From Pion to DiPion LCDAs
Opportunities and Challenges

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Overview

The emergent phenomena of QCD

Conformal Algebra

LCDAs: Pion and DiPion

- Extraction of a_n^π and m_0^π from $F_\pi(q^2)$
- Double expansion coefficients $B_{nl}^{\parallel(\perp)}(q^2, s)$

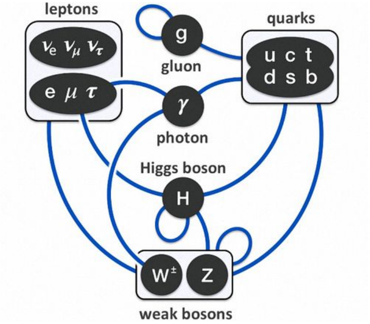
DiPion LCDAs in $D_s^{(*)}$ weak decays

- $D_s \to (f_0 \to) [\pi^+\pi^-]_{\mathrm{S}}\, e^+\nu_e$
- D_s^* weak decay

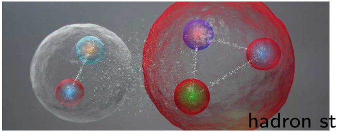
Conclusion

Emergent phenomena of QCD

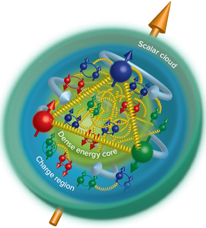
In philosophy, systems theory, science, and art, emergence occurs when an entity is observed to have properties its parts do not have on their own, properties or behaviors which emerge only when the parts interact in a wider whole.   Wiki



Parts



hadron structure



hadron formation

Observed entity

QCD inverse problem

QCD is believed to confine, that is, its physical states are color singlets with internal quark and gluon degrees of freedom

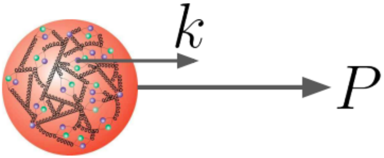
- QCD allow us to study hadron structures in terms of partons
- Factorization theorem to separate the hard partonic physics out of the hadronic physics (soft, nonperturbative objects)
- Define hadron structures by quantum field theories
- Identify theoretical observables in factorizable formulism

$$\frac{d\sigma}{d\Omega} = \int_x^1 \frac{d\zeta}{\zeta} \mathcal{H}(\zeta) f(\frac{x}{\zeta})$$

- Perform global QCD analysis and fit the universal nonperturbative objects
- Complicated inverse problem !
- ▶ Evaluation QCD on a supercomputer (LQCD) is another story

Definition of hadron structures

Parton distribution amplitudes



One dimension PDF

△ $f_i(\zeta) = \int \frac{dz^-}{4\pi} e^{-i\zeta P^+ z^-} \langle N|\bar{\psi}_i(0, z^-, \mathbf{0}_T)\gamma^+\psi_i(0)|N\rangle$

△ $\zeta = \frac{k^+}{P^+}$, the parton momentum fraction

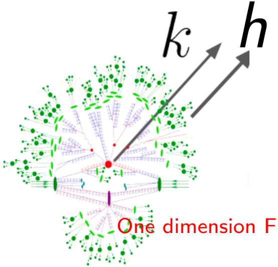
△ $f_i(\zeta) \sim \sum_\alpha \int dk_T^2 \langle N|b^\dagger_{k,\alpha} b_{k,\alpha}(\zeta P^+, k_T, \alpha)|N\rangle$

number operator

△ transversal momentum distributions (TMD) $f(\zeta, k_T)$

△ Generalized parton distributions (GPDs) $f(\zeta, b_T)$

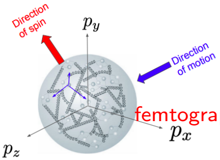
Fragmentation functions



One dimension FF

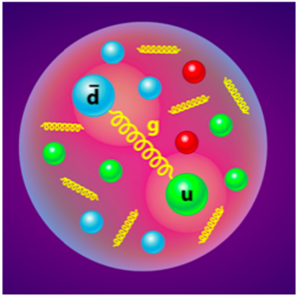
△ $d_{h/i}(\zeta) \propto \sum_X \langle 0|\bar{\psi}_i|P_h, X\rangle\langle P_h, X|\psi_j(0)|0\rangle$

△ X: all states except detected hadron h

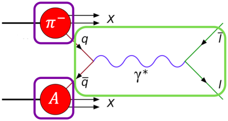


femtography of hadron

Take Pion as the example

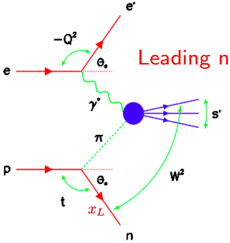


Experiment probe



Drell-Yan

$$\sigma \propto \sum_{i,j} f_i^{\pi}(x_\pi, \mu) \otimes f_j^{A}(x_A, \mu) \otimes C_{i,j}(x_\pi, x_A, Q/\mu)$$



Leading neutrons

- made up of q and $\bar{q}$ constituents
- the Goldstone boson
  spontaneous symmetry breaking of chiral $SU(2)_L \times SU(2)_R$ symmetry
- DA can be extracted from fixed target πA data
- deeply virtuality compton scattering (DVCS)
  deeply virtuality meson production (DVMP)

△ TDIS at 12GeV JLab, leading proton observable, fixed target instead of collider (HERA);
△ EIC, EIcC, great integrated luminosity to reduce the systematics uncertainties;
△ COMPASS++/AMBER give π-induced DY data.

Conformal Algebra

Conformal Algebra

Pion DA can also be studied in the light-cone dominated processes

- DA is expressed by the MEs of gauge invariant non-local operators

$$\langle 0|\bar{u}(x)\Gamma[x,-x]d(-x)|\pi^-(P)\rangle \tag{1}$$

△ Γ is a Dirac matrix, the path-ordered gauge factor along the straight line

$$[x,y] = \mathrm{P}\exp\left[ig\int_0^1 dt(x-y)_\mu A^\mu(tx+\bar{t}y)\right] \tag{2}$$

- introduce light-like vectors p and z such that $P \to p$ in the limit $m_\pi^2 = 0$ and $x \to z$ for $x^2 = 0$

$$P^2 = m_\pi^2,\ p^2 = 0,\ z^2 = 0 \tag{3}$$

△ expansion in power of large momentum transfer is governed by contributions from small transversal separations (derivation from the light-cone) between constituents x^2

$$z_\mu = x_\mu - \frac{P_\mu}{m_\rho^2}\left[xP - \sqrt{(xP)^2 - x^2 m_\pi^2}\right] = x_\mu\left[1 - \frac{x^2 m_\pi^2}{4(z\cdot p)^2} - \frac{p_\mu}{2}\frac{x^2}{z\cdot p} + \mathcal{O}(x^4)\right]$$

$$p_\mu = P_\mu - \frac{z_\mu}{2}\frac{m_\pi^2}{p\cdot z} \quad \Rightarrow \quad z\cdot P = z\cdot p = \left[(xP)^2 - x^2 m_\pi^2\right]^{1/2} \tag{4}$$

