$ scattering (2)

$$\boxed{D = \sum_d V_d(d-2) + 2L + 2}$$

- $D = 6$
  - ☞ $L = 0$, tree-level diagram with one insertion from $\mathcal{L}^{(6)}$
  - ☞ $L = 0$, tree-level diagram with two insertions from $\mathcal{L}^{(4)}$
  - ☞ $L = 1$, one-loop diagram with one insertion from $\mathcal{L}^{(4)}$
  - ☞ $L = 2$, two-loop diagram with $V_d = 2$ vertices



Chiral symmetry breaking scale Λ_χ

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\left(\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle + \frac{1}{\Lambda_\chi^2}\tilde{\mathcal{L}}^{(4)} + \frac{1}{\Lambda_\chi^4}\tilde{\mathcal{L}}^{(6)} + \ldots\right)$$

- What is the scale Λ_χ?
  - ☞ hard energy scale where the momentum expansion definitely fails
  - ☞ uncertainty estimate: higher-order corrections are suppressed by $\sim p^2/\Lambda_\chi^2$

    $\Rightarrow$ how big?

- Estimate with resonance masses:

  The only dynamical degrees of freedom are the GBs, no resonances

  $\Rightarrow$ CHPT must fail once the energy reaches the resonance region: a perturbative momentum expansion cannot generate a pole

$$\Lambda_\chi \approx M_{\text{res}}$$

  Lowest narrow resonance $M_\rho \approx 770$ MeV,

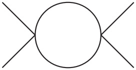
  typically $M_{\text{res}} \sim 1$ GeV

Chiral symmetry breaking scale Λ_χ (2)

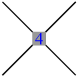
$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\left(\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle + \frac{1}{\Lambda_\chi^2}\tilde{\mathcal{L}}^{(4)} + \frac{1}{\Lambda_\chi^4}\tilde{\mathcal{L}}^{(6)} + \ldots\right)$$

- Naturalness argument: Manohar, Georgi (1984)

  Compare



$$\propto \left(\frac{p^2}{F^2}\right)^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2)^2} \overset{\text{dim.reg}}{\propto} \frac{1}{(4\pi)^2}\frac{p^4}{F^4}\log\mu$$



$$\propto \frac{F^2}{\Lambda_\chi^2}\frac{p^4}{F^4}\tilde{\ell}_i(\mu)$$

- Scale-dependent LEC $\tilde{\ell}_i(\mu)$ compensates for $\log\mu$ dependence of the loop graph
- If no accidental fine-tuning (naturalness), $\tilde{\ell}_i(\mu)$ should be at least of similar size as the shift induced by change in scale μ $\Rightarrow$

$$\Lambda_\chi \approx 4\pi F \approx 1.2 \text{ GeV}$$

- Using the naturalness assumption, the size of L_i^r in $\mathcal{L}^{(4)}$: $\sim \mathcal{O}\left(\frac{1}{(4\pi)^2}\right)$

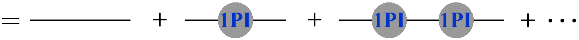
Meson masses at NLO (1)

Take the pion mass as an example of NLO calculations.

- Mass: pole in the two point correlation function

  At LO, $M_\pi^2 = M^2 \equiv B(m_u + m_d)$
- Higher orders: self-energy

$$i\,\delta^{ab}\Delta_\pi(p) = \int d^4x e^{-ipx} \left\langle 0 \left| T\left[\pi^a(x)\pi^b(0)\right] \right| 0 \right\rangle$$



$$= \frac{i}{p^2 - M^2 + i\epsilon} + \frac{i}{p^2 - M^2 + i\epsilon}\left[-i\Sigma(p^2)\right]\frac{i}{p^2 - M^2 + i\epsilon} + \ldots$$

$$= \frac{i}{p^2 - M^2 - \Sigma(p^2) + i\epsilon} = \frac{i\,Z_\pi}{p^2 - M_\pi^2 + i\epsilon} + \text{non-singular terms}$$

☞ M_π: physical pion mass, solution of the equation

$$M_\pi^2 - M^2 - \Sigma(M_\pi^2) = 0$$

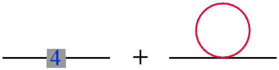
☞ Exercise: show that the wave function renormalization constant is

$$Z_\pi = \frac{1}{1 - \Sigma'(M_\pi^2)}, \qquad \text{with } \Sigma'(M_\pi^2) \equiv \left.\frac{d\Sigma(p^2)}{dp^2}\right|_{p^2 = M_\pi^2}$$

Meson masses at NLO (2)

- Self-energy contains two parts:

  π, K, η tadpole loops and counterterms in $\mathcal{L}^{(4)}$



  Assuming isospin symmetry ($m_u = m_d = \hat{m}$), the pion mass at NLO:

$$M_\pi^2 = M^2 \left[1 + \frac{I(M^2)}{2F^2} - \frac{I(M_{\eta,2}^2)}{2F^2} + \frac{8M^2}{F^2}(2L_8 - L_5) + \frac{24M_{\eta,2}^2}{F^2}(2L_6 - L_4) \right]$$

  here $M^2 = 2B\hat{m}, \;\; M_{\eta,2}^2 = \frac{2}{3}B(2\hat{m} + m_s)$

- Loop is divergent, in dimensional regularization (good for preserving symmetry)

$$I(M^2) = i\mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - M^2 + i\epsilon} = \frac{M^2}{16F^2} \left(\lambda + \log \frac{M^2}{\mu^2} \right)$$

$$\lambda = \frac{2}{d-4} - [\log(4\pi) + \Gamma'(1) + 1], \qquad \text{divergent for } d = 4\,!$$

