Dispersive determination of fermion masses

Hsiang-nan Li

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2309.15602

All start with D mixing

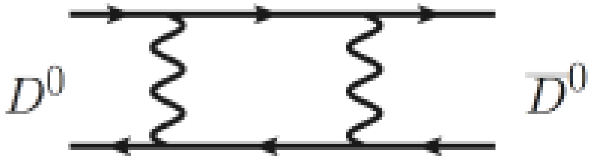
Dec. 2010, HFCPV, Sanya, charm physics

Aug. 2011, IHEP, FAT approach

Mar. 2012, prediction for direct CPV in charm decays

May 2017, 1st attempt to D mixing in FAT

$D^0 - \overline{D}^0$ Mixing



- The time evolution

$$i\frac{\partial}{\partial t}\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix} = \left(\mathbf{M} - \frac{i}{2}\mathbf{\Gamma}\right)\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix}$$

- Mass eigenstates in terms of weak eigenstates

$$|D_{1,2}\rangle = p|D^0\rangle \pm q|\overline{D}^0\rangle$$

- Mass difference and Width difference

$$\boxed{x \equiv \frac{\Delta m}{\Gamma} = \frac{m_1 - m_2}{\Gamma}}$$

$$\boxed{y \equiv \frac{\Delta\Gamma}{2\Gamma} = \frac{\Gamma_1 - \Gamma_2}{2\Gamma}}$$

Notorious puzzle

- Long-standing (2-decade) challenge---how to understand large D meson mixing?

Bobrowski et al, 2009

y_D	no GIM	with GIM
$D = 6, 7$	$2 \cdot 10^{-2}$	$5 \cdot 10^{-7}$
$D = 9$	$5 \cdot 10^{-4}$	?
$D = 12$	$2 \cdot 10^{-5}$	?

- Inclusive approach: all HQE calculations predicted x, y < 10E-5, but data x, y > 10E-3
- Exclusive approach: FAT accounted for only 1/3 of y, even after including VV. Nonresonant contribution? Work for y only

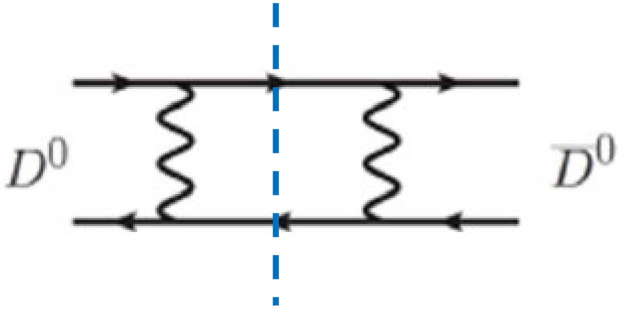
Jiang, Yu, Qin, Li, Lu, 1705.07335

- Nonperturbative dynamics important, but lattice and sum rules apply only to matrix elements of $\Delta C = 2$ operators (bag parameters)
- All approaches led to dead ends
- Came across 0402204 by Falk et al, which used dispersion relation

Dispersion relation---just an identity?

- Example: mixing of D meson with mass squared s

$$M_{12}(s) - \frac{i}{2}\Gamma_{12}(s) = \langle D^0|\mathcal{H}|\bar{D}^0\rangle$$

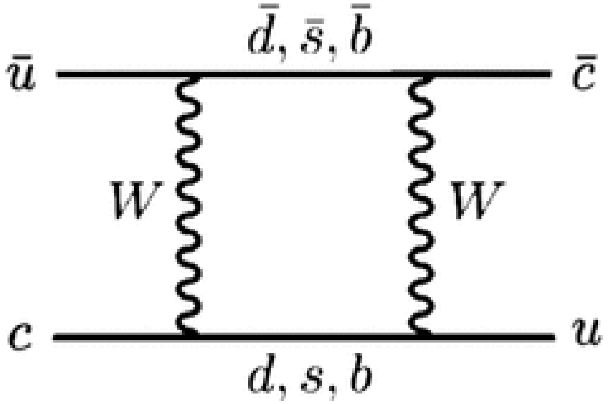


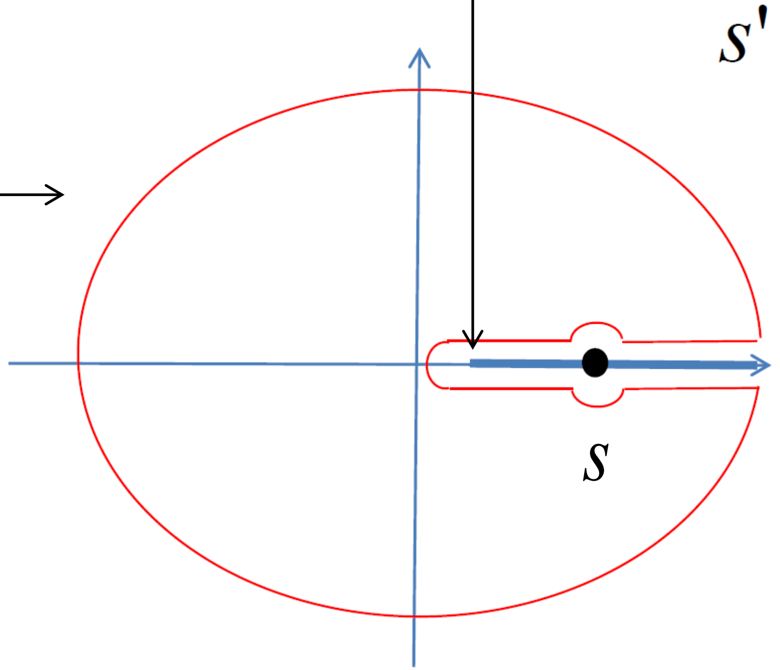
imaginary part

y(s’)

real part $$M_{12}(s) = \frac{P}{2\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Gamma_{12}(s')}{s-s'}$$

x(s)





large circle contribution suppressed

branch cut caused by intermediate real states

Key---how to use dispersion relation?

- Falk et al guessed y(E), then predicted x(mD) from dispersive integral

$$g_F(E) \propto \frac{\Gamma_F(E)}{\Gamma_F(m_D)}$$

chiral limit

$$g_{PP}(E) = \begin{cases} E^3/(m_1^2 m_2) & \text{for } E < m_1\,; \\ E/m_2 & \text{for } m_1 < E < m_2 \\ 1 & \text{for } E > m_2\,, \end{cases}$$

free parameters

heavy-quark limit

guess

- Inconclusive, though varying m1, m2 explained order of magnitude
- Important observation: hadronic thresholds break SU(3)
- First followed their approach, but used data input for y(s)
- Only 4 neutral mesons mix, 4 points not enough for fixing y(s)
- Got idea in summer 2018, when Hiroyuki and Fanrong joined

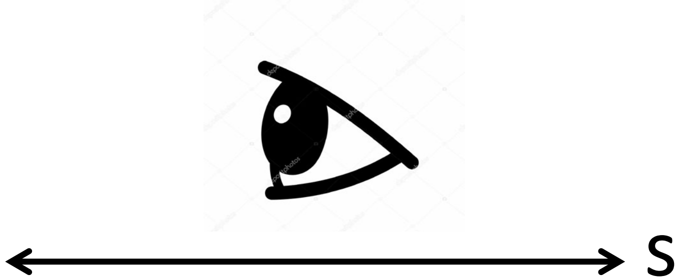
Turn dispersion relation into inverse problem

- $s > \Lambda^2$ known heavy-quark input
- $s < \Lambda^2$ unknown to be solved
- Divide both sides by measured total width

$$\int_{4m_\pi^2}^{\Lambda^2} ds' \frac{y(s')}{s-s'} = \pi x(s) + \int_{\Lambda^2}^{m_W^2/2} ds' \frac{y(s')}{s'-s} \equiv \omega(s)$$

"charge distribution" at low s

"potential" at high s



Ill-posed inverse problem

- Discretize integral equation

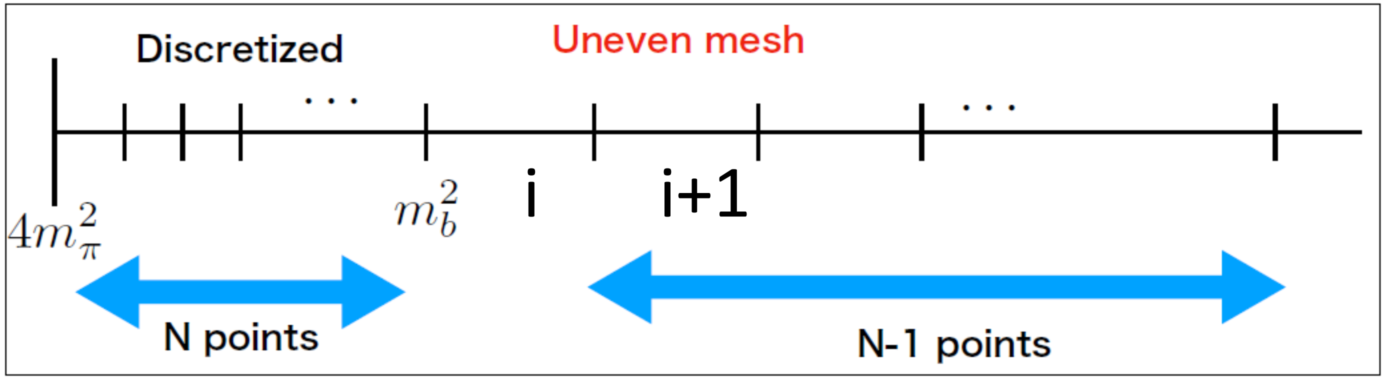
$$\frac{1}{\pi}\sum_{j=2}^{N} M_{ij} y_j = x(s_{N'+i}) + \frac{1}{\pi} P\int_{m_b^2}^{M_W^2} \frac{y(s)ds}{s-s_{N'+i}}, \quad (\text{for } i=1,2,\cdots,N-1)$$