Conformal Algebra

- also need the projector onto the directions orthogonal to p and z

$$g_{\mu\nu}^{\perp} = g_{\mu\nu} - \frac{1}{p \cdot z}\left(p_\mu z_\nu + p_\nu z_\mu\right) \tag{5}$$

  and the notations for an arbitrary Lorentz vector a_μ and b_μ

$$a_z \equiv a_\mu z^\mu, \quad a_p \equiv a_\mu p^\mu, \quad b_{\mu z} \equiv b_{\mu\nu} z^\nu, \quad \cdots \tag{6}$$

- introduce LCDAs by the MEs of non-local operators on the light-cone

$$\langle 0|\bar{u}(z)\Gamma[z,-z]d(-z)|\pi^-(P)\rangle \propto \phi_t(u,\mu) \tag{7}$$

- consider the light-cone expansion up to (power) twist 3 of lowest Fock wave function, $\zeta = 2u - 1$, $m_0^\pi = m_\pi^2/(m_0 + m_d)$

$$\begin{aligned}
\langle 0|\bar{u}(z)\gamma_z\gamma_5 d(-z)|\pi^-(P)\rangle &= if_\pi p_z \int_0^1 du\, e^{i\zeta p\cdot z}\phi(u,\mu) \\
\langle 0|\bar{u}(z)i\gamma_5 d(-z)|\pi^-(P)\rangle &= f_\pi m_0^\pi \int_0^1 du\, e^{i\zeta p\cdot z}\phi^p(u,\mu) \\
\langle 0|\bar{u}(z)i\sigma_{\mu\nu}\gamma_5 d(-z)|\pi^-(P)\rangle &= -\frac{if_\pi m_0^\pi}{3}\left(p_\mu z_\nu - p_\nu z_\mu\right)\int_0^1 du\, e^{i\zeta p\cdot z}\phi^\sigma(u,\mu)
\end{aligned} \tag{8}$$

Conformal Algebra

- LCDAs $\phi s(u, \mu)$ are dimensionless functions of u
- describe the probability amplitudes to find the π in a state with minimal number of constitutes and have small transversal separation of order $1/\mu$

  △ the nonlocal operators on the lhs are renormalized at scale μ with the factor $Z_2(\mu)$

$$\phi_2(u, \mu) = Z_2(\mu) \int^{|k_\perp| < \mu} d^2 k_\perp \, \phi_{\mathrm{BS}}(u, k_\perp) \tag{9}$$

- decay constant is defined by the local matrix element

$$\langle 0 | \bar{u}(0) \gamma_z \gamma_5 d(0) | \pi^-(P) \rangle = i f_\pi p_\mu \tag{10}$$

- normalization follows from the requirement that the local limit $z \to 0$ of definition in Eq. (13) reproduce the above definition, $\Phi = \{\phi, \phi^p, \phi^\sigma\}$

$$\int_0^1 du \, \phi(u) = 1 \tag{11}$$

Conformal Algebra

- the (power) twist classification in the infinite momentum frame

  [Perturbative Chromodynamics, ed. A.H. Muller (World Scientific, Singapore, 1989)]

  △ π is moving in the positive $\hat{e}_3$ direction $\Rightarrow$ p^+ and z^- are the only non-zero component

  △ $p^+ \sim Q \to \infty$ with a fixed $p \cdot x \sim 1 \Rightarrow z^- \sim 1/Q$

- the power counting of all terms in the RHS of Eqs. (8)

Twist	2, $\mathcal{O}(Q)$	3, $\mathcal{O}(1)$
	ϕ	ϕ^p,ϕ^σ

- the definition of twist based on the power counting in the infinite momentum frame is convenient

  △ directly related to the power of $1/Q$ △ the corresponding distributions appear in the physical scattering amplitudes △ is frequently employed in hard scattering processes

- the definition is not Lorentz invariant and does not match the conventional definition of twist as "dimension minus spins"

  ie., twist 3 DAs $\phi_\pi^{p,\sigma}$ defined by power counting in fact relates to the three-particle DA $\psi_{3\pi}$ by EOM

  ie., $g_{\rho,\perp}^{(v,a)}$ ($h_{\rho,\parallel}^{(t,s)}$) have contributions from $\phi_{\rho,\parallel}$ ($\phi_{\rho,\perp}$), $\Psi_{3\rho}$ and quark mass terms

Conformal Algebra

- a mathematically similar approach to twist counting is based on the light-cone quantization formalism [S.J.Brodsky and et.al, Phys.Rep.301(1998)299]
- quark fields are decomposed into "good" and 'bad' components

$$\psi = \psi_+ + \psi_-, \quad \psi_+ = \frac{1}{2}\frac{\gamma_\nu p^\nu}{p\cdot z}\gamma_\mu z^\mu \psi, \quad \psi_- = \frac{1}{2}\gamma_\mu z^\mu \frac{\gamma_\nu p^\nu}{p\cdot z}\psi \qquad (12)$$

- a "bad" component introduces one unit of twist, for a $\bar{u}d$ operator

Twist	2	3	4
	$\bar{u}_+ d_+$	$\bar{u}_+ d_-, \bar{u}_- d_+$	$\bar{u}_- d_-$

- ψ_+ represents an independent dof corresponding to the particle content of the Fock state
- ψ_- are not dynamically independent
- only twist 2 DA correspond to the valence $q\bar{q}$ component

Conformal Algebra

- three sources of the power suppressed contributions to exclusive processes in QCD
  - ‡ bad component (wrong spin projection) in the wave function
  - ‡ transversal motion of q $(\bar{q})$ in the leading twist components
    △ given by integrals as in Eq. (9) with additional $k_\perp^2$
  - ‡ higher Fock states with additional gluons and/or $q\bar{q}$ pairs

- define the LCDAs with the Lorentz and gauge invariant ME in Eq. (1)
  △ even it is perhaps less intuitive than the infinite momentum frame formulation in Eq. (7)

$$
\begin{aligned}
\langle 0|\bar{u}(x)\gamma_\mu\gamma_5 d(-x)|\pi^-(P)\rangle &= f_\pi \int_0^1 du\, e^{i\zeta P\cdot x}\left[iP_\mu\left(\phi(u)+\frac{x^2}{4}g_1(u)\right)+\left(x_\mu-\frac{x^2 P_\mu}{2P\cdot x}\right)g_2(u)\right]\\
\langle 0|\bar{u}(x)i\gamma_5 d(-x)|\pi^-(P)\rangle &= f_\pi m_0^\pi \int_0^1 du\, e^{i\zeta P\cdot x}\phi^p(u)\\
\langle 0|\bar{u}(x)i\sigma_{\mu\nu}\gamma_5 d(-x)|\pi^-(P)\rangle &= -\frac{if_\pi m_0^\pi}{3}\left(P_\mu x_\nu - P_\nu x_\mu\right)\int_0^1 du\, e^{i\zeta P\cdot x}\phi^\sigma(u) \qquad (13)
\end{aligned}
$$

