Scattering Amplitudes

@ intersection of QFT, Strings & Maths

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Institute of Theoretical Physics
Chinese Academy of Sciences

review based on works with F. Cachazo, E. Y. Yuan
PRL 113 (2014) PRD90 (2014) JHEP 1407 (2014) JHEP 1501 (2015)
JHEP 1507 (2015) PRD92 (2015) JHEP 1608 (2016) …

with N. Arkani-Hamed, Y. Bai, G.Yan JHEP 1805 (2018) 096
N. Arkani-Hamed, T. Lam JHEP (2021), SIGMA (2022) …
N. Arkani-Hamed, T. Lam, G. Salvatori, H. Thomas JHEP (2022) …

with Chia-Kai Kuo, Zhenjie Li, Yaoqi Zhang
2204.08297 (PRL), 2303.03035 (JHEP) …

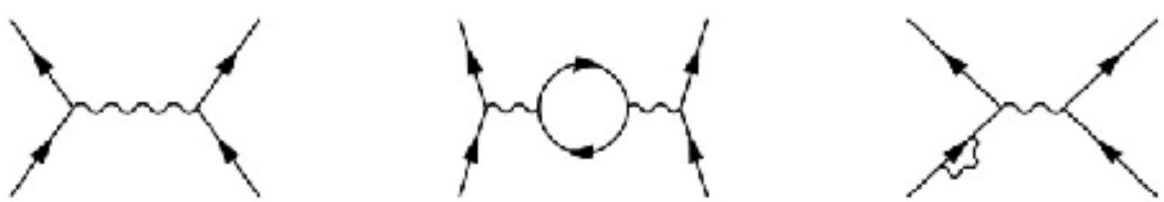
IHEP-CAS
Sep 14, 2023

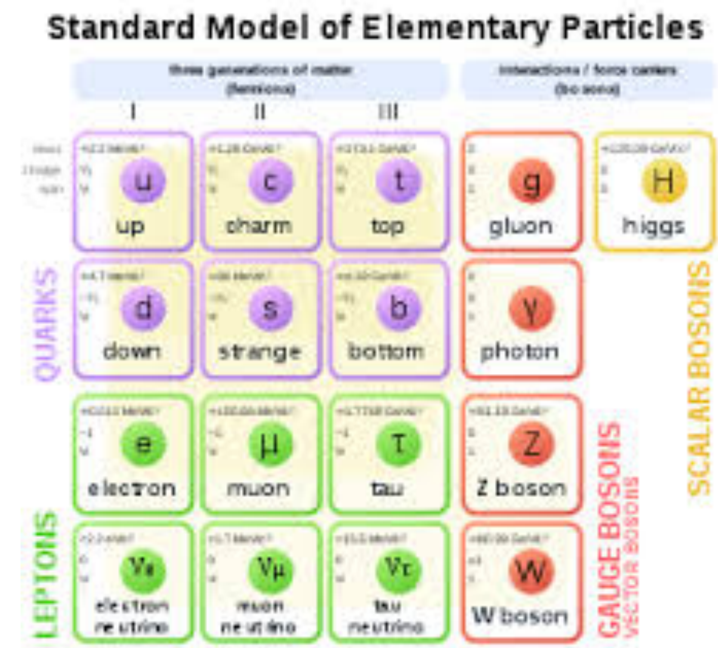
Quantum Field Theory (QFT)

Most successful theoretical framework to describe Nature:
particle physics, condensed matter, cosmology, strings

inevitable & universal: consequence of QM & relativity!
fundamental interactions unified @ high energy

simple picture in perturbation theory: Feynman diagrams

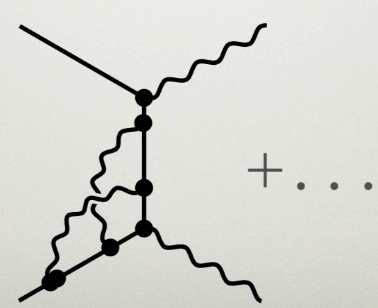




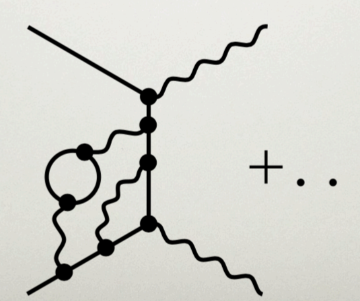
incredible accuracy!
e.g. g-factor of electron
magnetic dipole moment

$$g_e^{\text{theory}} = 2 + \frac{\alpha}{\pi} + \ldots \quad [1947]$$
$$g_e^{\text{expt}} = 2.0023\ldots \quad [1947]$$

$$g_e^{\text{theory}} = 2.0023193\ldots \quad [1957]$$
$$g_e^{\text{expt}} = 2.00231931\ldots \quad [1972]$$



$$g_e^{\text{theory}} = 2.0023193044\ldots \quad [1990]$$
$$g_e^{\text{expt}} = 2.00231931\ldots \quad [1972]$$



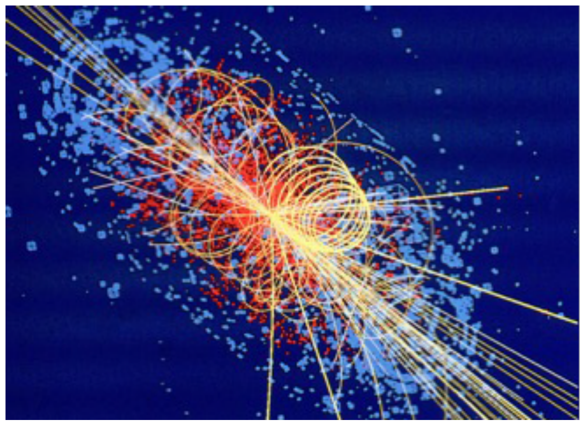
$$g_e^{\text{theory}} = 2.00231930435801\ldots \quad [2012]$$
$$g_e^{\text{expt}} = 2.002319304361\ldots \quad [2011]$$

Inevitability of QFT

- Combining QM & SR turns out to be extremely constraining!
- Whatever "final theory" is, at long distance, QM+SR → **for massless particles, interact via "simple" vertices with a simple menu s=0,1/2,1 (=gauge boson),3/2 (=gravitino) ,2 (=graviton)**
- Effective Field Theories: higher vertices suppressed at "low" energy → essentially 3-pt interactions: on-shell only need 3-pt amplitude!
  - s=2: unique & universal coupling=> gravitons, GR (pert. around Minkowski)
  - s=1: coupling f_{abc} Lie algebra => "gluons", gauge theories (YM+ matters)

S-matrix in QFT

- Colliders at high energies need amplitudes of many gluons/quarks



gg → gg … g

- Fundamental level our understanding of QFT & gravity incomplete: strong coupling, dualities, hidden symmetries, quantum gravity & cosmology…

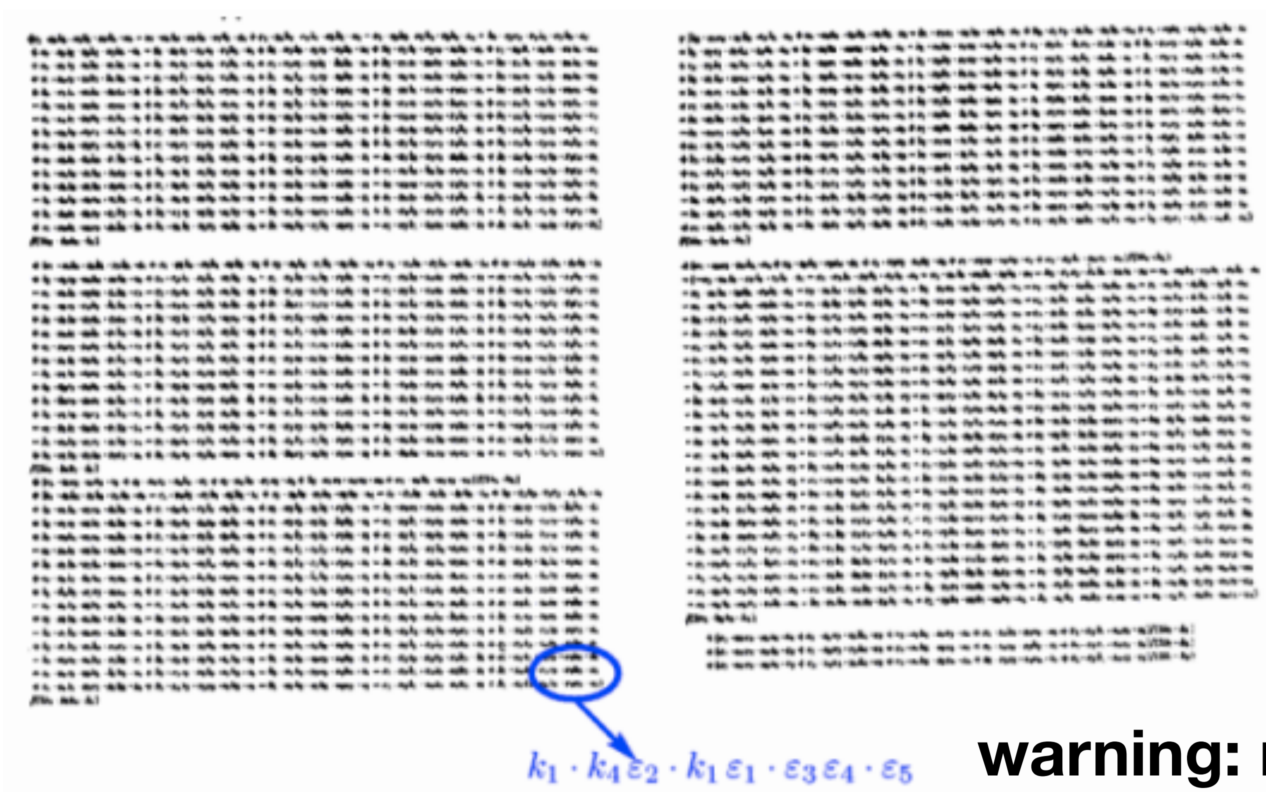
simplicity, new structures & relations seen in perturbative scattering amplitudes!

- Goal: new tools, ideas & theories for QFT+gravity from studying S-matrix

Impossible computations?

Feynman diagrams manifest locality & unitarity, but usually no manifest symmetry

Challenging for more legs/loops: many diagrams, lots of terms, huge redundancy



Process	N_{FG}
$gg \to 2g$	4
$gg \to 3g$	25
$gg \to 4g$	220
$gg \to 5g$	2485
$gg \to 6g$	34300
$gg \to 7g$	559405
$gg \to 8g$	10525900
$gg \to 9g$	224449225
$gg \to 10g$	5348843500

warning: not with your bare hands!

Gluons: 2 states $h = \pm$, but manifest locality requires 4 states (huge redundancies)
Much worse for graviton scattering: redundancies from diff invariance

A prior no reason to expect any simplicity or structures in the S-matrix

Parke-Taylor formula

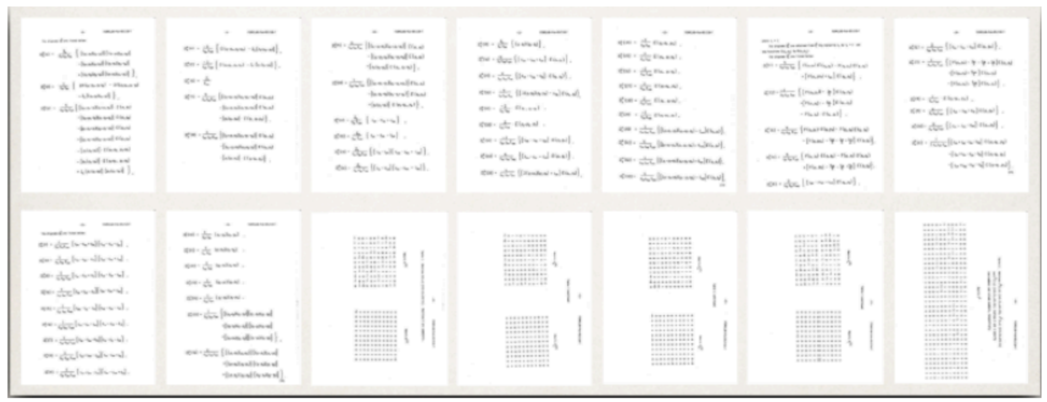
1985: heroic calculation of tree amp gg → gggg (results ~10 pages)

GLUONIC TWO GOES TO FOUR

Stephen J. Parke and T.R. Taylor
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510
U.S.A.