unknowns input

$$M_{ij} = \begin{cases} 1/(i-j), & i \neq j \\ 0, & i = j \end{cases}$$

- initial condition $y_1 = y(4m_\pi^2) = 0$
- Inverse matrix to get $y(m_c^2)$, and then $x(m_c^2)$

rows M_{ij} and $M_{(i+1)j}$ become almost identical, when mesh gets finer



M is singular

One step back---parametrization

~10%, convergent

- Propose

$$y(s) = \frac{Ns[b_0 + b_1(s-m^2) + b_2(s-m^2)^2]}{[(s-m^2)^2 + d^2]^2}$$

pion mass neglected

- Obeys boundary condition y(0)=0, and distributes around scale m in narrow width d

Li, Umeeda, Xu, Yu
2001.04079

- Normalization constant N chosen such that

$Ns/[(s-m^2)^2+d^2]^2 \to \delta(s-m^2)$ as $d \to 0.$

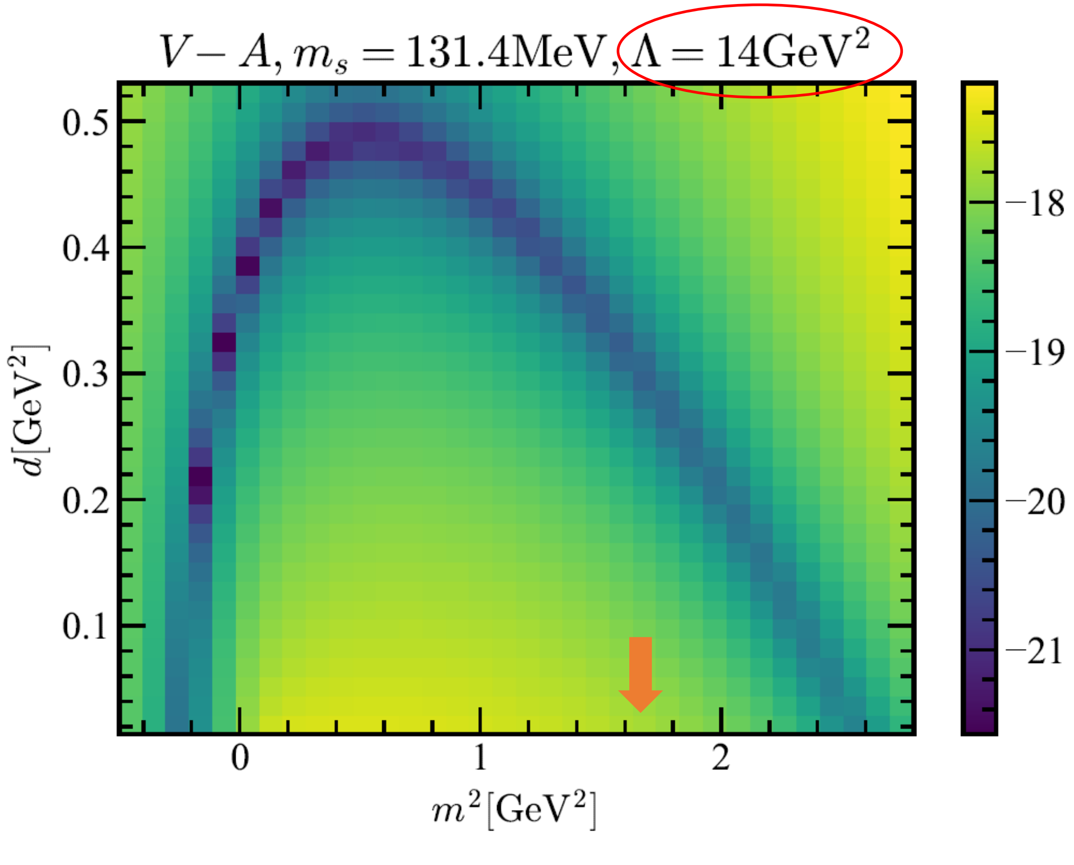
- For given m, d, tune parameters to minimize

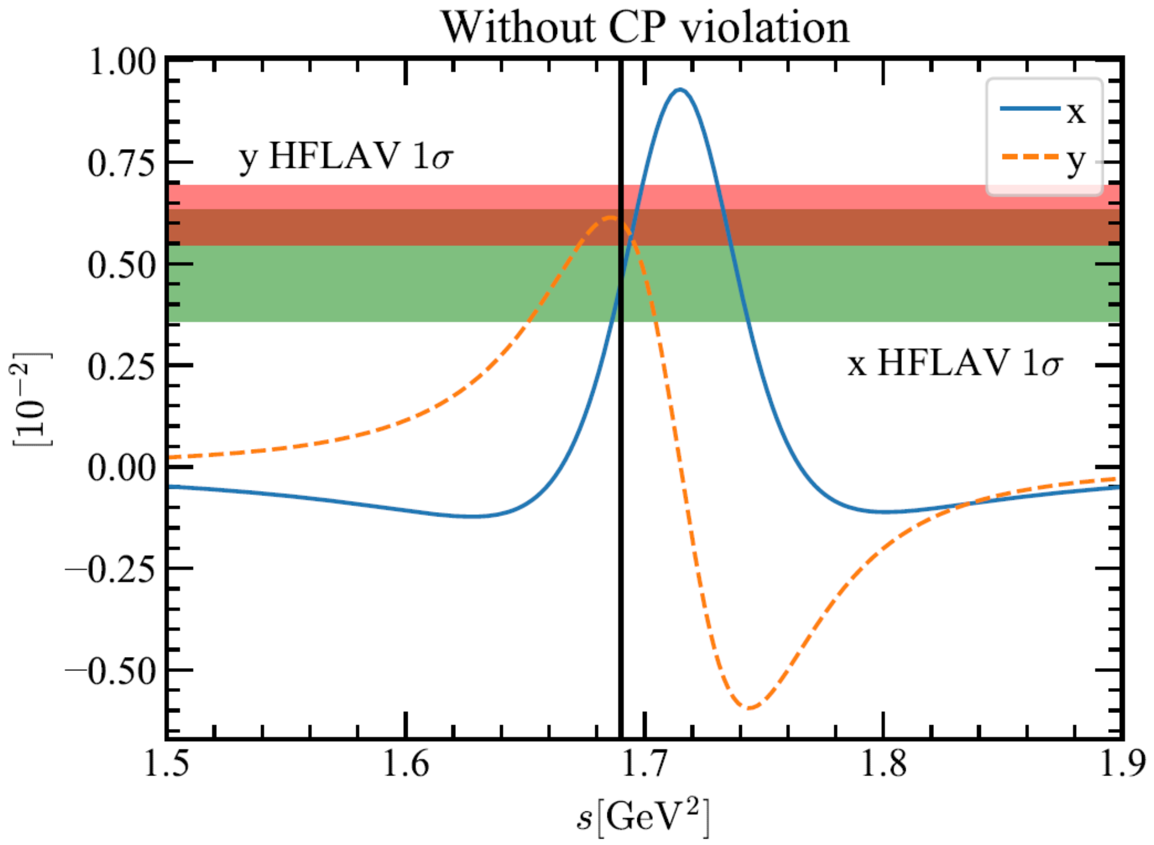
$$\sum_{i=1}^{200} \left| \int_0^{\Lambda} ds' \frac{y(s')}{s_i - s'} - \omega(s_i) \right|^2 \qquad 30\ \mathrm{GeV}^2 < s_i < 250\ \mathrm{GeV}^2$$

goodness of fit

Minima & solution for x(s), y(s)

- Distribution of minima for V-A operator gives large d at charm scale
- Smooth distribution in s; negligible contribution to D mixing





Unsatisfactory points

- Cutoff, or UV truncation scale, dependence may not avoidable
- Hard cutoff makes difficult continuity condition of solution
- Parametrization excludes solutions with more complicated form
- More oscillations should appear at one kaon, di-kaon thresholds
- Numerous minima allowed (uniqueness of solution?)