  On the other hand, L_i's are divergent either $\quad L_i = L_i^r + \frac{\Gamma_i}{32\pi^2}\lambda$

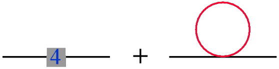
$$\Gamma_4 = \frac{1}{8}, \qquad \Gamma_5 = \frac{3}{8}, \qquad \Gamma_6 = \frac{11}{144}, \qquad \Gamma_8 = \frac{5}{48}$$

  Useful formulae in dim.reg.: the Peskin & Schroeder QFT book, App. A.4, p.807

Meson masses at NLO (3)

- Self-energy contains two parts:

  π, K, η tadpole loops and counterterms in $\mathcal{L}^{(4)}$



  Assuming isospin symmetry ($m_u = m_d = \hat{m}$), the pion mass at NLO:

$$M_\pi^2 = M^2 \left[1 + \frac{I(M^2)}{2F^2} - \frac{I(M_{\eta,2}^2)}{2F^2} + \frac{8M^2}{F^2}(2L_8 - L_5) + \frac{24M_{\eta,2}^2}{F^2}(2L_6 - L_4) \right]$$

  here $M^2 = 2B\hat{m}, \;\; M_{\eta,2}^2 = \frac{2}{3}B(2\hat{m} + m_s)$

- Renormalization: divergences from LECs (i.e., counterterms) and loops cancel out (as well as the μ-dependence)

  $\Rightarrow$ the pion mass is finite and μ-independent

$$\begin{aligned} M_\pi^2 \; = \; M^2 \Bigg[1 &+ \frac{1}{32\pi^2 F^2} M^2 \log\frac{M^2}{\mu^2} - \frac{1}{96\pi^2 F^2} M_{\eta,2}^2 \log\frac{M_{\eta,2}^2}{\mu^2} \\ &+ \frac{8M^2}{F^2}(2L_8^r - L_5^r) + \frac{24M_{\eta,2}^2}{F^2}(2L_6^r - L_4^r) \Bigg] \end{aligned}$$

- M_π^2 vanishes in chiral limit, loop correction does not generate a non-zero mass
- Chiral logarithm: non-analytic in the light quark masses

Quark mass ratios revisited (1)

- Calculate pion/kaon masses beyond leading order:

$$M_{\pi^+}^2 = B(m_u + m_d)\Big[1 + \mathcal{O}\left(\hat{m}, m_s\right)\Big]$$

$$M_{K^+}^2 = B(m_u + m_s)\Big[1 + \mathcal{O}\left(\hat{m}, m_s\right)\Big]$$

- Form dimensionless ratios:

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{m_u + m_d}\Big[1 + \Delta_M + \mathcal{O}\left(m_q^2\right)\Big]$$

$$\frac{(M_{K^0}^2 - M_{K^+}^2)_{\text{strong}}}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\Big[1 + \Delta_M + \mathcal{O}\left(m_q^2\right)\Big]$$

$$\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8^r - L_5^r) + \text{ chiral logs}$$

- Double ratio Q^2 particularly stable:

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{(M_{K^0}^2 - M_{K^+}^2)_{\text{strong}}}\Big[1 + \mathcal{O}\left(m_q^2\right)\Big]$$

Quark mass ratios revisited (2)

- Leutwyler's ellipse: Leutwyler (1996)

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$

- Recall Dashen's theorem:

$$(M_{K^+}^2 - M_{K^0}^2)_{\text{em}} = (M_{\pi^+}^2 - M_{\pi^0}^2)_{\text{em}} + \mathcal{O}\left(e^2 m_q\right)$$

$$\boxed{Q_{\text{Dashen}} = 24.2}$$

- Value extracted from lattice simulations FLAG, EPJC77, 112 (2017); arXiv:2111.09849 [hep-lat]

$N_f = 2$: $24.3 \pm 1.4 \pm 0.6$

$N_f = 2+1$: 23.3 ± 0.5

$N_f = 2+1+1$: 22.5 ± 0.5

$\pi\pi$ scattering beyond LO

- S-wave scattering lengths:

	a_0^0	a_0^2	
LO	0.16	-0.045	Weinberg (1966)
NLO (one-loop)	0.20 ± 0.01	-0.042 ± 0.002	Gasser, Leutwyler (1983)
NNLO (two-loop)	$0.217 \pm \ldots$	$-0.041 \pm \ldots$	Bijnens *et al.* (1996)
NNLO + Roy eq.	0.220 ± 0.005	-0.0444 ± 0.0010	Colangelo *et al.* (2001)

- Compare again with the modern data from NA48/2 (K_{e4} decays & $K^\pm \to \pi^0\pi^0\pi^\pm$) B.Bloch-Devaux, Kaon09

$$\begin{aligned} a_0^0 &= 0.2210 \pm 0.0047_{\text{stat}} \pm 0.0015_{\text{syst}} \\ a_0^2 &= -0.0429 \pm 0.0044_{\text{stat}} \pm 0.0016_{\text{syst}} \end{aligned}$$

Exercises

2-5) Derive the $\pi^0\eta$ mixing angle:

$$\epsilon_{\pi^0\eta} = \frac{1}{2}\arctan\left(\frac{\sqrt{3}(m_d - m_u)}{2m_s - m_u - m_d}\right) \simeq \frac{\sqrt{3}}{2}\,\frac{m_d - m_u}{2m_s - m_u - m_d}.$$

2-6) One possible solution of the strong CP problem is the Peccei-Quinn (PQ) mechanism which introduces a global U(1) symmetry, called the PQ symmetry. Axion is the pseudo-Goldstone boson of the spontaneous breaking of this symmetry. Its properties can be studied in CHPT by changing the quark mass matrix $\mathcal{M}$ to $\mathcal{M}e^{i\,Xa/f_a}$ with a the axion field, f_a the axion decay constant, and X satisfying $\langle X\rangle = 1$. Consider the LO mass term of the SU(2) version of CHPT with axion,

$$\mathcal{L}_a^{(2)} = \frac{F^2}{2}B\langle \mathcal{M}e^{i\,Xa/f_a}U^\dagger + \text{h.c.}\rangle,$$

where h.c. represents the Hermitian conjugated term.