ABSTRACT

The cross section for two gluon to four gluon scattering is given in a form suitable for fast numerical calculations.



Our result has succesfully passed both these numerical checks. Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

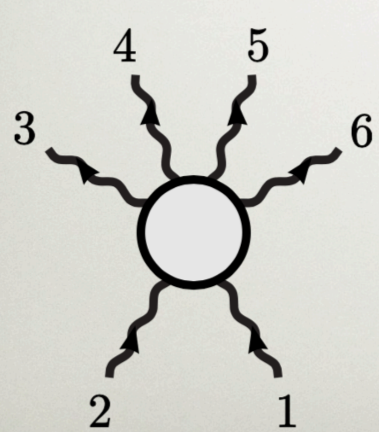
MHV: Maximally helicity violating
(all out-going) amps for all + or one - vanish!

Spinor-helicity variables

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \tilde{\lambda}_{\dot{a}}$$
$$\langle 12 \rangle = \epsilon_{ab} \lambda^{(1)}_a \lambda^{(2)}_b$$
$$[12] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{(1)}_{\dot{a}} \tilde{\lambda}^{(2)}_{\dot{b}}$$

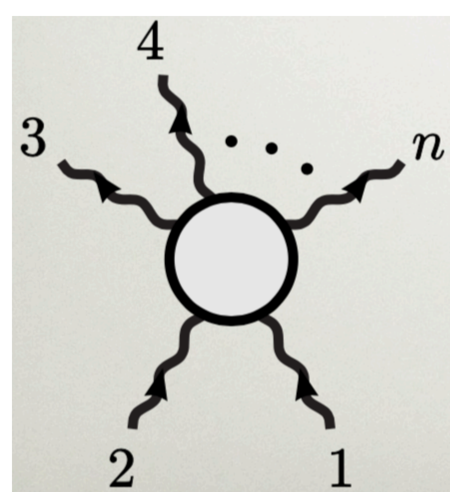
(Mangano, Parke, Xu 1987)

6 months later they realized



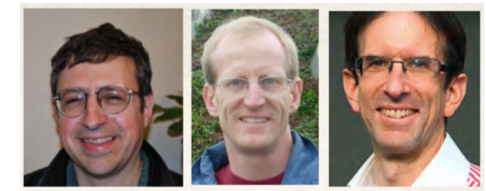
$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle}$$

conjecture for n-pt MHV amp [Parke,Taylor]



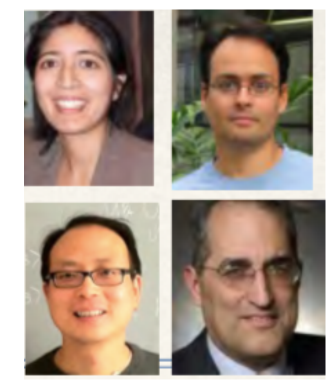
$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle \cdots \langle n\,1\rangle}$$

A very selective history



- 1986 - 2000: **spinor-helicity + generalized unitarity**

→ tree & one-loop gluon amps in QCD & N=4 SYM… powerful generalized unitarity method: cuts of loops = products of tree amp



- Twistor strings (2003) … **BCFW recursion**: all trees in QCD

new unitarity methods → one-loop QCD & more

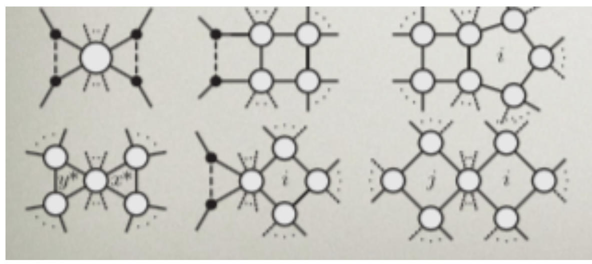
→NLO revolution -> NNLO, loop integrands, integrals & polylogs, …

New math structures (2009-): Grassmannian for all-loop integrands in N=4 SYM (hydrogen atom of QFT) + bootstrap, integrability, AdS/CFT…

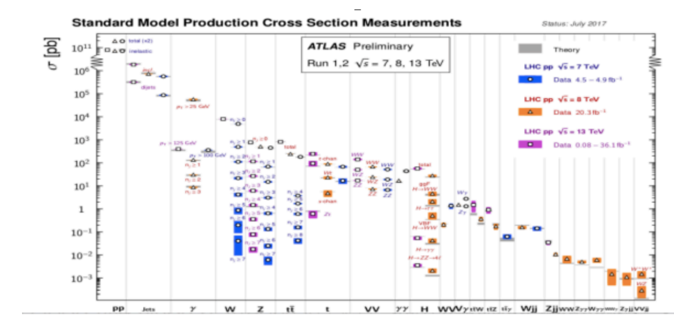
double copy (gauge theories, gravity & strings)—> CHY formulation etc.
geometric pictures for QFT & strings —> amplituhedron, associahedron, etc.

(numerous topics & names omitted here…)

Who do we connect to?



Multi-loop integrands, integrals,…



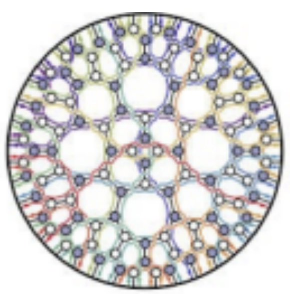
Formal QFT

Mathematics

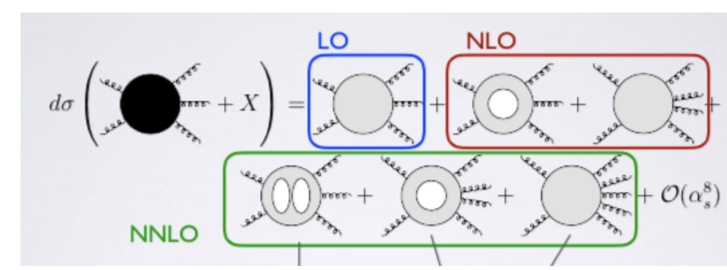
Amplitudes

Collider Phenomenology

precision frontier

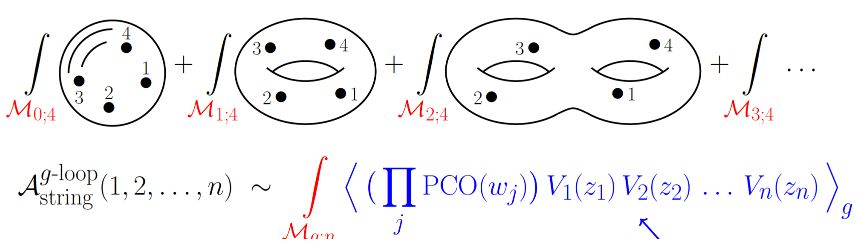


Geometries, combinatorics,..

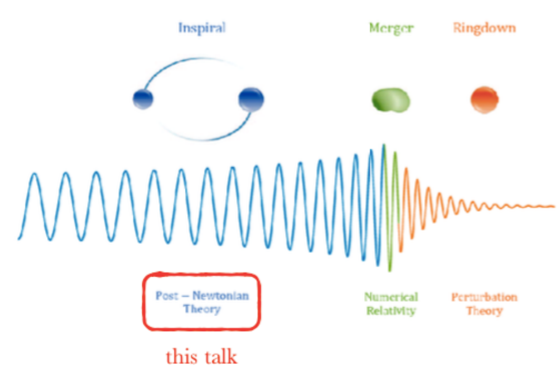


String Theory

Gravity



CFT, string perturbation,…

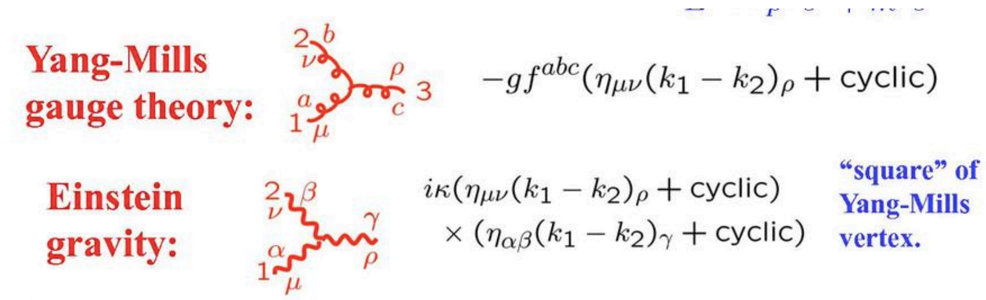


gravity amps
-> post-Newtonian

Gravity=(Gauge Theory)^2

1985: Kawai, Lewellen, Tye (KLT): **“closed string amp=open-string amp^2”**

Field-theory limit:



2008: Bern, Carrasco, Johansson (BCJ): **double-copy construction**

$$\mathscr{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$$n_s + n_t + n_u = 0$$

$$\mathscr{A}_4^{\text{tree}}\Big|_{c_i\to n_i} \equiv \mathscr{M}_4^{\text{tree}} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

If you have a set of duality satisfying numerators.
To get:

gauge theory → gravity theory

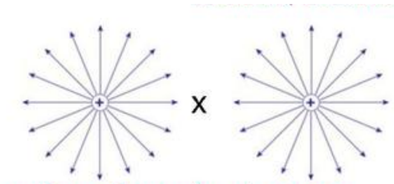
simply take

color factor → kinematic numerator

extended to classical solutions, curved background etc.-> hidden symmetry & structure of classical gravity!

black hole

↔

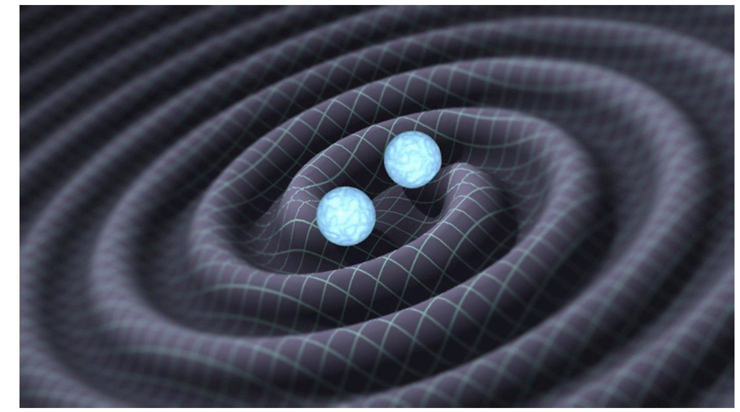


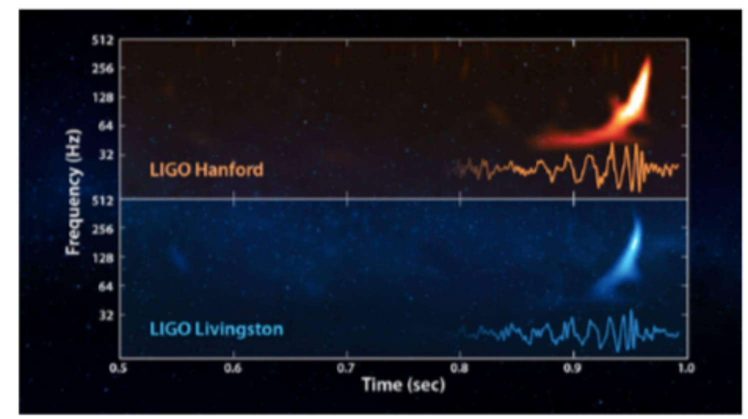
point electric charges

Schwarzschild ~ (Coulomb)²

Gravitational waves

How to help calculations needed for LIGO (inspiral)?