See 傲昇’s talk

- Fit y(mD) by tuning ms, then predict x(mD), low predictive power
- Sensitivity to ms (ill-posedness not completely removed?)
- Improvement needed, tried maximal entropy method, still not good

Breakthrough

- Parametrization + fitting applied to g-2, rho resonances (sum rules)…
- 2109.04956 marks breakthrough
- M becomes singular, as dimension large; finite for finite dimension
- Decompose solution by orthogonal polynomials?
- Turn unknown solution into unknown coefficients of polynomials
- Dimension of M greatly reduced; hope finite dimension works
- Three types of polynomials according to support (-1,1), (-infinity, infinity), (0,infinity)---choose the last one
- Last one contains Laguerre with weight exp(-y) and Bessel with weight exp(-2/y), which suppress low y (energy) region---choose Laguerre

Strategy

- Typical integral equation

unknown spectral density to be solved

$$\int_0^\infty dy \frac{\rho(y)}{x-y} = \omega(x)$$ ← Input in powers of 1/x

1st kind of Fredholm integral equation

- Suppose $\rho(y)$ decreases quickly enough, expansion into powers of 1/x justified

$$\frac{1}{x-y} = \sum_{m=1}^{N} \frac{y^{m-1}}{x^m} \qquad \omega(x) = \sum_{n=1}^{N} \frac{b_n}{x^n}$$

e.g., true for OPE in sum rules

depend on $\rho(y)$ at $y \to 0$

- Decompose

$$\rho(y) = \sum_{n=1}^{N} a_n y^\alpha e^{-y} L_{n-1}^{(\alpha)}(y)$$ ← Generalized Laguerre polynomials

- Orthogonality

$$\int_0^\infty \underline{y^\alpha e^{-y}} L_m^{(\alpha)}(y) L_n^{(\alpha)}(y) dy = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}$$

Inverse matrix method

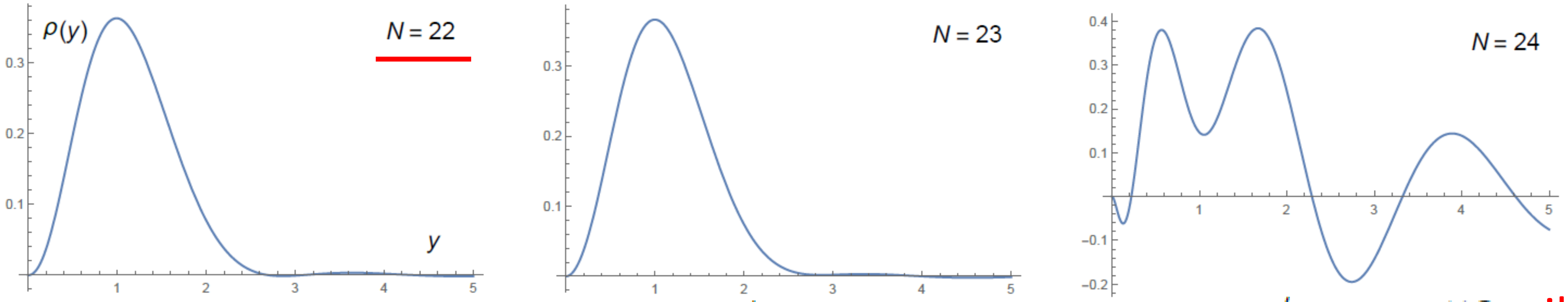
- Equating coefficients of $1/x^n$

matrix unknown $a = (a_1, a_2, \cdots, a_N)$

$$M_{mn} = \int_0^\infty dy y^{m-1+\alpha} e^{-y} L_{n-1}^{(\alpha)}(y) \qquad Ma = b \leftarrow \text{input } b = (b_1, b_2, \cdots, b_N)$$

- Solution $a = M^{-1}b$ (easily done by Mathematica)
- True solution approached by increasing N, but M^{-1} diverges with N
- Neglected polynomials give $1/x^{N+1}$ correction due to orthogonality
- Test mock data generated from $\rho(y) = y^2 e^{-y^2}$ $\quad b_n = \int_0^\infty dy y^{n-1} y^2 e^{-y^2}$

$\alpha = 2$



$a_{22}/a_{21} \approx 1$ $\quad a_{23}/a_{22} \approx 2$ $\quad a_{24}/a_{23} \approx 58$, ill-posed

key formulas

- Decomposition $$\Gamma_{12}(s) = \sum_{i,j=d,s,b} \lambda_i \lambda_j \Gamma_{ij}(s) \qquad \lambda_k \equiv V_{ck} V_{uk}^*,\ k = d, s, b,$$
- Dispersion relation $$M_{ij}(s) = \frac{1}{2\pi} \int_{m_{IJ}}^{R} ds' \frac{\Gamma_{ij}(s')}{s-s'} + \frac{1}{2\pi i} \int_{C_R} ds' \frac{\Pi_{ij}^{\text{box}}(s')}{s-s'}$$

physical threshold

SU(3) breaking

$$m_{\pi\pi} = 4m_\pi^2 \qquad m_{\pi K} = (m_\pi + m_K)^2,\ m_{KK} = 4m_K^2$$

- Box contribution $$M_{ij}^{\text{box}}(s) = \frac{1}{2\pi} \int_{m_{ij}}^{R} ds' \frac{\Gamma_{ij}^{\text{box}}(s')}{s-s'} + \frac{1}{2\pi i} \int_{C_R} ds' \frac{\Pi_{ij}^{\text{box}}(s')}{s-s'}$$

quark-level threshold ~ 0

arbitrary scale introduced

---it is good

- Equality of M(s) at high s
- UV regularization

$$\int_{m_{IJ}}^{R} ds' \frac{\Gamma_{ij}(s')}{s-s'} = \int_{m_{ij}}^{R} ds' \frac{\Gamma_{ij}^{\text{box}}(s')}{s-s'}$$

$$\Delta\Gamma_{ij}(s,\Lambda) = \Gamma_{ij}(s) - \Gamma_{ij}^{\text{box}}(s)\{1 - \exp[-(s - m_{IJ})^2/\Lambda^2]\}$$

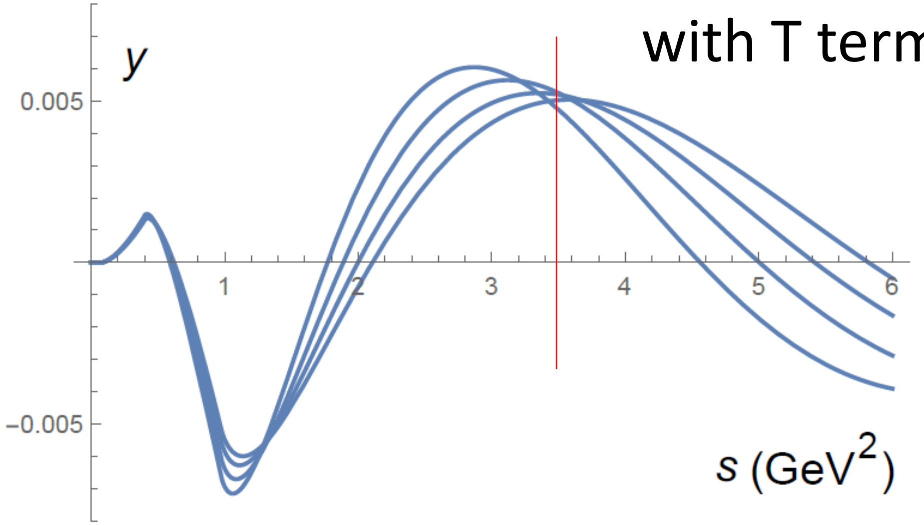
$$\int_{m_{IJ}}^{\infty} ds' \frac{\Delta\Gamma_{ij}(s',\Lambda)}{s-s'} = \int_{m_{IJ}}^{\infty} ds' \frac{\Gamma_{ij}^{\text{box}}(s') \exp[-(s' - m_{IJ})^2/\Lambda^2]}{s-s'} + \int_{m_{ij}}^{m_{IJ}} ds' \frac{\Gamma_{ij}^{\text{box}}(s')}{s-s'}$$

T

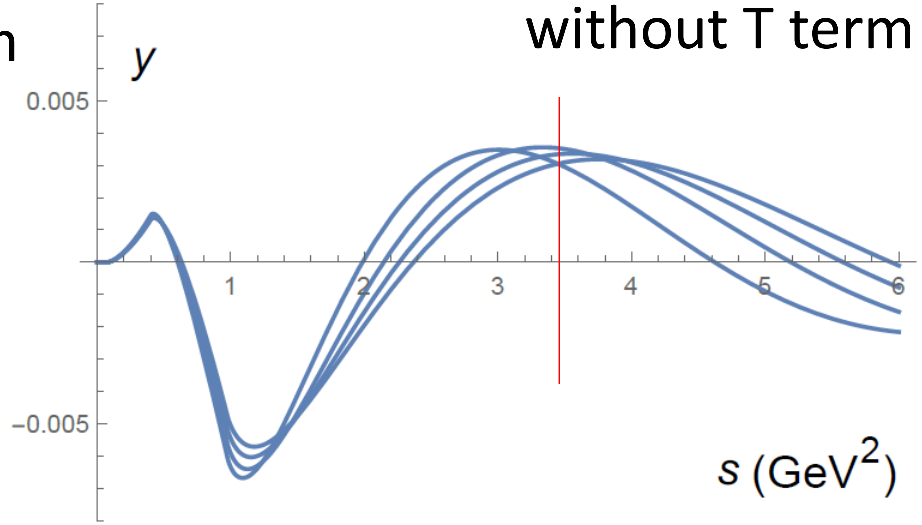
Parameter y

$\Lambda = 4.0\ \mathrm{GeV}^2,\ 4.5\ \mathrm{GeV}^2,\ 5.0\ \mathrm{GeV}^2$ and $5.5\ \mathrm{GeV}^2$ from left to right