1) show that there will be no a-π^0 mixing if we choose $X = \mathcal{M}^{-1}/\langle\mathcal{M}^{-1}\rangle$;

2) show that the axion mass squared is given by $m_a^2 = \frac{F^2M_\pi^2}{f_a^2}\,\frac{m_u m_d}{(m_u+m_d)^2}$.

Exercises

2-7) Show that the leading order chiral amplitude for $\pi^+(p_1)\pi^-(p_2) \to \pi^0(p_3)\pi^0(p_4)$ in the chiral limit is given by

$$\mathcal{A}(s,t,u) = \frac{s}{F^2}$$

and show that the exponential and square-root representations are related by the following field redefinition:

$$\vec{\pi}' = \vec{\pi}\frac{F}{|\vec{\pi}|}\sin\left(\frac{|\vec{\pi}|}{F}\right) = \vec{\pi} + \text{nonlinear terms}, \quad \text{here } |\vec{\pi}| \equiv \sqrt{\vec{\pi}^2}.$$

2-8) The pion mass is given by the pole of the pion propagator,

$$i\,\delta^{ab}\Delta_\pi(p) = \frac{i}{p^2 - M^2 - \Sigma(p^2) + i\epsilon} = \frac{i\,Z_\pi}{p^2 - M_\pi^2 + i\epsilon} + \text{non-singular terms},$$

where $\Sigma(p^2)$ is the pion self-energy. Show that the wave function renormalization constant is

$$Z_\pi = \frac{1}{1 - \Sigma'(M_\pi^2)}, \qquad \text{with } \Sigma'(M_\pi^2) \equiv \left.\frac{d\Sigma(p^2)}{dp^2}\right|_{p^2=M_\pi^2}$$

Further Reading: Heavy meson CHPT

Transformations of heavy meson fields (1)

For SU(3), the charmed meson ground state flavor anti-triplet (a: light flavor index):

$$P_a = \left(D^0, D^+, D_s^+\right)_a, \quad P_a^* = \left(D^{*0}, D^{*+}, D_s^{*+}\right)_a \qquad \left[\left(c\bar{u}, c\bar{d}, c\bar{s}\right)\right]$$

Let H denote heavy mesons. It transforms under the global unbroken $\mathrm{SU}(3)_V$ as

$$H \xrightarrow{V\in\mathrm{SU}(3)_V} H\,V^\dagger$$

- Representation independence:

  free to choose how H transforms under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as long as it reduces to the above under $\mathrm{SU}(3)_V$

- Example:

  describe the heavy mesons by H_1 or H_2, under

  $g = (L, R) \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$

  $$H_1 \xrightarrow{g} H_1\,L^\dagger, \quad H_2 \xrightarrow{g} H_2\,R^\dagger$$

  both transform as an anti-triplet under $(V, V) \in \mathrm{SU}(3)_V$

Transformations of heavy meson fields (2)

- Example:

  describe the heavy mesons by H_1 or H_2, under

  $g = (L, R) \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$

  $$H_1 \xrightarrow{g} H_1\, L^\dagger, \quad H_2 \xrightarrow{g} H_2\, R^\dagger$$

  both transform as an anti-triplet under $(V, V) \in \mathrm{SU}(3)_V$

- Related to each other through field redefinition:

  $$H_2 = H_1 U = H_1 + \frac{i}{F} H_1 \phi + \ldots, \qquad U = \exp\left(\frac{i}{F}\phi\right) \xrightarrow{g} L\, U\, R^\dagger$$

- But $H_{1,2}$ are inconvenient: complicated parity transformation (P)

  $L(L^\dagger)$ needs to be replaced by $R(R^\dagger)$ under parity $\quad \Rightarrow$

  $$H_{1,a}(t, \vec{x}) \xrightarrow{P} \gamma^0 H_{1,b}(t, -\vec{x}) \gamma^0 U_{ba} \xrightarrow{g} \gamma^0 H_{1,b}(t, -\vec{x}) \gamma^0 U_{ba} R^\dagger$$

  [recall for a spinor: $\psi(t, \vec{x}) \xrightarrow{P} \gamma^0 \psi(t, -\vec{x})$]

Transformations of heavy meson fields (3)

- An elegant/convenient way:

  introduce

  $$u = \exp\left(\frac{i\phi}{2F}\right) \quad \text{or} \quad u^2 = U$$

  it transforms under $g \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as (recall $U \xrightarrow{g} L\,U\,R^\dagger$)

  $$u \xrightarrow{g} L\,u\,h^\dagger(L,R,\phi) = h(L,R,\phi)\,u\,R^\dagger$$

  $h(L,R,\phi)$: space-time dependent nonlinear function, called compensator field.