Conformal Algebra

- LCDAs defined in the three-particle wave functions

$$\begin{aligned}
\langle 0|\bar{u}(x)\sigma_{\mu\nu}\gamma_5 gG_{\alpha\beta}(vx)d(-x)|\pi^-(P)\rangle &= if_{3\pi}\left[P_\alpha\left(P_\mu\delta_{\nu\beta}-P_\nu\delta_{\mu\beta}\right)-(\alpha\leftrightarrow\beta)\right] \\
&\quad\cdot\int\mathcal{D}\alpha_i e^{\alpha_2-\alpha_1+v\alpha_3}\psi_{3\pi}(\alpha_i) \\
\langle 0|\bar{u}(x)\gamma_\mu\gamma_5 gG_{\alpha\beta}(vx)d(-x)|\pi^-(P)\rangle &= f_\pi\int\mathcal{D}\alpha_i e^{\alpha_2-\alpha_1+v\alpha_3}\left\{P_\mu\left(P_\alpha x_\beta-P_\beta x_\alpha\right)\frac{\Phi_\parallel(\alpha_i)}{P\cdot x}\right. \\
&\quad\left.+\left[P_\beta\left(\delta_{\beta\mu}-\frac{x_\alpha P_\mu}{P\cdot x}\right)-\alpha\leftrightarrow\beta\right]\Phi_\perp(\alpha_i)\right\} \\
\langle 0|\bar{u}(x)\gamma_\mu ig\tilde{G}_{\alpha\beta}(vx)d(-x)|\pi^-(P)\rangle &= f_\pi\int\mathcal{D}\alpha_i e^{\alpha_2-\alpha_1+v\alpha_3}\left\{P_\mu\left(P_\alpha x_\beta-P_\beta x_\alpha\right)\frac{\Psi_\parallel(\alpha_i)}{P\cdot x}\right. \\
&\quad\left.+\left[P_\beta\left(\delta_{\beta\mu}-\frac{x_\alpha P_\mu}{P\cdot x}\right)-\alpha\leftrightarrow\beta\right]\Psi_\perp(\alpha_i)\right\} \qquad (14)
\end{aligned}$$

- gluon field strength tensor $G_{\mu\nu}=G^a_{\mu\nu}\frac{\lambda^a}{2}=\partial_\mu A_\nu-\partial_\nu A_\mu$
- dual gluon field strength tensor $\tilde{G}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$

Conformal Algebra

Conformal spin and collinear twist definition

- A convenient tool to study DAs is provided by conformal expansion
- the underlying idea of conformal expansion of LCDAs is similar to partial-wave expansion of wave function in quantum mechanism
- invariance of massless QCD under conformal trans. *VS* rotation symmetry
- longitudinal $\otimes$ transversal dofs *VS* angular $\otimes$ radial dofs for spherically symmetry potential
- the transversal-momentum dependence (scale dependence of the relevant operators) is governed by the RGE
- the longitudinal-momentum dependence (orthogonal polynomials) is described in terms of irreducible representations of the corresponding symmetry group collinear subgroup of conformal group $SL(2,R) \cong SU(1,1) \cong SO(2,1)$

Conformal Algebra

- Conformal algebra $SU(2,2) \cong O(4,2)$ is obtained by adding the generator D of dilatations and K_μ of the special conformal trans. to the Poincaré group

  △ the conformal algebra has 15 generators: translations $\mathbf{P}_\mu$, Lorentz rotation $\mathbf{M}_{\mu\nu}$, $\mathbf{D}$ and $\mathbf{K}_\mu$

  △ 10-parameter Lie algebra of the Poincaré group [V.M.Braun and et.al., PPNP51(2003) 311]

- examples: △ dilatation $x^\mu \to x'^\mu = \lambda x^\mu$ and inversion $x^\mu \to x'^\mu = \frac{x^\mu}{x^2}$

  △ sequential inversion $x^\mu \to x'^\mu = \frac{x^\mu + a^\mu x^2}{1+2a\cdot x + a^2 x^2}$

- A coordinate transformations that conserve the interval $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$ and only changes the scale of the metric $g'_{\mu\nu}(x') = \omega(x) g_{\mu\nu}(x)$ and, consequently, preserves the angles and leave the light-cone invariant.

- the commutation relations of the conformal algebra

$$
\begin{aligned}
&i[\mathbf{P}_\mu, \mathbf{P}_\nu] = 0, \quad i[\mathbf{M}_{\alpha\beta}, \mathbf{P}_\mu] = g_{\alpha\mu}\mathbf{P}_\beta - g_{\beta\mu}\mathbf{P}_\alpha, \\
&i[\mathbf{M}_{\alpha\beta}, \mathbf{M}_{\mu\nu}] = g_{\alpha\mu}\mathbf{M}_{\beta\nu} + g_{\beta\nu}\mathbf{M}_{\alpha\mu} - [\alpha \leftrightarrow \beta], \\
&i[\mathbf{D}, \mathbf{P}_\mu] = \mathbf{P}_\mu, \quad i[\mathbf{D}, \mathbf{K}_\mu] = -\mathbf{K}_\mu, \quad i[\mathbf{M}_{\alpha\beta}, \mathbf{K}_\mu] = g_{\alpha\mu}\mathbf{K}_\beta - g_{\beta\mu}\mathbf{K}_\alpha \\
&i[\mathbf{P}_\mu, \mathbf{K}_\nu] = -2g_{\mu\nu}\mathbf{D} + 2\mathbf{M}_{\mu\nu}, \quad i[\mathbf{D}, \mathbf{M}_{\mu\nu}] = i[\mathbf{K}_\mu, \mathbf{K}_\nu] = 0
\end{aligned}
\tag{15}
$$

Conformal Algebra

- act the generators on a fundamental field $\Phi(x)$ with an arbitrary spin

$$\delta_P^\mu \Phi(x) \equiv i[\mathbf{P}^\mu, \Phi(x)] = \partial^\mu \Phi(x), \quad \delta_D \Phi(x) \equiv i[\mathbf{D}, \Phi(x)] = (x\cdot\partial + l)\Phi(x),$$
$$\delta_M^{\mu\nu} \Phi(x) \equiv i[\mathbf{M}^{\mu\nu}, \Phi(x)] = (x^\mu \partial^\nu - x^\nu \partial^\mu - \Sigma^{\mu\nu})\Phi(x),$$
$$\delta_K^\mu \Phi(x) \equiv i[\mathbf{K}^\mu, \Phi(x)] = (2x^\mu x\cdot\partial + x^2\partial^\mu + 2lx^\mu - 2x_\nu \Sigma^{\mu\nu})\Phi(x) \qquad (16)$$

△ the generator of spin rotations:

$$\Sigma^{\mu\nu}\phi = 0, \quad \Sigma^{\mu\nu}\psi = \frac{\sigma^{\mu\nu}}{2}\psi, \quad \Sigma^{\mu\nu}A^\alpha = g^{\nu\alpha}A^\mu - [\mu \leftrightarrow \nu] \qquad (17)$$

△ l is the scaling dimension which specifies the field transformation under the dilatations

△ $l = l^{\mathrm{can}}$ in the free theory (classical level) $\Leftarrow$ the action of the theory is dimensionless

△ $l \neq l^{\mathrm{can}}$ in the quantum theory, called the anomalous dimension

- an ultra-relativistic particle (q/g) propagates close to the light-cone
- introduce the projections on the two independent light-like vectors $n_\mu, \bar{n}_\mu$

$$A_+ = A_\mu n^\mu, \quad A_- = A_\mu \bar{n}^\mu, \quad A^2 = 2A_+A_- - A_\perp^2$$
$$g_\perp^{\mu\mu} = g^{\mu\nu} - n^\mu \bar{n}^\nu - n^\nu \bar{n}^\mu$$

- consider the special conformal transformation with $a_\mu = a\bar{n}_\mu$

△ then $x_- \to x'_- = \frac{x_-}{1+2ax_-}$, which maps the light-ray in the x_- direction into itself