with T term



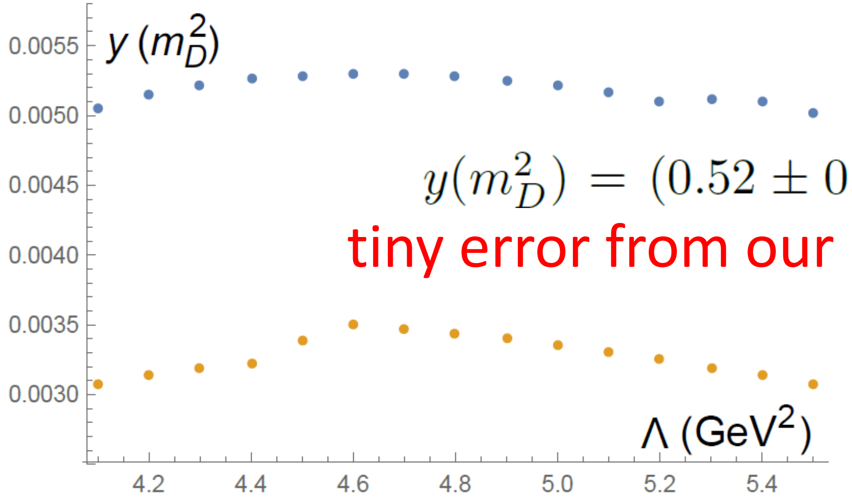
without T term



all curves cross at $s \approx m_D^2$

overlap area spreads

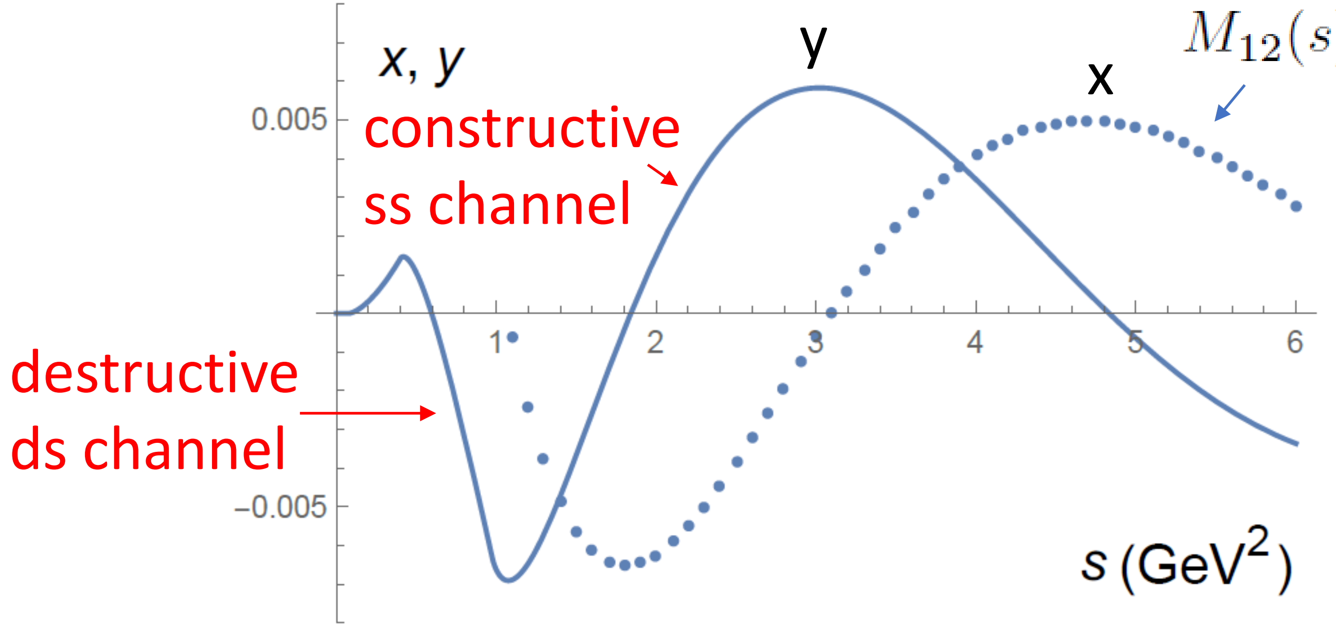
T term stabilizes solutions of $y(m_D^2)$ with respect to variation of Λ



$y(m_D^2) = (0.52 \pm 0.01)\%,$

tiny error from our method

Results of x, y



$$M_{12}(s) = \frac{P}{2\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Gamma_{12}(s')}{s-s'}$$

$$x(m_D^2) = (0.21^{+0.04}_{-0.07})\%, \quad y(m_D^2) = (0.52 \pm 0.03)\%$$

include errors from ms = 0.093 GeV,
10% change induces 0.6% effect

Data (CP conserving)

$$x = (0.44^{+0.13}_{-0.15})\%, \quad y = (0.63 \pm 0.07)\%$$

consistency expected to be improved by including higher-order inputs

Surprise:

all curves for different Λ

cross at D meson mass

Why?

D meson (charm quark) mass emerges…

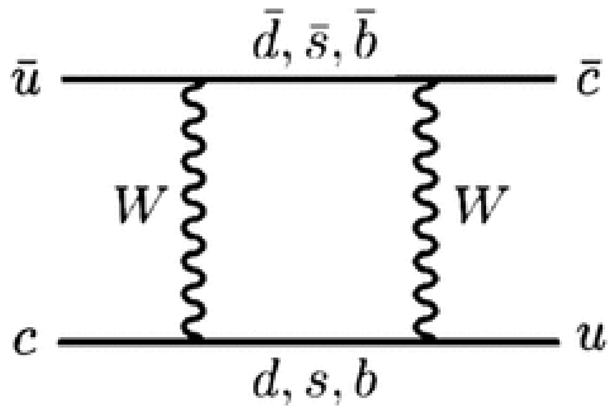
Try dispersive analysis of another physical observable, heavy quark decay widths

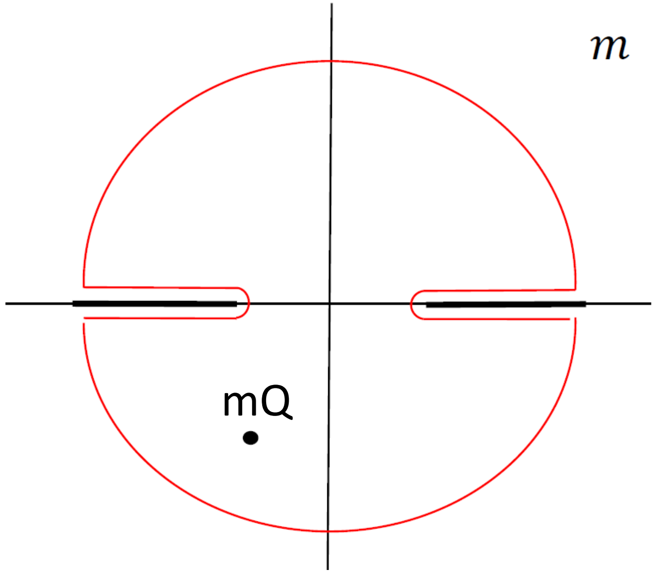
bottom quark mass emerges!

Li, 2302.01761

try top quark mass

Framework





- Contour integration

3 channels

$ij = db,\ sb$ and bb

big circle contributions cancel, because

$\mathrm{Im}\Pi_{ij}(m) \to \mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)$

unknowns to be solved

perturbative inputs from box diagrams

branch cuts along both m > 0, m < 0

$$\int_{M_{ij}^2}^{R^2} \frac{\mathrm{Im}\Pi_{ij}(m)}{m_Q^2 - m^2} dm^2 = \int_{m_{ij}^2}^{R^2} \frac{\mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)}{m_Q^2 - m^2} dm^2$$

due to analyticity

hadronic thresholds

$M_{db} = m_\pi + m_B,\ M_{sb} = m_K + m_B$ and $M_{bb} = 2m_B$

quark-level thresholds

$m_{db} = m_d + m_b,$

$m_{sb} = m_s + m_b$ and $m_{bb} = 2m_b.$

Box diagram inputs

- Box diagrams generate (V-A)(V-A), (S-P)(S-P) structures
- Focus on the former

intermediate quark masses

$$\Gamma_{ij}^{\rm box}(m_Q) \;\propto\; \frac{C_2(m_Q)}{m_Q^4}\,\frac{\sqrt{[m_Q^2-(m_i+m_j)^2][m_Q^2-(m_i-m_j)^2]}}{(m_W^2-m_i^2)(m_W^2-m_j^2)}$$
$$\times\left\{2\left(m_W^4+\frac{m_i^2 m_j^2}{4}\right)[m_Q^2-(m_i+m_j)^2][m_Q^2-(m_i-m_j)^2]\right.$$
$$\left.-3m_W^2 m_Q^2(m_i^2+m_j^2)(m_Q^2-m_i^2-m_j^2)\right\},$$