  From the above definition, we can express h in terms of L, R, U:

  $$h = \sqrt{LUR^\dagger}R\sqrt{U^\dagger} = \sqrt{RU^\dagger L^\dagger}L\sqrt{U}$$

- For for SU$(3)_V$ transformations ($L = R = V$), reduces to $h(L,R,\phi) = V$

- We can construct $H = H_1 u$ or $H = H_2 u^\dagger$, it transforms as

  $$H \xrightarrow{g} H\,h^\dagger$$

  Under parity transformation $h(t,\vec{x}) \xrightarrow{P} h(t,-\vec{x})$, and $H(t,\vec{x}) \xrightarrow{P} \gamma^0 H(t,-\vec{x})\gamma^0$.

Building blocks of the chiral Lagrangian

- Useful to introduce combinations of u whose transformations only involve h:

$$\Gamma^\mu = \frac{1}{2}\left[u^\dagger(\partial^\mu - i\,l^\mu)u + u(\partial^\mu - i\,r^\mu)u^\dagger\right]$$
$$u^\mu = i\left[u^\dagger(\partial^\mu - i\,l^\mu)u - u(\partial^\mu - i\,r^\mu)u^\dagger\right]$$

Γ^μ: chiral connection, vector; u^μ: chiral vielbein, axial vector

$$\Gamma^\mu \xrightarrow{g} h\,\Gamma^\mu h^\dagger + h\,\partial^\mu h^\dagger, \qquad u^\mu \xrightarrow{g} h\,u^\mu h^\dagger$$

- Introduce a covariant derivative:

$$\mathcal{D}^\mu H = \partial^\mu H - H\,\Gamma^\mu$$

which transform the same way as H under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$

$$\mathcal{D}^\mu H \xrightarrow{g} \mathcal{D}^\mu H\,h^\dagger$$

- Include the quark mass term $\chi = 2B(s + i\,p) = 2B\mathcal{M} + \ldots$ by introducing

$$\chi_+ = u^\dagger\chi u^\dagger + u\chi^\dagger u, \qquad \chi_+ \to h\,\chi_+ h^\dagger$$

All fields transform in terms of h, convenient to construct the effective Lagrangians

Simplified two-component notation

The superfield for pseudoscalar and vector heavy mesons: ($^{(4)}$ means 4-component)

$$H_a^{(4)} = \frac{1+\not{v}}{2}\left[P_a^{*\mu}\gamma_\mu - P_a\gamma_5\right]$$

In the rest frame of heavy meson, $v^\mu = (1,\mathbf{0})$. We take the Dirac basis

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Simplifications: $\frac{1+\not{v}}{2} = \frac{1+\gamma^0}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$

$$H_a^{(4)} = \begin{pmatrix} 0 & -(P_a + \boldsymbol{P}_a^* \cdot \boldsymbol{\sigma}) \\ 0 & 0 \end{pmatrix}, \qquad \bar{H}_a^{(4)} = \begin{pmatrix} 0 & 0 \\ (P_a^\dagger + \boldsymbol{P}_a^{*\dagger} \cdot \boldsymbol{\sigma}) & 0 \end{pmatrix}$$

Thus, it is convenient to simply use the two-component notation

$$H_a = P_a + \boldsymbol{P}_a^* \cdot \boldsymbol{\sigma}, \qquad H_a^{(4)} \to -H_a, \qquad \bar{H}_a^{(4)} \to H_a^\dagger$$

Heavy meson CHPT at LO

The LO Lagrangian [$\mathcal{O}\left(p\right)$]:

$$\mathcal{L}_{\text{HM}}^{(1)} = \underbrace{-i\,\mathrm{Tr}\left[\bar{H}_a v_\mu (\mathcal{D}^\mu H)_a\right]}_{\text{kinetic term}+D\pi\text{ scattering}+\dots} + \underbrace{\frac{g}{2}\mathrm{Tr}\left[\bar{H}_a H_b \gamma_\mu \gamma_5\right] u^\mu_{ba}}_{\text{terms for }D^*\to D\pi+\dots}$$

- invariant under Lorentz transformation, chiral symmetry, parity
- Tr: trace in the spinor space, a, b: indices in the light flavor space

- The chirally covariant kinetic term:

$$-i\,\mathrm{Tr}\left[\bar{H}_a^{(4)} v_\mu (\mathcal{D}^\mu H)_a^{(4)}\right] = i\,\mathrm{Tr}\left[H_a^\dagger\left(\partial^0 H_a - H_b \Gamma^0_{ba}\right)\right]$$

$$= 2i\left(P_a^\dagger \partial^0 P_a + P_a^{*i\,\dagger}\partial^0 P_a^{*i}\right) + \underbrace{\frac{-i}{4F^2}\left(P_a^\dagger P_b + \boldsymbol{P}_a^{*\dagger}\cdot\boldsymbol{P}_b^{*}\right)\left[\phi, \partial^0\phi\right]_{ba}}_{\text{scattering between }(D,D^*,\bar{B},\bar{B}^*)\text{ and GBs}(\pi,K,\eta)} + \dots$$

Universality of the LO, $\mathcal{O}\left(p\right)$, scattering amplitudes: completely determined by chiral symmetry (strength in term of F)! the Weinberg–Tomozawa term