△ together with the translations and dilatations along the same direction ($x_- \to x_- + c$, $x_- \to \lambda x_-$),

form a collinear subgroup $SL(2, R)$ of the full conformal group

Conformal Algebra

- hadrons states are replaced by a bunch of collinear partons ($\bar{n}_\mu$), only to consider the quantum field "living" on the light-ray $\Phi(x) \to \Phi(\alpha n) \equiv \Phi(\alpha)$
- chose primary quantum field $\Phi(\alpha)$ as the eigenstate of the spin operator on the "+" direction $\quad \Sigma_{+-}\Phi(\alpha x) = s\Phi(\alpha x) \quad (18)$
- four-dimension conformal group is reduced to the collinear subgroup that generates projective transformation on the line

$$\alpha \to \alpha' = \frac{a\alpha + b}{c\alpha + d}, \quad ad - bc = 1$$

$$\Phi(\alpha) \to \Phi'(\alpha) = (c\alpha + d)^{-2j}\Phi\left(\frac{a\alpha + b}{c\alpha + d}\right), \quad j = \frac{l + s}{2} \tag{19}$$

- this trans. are generated by the four generators $\mathbf{P}_+$, $\mathbf{M}_{-+}$, $\mathbf{D}$, and $\mathbf{K}_-$, which form a collinear subalgebra of the conformal algebra $SL(2, R)$
- introduce the linear combinations

$$\mathbf{L}_+ = \mathbf{L}_1 + i\mathbf{L}_2 = -i\mathbf{P}_+, \quad \mathbf{L}_- = \mathbf{L}_1 - i\mathbf{L}_2 = i/2\mathbf{K}_-,$$
$$\mathbf{L}_0 = i/2[\mathbf{D} + \mathbf{M}_{-+}], \quad \mathbf{E} = i/2(\mathbf{D} - \mathbf{M}_{-+}) \quad \Downarrow$$

- the commutation relations between the four generators

$$[\mathbf{L}_0, \mathbf{L}_\mp] = \mp\mathbf{L}_\mp, \quad [\mathbf{L}_-, \mathbf{L}_+] = -2\mathbf{L}_0 \tag{20}$$

Conformal Algebra

- act they on the quantum fields (be traded for the algebra of differential operators $\partial_\alpha = \frac{d}{d\alpha}$)

$$\begin{aligned}
[\mathbf{L}_+, \Phi(\alpha)] &= -\partial_\alpha \Phi(\alpha) = L_+ \Phi(\alpha), \\
[\mathbf{L}_-, \Phi(\alpha)] &= (\alpha^2 \partial_\alpha + 2j\alpha)\Phi(\alpha) = L_- \Phi(\alpha), \\
[\mathbf{L}_0, \Phi(\alpha)] &= (\alpha \partial_\alpha + j)\Phi(\alpha) = L_0 \Phi(\alpha)
\end{aligned} \tag{21}$$

satisfying the familar $SL(2)$ commutation relations

$$[L_0, L_\mp] = \pm L_\mp, \quad [L_-, L_+] = 2L_0 \tag{22}$$

- the remaining generator **E** counts the collinear twist of the field Φ

$$[\mathbf{E}, \Phi(\alpha)] = \frac{l-s}{2}\Phi(\alpha) \tag{23}$$

- **E** commutes with all $\mathbf{L}_i \Rightarrow$ is not relevant for most of discussions
- the collinear twist (dimension - spin projection on the plus direction) is different from the geometric twist (dimension - spin) in full conformal algebra respecting full Lorentz symmetry
- **DAs of definite collinear twist are well understood by the classification in terms of geometric twist**

Conformal Algebra

- field operator Φ with fixed spin projection s on the light-cone is an eigenstate of the quadratic operator $L^2 = L_0^2 + L_1^2 + L_2^2 = L_0^2 - L_0 + L_+ L_-$, $\quad [L^2, L_i] = 0$

$$\sum_{i=0,1,2} [\mathbf{L}_i, [\mathbf{L}_i], \Phi(\alpha)] = j(j-1)\Phi(\alpha) = L^2\Phi(\alpha) \tag{24}$$

- $\Phi(\alpha)$ is transformed under the projective trans. according to the $SL(2,R)$ group representation specified by the conformal spin $j = \frac{l+s}{2}$

- there is another collinear subgroup $\simeq SL(2,C)$ corresponding to trans. of the two-dimensional transverse plane $x_\perp^\mu$

‡ involves six generators $\mathbf{P}_\perp^\mu, \mathbf{M}_\perp^{\mu\nu}, \mathbf{D}$ and $\mathbf{K}_\perp^\mu$

‡ introduce complex coordinate $z = x_1 + ix_2$ and $\bar{z} = 1 - z$

‡ the group trans. in the transversal plane are

$$z \to \frac{az+b}{cz+d}, \quad \bar{z} \to \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}, \quad ad - bc = 1$$

$$\Phi(z,\bar{z}) \to \Phi'(z,\bar{z}) = (cz+d)^{-2h}(\bar{c}\bar{z}+\bar{d})^{-2\bar{h}}\Phi\left(\frac{a\alpha+b}{c\alpha+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right)$$

‡ $h = \frac{l+\lambda}{2}, \bar{h} = \frac{l-\lambda}{2}$ with the helicity of the field $\Sigma^{z\bar{z}}\Phi = \lambda\Phi$

- collinear and transversal subgroups share the same $\mathbf{D}$, not independent

Conformal Algebra

Examples of conformal operator in QCD

- Nonlocal operator built of q and $\bar{q}$ fields at light-like separation

$$\mathcal{Q}_\mu(\alpha_1, \alpha_2) = \bar{\psi}(\alpha_1)\gamma_\mu[\alpha_1, \alpha_2]\psi(\alpha_2) \tag{25}$$

with the Wilson line

$$[\alpha_1, \alpha_2] = \mathrm{P}\exp\left[ig\int_{\alpha_1}^{\alpha_2} dt\, A_+(t)\right] \tag{26}$$

- Expanding $\mathcal{Q}_\mu(\alpha_1, \alpha_2)$ at short distance gives rise to local operators built of q, $\bar{q}$ and covariant derivatives $\sim \bar{\psi}\left(\overleftarrow{D}_+\right)^{n_1}\gamma_\mu\left(\overrightarrow{D}_+\right)^{n_2}\psi$
- Define the "plus" and "minus" components of quark fields (the same decomposition as in light-cone quantization) with different spin projective operators

$$\begin{aligned} &\psi_+ = \Pi_+\psi, \quad \psi_- = \Pi_-\psi, \quad \psi = \psi_+ + \psi_- \\ &\Pi_+ = \frac{1}{2}\gamma_-\gamma_+, \quad \Pi_- = \frac{1}{2}\gamma_+\gamma_-, \quad \Pi_+ + \Pi_- = 1 \end{aligned} \tag{27}$$

- Spin rotation generators acting on spinor fileds Eq. (17) $\Rightarrow$ ψ_+ and ψ_- correspond to the spin projectors $s = 1/2$ and $-1/2$ in Eq. (18)

Conformal Algebra

- Canonical dimension of spinor field is $l_\psi = 3/2 \quad \Downarrow$

  conformal spin of ψ_+, ψ_- is $j = 1, 1/2$, and twist is $t = 1, 2$, respectively