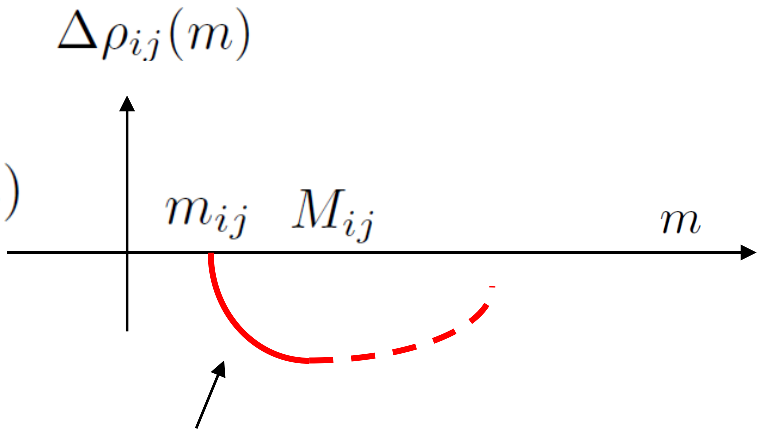
W boson mass

Initial conditions

- Move RHS to LHS, $$\Delta\rho_{ij}(m) \equiv \mathrm{Im}\Pi_{ij}(m) - \mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)$$

extended to infinity

$$\int_{m_{ij}^2}^{\infty} \frac{\Delta\rho_{ij}(m)}{m_Q^2 - m^2} dm^2 = 0$$



initial condition

$-\mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)$

- Threshold behaviors around $m_Q \sim m_{ij}$

$$\Gamma_{db}^{\mathrm{box}}(m_Q) \sim \frac{(m_Q^2 - m_b^2)^3}{m_Q^4},$$

$$\Gamma_{sb}^{\mathrm{box}}(m_Q) \sim \frac{\sqrt{[m_Q^2 - (m_b + m_s)^2][m_Q^2 - (m_b - m_s)^2]}^3}{m_Q^4}$$

$$\Gamma_{bb}^{\mathrm{box}}(m_Q) \sim \frac{\sqrt{m_Q^2 - 4m_b^2}^3}{m_Q}.$$

governed by 1st term in curly brackets

Integrands

- Motivated by threshold behaviors, select integrands

suppress low-m residues like $m = \pm(m_i + m_j)$ relative to $m = \pm m_Q$

$$\mathrm{Im}\Pi_{db}(m) = \frac{m^4\Gamma_{db}(m)}{(m^2 - m_b^2)^2},$$

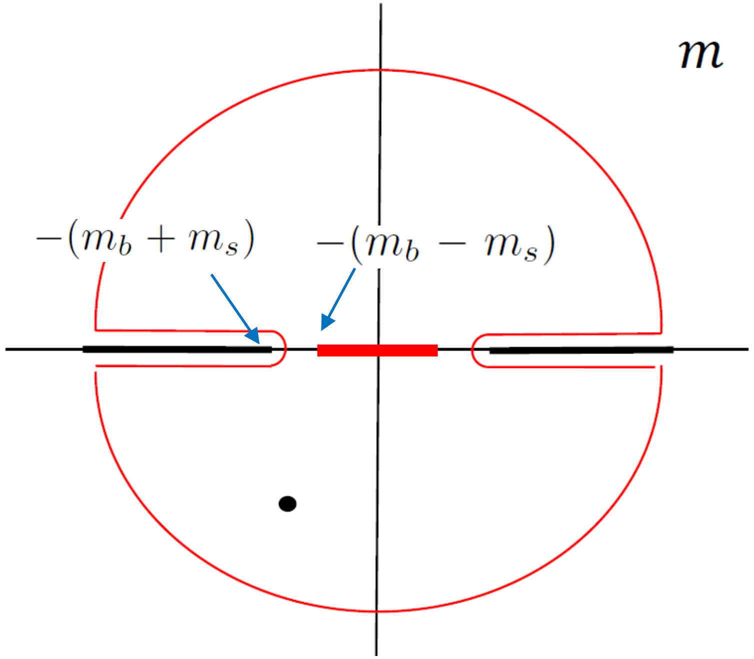
alleviate divergent behaviors in numerators

$$\mathrm{Im}\Pi_{sb}(m) = \frac{m^4\Gamma_{sb}(m)}{[m^2-(m_b+m_s)^2]^2\sqrt{m^2-(m_b-m_s)^2}^3}$$

$$\mathrm{Im}\Pi_{bb}(m) = \frac{m\Gamma_{bb}(m)}{m^2 - 4m_b^2},$$

additional branch cut

Does not contribute



odd power of m due to odd function $\Gamma_{bb}^{\mathrm{box}}(m)$ in m

- Definitions of $\mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)$ are self-evident

Solutions

- General form

$$\Delta\rho_{ij}(m_Q) \approx y_{ij}\left(\omega\sqrt{m_Q^2-(m_i+m_j)^2}\right)^{\alpha_{ij}} J_{\alpha_{ij}}\left(2\omega\sqrt{m_Q^2-(m_i+m_j)^2}\right)$$

arbitrary scale from scaling integration variable m^2

- Insensitivity to ω achieved by

vanish to get roots of m_Q

$$\Delta\rho_{ij}(m_Q) = \Delta\rho_{ij}(m_Q)|_{\omega=\bar{\omega}_{ij}} + \frac{d\Delta\rho_{ij}(m_Q)}{d\omega}\bigg|_{\omega=\bar{\omega}_{ij}}(\omega-\bar{\omega}_{ij}) + \frac{1}{2}\frac{d^2\Delta\rho_{ij}(m_Q)}{d\omega^2}\bigg|_{\omega=\bar{\omega}_{ij}}(\omega-\bar{\omega}_{ij})^2 + \cdots$$

fit to initial conditions
to fix $\bar{\omega}_{ij}$, α_{ij}, y_{ij}

minimal to maximize stability window in ω

Parameter fixing

$m_d = 0 \quad m_s = 0.1\ \mathrm{GeV} \quad m_b = 4.16\ \mathrm{GeV}$

$m_\pi = 0.14\ \mathrm{GeV} \quad m_K = 0.49\ \mathrm{GeV},\ m_B = 5.28\ \mathrm{GeV}$

- Initial conditions around $m_Q \sim m_{ij}$

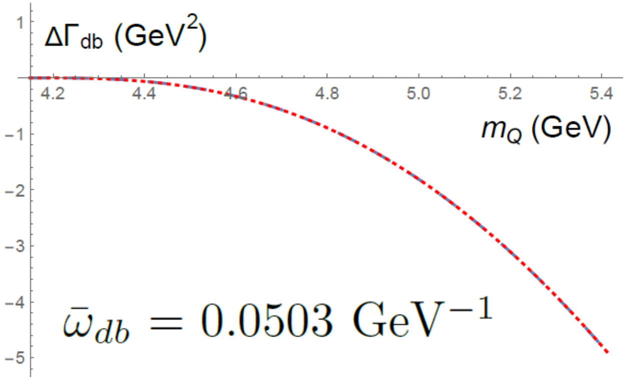
$$\begin{aligned}\Delta\rho_{db}(m_Q) &\sim m_Q^2 - m_b^2, \\ \Delta\rho_{sb}(m_Q) &\sim [m_Q^2 - (m_b + m_s)^2]^{-1/2} \\ \Delta\rho_{bb}(m_Q) &\sim (m_Q^2 - 4m_b^2)^{1/2}.\end{aligned}$$

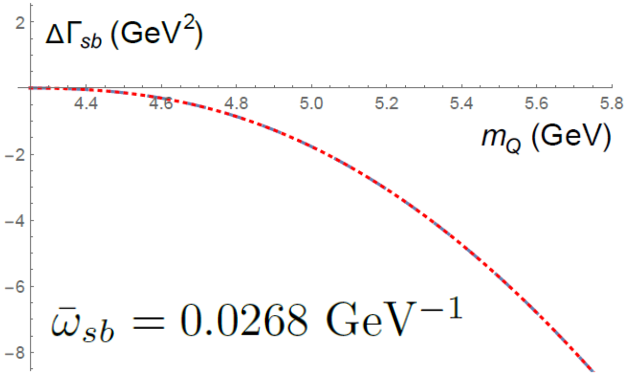
clear why considering complicated integrands: to have simple power of $m_Q^2 - (m_i + m_j)^2$

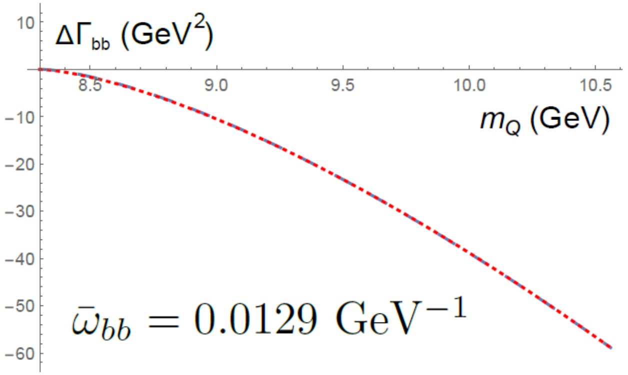
$\Rightarrow \quad \alpha_{db} = 1, \quad \alpha_{sb} = -1/2, \quad \alpha_{bb} = 1/2$