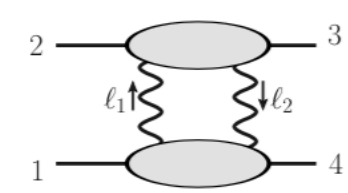
Classical limits from quantum scattering amplitudes

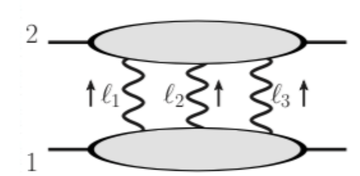
New tools e.g. double-copy simplifies GW calculations

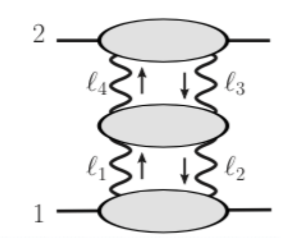
Post-Newtonian/Minkowski from (EFT) amplitudes

[Goldberger, Rothstein, Porto,…]
[Bern, Cheung, Roiban, Shen, Solon, Zeng; …]
[…]

		0PN	1PN	2PN	3PN	4PN	5PN	…
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	…
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	…
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	…
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	…
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	…
…						…	…	







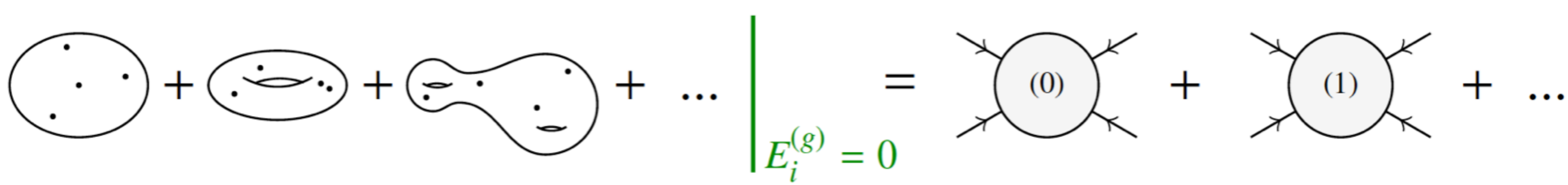
New formulation of QFT

- Twistor string theory [Witten 2003]: worldsheet model for N=4 SYM tree amps failed at loops, but led to BCFW, CSW & many new developments!

- How universal is Witten's twistor string? no SUSY? any spacetime dim? more general theories: (pure) Yang-Mils, gravity, effective field theories? loop level?

- CHY formulation: scattering of massless particles in any dim [Cachazo, SH, Yuan 2013]

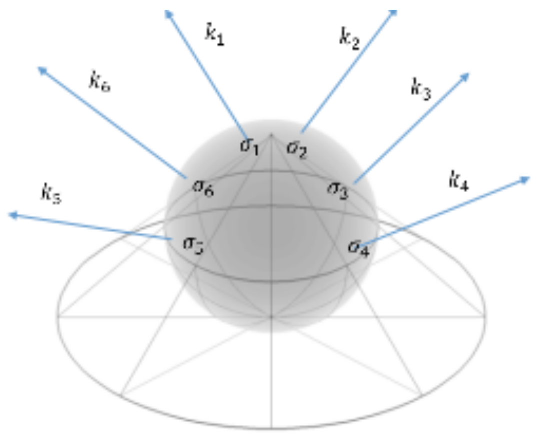
  - *compact formulas* for amps of gluons, gravitons, scalars, (fermions?!) etc.
  - *manifest* gauge (diff) invariance, soft theorems, double-copy & new relations, etc.
  - *worldsheet picture:* ambitwsitor strings etc. [Mason, Skinner; Adamo et al; Berkovits; Siegel…]



Scattering equations & CHY formulas

$$E_a := \sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \qquad a = 1, 2, \ldots, n$$ [CHY 2013]

$SL(2,\mathbb{C})$ symmetry:
n-3 variables, n-3 equations



- saddle-point equations of "Koba-Nielson" factor in string theory [Gross, Mende;...]
- moduli space of n-punctured Riemann sphere knows locality (& unitarity) of tree amps

$$M_n = \underbrace{\int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k,\epsilon,\sigma\}) = \sum_{\{\sigma\}\in\text{solns.}} \frac{\mathcal{I}(\{k,\epsilon,\sigma\})}{J(\{\sigma\})}$$

- New picture: scattering of massless particles via worldsheet correlators
- Feynman diagrams, Lagrangians, even spacetime itself become emergent

Gluons & Gravitons: gauge (diff.) invariance [CHY]

- Two basic building blocks: color & kinematics (polarization)

$$PT(\alpha) := \frac{1}{\sigma_{\alpha(1),\alpha(2)}\sigma_{\alpha(2),\alpha(3)}\cdots\sigma_{\alpha(n),\alpha(1)}}; \qquad \mathrm{Pf}'\Psi \sim \langle V^{(0)}(\sigma_1)\ldots V^{(-1)}(\sigma_i)\ldots V^{(-1)}(\sigma_j)\ldots V^{(0)}(\sigma_n)\rangle$$

- All tree amps in bi-adjoint scalar, Yang-Mills and Einstein Gravity!

$$\mathcal{M}_n^{\phi^3} = \int d\mu_n\, \mathcal{C}_n\, \mathcal{C}'_n$$

$$\mathcal{C}_n = \sum_\pi \mathrm{PT}(\pi)\mathrm{Tr}(T^{\pi(1)}\cdots T^{\pi(n)})$$

$$\mathcal{M}_n^{\mathrm{YM}} = \int d\mu_n\, \mathcal{C}_n\, \mathrm{Pf}'\Psi(\epsilon)$$

$$\mathcal{L}_{\phi^3} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{3!} f^{I\,J\,K} f^{I'\,J'\,K'}\, \phi^{I\,I'}\phi^{J\,J'}\phi^{K\,K'}$$

$$\mathcal{M}_n^{\mathrm{GR}} = \int d\mu_n\, \mathrm{Pf}'\Psi(\epsilon)\, \mathrm{Pf}'\Psi(\tilde{\epsilon})$$

hidden simplicity of gluons/gravitons in any dimension!

$$\mathrm{Pf}'\Psi := \frac{\mathrm{Pf}|\Psi|^{i,j}_{i,j}}{\sigma_{i,j}}$$

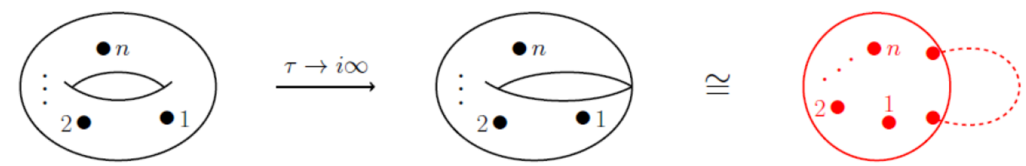
$$\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix},$$

$$A_{a,b} := \begin{cases} \frac{k_a\cdot k_b}{\sigma_{a,b}} & a\neq b \\ 0 & a=b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a\cdot\epsilon_b}{\sigma_{a,b}} & a\neq b \\ 0 & a=b \end{cases},$$

$$C_{a,b} := \begin{cases} \frac{\epsilon_a\cdot k_b}{\sigma_{a,b}} & a\neq b \\ -\sum_{c\neq a} C_{a,c} & a=b \end{cases}$$

Defining feature: Pfaffian is gauge invariant by SE -> gauge & diff. invariance!

Loops & (ambi-twistor) strings



- Ambitwistor strings (2d chiral CFT) [Mason,Skinner]: derive CHY formulas from CFT correlators

- Higher genus too difficult! -> loop amps from nodal Riemann sphere [Geyer, Mason, Monteiro,Tourkine,…]

- possible to obtain higher-genus string correlators from ambitwistor/CHY integrands [Geyer, Monteiro;…]

- another method: 1-loop CHY from forward limit of trees in higher dim, $\ell^2 \neq 0$ [SH, Yuan; CHY]

$$M^{(1)} = \int \frac{d^d\ell}{\ell^2} \lim_{k_\pm \to \pm\ell} \int \prod_{i=2}^{n} \delta\left(\frac{\ell \cdot k_i}{\sigma_i} + \sum_{j=1, j\neq i}^{n} \frac{k_i \cdot k_j}{\sigma_{ij}}\right) \hat{I}(\ell)$$

- 1-loop KLT formula for gauge theories + gravity, etc., manifest double copy [SH Schlotterer, 17 PRL] equivalence of two methods: both from superstring amps [SH Schlotterer, Y. Zhang 18] higher loops?

- New relations: QFT amps <-> string amps, also for bosonic/heterotic strings [SH, F. Teng Y. Zhang 19 PRL]

Goldstone particles from Adler zero

EFTs for Goldstone particles (symmetry breaking) e.g. pions, DBI, Galileon etc. [CHY 14] [Cheung et al 14]

What is special about them? Amplitudes vanish in soft limit: enhanced Adler zero!