- Different Lorentz projections of $\mathcal{Q}_\mu(\alpha_1, \alpha_2)$ correspond to take different quark field components

$$
\begin{array}{ll}
\text{twist two} & \mathcal{Q}_+ = \bar{\psi}_+\gamma_+\psi_+ \equiv \mathcal{O}^{1,1} \\
\text{twist three} & \mathcal{Q}_\perp = \bar{\psi}_+\gamma_\perp\psi_- + \bar{\psi}_-\gamma_\perp\psi_+ \equiv \mathcal{O}^{1,1/2} + \mathcal{O}^{1/2,1} \\
\text{twist four} & \mathcal{Q}_- = \bar{\psi}_-\gamma_-\psi_- \equiv \mathcal{O}^{1/2,1/2}
\end{array} \tag{28}
$$

- They have different properties under conformal trans., the corresponding local conformal operators are

$$
\begin{aligned}
\mathbb{O}_n^{1,1}(x) &= (i\partial)^n \left[\bar{\psi}(x)\gamma_+ C_n^{3/2}\left(\overleftrightarrow{D}_+/\overleftrightarrow{\partial}_+\right)\psi(x)\right] \\
\mathbb{O}_n^{1,1/2}(x) &= (i\partial)^n \left[\bar{\psi}(x)\gamma_+\gamma_\perp\gamma_- P_n^{1,0}\left(\overleftrightarrow{D}_+/\overleftrightarrow{\partial}_+\right)\psi(x)\right] \\
\mathbb{O}_n^{1/2,1/2}(x) &= (i\partial)^n \left[\bar{\psi}(x)\gamma_- C_n^{1/2}\left(\overleftrightarrow{D}_+/\overleftrightarrow{\partial}_+\right)\psi(x)\right]
\end{aligned} \tag{29}
$$

$\triangle$ $\overleftrightarrow{D}_+ = \overrightarrow{D}_+ - \overleftarrow{D}_+$, $\overleftrightarrow{\partial}_+ = \overrightarrow{D}_+ - \overleftarrow{D}_+$; $\quad P_n^{1,1} \sim C_n^{3/2}$, $P_n^{0,0} \sim C_n^{1/2}$

Pion LCDAs

a_n^π and m_0^π

Pion LCDAs

$$\langle 0|\bar{u}_i(0)d_j(z)|\pi^-(p)\rangle = -\frac{i}{4}\int_0^1 du e^{-iup\cdot z}\left[\not{p}\gamma_5\phi_\pi(u) + \gamma_5\phi_\pi^p(u) + \gamma_5(1-\not{n}\not{\bar{n}})\phi_\pi^t(u)\right]_{ji}$$

$$\phi_\pi(u,\mu) = 6u(1-u)\sum_{n=0} a_n^\pi(\mu) C_n^{3/2}(u)$$

$$\phi_\pi^p(u,\mu) = \frac{m_0^\pi(\mu)}{p^+}\left[1 + 30\eta_{3\pi} C_2^{1/2}(u) - 3\eta_{3\pi}\omega_{3\pi} C_4^{1/2}(u)\right]$$

$$\phi_\pi^\sigma(u) = \frac{m_0^\pi(\mu)}{p^+} 6u(1-u)\left[1 + 5\eta_{3\pi} C_2^{3/2}(u)\right]$$

- $\phi(x)$ and $\phi^{p,t}(u)$ are the twist two and twist three LCDAs, twist four $\cdots$
- $a_0^\pi = f_\pi$, $a_{n\geqslant 2}^\pi(\mu_0)$ and $m_0^\pi(\mu_0)$ obtained by non-pert. theory/lattice QCD
- μ dependences in a_n^π the integration over the transversal dof
- $C_n(u)$ are Gegenbauer polynomials $\sim$ Jacobi Polynomials $P_n^{j_1,j_2}\left(\frac{\overleftrightarrow{D}_+}{\overleftrightarrow{\partial}_+}\right)$ in the local collinear conformal expansion longitudinal dof
- Great achievements (high precision) in F_π, $B \to \pi$ et.al processes

Pion LCDAs

$\phi_\pi(u,\mu) = 6u(1-u)\sum_{n=0} a_n^\pi(\mu) C_n^{3/2}(u)$

- QCD definition $\quad a_n^\pi(\mu) = \langle \pi | q(z)\bar{q}(z) + z_\rho \partial_\rho q(z)\bar{q}(z) + \cdots |0\rangle$
- LQCD: 0.334 ± 0.129[UKQCD 2010], 0.135 ± 0.032[RQCD 2019], $0.258^{+0.079}_{-0.052}$[LPC 2022]

  △ default scale at 1 GeV △ a_4^π is not available ← the growing number of derivatives in $q\bar{q}$ operator

  △ new technique is being developed[RQCD 2017, 2018].
- QCDSR: 0.19 ± 0.06[Chernyak 1984], $0.26^{+0.21}_{-0.09}$[Khodjamirian 2004], 0.28 ± 0.08[Ball 2006]

  △ nonlocal vacuum condensate is introduced and modeled for $a_{n>2}^\pi$ [Bakulev 2001]

  △ QCD sum rules as an inverse problem[Li 2020, Yu 2022]

  quark-hadron duality → Laguerre Polynomials to construct spectral density

- LCSRs: data-driven of $a_2^\pi(1\,\mathrm{GeV})$

△ $F_{B\to\pi}$: large error from B meson

△ $F_{\pi\gamma\gamma^*}$: 0.14[Agaev 2010] BABAR+CLEO

0.10[Agaev 2012] Belle+CLEO

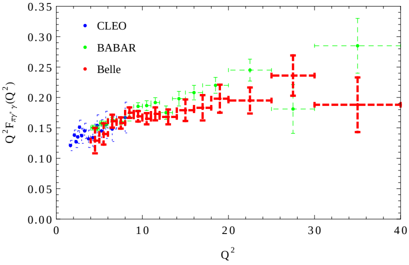
large uncertainty of $a_{n>2}^\pi$, discrepancy at large Q^2

[Gao 2022] N^{2}LO ∼ NLO

△ F_π: 0.24 ± 0.17[Bebek1978] Wilson Lab+NA7

0.20 ± 0.03[Agaev 2005] Wilson Lab+JLab,

large uncertainty of $a_{n>2}^\pi$, data only in small Q^2



Pion LCDAs

$\phi_\pi^{p/\sigma}(u,\mu) \propto m_0^\pi(\mu)/P^+$

- QCD definition $\langle \pi^+ | \bar{u}(0)(-i\gamma_5)d(0)|0\rangle = f_\pi m_0^\pi(\mu)$
- $m_0^\pi(1\,\mathrm{GeV}) = 1.892$ GeV is obtained from $\chi\mathrm{PT}$[Leutwyler 1996]
- $\phi_\pi^{p/\sigma}$ (m_0^π) are not involved in F_π^{LCSRs} due to the chiral symmetry limit
- the dominant contribution in F_π^{pQCD} $\Leftarrow$ chiral enhancement $\mathcal{O}(m_0^\pi/Q^2)$

Table 1. Input of m_0^π in the previous pQCD calculations.