Axial coupling

- The axial coupling term ($u^k = -\frac{1}{F}\partial^k\phi + \ldots$):

$$\frac{g}{2}\text{Tr}\left[\bar{H}_a^{(4)}H_b^{(4)}\gamma_\mu\gamma_5\right]u_{ba}^\mu = -\frac{g}{2}\text{Tr}\left[H_a^\dagger H_b\sigma^i\right]u_{ba}^i$$

$$= -\frac{g}{2}\text{Tr}\left[\left(P_a^{*i\,\dagger}\sigma^i + P_a^\dagger\right)\left(P_b^{*j}\sigma^j + P_b^\dagger\right)\sigma^k\right]u_{ba}^k$$

$$= \underbrace{\frac{g}{F}P_a^\dagger P_b^{*i}\partial^i\phi_{ba}}_{\text{term for } D^*\to D\pi} + \frac{g}{F}P_a^{*i\,\dagger}P_b\partial^i\phi_{ba} + \underbrace{i\frac{g}{F}\epsilon^{ijk}P_a^{*i\,\dagger}P_b^{*j}\partial^k\phi_{ba}}_{D^*D^*\pi\text{ coupling}} + \mathcal{O}\left(\frac{\phi^3}{F^3}\right)$$

- Decay amplitude:

$$\mathcal{A}(D^{*+}\to D^0\pi^+) = \frac{i\sqrt{2}g}{F}\boldsymbol{\varepsilon}_{(\lambda)}\cdot\boldsymbol{q}_\pi \quad \underbrace{\sqrt{M_{D^*}M_D}}_{\text{accounts for NR normalization}}$$

and the two-body decay width

$$\Gamma(D^{*+}\to D^0\pi^+) = \frac{1}{8\pi}\frac{|\boldsymbol{q}_\pi|}{M_{D^*}^2}\frac{1}{3}\sum_\lambda|\mathcal{A}|^2 = \frac{g^2M_D|\boldsymbol{q}_\pi|^3}{12\pi F^2M_{D^*}}$$

where we used $\sum_\lambda \varepsilon_{(\lambda)}^i\varepsilon_{(\lambda)}^j = \delta_{ij}$

Determination of g

- Measured D^* widths: PDG2018

$$\Gamma(D^{*0}) < 2.1 \text{ MeV}, \qquad \Gamma(D^{*\pm}) = (83.4 \pm 1.8) \text{ keV}$$

$$\mathcal{B}(D^{*+} \to D^0\pi^+) = (67.7 \pm 0.5)\%, \qquad \mathcal{B}(D^{*+} \to D^+\pi^0) = (30.7 \pm 0.5)\%$$

- The two-body decay width

$$\Gamma(D^{*+} \to D^0\pi^+) = \frac{g^2 M_D |\boldsymbol{q}_\pi|^3}{12\pi F^2 M_{D^*}} \quad \Rightarrow \quad |g| \simeq 0.57$$

For D^{*0}, with measured branching fraction $\mathcal{B}(D^{*0} \to D^0\pi^0) = (64.7 \pm 0.9)\%$, we can predict:

$$\Gamma(D^{*0} \to D^0\pi^0) = \frac{g^2 M_D |\boldsymbol{q}_\pi|^3}{24\pi F^2 M_{D^*}} \quad \Rightarrow \quad \Gamma(D^{*0}) = (55.3 \pm 1.4) \text{ keV}$$

- HQFS: g should be approximately the same in bottom sector with a relative uncertainty of $\sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{m_c} - \frac{\Lambda_{\text{QCD}}}{m_b}\right) \sim \mathcal{O}(20\%)$

Lattice QCD results:

$g_b = 0.492 \pm 0.029$ ALPHA Collaboration, Phys. Lett. B **740** (2015) 278

$g_b = 0.56 \pm 0.03 \pm 0.07$ RBC and UKQCD Collaborations, Phys. Rev. D **93** (2016) 014510

Mass splittings among heavy mesons (2)

- Light quark mass-dependent terms in two-component notation:

$$\mathcal{L}_\chi = -\lambda_1 \mathrm{Tr}\left[H_a^\dagger H_b\right]\chi_{+,ba} - \lambda_1' \mathrm{Tr}\left[H_a^\dagger H_a\right]\chi_{+,bb}$$

here, $\chi_+ = u^\dagger \chi u^\dagger + u\chi^\dagger u = 4B\mathcal{M} - \frac{B}{2F^2}\{\phi,\{\phi,\mathcal{M}\}\} + \ldots$

  - ☞ SU(3) mass differences:

$$M_{D_s^+} - M_{D^+} = 4\lambda_1 B(m_s - m_d) = 4\lambda_1\left(M_{K^\pm}^2 - M_{\pi^\pm}^2\right)$$

$$\Rightarrow \quad \lambda_1 \simeq 0.11 \text{ GeV}^{-1}$$

  - ☞ Isospin splitting induced by $m_d - m_u$:

$$\begin{aligned}\left(M_{D^0} - M_{D^+}\right)_{\text{quark mass}} &= 4\lambda_1 B(m_u - m_d)\\ &= 4\lambda_1\left(M_{K^\pm}^2 - M_{K^0}^2 - M_{\pi^\pm}^2 + M_{\pi^0}^2\right)\\ &= -2.3 \text{ MeV}\end{aligned}$$

Exp. value: $M_{D^0} - M_{D^+} = -(4.77 \pm 0.08)$ MeV

$$= \left(M_{D^0} - M_{D^+}\right)_{\text{quark mass}} + \left(M_{D^0} - M_{D^+}\right)_{\text{e.m.}}$$

- $\mathcal{L}_\chi$ also contributes to scattering between a heavy meson and the lightest pseudoscalar mesons (GBs)