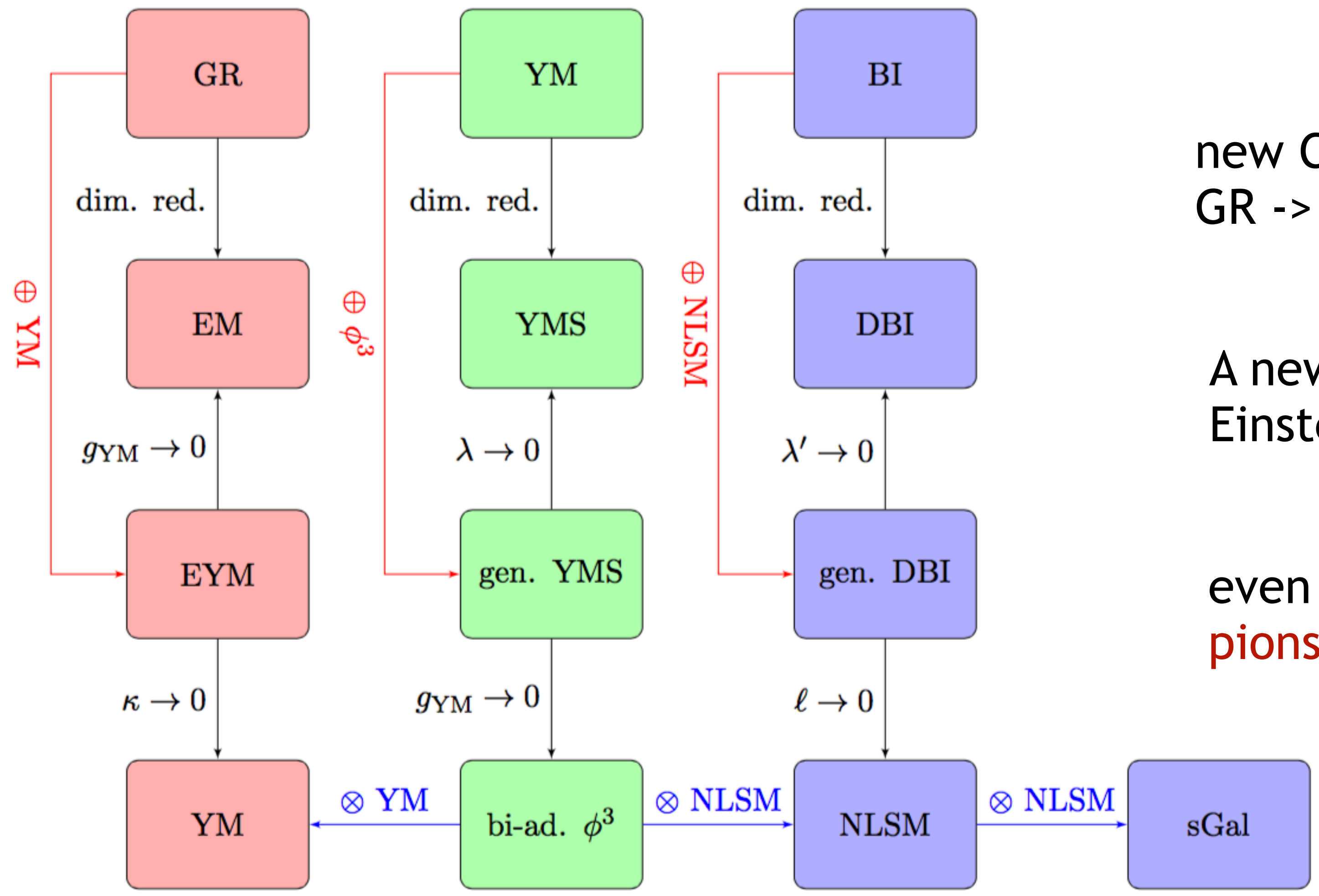
From CHY: a new ingredient with Adler zero $(\det{}' A_n)|_{p_i \sim \mathcal{O}(\tau)} = \mathcal{O}(\tau^2)$.

- $M_n = \int d\mu_n \, (\mathrm{Pf}' A)^2 \, \mathrm{PT}$, adjoint scalars with two derivative coupling?
  U(N) NLSM (the chiral Lagrangian) $\mathcal{L} = \mathrm{Tr}(\partial_\mu U^+ \partial^\mu U)$
- $M_n = \int d\mu_n \, (\mathrm{Pf}' A)^2 \, \mathrm{Pf}' \Psi$, higher-derivative-coupled photons?
  Born-Infeld theory (BI) & DBI by dim reduction $\mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} - \ell F_{\mu\nu} - \ell^2 \partial_\mu \phi \partial_\nu \phi)}$
- a special Galileon (single scalar with many derivatives) $M_n^{\mathrm{sGal}} = \int d\mu_n \, (\mathrm{Pf}' A)^4$

	Gauge Theories		
I	GR (s=2)	YM (s=1)	BI (s=1)
II	YM (s=1)	ϕ^3 (s=0)	NLSM (s=0)

	Effective Field Theories			
I	sGal (τ^3)	NLSM (τ^1)	BI (τ^1)	DBI (τ^2)
II	NLSM (τ^1)	ϕ^3 (τ^{-1})	YM (τ^{-1})	YMs (τ^0)

A landscape of massless theories



new CHY from old ones by e.g. dim reduction
GR -> Einstein-Maxwell, YM -> YM-scalar

A new operation as direct sum of two particles ->
Einstein-Yang-Mills, Yang-Mills + bi-adjoint scalars

even more interesting relations [CHY 14][Cheung et al]:
pions from special dimension reduction of gluons!

These amplitudes are strongly constrained (even uniquely determined) by symmetries:
gauge invariance & Adler zero; deeply connected to each other!

Double-copy as direct product

- (Tree-level) double copy explained by CHY: $GR = YM^2/\phi^3$ (inverse of bi-adjoint amps)
- Direct product of amplitudes in two theories: discover new double-copies

Double copies from CHY

$$A \equiv L \otimes R = \int d\mu_n \, I_L \, I_R$$

$$A_L(\alpha) = \int d\mu_n \, I_L \, PT(\alpha)$$

$$A_R(\beta) = \int d\mu_n \, I_R \, PT(\beta)$$

$$m(\alpha|\beta) = \int d\mu_n \, PT(\alpha) \, PT(\beta)$$

$$A = \sum_{\alpha,\beta \in S_{n-3}} A_L(\alpha) m^{-1}(\alpha|\beta) A_R(\beta)$$

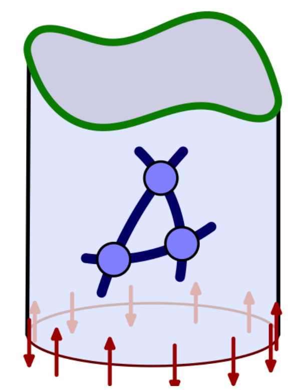
L⊗R	L	R
GR	YM	YM
BI	YM	NLSM
DBI	YMS	NLSM
sGal	NLSM	NLSM

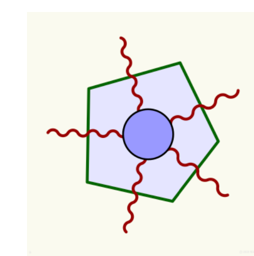
- None of these can be seen from Lagrangian/Feynman diagrams: deeper reason?
- Special cases in D=3,4,6, e.g. D=6 amps for M5/D5 brane & D=3 super-gravity=(ABJM)^2
- “Interpolating” between ⊗ and ⊕: expand GR/YM amps into EYM/YMS … [w. Dong, Hou]

The simplest QFT

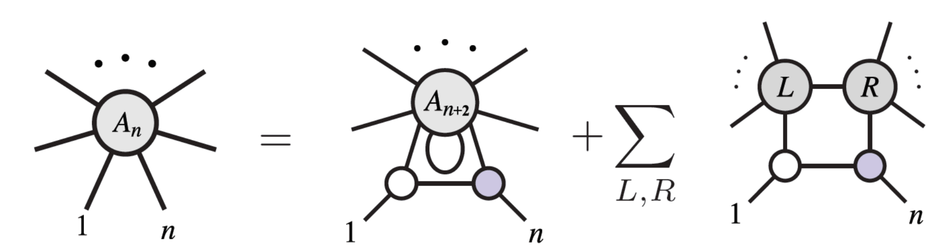
Harmonic oscillator of 21st century: hidden simplicity & structure in $\mathcal{N} = 4$ SYM (planar limit)

Integrability (planar limit): strong coupling via AdS/CFT, Wilson loops, Yangian symmetry … Ising model of gauge theories!





All-loop integrands <-> positive Grassmannian + amplituhedron [Arkani-Hamed, Trnka]



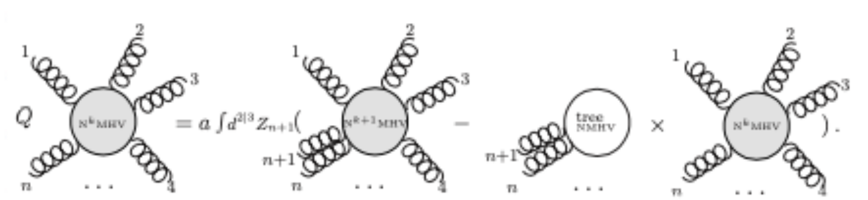
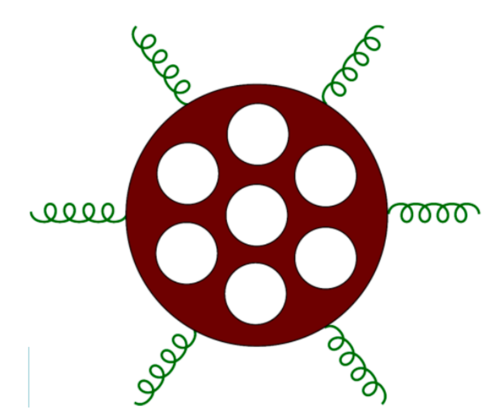


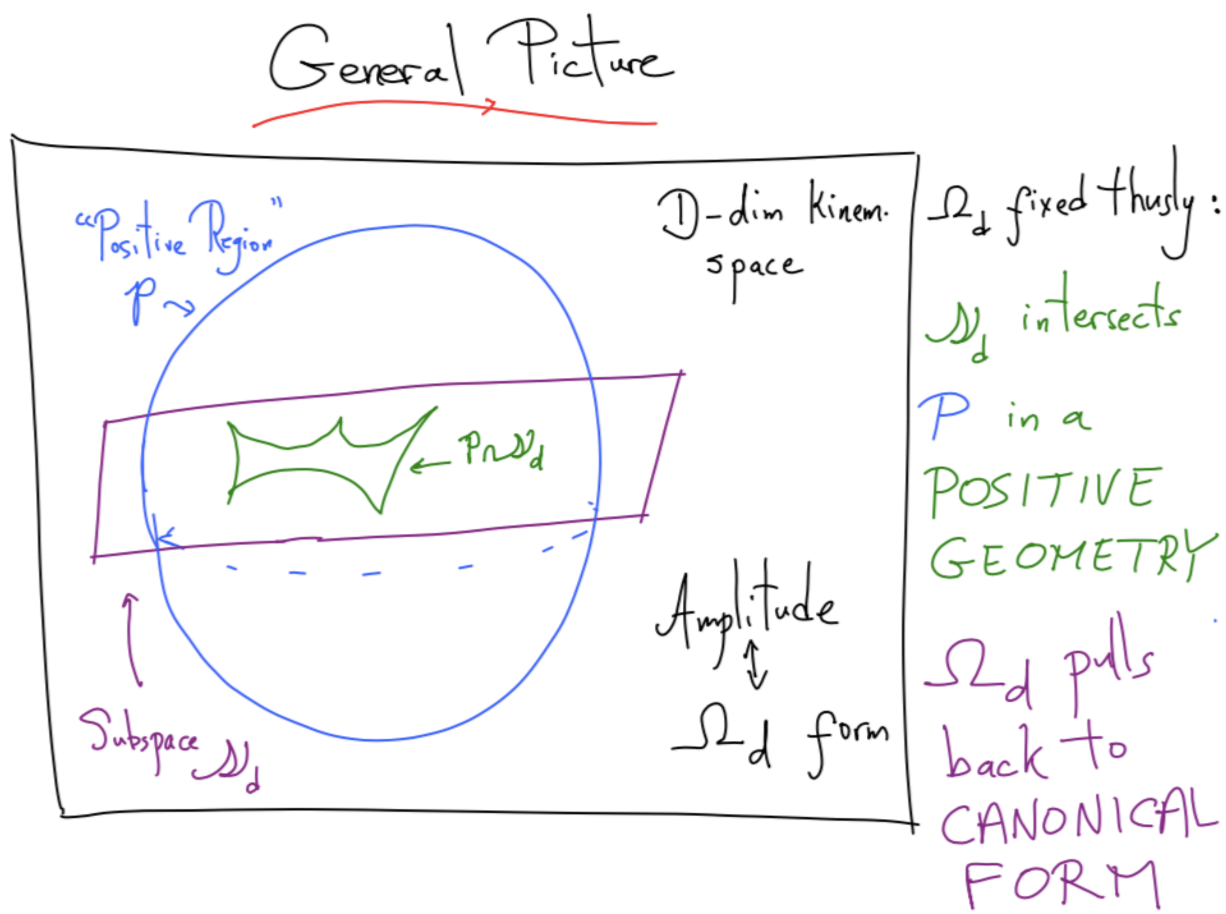
Figure 1. All-loop equation for planar $\mathcal{N} = 4$ S-matrix.