- Boundary conditions $\Delta\rho_{ij}(m_Q)$ set coefficients

$$y_{ij} = -\mathrm{Im}\Pi_{ij}^{\mathrm{box}}(M_{ij})\left[\left(\omega\sqrt{M_{ij}^2 - (m_i + m_j)^2}\right)^{\alpha_{ij}} J_{\alpha_{ij}}\left(2\omega\sqrt{M_{ij}^2 - (m_i + m_j)^2}\right)\right]^{-1}$$





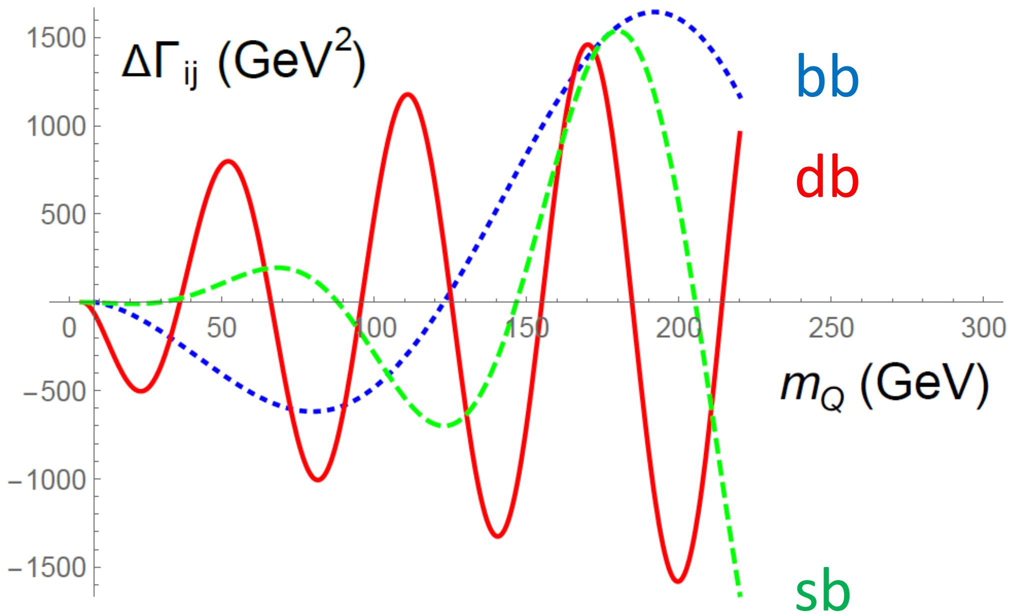


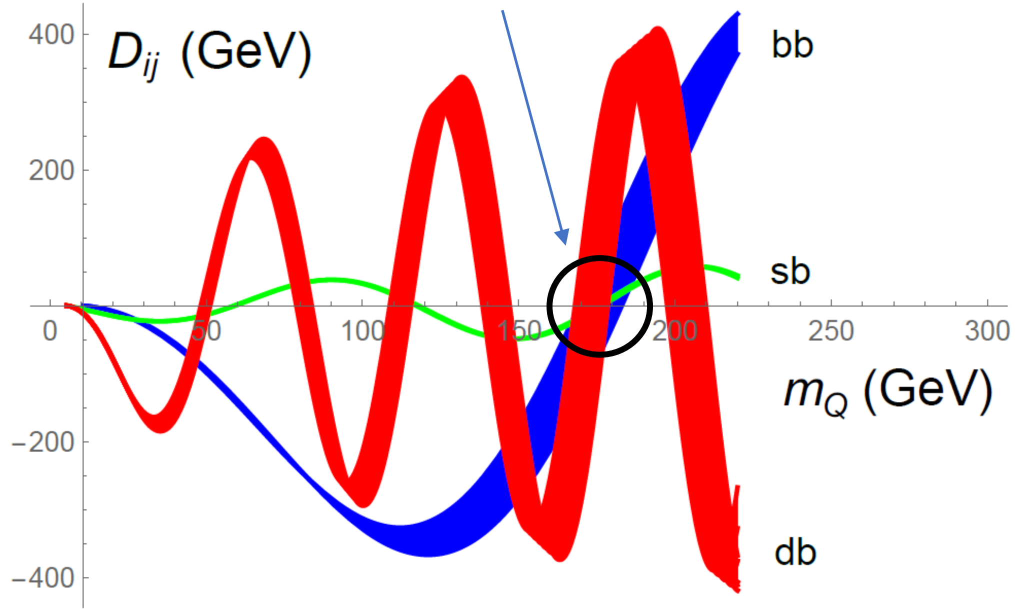
Roots

- Solutions of unknowns

higher roots, larger 2^{nd} derivative

3 derivatives first vanish simultaneously at $m_t = (173 \pm 3)$ GeV





1st peak of bb, 2nd peak of sb, 3rd peak of db overlap around mQ ~ 180 GeV

$$D_{ij}(m_Q) \equiv \frac{d}{d\omega} \frac{J_{\alpha_{ij}}\left(2\omega\sqrt{m_Q^2 - (m_i+m_j)^2}\right)}{J_{\alpha_{ij}}\left(2\omega\sqrt{M_{ij}^2 - (m_i+m_j)^2}\right)}\Bigg|_{\omega=\bar{\omega}_{ij}}$$

uncertainties from $m_b = (4.16 \pm 0.01)$ GeV and different ways of fixing $\bar{\omega}_{ij}$

Conjecture

- Dispersive analyses of heavy quark decay widths, neutral meson mixing, etc. indicated that scalar sector of SM may not be free
- Higgs mass, fermion masses, mixing angles constrained
- Bold conjecture: SM contains only three fundamental (gauge) parameters, and other parameters, governing interplay among various generations of fermions, are fixed by SM dynamics itself
- To maintain simplicity and beauty, natural extension of SM is to introduce sequential fourth generation of fermions, since associated parameters in scalar sector are not free
- Stay tuned…

Back-up slides

Merits of SM4 and experimental exclusion

- condensates of 4th generation quarks and leptons as responsible mechanism of dynamical electroweak symmetry breaking
- 1st-order phase transition for electroweak baryogenesis realized
- provide source of CP violation for baryon asymmetry of the Universe
- But SM4 ruled out by data of Higgs production via gluon fusion and decay into photon pairs
- Will show b’ mass 2.7 TeV and t’ mass 200 TeV, so heavy that bound states formed in Yukawa potential
- These bound states could bypass experimental constraints

b' mass

- Similar box diagrams with ut, ct channels (t does not hadronize)

$$m_{ut} = m_t \ (m_u = 0) \qquad M_{ut} = m_\pi + m_t \qquad m_{ct} = m_c + m_t \qquad M_{ct} = m_D + m_t$$

- Threshold behaviors

$$\Gamma^{\rm box}_{ut}(m_Q) \sim \frac{(m_Q^2 - m_t^2)^2}{m_Q^2},$$

$$\Gamma^{\rm box}_{ct}(m_Q) \sim \frac{m_Q^2 - m_t^2 - m_c^2}{m_Q^2}\sqrt{[m_Q^2 - (m_t + m_c)^2][m_Q^2 - (m_t - m_c)^2]}.$$

governed by 2nd term in curly brackets

- Integrands

$$\mathrm{Im}\Pi_{ut}(m) = \frac{m^2\Gamma_{ut}(m)}{m^2 - m_t^2},$$

$$\mathrm{Im}\Pi_{ct}(m) = \frac{m^2\Gamma_{ct}(m)}{[m^2 - (m_t + m_c)^2](m_Q^2 - m_t^2 - m_c^2)\sqrt{m^2 - (m_t - m_c)^2}}$$

Parameter fixing and roots

$m_D = 1.87\ \text{GeV} \qquad m_t = (173 \pm 3)\ \text{GeV}$

- Initial conditions around $m_Q \sim m_{ij}$

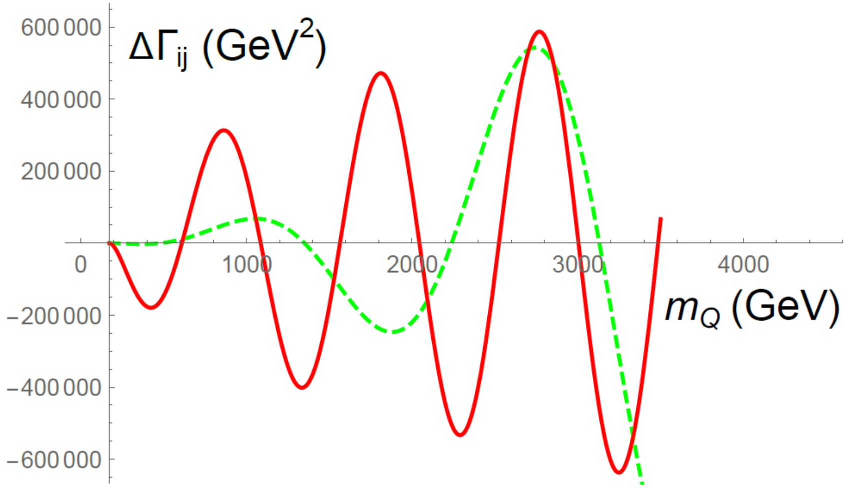
$$m_c(m_t) = m_c(m_c)\left[\frac{\alpha_s(m_t)}{\alpha_s(m_c)}\right]^{4/\beta_0} \approx 0.7\ \text{GeV}$$