Baryon CHPT at NLO

Baryon CHPT beyond LO

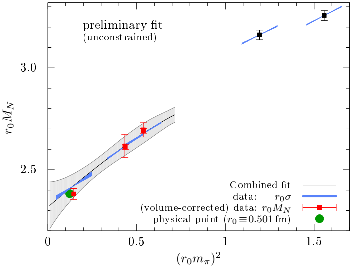
- CHPT is useful in the low-energy region
- Remember:

  the nucleon mass does not vanish in the chiral limit ($M_\pi = 0$)

$$m_N\Big|_{M_\pi=0} \sim m_N\Big|_{M_\pi=M_\pi^{\text{phys.}}}$$

$$\sim \Lambda_\chi = \mathcal{O}\left(1\text{ GeV}\right)$$

  $\Rightarrow$ a large mass scale $\gg$ low-momenta, M_π



G.Bali *et al.*, Lattice2013

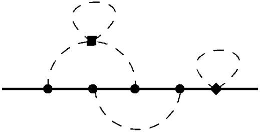
- As a result, the power counting of baryon CHPT is different:

☞ $$q^2 - m^2 = \left(q^0 + \sqrt{\vec{q}^{\,2} + m^2}\right)\left(q^0 - \sqrt{\vec{q}^{\,2} + m^2}\right) = \mathcal{O}\left(p\right)$$

☞ $\mathcal{L}_{\pi N} = \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(3)} + \ldots$ not only even powers

Weinberg's power counting for baryon CHPT (1)

- Consider an arbitrary L-loop 1-baryon diagram with $V_d^{\pi\pi}$ meson–meson vertices of order d, $V_{d'}^{\pi N}$ meson–baryon vertices of order d', and $I_\pi (I_N)$ internal meson (baryon) lines.



- Chiral dimension D is

$$D = 4L - 2I_\pi - I_N + \sum_d V_d^{\pi\pi} d + \sum_{d'} V_{d'}^{\pi N} d'$$

☞ Again, topological relation: $L = I_\pi + I_N - \sum_d V_d^{\pi\pi} - \sum_{d'} V_{d'}^{\pi N} + 1$

☞ Baryon number conservation $\Rightarrow$ $\sum_{d'} V_{d'}^{\pi N} = I_N + 1$

- We get

$$\boxed{D = 2L + 1 + \sum_d V_d^{\pi\pi}(d-2) + \sum_{d'} V_{d'}^{\pi N}(d'-1)}$$

Weinberg’s power counting for baryon CHPT (2)

$$\boxed{D = 2L + 1 + \sum_d V_d^{\pi\pi}(d-2) + \sum_{d'} V_{d'}^{\pi N}(d'-1)}$$

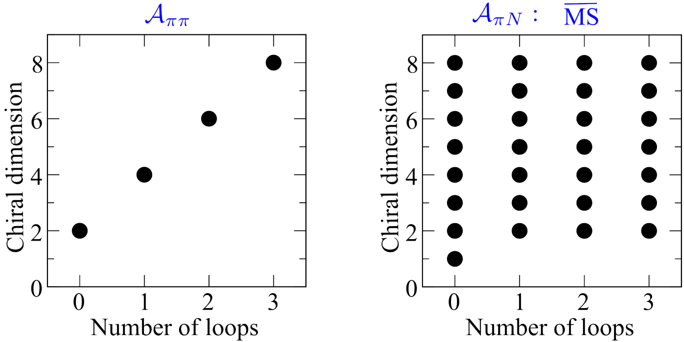
Note again: $d \geq 2$, $d' \geq 1 \quad \Rightarrow \quad D \geq 1$

- Therefore,
  $\mathcal{O}\left(p^1\right)$, $\mathcal{O}\left(p^2\right)$: tree-level only
  $\mathcal{O}\left(p^3\right)$, $\mathcal{O}\left(p^4\right)$: tree-level + one-loop
  $\mathcal{O}\left(p^5\right)$, $\mathcal{O}\left(p^6\right)$: tree-level + one-loop + two-loop
  ...
- But, the problem is it does not naively work :
  e.g., $\mathcal{O}\left(p^2\right)$ receives contribution from any loop diagram if we use dimensional regularization with the $\overline{\text{MS}}$ scheme (subtracting $\lambda = \frac{2}{d-4} - [\log(4\pi) + \Gamma'(1) + 1]$)!

Problem of power counting in baryon CHPT (1)

- In Goldstone boson sector, all masses are small quantities: $M \sim p \ll \Lambda_\chi$
- With baryons, loop integration picks up momenta of order $m_N \sim \Lambda_\chi$
- Schematically,

Gasser, Sainio, Švarc (1988)



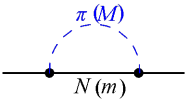
☞ "One no longer has a one-to-one mapping between the loop and small-momentum expansion"

Bernard, Kaiser, Kambor, Meißner (1992)

☞ Higher-order loops renormalize lower-order couplings

Problem of power counting in baryon CHPT (2)

- To see the problem, consider the nucleon self-energy



$$\Sigma_N \sim M^2 I_{\pi N}(p_N^2) + \dots$$

$$I_{\pi N}(p_N^2) = \mu^{4-d} i \int \frac{d^d l}{(2\pi)^d} \frac{1}{[(p_N - l)^2 - m^2 + i\epsilon]\,(l^2 - M^2 + i\epsilon)}$$

$$= \frac{1}{16\pi^2}\left[\lambda \underbrace{-1}_{=\mathcal{O}(1)} + \mathcal{O}(M)\right]$$

⇒ nucleon mass to $\mathcal{O}\left(p^3\right)$: $m_N = m \underbrace{-4c_1 M^2}_{\text{from } \mathcal{L}_{\pi N}^{(2)}} + \underbrace{\overbrace{\frac{3g_A^2 M^2 m}{32\pi^2 F^2}}^{\text{breaks power counting}} - \frac{3g_A^2 M^3}{32\pi F^2}}_{\text{from one-loop}}$