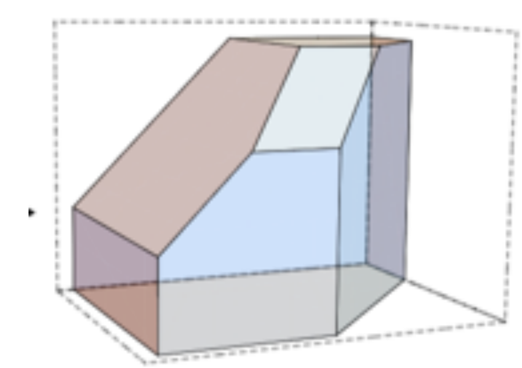


(Integrated) amplitudes + Feynman Integrals: extremely rich laboratory for perturbative QFT!

=> new methods: Qbar eqs + Wilson loops, bootstrap, UT integral basis, differential eqs …
new maths: iterated integrals, intersection theory, elliptic -> Calabi-Yau geometries …

Amplitudes as differential forms





Generalize amplituhedron to general theories in any dimensions (even phi^3)!

Bi-adjoint scalar: Amp (form)="volume" of associahedron in kinematic space

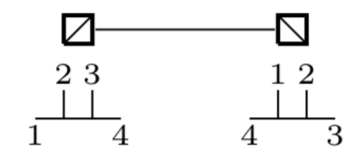
Geometrize color & its duality to kinematics, forms for gluon/pion amps etc.

Locality & unitarity emerges purely from geometries @ infinity of spacetime!

Kinematic associahedron

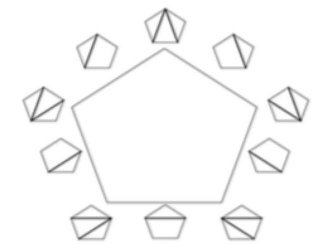
[Arkani-Hamed, Bai, SH, Yan, 2018]

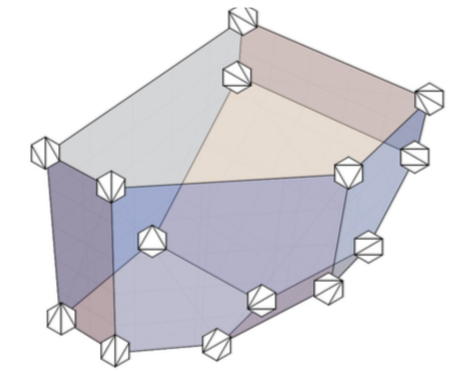
Associahedron of dim. (n-3): faces 1:1 corresp. with triangulation of n-gon



Positive region Δ_n: all planar variables $s_{i,i+1,\cdots,j} \geq 0$ (top-dimension)

Subspace H_n: $-s_{i\,j} = c_{i,j}$ as *positive constants*, for all non-adjacent pairs $1 \leq i,j < n$; we have $\frac{(n-2)(n-3)}{2}$ conditions $\Longrightarrow$ dim $H_n = n-3$

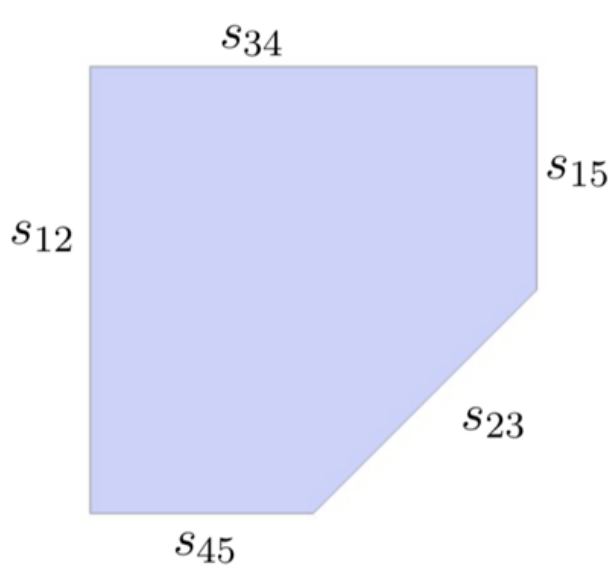


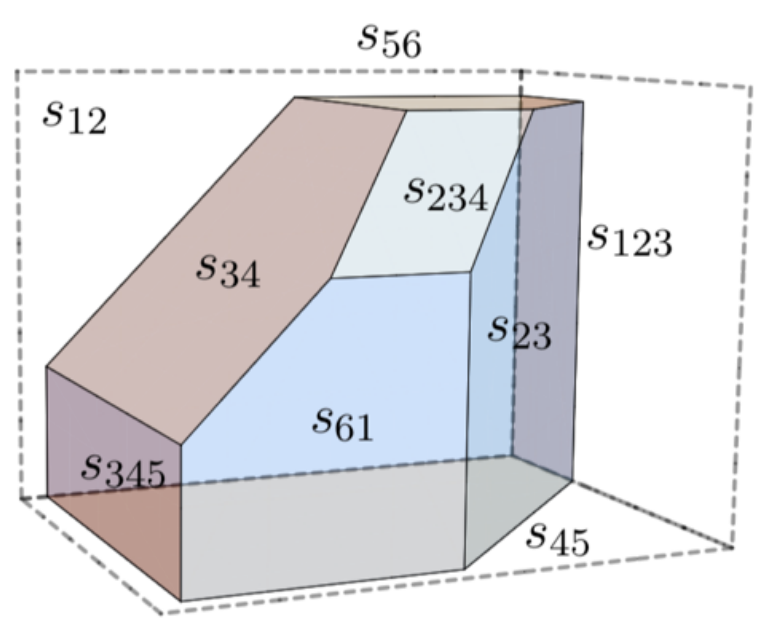


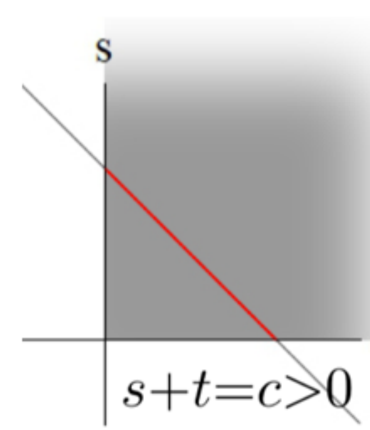
Kinematic Associahedron is their intersection! $\mathcal{A}_n := \Delta_n \;\cap\; H_n$

$e.g.\ \mathcal{A}_4 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$

encode singularities of any (colored) massless amplitudes at tree level: gluons, pions, etc.







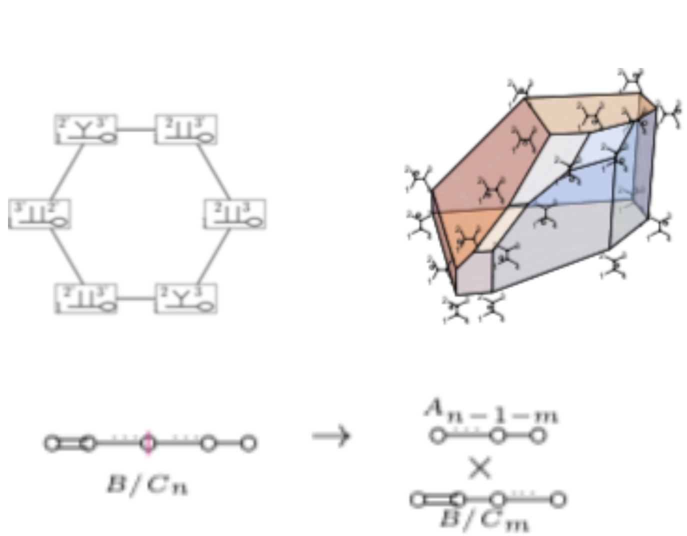
Amplitudes as "volume" [Arkani-Hamed, SH, Salvatori, Thomas, 2019]

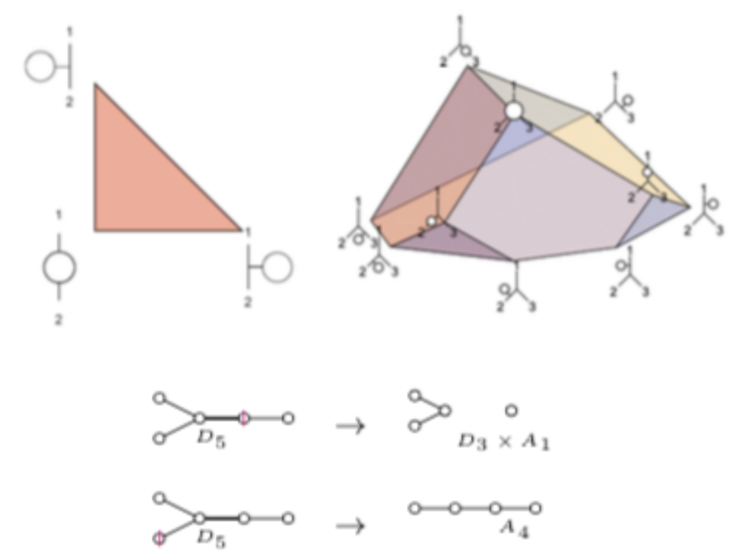
Canonical form of $\mathcal{A}_n$ = Pullback of Ω_n to H_n $\propto$ planar ϕ^3 amplitude!

$$e.g. \qquad \Omega(\mathcal{A}_4) = \Omega_4^{(1)}|_{H_4} = \left(\frac{ds}{s} - \frac{dt}{t}\right)\Big|_{-u=c>0} = \left(\frac{1}{s} + \frac{1}{t}\right) ds$$

$$\Omega(\mathcal{A}_5) = \Omega_5^{(2)}|_{H_5} = \left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right) ds_{12} \wedge ds_{34}$$

- Associahedron is the (tree) "amplituhedron" for scalar: amps="volume"
- Feynman-diagram expansion=special triangulation -> many new representations





- Extend to "cluster polytope" of finite types: B/C: cyclohedron <—> tadpoles; type D <—> one-loop planar phi^3 (all with "factorizing" boundaries)
- Hidden symmetry (invisible in FD's) manifest by geometry (analog in N=4 SYM)

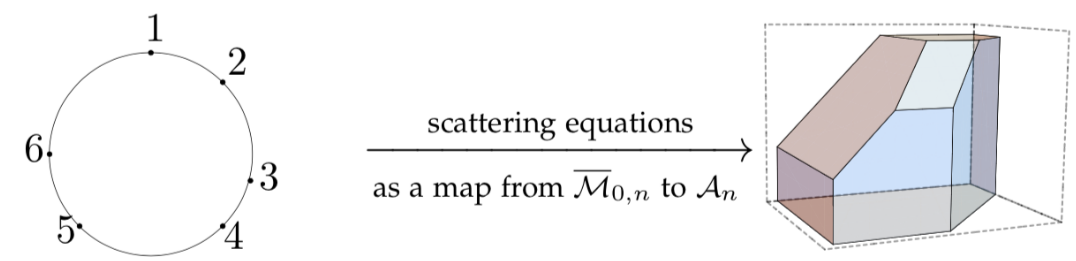
Generalized string amps [Arkani-Hamed, SH, Lam, Thomas, 2019]

A well-known associahedron: minimal blow-up of the open-string worldsheet $\mathcal{M}^+_{0,n} := \{\sigma_1 < \sigma_2 < \cdots < \sigma_n\}/\mathrm{SL}(2,\mathbb{R})$ [Deligne, Mumford]

a geometric origin of scattering eqs & CHY

The *canonical form* of $\overline{\mathcal{M}}^+_{0,n}$ is the "Parke-Taylor" form