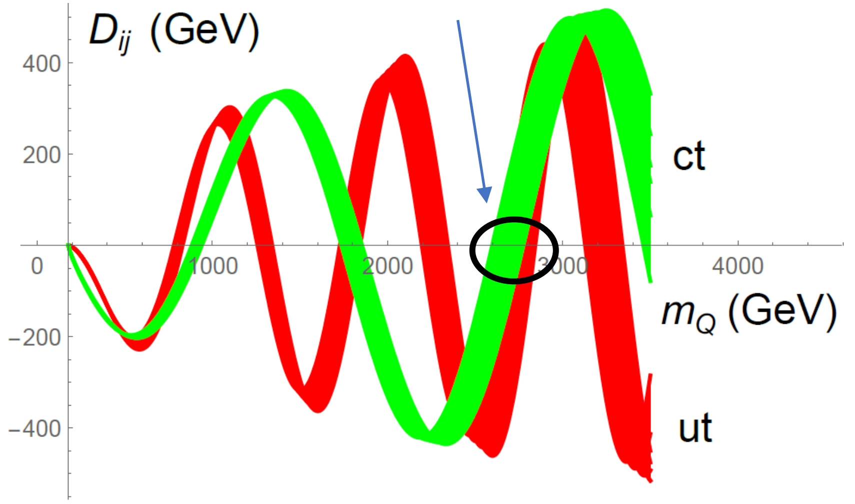
$$\begin{aligned}\Delta\rho_{ut}(m_Q) &\sim m_Q^2 - m_t^2,\\ \Delta\rho_{ct}(m_Q) &\sim [m_Q^2 - (m_t + m_c)^2]^{-1/2}\end{aligned} \quad \Rightarrow \quad \alpha_{ut} = 1, \quad \alpha_{ct} = -1/2.$$

- Same forms of solutions and coefficients
- Fits to initial conditions give $\bar{\omega}_{ut} = 0.00326\ \text{GeV}^{-1}$ and $\bar{\omega}_{ct} = 0.00176\ \text{GeV}^{-1}$

$m_{b'} = (2.7 \pm 0.1)\ \text{TeV}$

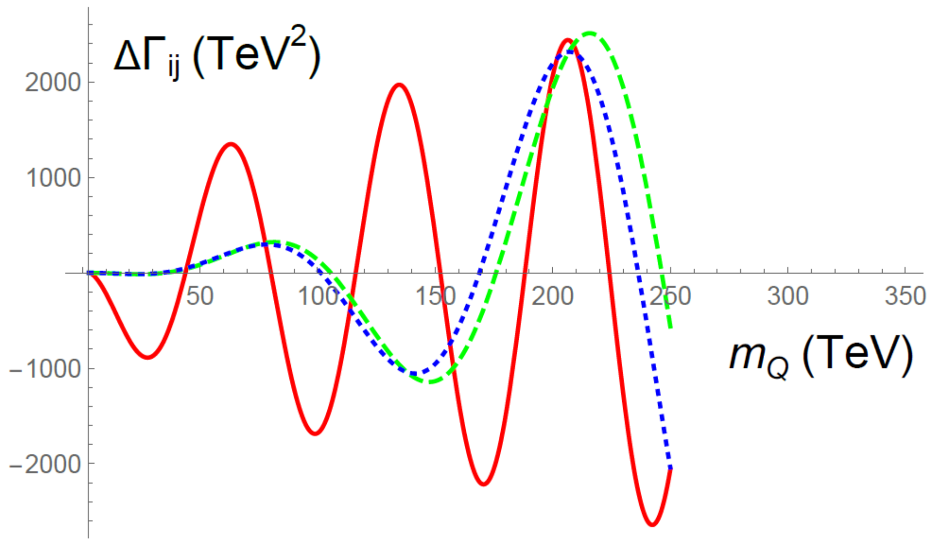
2nd peak of ct, 3rd peak of ut overlap at mQ ~ 2.7 TeV



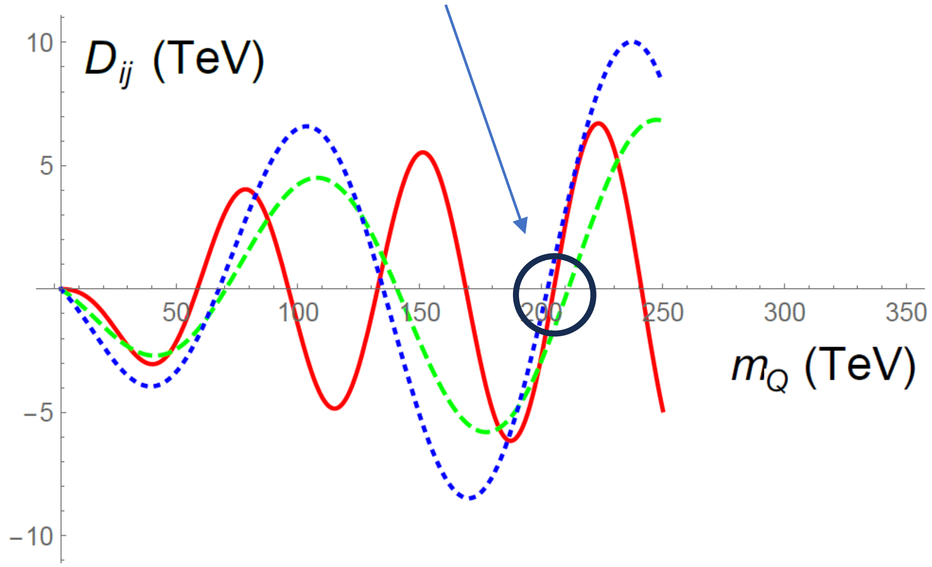


t’ mass

- Similar box diagrams with db’, sb’, bb’ channels
- Same analysis
- sb’, bb’ curves close in shape



$m_{t'} \approx 200$ TeV



$\bar{b}'b'$ bound states

$v = 246\ \mathrm{GeV}$

- As 4th generation quark mass meets criterion $K_Q \doteq m_Q^3/(4\pi v^2 m_H) > 1.68$ bound states formed

Hung, Xiong 2011

- Binding energy for $m_Q^* \approx 1.26\ \mathrm{TeV}$ and $m_H^* \approx 1.45\ \mathrm{TeV}$ at fixed point of RG evolution in SM4 estimated to be -4.9 GeV
- With b' mass 2.7 TeV, $\bar{b}'b'$ bound states formed definitely
- Should analyze gluon fusion involving internal b' in effective theory
- Gluon fusion into S via effective operator $A^\mu A^\nu S$, coupling $\sqrt{s} g_{ggS}$
- Scalar S propagates according to BW factor $1/(s - m_S^2 - i\sqrt{s}\Gamma_S)$
- S transforms into H with magnitude $s g_{SH}$
- Total amplitude

$$\mathcal{M} \sim \frac{\sqrt{s}^3 g_{ggS} g_{SH}}{s - m_S^2 - i\sqrt{s}\Gamma_S}$$

Heavy quarkonia in Yukawa potential

- Yukawa potential $$V(r) = -\alpha_Y \frac{e^{-m_H^* r}}{r}$$ ← b' mass higher than fixed point