- Solutions:
  - ☞ Heavy baryon CHPT — Jenkins, Manohar (1991); Bernard, Kaiser, Kambor, Meißner (1992)
  - ☞ Infrared regularization — Becher, Leutwyler (1999)
  - ☞ Extended on-mass-shell scheme — Gegelia, Japaridze (1999); Fuchs *et al.* (2003)

Heavy baryon CHPT (1)

- $m_N \gg M_\pi \Rightarrow$ in low-momentum region, treat nucleons as heavy
- Analogous to heavy quark effective theory, decompose baryon momentum according to (nearly on-shell)

$$p_\mu = \underbrace{m_N v_\mu}_{\text{large}} + \underbrace{k_\mu}_{\text{residual}} , \quad v^2 = 1 , \quad v \cdot k \ll m_N$$

- Decompose the nucleon field into large (N_v) and small (n_v) components

$$N(x) = e^{-imv\cdot x}[N_v(x) + n_v(x)],$$

with $N_v = e^{imv\cdot x}\frac{1}{2}(1+\not{v})N(x), \quad n_v = e^{imv\cdot x}\frac{1}{2}(1-\not{v})N(x)$

- Using the EOM for $n_v(x) \propto \frac{1}{m}N_v(x)$, we can eliminate n_v.

$\mathcal{L}_{\pi N}^{(1)}$ becomes:

$$\mathcal{L}_{\pi N}^{(1)} = \bar{N}\left(i\gamma_\mu \mathcal{D}^\mu - m + \frac{g_A}{2}\gamma_\mu\gamma_5 u^\mu\right)N$$

$$= \bar{N}_v\big(iv\cdot\mathcal{D} + g_A S\cdot u\big)N_v + \mathcal{O}\left(m_N^{-1}\right)$$

$S_\mu = \frac{i}{2}\gamma_5\sigma_{\mu\nu}v^\nu$: Pauli-Lubanski spin vector

Heavy baryon CHPT (2)

- Nucleon mass gone from $\mathcal{L}_{\pi N}^{(1)}$, nucleon propagator becomes

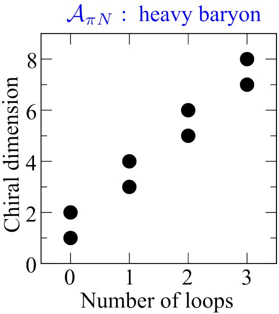
$$\frac{i}{v \cdot k + i\epsilon}$$

  the mass scale m was eliminated!

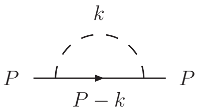
- $1/m_N$ corrections can be constructed systematically on Lagrangian level

  Thus, heavy baryon CHPT is a two-fold expansion in $\left(\frac{p}{\Lambda_\chi}\right)^n$, $\left(\frac{p}{m_N}\right)^n$ (but treated as one)

- Power counting works as in the meson sector: each loop only contributes at one momentum power



Infrared (IR) regularization



$$a = k^2 - M^2 + i\epsilon\,, \quad b = (P-k)^2 - m^2 + i\epsilon$$

$$H = \frac{1}{i}\int \frac{d^dk}{(2\pi)^d}\frac{1}{ab} = \int_0^1 dz\,\frac{1}{i}\int \frac{d^dk}{(2\pi)^d}\frac{1}{[(1-z)a+zb]^2}$$

- The integral can be separated into two parts:

$$H = I + R, \quad I = \int_0^\infty dz\ \ldots\,, \quad R = -\int_1^\infty dz\ \ldots$$

  - ☞ IR singular part I: generated by momenta of order of the pion mass
    obeys power counting
    contains the chiral physics like chiral logs etc
  - ☞ IR regular part R: generated by momenta of order of the nucleon mass
    violates power counting
    polynomial in pion mass and external momenta
    ⇒ can be absorbed into redefinition of LECs

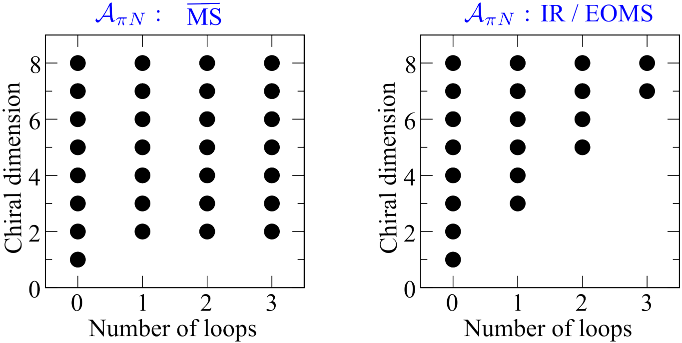
- Practical recipe: replace any one-loop integral by the IR singular part

Extended on-mass-shell regularization

- Idea: Perform additional subtractions beyond the $\overline{\text{MS}}$ scheme so that renormalized diagrams satisfy the power counting
- Method: Expand loop integrand in small quantities, and subtract those power counting violating terms
- Similar to the IR regularization:

  power counting violating terms are analytic in M_π and external momenta

  $\Rightarrow$ can be absorbed in a renormalization of the counterterms



Large N_c

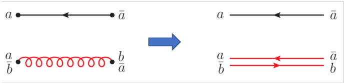
Large N_c (1)

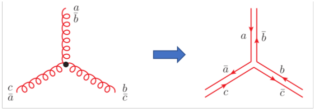
- SU(N_c): N_c quark colors, $N_c^2 - 1$ gluons; N_c regarded as a parameter of QCD proposed by 't Hooft NPB72(1974)461