Physical quantity (Accuracy)	m_0^π	Refs
Pion EM FF (NLO, 2p, twist-3)	1.74	[12–15]
Pion EM FF (NLO, 3p, twist-3)	1.74	[16]
Pion EM FF (NLO, 3p, twist-4)	1.90(1 GeV)	[9]
$B \to \pi$ FF (LO, 3p)	1.4	[17]
$B \to \pi$ FF (twist-2 NLO, 2p)	$1.74^{+0.67}_{-0.38}$	[18]
$B \to \pi$ FF (twist-3 NLO, 2p)	1.4	[19]

△ usually chosen at a fixed value in the previous pQCD study
△ maybe the largest error source of pQCD predictions of F_π
△ the corresponding large uncertainty is formerly disregarded

Pion LCDAs from F_π

- $F_\pi^{\rm LCSRs}(q^2)$ is applicable when $q^2 \in [-10, -1]$ GeV^2[Braun 1994, 2000, Bijnens 2002]
- $F_\pi^{\rm pQCD}(q^2)$ is applicable when $|q^2| > 10$ GeV^2[Li 2001, 2012, **SC** 2014]
- spacelike data is only available in the small region $q^2 \in [-2.5, 0]$ GeV^2
- the mismatch destroys the direct extracting programme from $F_\pi(q^2 < 0)$

  △ $a_{n>2}$ terms become important in the intermediate and large transfers in LCSRs/*powerlessness*

- timelike form factor $F_\pi(q^2 > 0)$ provides another opportunity

  △ BABAR: $e^+e^- \to \pi^+\pi^-(\gamma)$, $4m_\pi^2 \leqslant q^2 \lesssim 9$ GeV^2[BABAR 2012]

  △ Belle: $\tau \to \pi\pi\nu_\tau$, $4m_\pi^2 \leqslant q^2 \leqslant 3.125$ GeV^2[Belle 2008]

  △ BESIII: $e^+e^-(\gamma) \to \pi^+\pi^-$, $0.6 \leqslant Q^2 \leqslant 0.9\,\mathrm{GeV}^2$ with ISR[BESIII 2016]

- TL measurement and SL predictions are related by dispersion relation
- the standard dispersion relation and the modulus representation[**SC** 2020]

$$F_\pi(q^2 < s_0) = \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{\mathrm{Im}F_\pi(s)}{s - q^2 - i\epsilon}$$

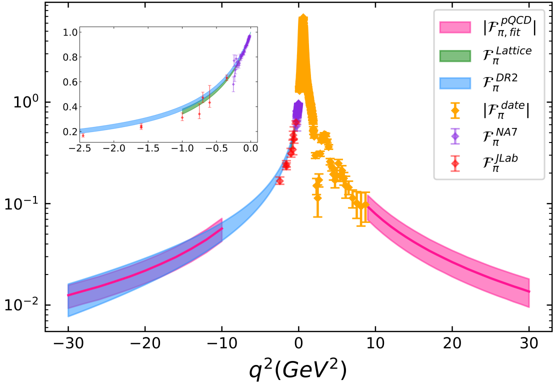
$$\Downarrow \qquad F_\pi(q^2 < s_0) = \exp\left[\frac{q^2\sqrt{s_0 - q^2}}{2\pi}\int_{s_0}^{\infty} \frac{ds\ \ln|F_\pi(s)|^2}{s\sqrt{s - s_0}\,(s - q^2)}\right]$$

Pion LCDAs from F_π [arXiv:2209.13312, 2007.05550]

- $a_2 = 0.275 \pm 0.055, \quad a_4 = 0.185 \pm 0.065, \quad m_0^\pi(1\,\mathrm{GeV}) = 1.37^{+0.29}_{-0.32}$

  △ Pion deviates from the purely asymptotic one △ a_2^π is not enough, more inner structures

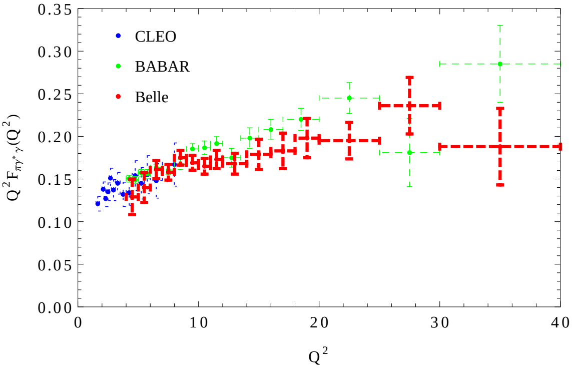
  △ LQCD: 0.334 ± 0.129[UKQCD 2010], 0.135 ± 0.032[RQCD 2019], $0.258^{+0.079}_{-0.052}$[LPC 2022]



- a slight derivation to the measurement of SL form factor intermediate region

Pion LCDAs from F_π

- a slight derivation to the measurement of SL form factor intermediate region



- $F_{\pi\gamma\gamma^*}$ is the theoretically most clean observable $\propto a_n^\pi$
- the measurement discrepancy starts from ~ 7 GeV^2
- BEPC up to 5.4 GeV (2023-2024), STCF(2-7 GeV), Belle-II(4+7 GeV), to solve the "fat pion" issue ?

DiPion LCDAs

$$B_{nl}^{\parallel(\perp)}(s)$$

Why DiPion ?

The width effect encountered in Flavor Physics

- long standing $|V_{ub}|$ tension [PDG 2022]
  - ‡ $|V_{ub}| = (3.82 \pm 0.20) \times 10^{-3}$, mainly extracted from $B \to X_u l\nu$ and $B \to \pi l\nu$
  - ‡ $|V_{ub}|_{\rm in} = (4.13 \pm 0.25) \times 10^{-3}, |V_{ub}|_{\rm ex} = (3.70 \pm 0.16) \times 10^{-3}$, $\sim 2.5\sigma$
  - ‡ enlarge the set of exclusive processes to determine $|V_{ub}|$, a candidate is $B \to \rho l\nu$
    - △ ρ is reconstructed by $\pi\pi$ invariant mass spectral, width effect/nonresonant contribution ?
    - △ the underlying consideration is $B \to \pi\pi l\bar{\nu}_l$ (B_{l4})[Faller 2014]

- V_{cs} issue
  - ‡ $|V_{cs}| = 0.975 \pm 0.006$, mainly extracted from the (semi)leptonic $D_{(s)}$ decays
  - ‡ $|V_{cs}| = 0.972 \pm 0.007, |V_{cs}| = 0.984 \pm 0.012$, $\sim 1.5\sigma$ derivation
  - ‡ $\sim 3\sigma$ tension two years ago, 0.939 ± 0.038 and 0.992 ± 0.012
  - ‡ new channels like semileptonic $D_s^{(*)}$ decays are highly anticipated
    - △ problems encountered, $D_s \to f_0 l\nu$ has large uncertainty due to the width and complicate structure
    - △ $D_s^* \to \phi l\nu$

- Phenomena: multibody decays, $B \to [\pi\pi]\, l\nu$, R_{D^*} anomaly, $B \to D^*$ form factors, $D\pi$ system $\cdots$

DiPion LCDAs

- Chiral-even LC expansion with gauge factor $[x, 0]$[Polyakov 1999, Diehl 1998]

$$\langle \pi^a(k_1)\pi^b(k_2)|\overline{q}_f(zn)\gamma_\mu \tau q_{f'}(0)|0\rangle = \kappa_{ab}\, k_\mu \int dx\, e^{iuz(k\cdot n)}\, \Phi_{\parallel}^{ab,ff'}(u,\zeta,k^2)$$