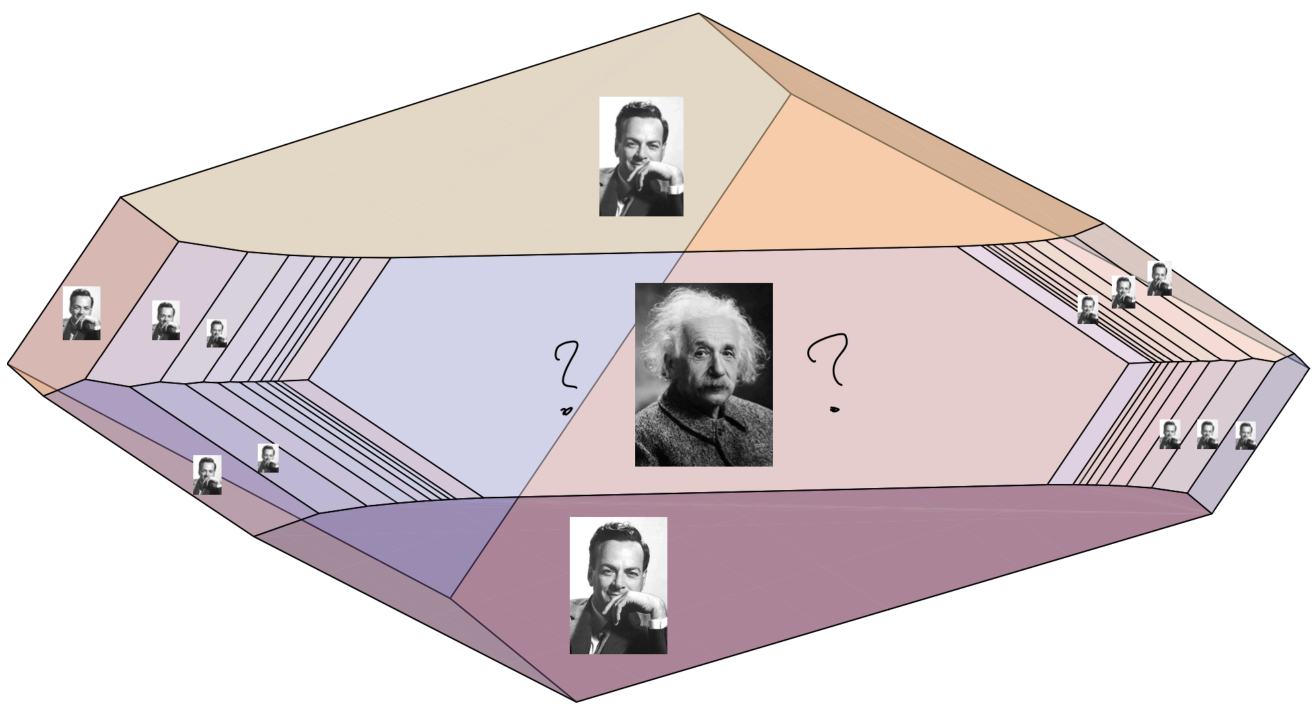
$$\omega_n^{\mathrm{WS}} := \frac{1}{\mathrm{vol}\,[\mathrm{SL}(2)]} \prod_{a=1}^{n} \frac{d\sigma_a}{\sigma_a - \sigma_{a+1}} := \mathrm{PT}(1,2,\cdots,n)\, d\mu_n$$



- Generalize $\mathscr{M}_{0,n}$(worldsheet associahedron) to other types: binary geometries
- Natural "string integrals" for all finite types: α'-deform. of loop ϕ^3 amps
- Field-theory (particles) $\alpha' \to 0$ = CHY formula with $\alpha' \to \infty$ (saddle points)
- Higher-genus surfaces vs. higher-loop amps?

Particles & strings from geometries

- Surfacehedra [Arkani-Hamed et al]: curves on surface w. any genus (all loops!)
- Infinite polytopes: truncations <-> (infinite) cluster algebras & quivers
- Canonical forms -> all-loop non-planar tr(phi^3) integrand

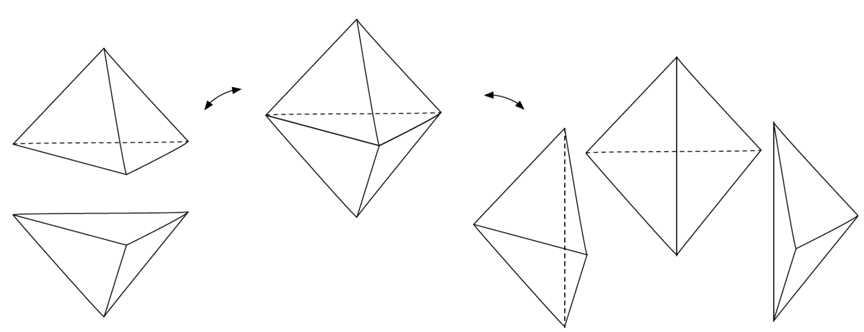


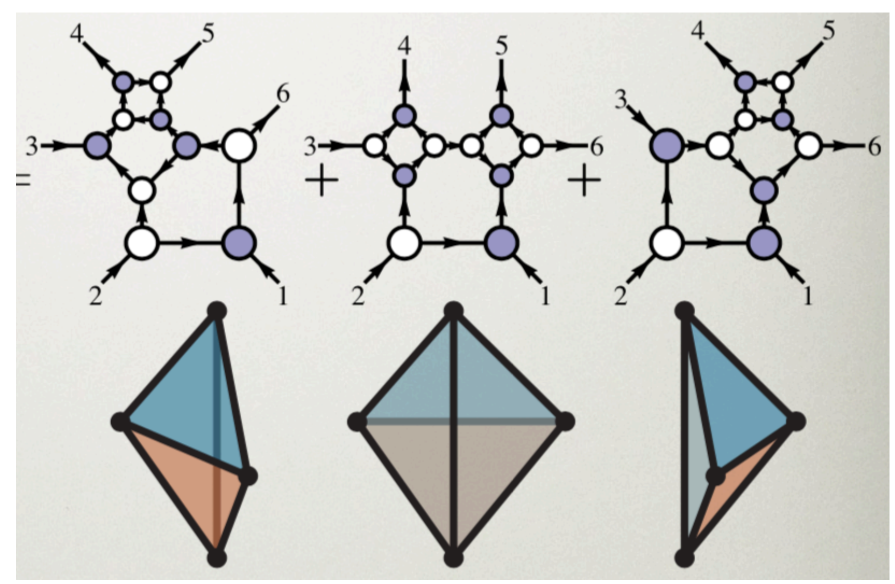
- Now also natural string-like integrals for surfacehedra (infinite product!)
- Appearance of “gravity” (like closed-string) from positivity (open-string)
- Goal: strings (& particles) without string (worldsheet) <— new geometries

Amplituhedron

All-loop recursions for planar N=4 SYM
each term = on-shell diagram from $Gr_{+}(k,n)$
satisfies Yangian symmetry <—> integrability

Amplituhedron tells us how they are combined
into tree amplitudes & all-loop integrands
—> geometry encoding QM & relativity!



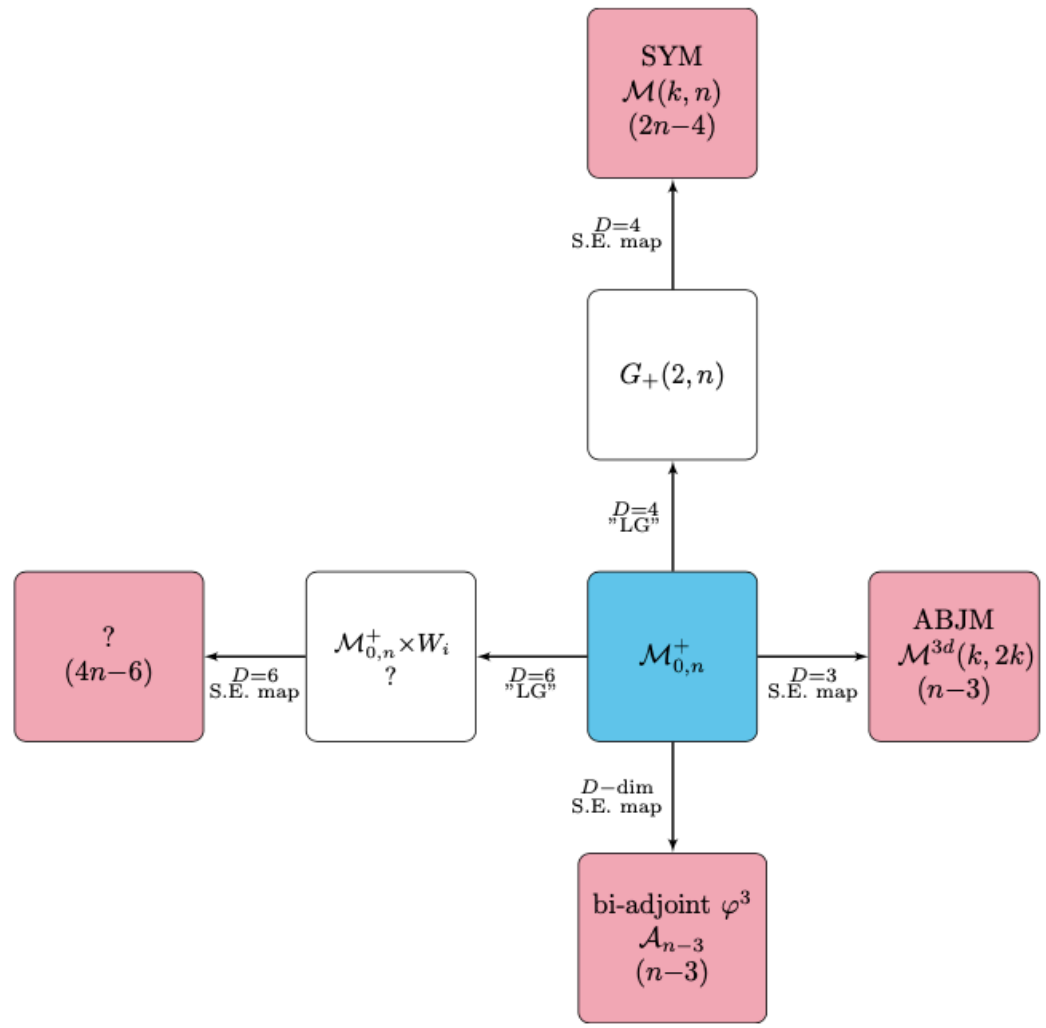


Key objects: positive geometry (real)
with a unique canonical form (complex):
only logarithmic singularities
@ boundaries (residues recursively defined)

momentum amplituhedron: (tree) amplituhedron directly in momentum space, T-duality etc.
[Damagaard, Ferro, Lukowski, Parisi; ......]