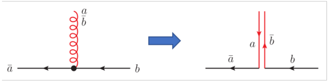
  extension to baryons by Witten NPB160(1979)57 [strongly recommended reading]
- Large N_c limit keeping Λ_{QCD} independent of N_c requires $g_s^2 N_c \equiv \lambda = \mathcal{O}\left(N_c^0\right)$:

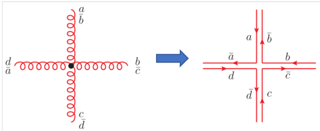
$$\alpha_s(\mu) = \frac{4\pi}{\left(\frac{11N_c}{3} - \frac{2N_f}{3}\right)\log\frac{\mu^2}{\Lambda_{\text{QCD}}^2}},$$

- At leading order, we can approximate $N_c^2 - 1$ by N_c^2 or SU(N_c) by U(N_c); double-line representation of gluons:







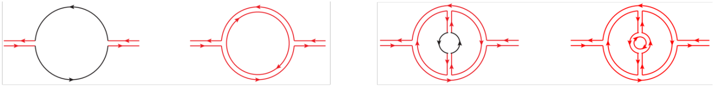


here most diagrams are taken from Lucha, Melikhov, Sazdjian, arXiv:2012.02542

Large N_c (2)

- Planar diagrams: can be mapped on a plane.

  Each order contains infinite diagrams, e.g., gluon self-energies



(a) quark loop $\mathcal{O}(N_c^{-1})$ (b) gluon loop $\mathcal{O}(N_c^{0})$ (c) $\mathcal{O}(N_c^{-1})$ (d) $\mathcal{O}(N_c^{0})$

  The next-to-leading diagrams (a) and (b) contain a quark loop, topologically a hole, suppressed by $\mathcal{O}\left(N_c^{-1}\right)$

- Nonplanar diagrams: suppressed. E.g., consider the 2-point correlation function: $\langle j_{\bar{k}l}(x)j^\dagger_{\bar{k}l}(y)\rangle$ with $j_{\bar{k}l} = \bar{q}_{a,k}q_l^a$, where a: color index, k, l: flavor indices

(a) $\mathcal{O}(N_c)$ (b) $\mathcal{O}(N_c)$ (c) $\mathcal{O}(N_c)$ (d) $\mathcal{O}(N_c^{-1})$

  characterized by a topological invariant: number of handles

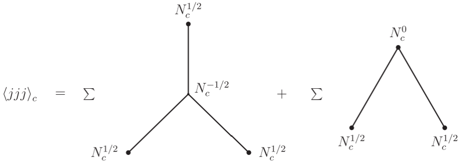
- General formula of the large N_c behavior for a diagram: N_c^{2-B-2H}, where B: number of holes; H: number of handles

Mesons in large N_c

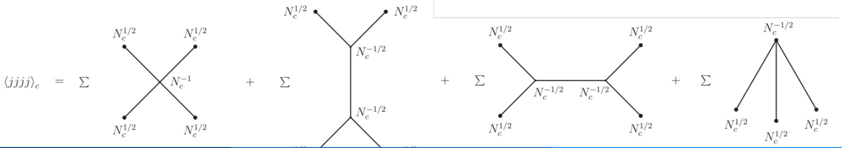
- The leading diagrams of the 2-point correlation function for a meson contain only 2 quark lines: sum of infinite ordinary mesons, $M_n = \mathcal{O}\left(N_c^0\right)$, $F_n = \mathcal{O}\left(\sqrt{N_c}\right)$

$$\int d^4x e^{ip\cdot x}\langle j(x)j^\dagger(0)\rangle = \sum_n \frac{i\,F_n^2}{p^2 - M_n^2 + i\,\epsilon} = \mathcal{O}\left(N_c\right), \quad F_n \equiv \langle 0|j|n\rangle$$

- Leading three-meson coupling scales as $\mathcal{O}\left(N_c^{-1/2}\right)$, meson decay width scales as $\mathcal{O}\left(N_c^{-1}\right)$: consider a 3-point correlation function



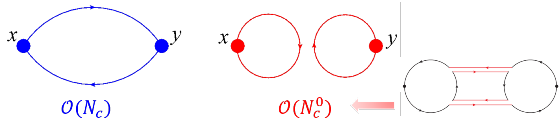
- Leading meson-meson scattering amplitude scales as $\mathcal{O}\left(N_c^{-1}\right)$



OZI rule

- OZI (Okubo-Zweig-Iizuka) rule: drawing only quark lines, any diagrams that can be separated into color-singlet clusters are suppressed

  $$\langle \bar{u}(y)u(y)\,\bar{u}(x)u(x)\rangle = -\langle S_u(y,x)S_u(x,y)\rangle + \langle S_u(y,y)\rangle\,\langle S_u(x,x)\rangle$$



- For three light quark flavors, mesons generally form nonets of U(3), rather than octets and singlets separately of SU(3); e.g., ω: $\bar{u}u + \bar{d}d$; ϕ: $\bar{s}s$
- Meson decays:

  ϕ decays into $K\bar{K}$, amplitude for the decay to $\pi\pi$ is relatively suppressed by N_c^{-1} (and by isospin breaking);

  widths of $\bar{c}c$ below open-charm thresholds are small since decays into light hadrons violate the OZI rule
- Meson-meson scattering:

  the scattering of mesons with different quark flavors violates the OZI rule; e.g., $D_s^+\pi^+ \to D_s^+\pi^+$