△ $n^2 = 0$, △ index f, f' respects the (anti-)quark flavor, △ a, b indicates the electric charge

△ coefficient $\kappa_{+-/00} = 1$ and $\kappa_{+0} = \sqrt{2}$, △ $k = k_1 + k_2$ is the invariant mass of dipion state

△ $\tau = 1/2, \tau^3/2$ corresponds to the isoscalar and isovector 2πDAs,

△ higher twist $\propto 1, \gamma_\mu\gamma_5$ have not been discussed yet, γ_5 vanishes due to P-parity conservation

† Three independent kinematic variables

△ momentum fraction z carried by anti-quark with respecting to the total momentum of DiPion state,

△ longitudinal momentum fraction carried by one of the pions $\zeta = k_1^+/k^+$, $2q\cdot\bar{k}\,(\propto 2\zeta - 1)$ △ k^2

† Normalization conditions

$$\int_0^1 \Phi_{\parallel}^{I=1(0)}(u,\zeta,k^2) = (2\zeta-1)F_\pi(k^2)$$

$$\int_0^1 dz\,(2z-1)\Phi_{\parallel}^{I=0}(z,\zeta,k^2) = -2M_2^{(\pi)}\zeta(1-\zeta)F_\pi^{\mathrm{EMT}}(k^2)$$

△ $F_\pi^{em}(0) = 1$, △ $F_\pi^{\mathrm{EMT}}(0) = 1$,

△ $M_2^{(\pi)}$ is the momentum fraction carried by quarks in the pion associated to the usual quark distribution

DiPion LCDAs

- 2πDAs is decomposed in terms of $C_n^{3/2}(2z-1)$ and $C_\ell^{1/2}(2\zeta-1)$

$$\Phi^{I=1}(z,\zeta,k^2,\mu) = 6z(1-z) \sum_{n=0,\text{even}}^{\infty} \sum_{l=1,\text{odd}}^{n+1} B_{n\ell}^{I=1}(k^2,\mu) C_n^{3/2}(2z-1) C_\ell^{1/2}(2\zeta-1)$$

$$\Phi^{I=0}(z,\zeta,k^2,\mu) = 6z(1-z) \sum_{n=1,\text{odd}}^{\infty} \sum_{l=0,\text{even}}^{n+1} B_{n\ell}^{I=0}(k^2,\mu) C_n^{3/2}(2z-1) C_\ell^{1/2}(2\zeta-1)$$

- $B_{n\ell}(k^2,\mu)$ have similar scale dependence as the a_n of π, ρ, f_0 mesons

$$B_{n\ell}(k^2,\mu) = B_{n\ell}(k^2,\mu_0) \left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{[\gamma_n^{(0)}-\gamma_0^{(0)}]/[2\beta_0]}$$

$$\gamma_n^{\perp(\parallel),(0)} = 8C_F \left(\sum_{k=1}^{n+1} \frac{1}{k} - \frac{3}{4} - \frac{1}{2(n+1)(n+2)}\right)$$

- Soft pion theorem relates the chirarlly even coefficients with a_n^π

$$\sum_{\ell=1}^{n+1} B_{n\ell}^{\parallel,I=1}(0) = a_n^\pi, \quad \sum_{\ell=0}^{n+1} B_{n\ell}^{\parallel,I=0}(0) = 0$$

- 2πDAs relate to the skewed parton distributions (SPDs) by crossing

△ express the moments of SPDs in terms of $B_{nl}(k^2)$ in the forward limit as

$$M_{N=\text{odd}}^{\pi} = \frac{3}{2}\frac{N+1}{2N+1} B_{N-1,N}^{I=1}(0), \quad M_{N=\text{even}}^{\pi} = 3\frac{N+1}{2N+1} B_{N-1,N}^{I=0}(0)$$

DiPion LCDAs

- In the vicinity of the resonance, 2πDAs reduce to the DAs of ρ/f_0

△ relation between the a_n^ρ and the coefficients $B_{n\ell}$

$$a_n^\rho = B_{n1}(0)\,\mathrm{Exp}\left[\sum_{m=1}^{N-1} c_m^{n1}\, m_\rho^{2m}\right], \quad c_m^{(n1)} = \frac{1}{m!}\frac{d^m}{dk^{2m}}\left[\ln B_{n1}(0) - \ln B_{01}(0)\right]$$

△ f_ρ relates to the imaginary part of $B_{nl}(m_\rho^2)$ by $\langle \pi(k_1)\pi(k_2)|\rho\rangle = g_{\rho\pi\pi}(k_1-k_2)^\alpha \epsilon_\alpha$

$$f_\rho^{\parallel} = \frac{\sqrt{2}\,\Gamma_\rho\,\mathrm{Im}B_{01}^{\parallel}(m_\rho^2)}{g_{\rho\pi\pi}}, \quad f_\rho^{\perp} = \frac{\sqrt{2}\,\Gamma_\rho\, m_\rho\,\mathrm{Im}B_{01}^{\perp}(m_\rho^2)}{g_{\rho\pi\pi}\, f_{2\pi}^{\perp}}$$

- The subtraction constants of $B_{n\ell}(s)$ [Polyakov 1999, SC 2019, 2023]

(nl)	$B_{n\ell}^{\parallel}(0)$	$c_1^{\parallel,(nl)}$	$\frac{d}{dk^2}\ln B_{n\ell}^{\parallel}(0)$	$B_{n\ell}^{\perp}(0)$	$c_1^{\perp,(nl)}$	$\frac{d}{dk^2}\ln B_{n\ell}^{\perp}(0)$
(01)	1	0	1.46 $\to$ 1.80	1	0	0.68 $\to$ 0.60
(21)	-0.113 $\to$ 0.218	-0.340	0.481	0.113 $\to$ 0.185	-0.538	-0.153
(23)	0.147 $\to$ -0.038	0	0.368	0.113 $\to$ 0.185	0	0.153
(10)	-0.556	-	0.413	-	-	-
(12)	0.556	-	0.413	-	-	-

△ firstly studied in the effective low-energy theory based on instanton vacuum

DiPion LCDAs

- What's the evolution from $4m_\pi^2$ to large invariant mass $k^2 \sim \mathcal{O}(m_c^2)$?
- Watson theorem of π-π scattering amplitudes

  △ implies an intuitive way to express the imaginary part of 2πDAs

  △ leads to the Omnés solution of N-subtracted DR for the coefficients

$$B^I_{n\ell}(k^2) = B^I_{n\ell}(0)\,\mathrm{Exp}\left[\sum_{m=1}^{N-1} \frac{k^{2m}}{m!}\frac{d^m}{dk^{2m}} \ln B^I_{n\ell}(0) + \frac{k^{2N}}{\pi}\int_{4m_\pi^2}^{\infty} ds\, \frac{\delta^I_\ell(s)}{s^N(s-k^2-i0)}\right]$$

- **2πDAs in a wide range of energies is given by δ^I_ℓ and a few subtraction constants**
- Above discussions are all at leading twist level, subleading twist LCDAs are still in lack

‡ Would the chiral EFT help us to set down $B_{nl}(k^2 \sim 4m_\pi^2)$ for both leading and subleading twist LCDAs ?