Universality of positive geometries

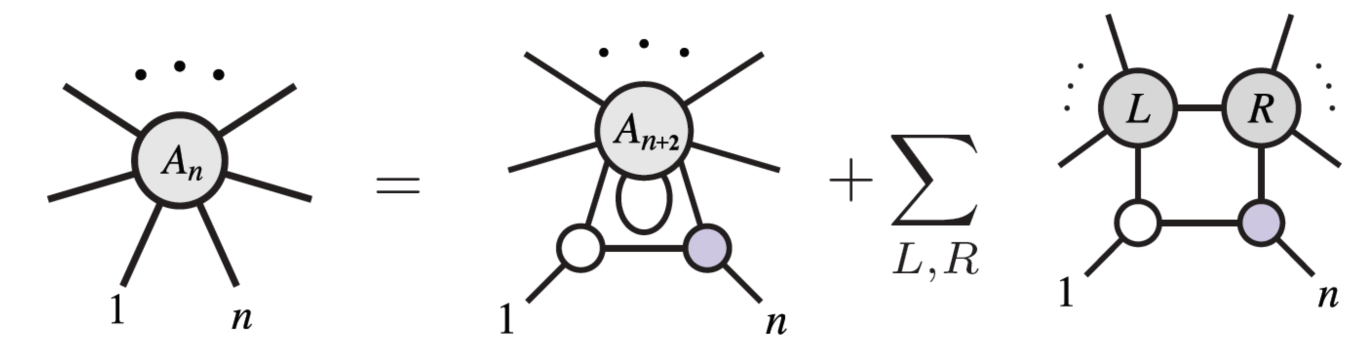
- Polytopes as "amplituhedron" for other scalar amps, Ψ_{universe} (cosmological polytopes) + stringy & Feynman integrals + other contexts e.g. EFThedron
- Q: simplified model of SYM amplituhedron to all loops?
- Q: any all-loop amplituhedron (not just polytopes) in a different theory? perhaps D=3 ABJM (N=6 Chern-Simons-matter)?



- ABJM tree momentum orthogonal amplituhedron with OG(n/2, n) [Huang et al, 21]
- equivalently, pushforward from moduli space to tree amplituhedron of ABJM [w. Kuo, Zhang, 21]
- Unified picture for D=3,4 tree momentum amplituhedra: ABJM amplituhedra from dimensional reduction of SYM ones

Loops in SYM & ABJM

SYM: duality to (super) Wilson Loops (& correlators in null limits) -> integrand=WL with Lagrangian insertions —> all-loop recursion —> amplituhedron!

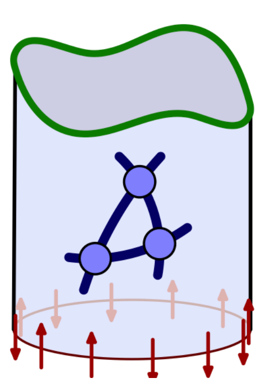


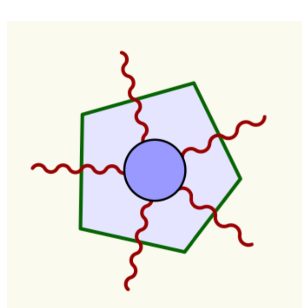
e.g. computed explicitly to L=3 for all n,k;

L=10 for n=4 & L=6 for n=5

+ highly-nontrivial all-loop cuts from amplituhedron

ABJM: duality to WL beyond n=4 unknown (k+2=n/2 sector only), def. of integrand for WL?
n=4: duality to WL checked to L=2, integrands conjectured up to L=3 [Bianchi et al]
amplitudes computed to L=2, n=6,8,10(?) [Caron-Huot, Huang; w. Kuo, Huang, Li 22]





In both theories: extremely rich structure from strong coupling (AdS/CFT) & integrability, especially for 2,3-pt function etc.

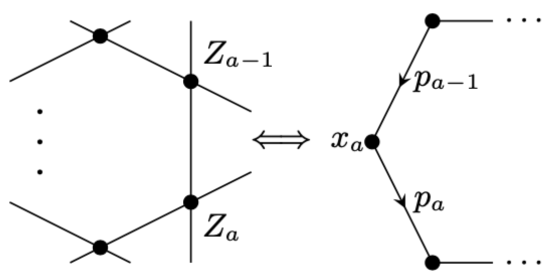
Q: can we improve perturbative (amps/WL) side of ABJM?

Are there ABJM amplituhedron? Any connections to SYM?

Reduced amplituhedron for ABJM

The simplest guess works! Reducing external & loop momenta to D=3 gives a new geometry reduced amplituhedron —> all-loop integrands in ABJM!

Momentum twistors [Hodges]: "light rays" of dual spacetime, inspired by duality of $\mathcal{N}=4$ SYM planar amplitudes with Wilson loops [Alday et al; Brandhuber et al; ...]



- $Z^I = (\lambda^\alpha, \mu^{\dot\alpha} := x^{\alpha,\dot\alpha}\lambda_\alpha)$: manifest dual conformal symmetry [Drummond et al]
- null polygon: $\lambda_a\tilde\lambda_a = p_a = x_{a+1} - x_a \leftrightarrow \{Z_1, \cdots, Z_n\}$ for n edges; $x_a := (Z_{a-1}, Z_a)$ is a line in twistor space

- (dual) loop momentum $x_0 \leftrightarrow$ a line (AB) in twistor space

The n-point L-loop amplituhedron: $Z_{a=1,\cdots,n}$ for external kinematics and $(AB)_{i=1,\cdots,L}$ for loop momenta

For $n=4$ (only $k=0$): a $4L$-dim geometry in $(AB)_i$ space (Z's fixed):

$$\langle (AB)_i 12\rangle > 0, \quad \langle (AB)_i 23\rangle > 0, \quad \langle (AB)_i 34\rangle > 0, \quad \langle (AB)_i 14\rangle > 0,$$
$$\langle (AB)_i 13\rangle < 0, \quad \langle (AB)_i 24\rangle < 0$$

as well as mutual positivity: $\langle (AB)_i (AB)_j\rangle > 0$ [Arkani-Hamed, Trnka 13]

External & loop momenta in $D=3$: twistor-space lines with symplectic conditions (in momentum space: $\lambda=\tilde{\lambda}$) [Elvang et al 14]:

$$\mathbf{\Omega}_{IJ}Z_a^I Z_{a+1}^J = \mathbf{\Omega}_{IJ}A_i^I B_i^J = 0\,, \quad \text{with } \mathbf{\Omega} = \begin{pmatrix} 0 & \epsilon_{2\times 2} \\ \epsilon_{2\times 2} & 0 \end{pmatrix}.$$

$(a=1,2,\cdots,n$ and $i=1,\cdots,L) \to$ reduced amplituhedron

Focusing on $n=4$: a $3L$-dim geometry in constrained $(AB)_i$ for $D=3$

With parametrization $Z_{A_i} = Z_1 + x_i Z_2 - w_i Z_4, \quad Z_{B_i} = y_i Z_2 + Z_3 + z_i Z_4$
$\Longrightarrow$ def. of $n=4$ reduced amplituhedron:

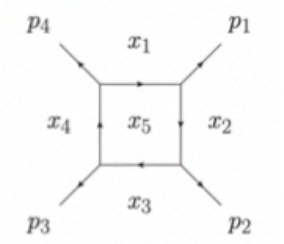
$$\forall i: x_i, y_i, z_i, w_i > 0, \quad x_i z_i + y_i w_i = 1,$$
$$\forall i,j: (x_i - x_j)(z_i - z_j) + (y_i - y_j)(w_i - w_j) < 0$$

First look at $L=1$: the *canonical form* in $D=4$ = box integral

$$\Omega_1^{(D=4)} = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w} = \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}$$

Dim. reduction $\to D=3$ box with ϵ num. = one-loop ABJM integrand [Chen, Huang 11]:

$$\Omega_1 = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w}\delta(xz+yw-1) = \frac{d^3(AB)\langle 1234\rangle^{3/2}(\langle AB13\rangle\langle AB24\rangle)^{1/2}}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}$$



$$= \int \mathrm{d}^3 x_5 \frac{\epsilon(5,1,2,3,4)}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}, \qquad \epsilon(i,j,k,l,m) \equiv \epsilon_{\mu\nu\rho\sigma\tau} x_i^\mu x_j^\nu x_k^\rho x_l^\sigma x_m^\tau$$

All-loop cuts from geometries

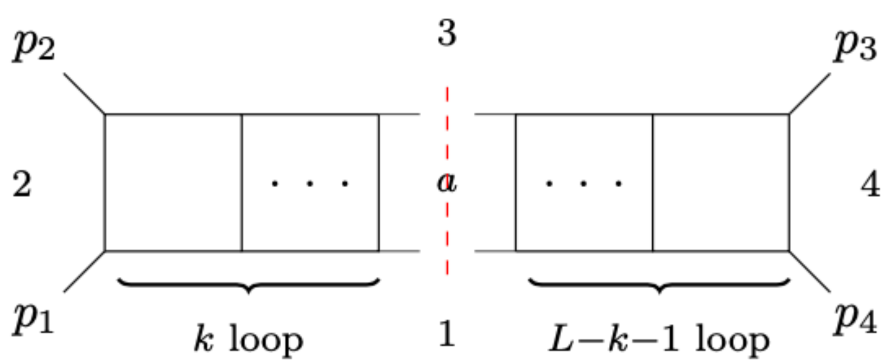
Nicely make some all-loop cuts of ABJM manifest from geometries (more later)

- Soft cut: *e.g.* $\langle \ell_i 12\rangle = \langle \ell_i 23\rangle = \langle \ell_i 34\rangle = 0$ or $y_i = z_i = w_i = 0 \implies$ manifest mutual positivity $D_{i,j} > 0$ for any j, residue = $(L-1)$-loop

- Vanishing cut: any cut isolating odd-point amplitude, *e.g.* $w_i = w_j = D_{i,j} = 0$ (triple cut) $\implies D_{i,j} \leq 0$, the residue vanishes; similarly, five-point cut $w_i = y_j = D_{i,j} = 0$ vanishes

- Double (unitarity) cut: $\langle \ell 14\rangle = \langle \ell 23\rangle = 0$

$$\sum_{k=0}^{L-1}$$



Shorthand notation: *e.g.* $\underline{\Omega}_1 = \frac{c\epsilon_1}{s_1 t_1}$ (strip off $d^3\ell$)

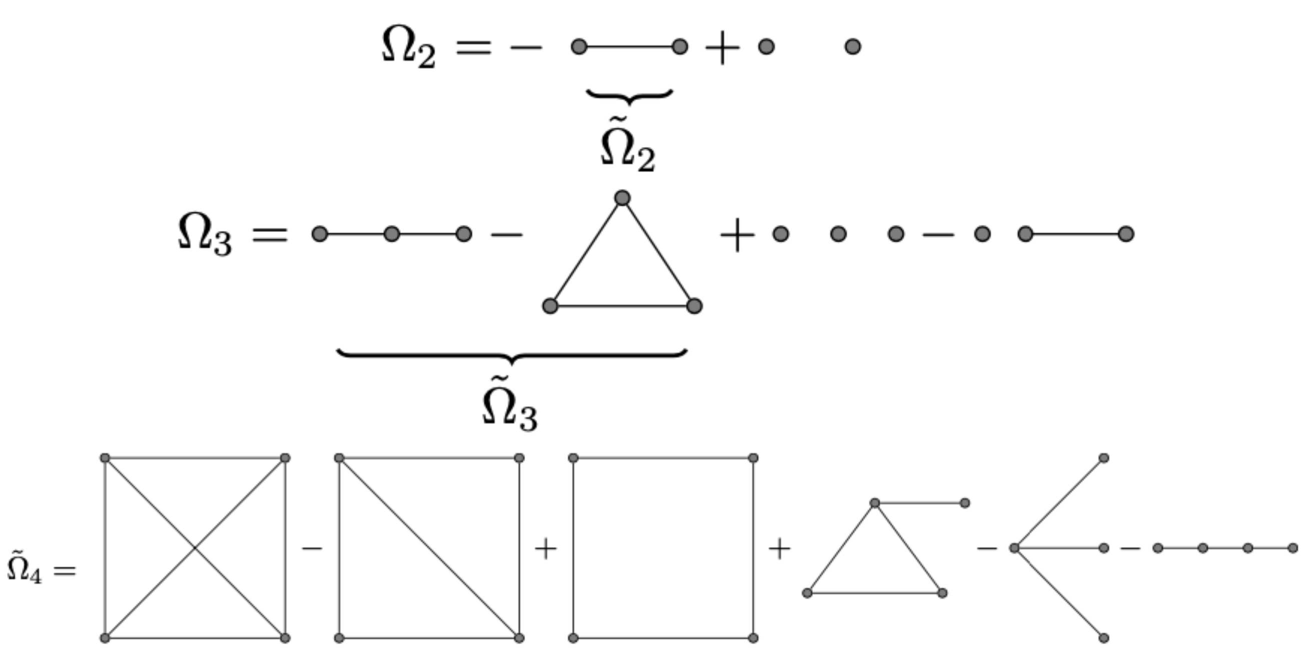
$$\ell_i \equiv (AB)_i, \qquad c \equiv \langle 1234\rangle, \qquad \epsilon_i \equiv (c\langle \ell_i 13\rangle\langle \ell_i 24\rangle)^{1/2};$$
$$s_i \equiv \langle \ell_i 12\rangle\langle \ell_i 34\rangle \sim y_i w_i, \qquad t_i \equiv \langle \ell_i 23\rangle\langle \ell_i 14\rangle \sim z_i w_i, \qquad D_{ij} \equiv -\langle \ell_i \ell_j\rangle.$$