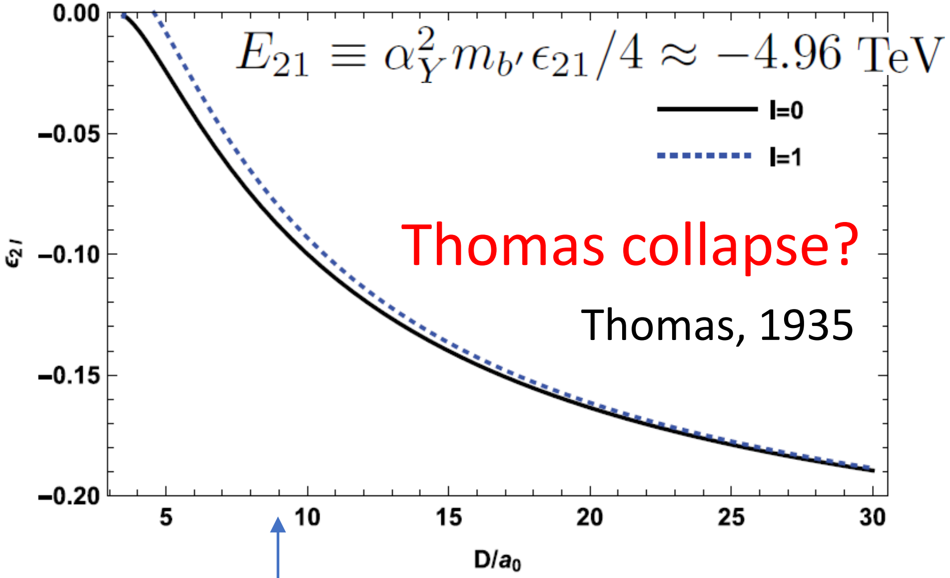
$$\alpha_Y = m_{b'}^2/(4\pi v^2)$$

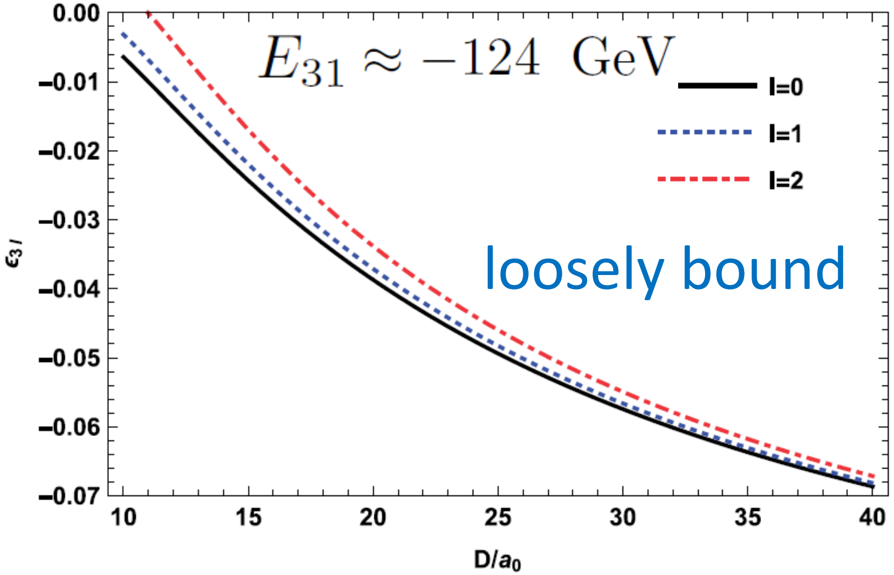
Napsuciale, Rodriguez 2021

pseudoscalar or vector

P-wave scalars

- Only 5 bound states exist $(n, l) = (1, 0),\ (2, 0),\ (2, 1),\ (3, 0)$ and $(3, 1)$





$1/(m_H^* a_0) = 8.9,\ a_0 \equiv 2/(\alpha_Y m_{b'})$ being Bohr radius

Contribution to Higgs production

Lansberg, Pham 2009

- Width approximated by (call for relativistic calculation)

1st derivative of radial wave function at origin

$$\Gamma_S = 48\alpha_S^2(2m_{b'})\frac{|R'_{21}(0)|}{m_S} \approx 570\ \text{GeV} \quad > \quad m_S = 2m_{b'} + E_{21} \approx 440\ \text{GeV}$$

- Imagine fictitious Higgs with $s \approx m_S^2$, matched to fundamental theory

Georgi et al. 1978; Spira et al. 1995

$$\left|\frac{v}{s}\frac{\sqrt{s}^3 g_{ggS}g_{SH}}{s - m_S^2 - i\sqrt{s}\Gamma_S}\right|^2 \approx \left(\frac{v g_{ggS}g_{SH}}{\Gamma_S}\right)^2 \approx \left(\frac{3}{2}\right)^2 \quad \Rightarrow \quad g_{ggS}g_{SH} = (2/3)\Gamma_S/v$$

- Extrapolate to $s = m_H^2$, relative to top-loop contribution in SM

$$\left|\frac{v}{s}\frac{\sqrt{s}^3 g_{ggS}g_{SH}}{s - m_S^2 - i\sqrt{s}\Gamma_S}\right|^2 \approx \left(\frac{2}{3}\frac{m_H\Gamma_S}{m_S^2}\right)^2 \approx 6.2\%$$

down by m_S^{-4}

Contribution to Higgs production

- Contribution of $(n,l) = (3,1)$ $\left(\frac{2}{3}\frac{m_H \Gamma_S}{m_S^2}\right)^2 \approx 4.3 \times 10^{-6}$

$$\Gamma_S = 48\alpha_S^2(2m_{b'})\frac{|R'_{31}(0)|}{m_S} \approx 694 \;\; \mathrm{GeV} \qquad m_S = 2m_{b'} + E_{31} \approx 5.28 \;\mathrm{TeV}$$

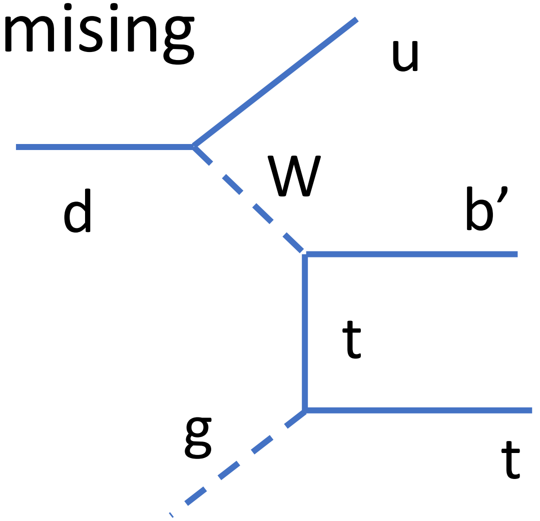
- Relativistic calculation---solving Dirac (not Schrodinger) equation
- Crude approximation, spectrum degenerate in l

Ikhdair, 2012

- Ground state mass 3.23 TeV, n=2 mass 4.45 TeV, n=3 mass < 5.4 TeV
- n=3 state indeed loosely bound
- n=2 state contributes at 10E-3 level, assuming width insensitive to bound state masses
- Conclusion: new scalar contribution to Higgs production negligible

Search modes

- Impossible to detect t' in near future
- Gluon fusion into $\bar{b}'b'$ ground state of mass 3.2 TeV not efficient owing to small gluon PDFs
- Weak boson fusion $qq \to WW, ZZ \to S$ more promising
- For single b' production, consider associated production $dg \to u\bar{t}b'$, power enhanced by one fewer virtual weak boson, but down by gluon PDFs. Similar to vector-like quark search
- Another single b' production $ug \to W^+b'$ down by diminishing 4X4 CKM matrix element $V_{ub'}$



Conclusion

- Dispersion relations physical observables must obey impose stringent constraints on dynamics at various scales
- Analyticity dictates scalar sector that couples generations
- Tested formalism by finding common solution for top mass from 3 channels, highly nontrivial and convincing
- Predicted b’ mass 2.7 TeV and t’ mass 200 TeV
- Bound states formed with huge Yukawa couplings, their contributions to Higgs production via gluon fusion tiny
- Worthwhile to contibue search of b’ quarks and ground state of mass 3.2 TeV

Polynomial expansion

arbitrary scale

- Introduce dimensionless variables, $m_S^2 - 4m_b^2 = u\Lambda$, $m^2 - 4m_b^2 = v\Lambda$

$$\int_0^\infty dv \frac{\Delta\rho(v)}{u-v} = 0$$

$\Delta\rho(v) \to 0$ at large v, because $\mathrm{Im}\Pi(m) \to \mathrm{Im}\Pi^{\mathrm{p}}(m)$

power series in $1/u$ using $1/(u-v) = \sum_{i=1}^\infty v^{i-1}/u^i$

- Start with case of N vanishing coefficients, N large

contained in $L_0^{(\alpha)}(v), L_1^{(\alpha)}(v), \cdots, L_{N-1}^{(\alpha)}(v)$

$$\int_0^\infty dv v^{i-1} \Delta\rho(v) = 0, \quad i = 1, 2, 3 \cdots, N$$

- Imply expansion in generalized Laguerre polynomials because of orthogonality

weight

$$\Delta\rho(v) = \sum_{j=N}^{N'} a_j \underline{v^\alpha e^{-v}} L_j^{(\alpha)}(v), \quad N' > N \qquad \int_0^\infty \underline{y^\alpha e^{-y}} L_m^{(\alpha)}(y) L_n^{(\alpha)}(y) dy = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}$$

fixed by initial condition in principle, needs not be infinite

Solution

- Large j approximation, subject to correction of $1/\sqrt{j}$

$$L_j^{(\alpha)}(v) \approx j^{\alpha/2} v^{-\alpha/2} e^{v/2} J_\alpha(2\sqrt{jv})$$

- Solution

arbitrary degree and scale appear in ratio

$$\Delta\rho(m) \approx \sum_{j=N}^{N'} a_j \sqrt{\frac{j(m^2-4m_b^2)}{\Lambda}}^{\alpha} e^{-(m^2-4m_b^2)/(2\Lambda)} J_\alpha\left(2\sqrt{\frac{j(m^2-4m_b^2)}{\Lambda}}\right)$$

- Scaling variable $\omega \equiv \sqrt{N/\Lambda}$, large N limit $N'/\Lambda = \omega^2 + (N'-N)/N \approx \omega^2$

$$J_\alpha(2\sqrt{j(m^2-4m_b^2)/\Lambda}) \approx J_\alpha(2\omega\sqrt{m^2-4m_b^2}) \qquad e^{-(m^2-4m_b^2)/(2\Lambda)} = e^{-\omega^2(m^2-4m_b^2)/(2N)}$$

≈ 1

$$\Delta\rho(m) \approx y\left(\omega\sqrt{m^2-4m_b^2}\right)^{\alpha} J_\alpha\left(2\omega\sqrt{m^2-4m_b^2}\right)$$

$\sum_{j=N}^{N'} a_j$

solution in terms of single Bessel function