Negative geometries

L-loop space: amplituhedron = complete graph with positive mutual conditions -> sum of all graphs with negative mutual conditions

Decomposition into sum of negative geometries: 

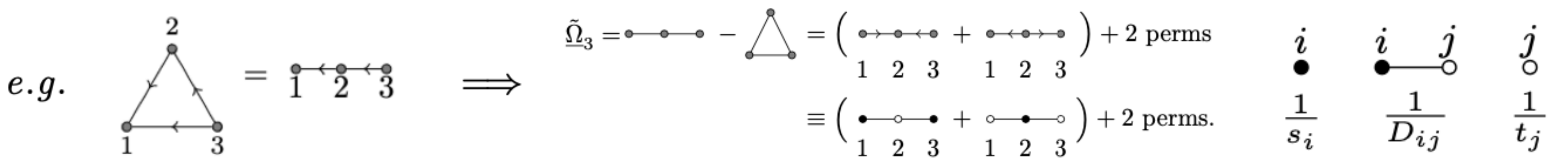
The sum of connected graphs gives logarithm of amplitudes [Arkani-Hamed et al], *e.g.*



easier to compute form/integrand of such negative geometries, e.g. all tree forms are known!

easier to integrate (only 1-loop divergence); add up to log of amps -> $\Gamma_{\rm cusp}$

—> huge simplifications in D=3: only bipartite graphs (with source & sinks) survive!
e.g. no triangle for L=3, only 2 types of trees + box for L=4



A tiny fraction ($\to 0$ as $L \to \infty$) of graphs remain (relatively simple ones):

L	top. of G	top. of g	directed acyclic graphs	bipartite g
2	1	1	2	1
3	2	1	18	3
4	6	3	446	19
5	21	5	26430	195
6	112	17	3596762	3031
7	853	44	1111506858	67263

ABJM amplituhedron gives new results for n=4 integrands up to L=5 (L=6 in progress)!

nicely generalize to n-pt and a new picture: loops as ``fibrations" over tree regions =>
e.g. 1-loop any n, 2-loop up to n=8 and even 3-loop up to n=6 [w. Y. Huang, C. Kuo 2023]

Integrating forms in SYM [Arkani-Hamed, Henn, Trnka]

A finite function of 1 var. (and coupling g) from log of (n=4) amps => Γ_{cusp} of SYM

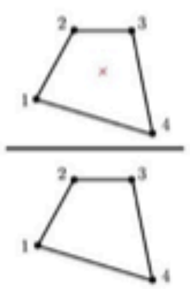
- IR divergence of log of amplitude

$$\log M = -\sum_{L\geq 1} g^{2L} \frac{\Gamma^{(L)}_{\text{cusp}}}{(L\epsilon)^2} + O(1/\epsilon)$$

- With one loop frozen and integrate over others

$$F_\Gamma\,(AB_0) = \int d\mu_{AB_1}\ldots d\mu_{AB_{L-1}}\Omega_\Gamma \text{ is IR finite}$$

This object is related to the Wilson loop with Lagrangian insertion



$$= \frac{1}{\pi^2}\frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} F(g;z), \qquad z = \frac{x_{20}^2 x_{40}^2 x_{13}^2}{x_{10}^2 x_{30}^2 x_{24}^2}$$

- Extract Γ-Cusp from $F(g,z)$

$$g\frac{\partial}{\partial_g}\Gamma_{\text{cusp}}\,(g) = -2I[F(g,z)] \quad \text{where} \quad I\,[z^p] = \frac{\sin(\pi p)}{\pi p}$$

Q: How to integrate forms of (n=4) amplituhedron (simplified by negative geometries)?

Could we resum some integrated functions =>non-pertubative info?

By using "boxing" DE, all ladder graphs resum to $\Gamma_{\text{ladder}} = \frac{4}{\pi}\log\cosh(\sqrt{2}\pi g)$

even all trees resum nicely to $\Gamma_{\text{tree}}(g) = A(\frac{4}{\pi}\tan\frac{\pi A}{2} - A)$ where $\frac{A}{2g\cos\frac{\pi A}{2}} = 1$

Integrating forms in ABJM [w. Kuo, Li, Zhang, 2023; see also Henn, Lagares, Zhang]

Exactly the same as in SYM: integrating L-1 loops of L-loop forms (of negative geometries)

$$\mathcal{W}_L(\ell_1,1,2,3,4) := \int \prod_{i=2}^{L} d^3\ell_i\, \underline{\tilde{\Omega}}_L,$$


$$= \frac{\epsilon(\ell_1,1,2,3,4)}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_1\cdot 4)}$$

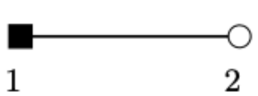
($F_0 = 1$ by definition)

Stripping off a prefactor (different for L odd or even) -> function of 1 variable

$$\mathcal{W}_L = \begin{cases} \frac{\epsilon(\ell_1,1,2,3,4)}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_1\cdot 4)}\, F_{L-1}(z)\,, & L \text{ odd} \\ \left(\frac{(1\cdot 3)(2\cdot 4)}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_1\cdot 4)}\right)^{3/4} F_{L-1}(z), & L \text{ even} \end{cases}$$

$$z = \frac{(\ell_1\cdot 2)(\ell_1\cdot 4)(1\cdot 3)}{(\ell_1\cdot 1)(\ell_1\cdot 3)(2\cdot 4)}\,.$$

e.g. L=2, weight-0 (algebraic) function $\times\,\pi$ by integrating out "triangle"



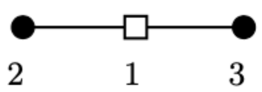
$$= -2\frac{(1\cdot 3)(2\cdot 4)}{(\ell_1\cdot 2)(\ell_1\cdot 4)}\int_{\ell_2}\frac{1}{(\ell_2\cdot\ell_1)(\ell_2\cdot 1)(\ell_2\cdot 3)}$$

$$= -\frac{\sqrt{(1\cdot 3)}(2\cdot 4)}{\sqrt{(\ell_1\cdot 1)}\sqrt{(\ell_1\cdot 3)}(\ell_1\cdot 2)(\ell_1\cdot 4)}\times\pi$$

$$= -\left(\frac{(1\cdot 3)(2\cdot 4)}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_4\cdot 4)}\right)^{\frac{3}{4}}\times \pi z^{1/4}$$

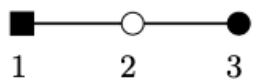
$$F_1(z) = -\pi(z^{1/4} + z^{-1/4}).$$

L=3: only 2 topologies => weight-2 (dilog) functions by integrating out 2 loops



$$= \frac{\epsilon(\ell_1,1,2,3,4)}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_1\cdot 4)} \times \pi^2.$$

In total:



$$= \frac{4\,(2\cdot 4)}{(\ell_1\cdot 2)(\ell_1\cdot 4)} \int_{\ell_2,\ell_3} \frac{\epsilon(\ell_2,1,2,3,4)}{(\ell_2\cdot \ell_1)(\ell_2\cdot 1)(\ell_2\cdot 3)(\ell_2\cdot \ell_3)(\ell_3\cdot 2)(\ell_3\cdot 4)}$$

$$F_2(z) = 4\left(f(z) + f\left(\frac{1}{z}\right) + \frac{\pi^2}{2}\right).$$

$$\frac{4\,\epsilon(\ell_1,1,2,3,4)}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_1\cdot 4)} \int_0^\infty \frac{dc}{4\pi\sqrt{c}} \frac{1}{1+(1+c)/z}\left(\frac{\pi^2}{2} + \frac{1}{2}\log^2\frac{1+c}{z}\right)$$

(similar integrals for higher-pt ABJM…)

$$f(z) := \frac{t-1}{t+1}\left(\frac{\pi^2}{2} + \mathrm{Li}_2(1-t) + \log(t)\log(t-1) - \frac{1}{4}\log(t)^2\right) \text{ with } t := \frac{\sqrt{1+z}+\sqrt{z}}{\sqrt{1+z}-\sqrt{z}}.$$

(prefactors: $\sqrt{\frac{z}{1+z}} = \frac{2\lambda}{1+\lambda^2}$ and $\sqrt{\frac{1}{1+z}} = \frac{1-\lambda^2}{1+\lambda^2}$, $\quad \frac{\lambda}{1-\lambda^2} \otimes \frac{(1+\lambda)^2}{(1-\lambda)^2}$ and $\frac{\lambda}{1-\lambda^2} \otimes \lambda^2$, : symbols)

Extracting Γ_{cusp} from ϵ^{-2} divergence (of last loop); work in progress for all loops!

$$\int_{\ell_1}\left(\frac{(1\cdot 3)(2\cdot 4)}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_1\cdot 4)}\right)^{\frac{3}{4}} z^p = (1\cdot 3)^{\frac{3}{4}+p}(2\cdot 4)^{\frac{3}{4}-p} \frac{\Gamma(3)}{\Gamma\left(\frac{3}{4}+p\right)^2 \Gamma\left(\frac{3}{4}-p\right)^2}$$

$$\times \int \frac{[d^3 a_1 a_2 a_3 a_4]}{\mathrm{vol}\,(\mathrm{GL}(1))} \int_{\ell_1} \frac{(a_1 a_3)^{-\frac{1}{4}+p}(a_2 a_4)^{-\frac{1}{4}-p}}{(\ell_1\cdot A)^3}$$

$$\mathcal{I}_o(F_{L-1}(z)) = 0.$$

$$\mathcal{I}_e\left(F_1(z)\right) = -\pi\left(\mathcal{I}_e(z^{1/4}) + \mathcal{I}_e(z^{-1/4})\right) = -1.$$

Summary & outlook

Scattering Amplitudes: one of the most exciting frontiers of hep-th
rich structures/applications to formal QFT, gravity, strings, math etc.

New Picture: general massless S-matrix via punctured Riemann spheres; higher-genus for loops. A (weak-weak) QFT/String duality for S-matrix?

New Relations: *gluons, pions, gravitons* … double copy for quantum gravity
Double copy beyond amps: *classical solutions*, *gravity waves*, ……

New Maths: geometries in kinematic space & amps as differential forms
"theory at infinity": geometry/combinatorics → Lorentz inv. + unitarity

"Marble statues in the Forest beyond Quantum Mechanics & Spacetime"
What will we see next?

Thank You!

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