DiPion LCDAs in $D_s^{(*)}$ weak decay

$D_s \to (f_0 \to) [\pi\pi]_{\mathrm{S}}\, e^+\nu_e$

- Semileptonic $D_{(s)}$ decays provide a clean environment to study scalar mesons

  △ $D_{(s)} \to a_0 e^+\nu$[BESIII 18, 21], $D^+ \to f_0/\sigma e^+\nu$[BESIII 19], $D_s \to f_0(\to \pi^+\pi^-)e^+\nu$[CLEO 09]

  △ $\mathcal{B}$ of $D_s \to f_0(\to \pi^0\pi^0, K_s K_s)e^+\nu$[BESIII 22], $D_s \to f_0(\to \pi^+\pi^-)e^+\nu$ form factor[BESIII 23]

$$\mathcal{B}(D_s \to f_0(\to \pi^0\pi^0)e^+\nu) = (7.9 \pm 1.4 \pm 0.3) \times 10^{-4}$$
$$\mathcal{B}(D_s \to f_0(\to \pi^+\pi^-)e^+\nu) = (17.2 \pm 1.3 \pm 1.0) \times 10^{-4}$$

  △ isospin symmetry expectation $\mathcal{B}(f_0 \to \pi^+\pi^-)/\mathcal{B}(f_0 \to \pi^0\pi^0) = 2$, possible ρ^0 pollution

  △ $f_+^{f_0}(0)|V_{cs}| = 0.504 \pm 0.017 \pm 0.035$

- Theoretical consideration $\frac{d\Gamma(D_s^+ \to f_0 l^+\nu)}{dq^2} = \frac{G_F^2|V_{cs}|^2\lambda^{3/2}(m_{D_s}^2, m_{f_0}^2, q^2)}{192\pi^3 m_{D_s}^3}|f_+(q^2)|^2$

- Observed in the $\pi\pi$ invariant mass spectral, improvement with the width effect

$$\frac{d\Gamma(D_s^+ \to [\pi\pi]_{\mathrm{S}}\, l^+\nu)}{ds dq^2} = \frac{1}{\pi}\frac{G_F^2|V_{cs}|^2}{192\pi^3 m_{D_s}^3}|f_+(q^2)|^2 \frac{\lambda^{3/2}(m_{D_s}^2, s, q^2)\, g_1^2 \beta_\pi(s)}{|m_{\mathrm{S}}^2 - s + i\left(g_1^2\beta_\pi(s)) + g_2^2\beta_K(s)\right)|^2}$$

$$\frac{d^2\Gamma(D_s^+ \to [\pi\pi]_{\mathrm{S}}\, l^+\nu)}{dk^2 dq^2} = \frac{G_F^2|V_{cs}|^2}{192\pi^3 m_{D_s}^3}\frac{\beta_{\pi\pi}(k^2)\sqrt{\lambda_{D_s}}q^2}{16\pi}\sum_{\ell=0}^{\infty} 2|F_0^{(\ell)}(q^2, k^2)|^2$$

- $D_s \to f_0$ ffs to $D_s \to [\pi\pi]_{\mathrm{S}}$ ffs

  [Hambrock 2015, SC 2017,19,20, Descotes-Genon 2019,23] in $B \to \pi\pi$, $B_s \to [KK]_{\mathrm{S}}$ cases

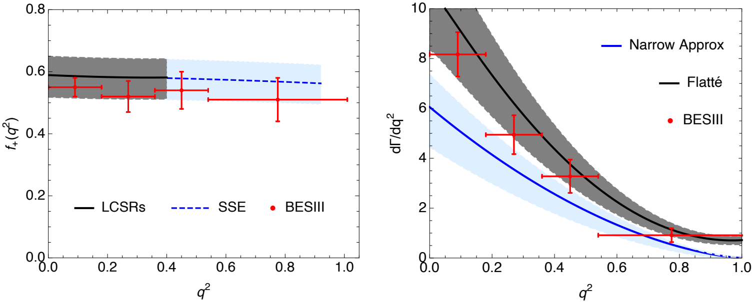
$D_s \to (f_0 \to) [\pi\pi]_S\, e^+\nu_e$ [arXiv:2307.02309]

- Definitions of $D_s \to f_0$ form factors

$$\langle f_0(p_1)|\bar{s}\gamma_\mu\gamma_5 c|D_s^+(p)\rangle = -i\left[f_+(q^2)\,(p+p_1)_\mu + f_-(q^2)q_\mu\right]$$

- Form factor and the differential decay width

△ $M^2 = 5.0 \pm 0.5$ GeV^2 and $s_0 = 6.0 \pm 0.5$ GeV^2 , △ $\tilde{f}_{f_0} = 335$ MeV, much larger than 180 MeV used in the previous LCSRs, △ $a_1^{s/\sigma}$ term contributions are considered for the first time , △ f_0 is not a pure $\bar{s}s$ state, the mixing angle is chosen at $20^\circ \pm 10^\circ$

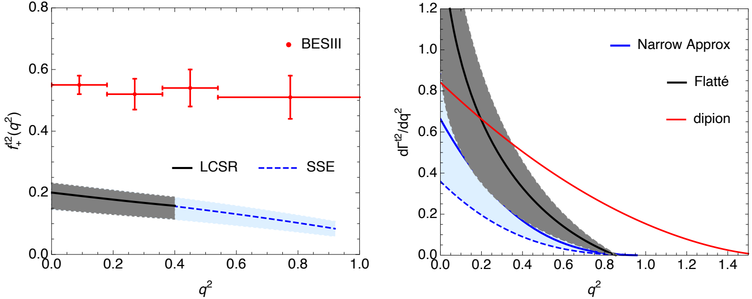


$D_s \to [\pi\pi]_{\rm S}\, e^+\nu_e$ [arXiv:2307.02309]

- Definitions of $D_s \to [\pi\pi]_{\rm S}$ form factors

$$\langle[\pi(k_1)\pi(k_2)]_{\rm S}\,|\bar{s}\gamma_\mu(1-\gamma_5)c|D_s^+(p)\rangle = -iF_t(q^2,s,\zeta)k_\mu^t - iF_0(q^2,s,\zeta)k_\mu^0 - iF_{\parallel}(q^2,s,\zeta)k_\mu^{\parallel}$$

- Form factor and the differential decay width at leading twist



- subleading twist LCDAs give dominate contribution in $D_s \to [\pi\pi]_{\rm S}$ transition
- shows relatively moderate evolution with larger allowed momentum transfer
- further measurements would help us to understand the dipion system, ρ, f_0
- different in B/Z case where the leading twist is dominate/overwhelming

D_s^* weak decay

- $\alpha_s : \alpha : G_F \sim \mathcal{O}(1) : \mathcal{O}(1/137) : \mathcal{O}(10^{-5})$
- very hard to measure weak decay from strong and EM interactions
- the total widths of heavy-light vector mesons are still in lack[PDG 2022]
  - △ $\Gamma_{D^{*+}} = 84.3 \pm 1.8\,\mathrm{keV}$ ($\to D^0\pi^+, D^+\pi^0, D^+\gamma$)
  - △ $\Gamma_{D^{*0}} < 2.1\,\mathrm{MeV}$ ($\to D^0\pi^0, D^0\gamma$), $\Gamma_{D_s^{*+}} < 1.9\,\mathrm{MeV}$ ($\to D_s^+\gamma, D_s^+\pi^0, D_s^+e^+e^-$)
  - △ $\Gamma_{B^*}, \Gamma_{B_s^*}$ no measurement
- but important to properties and $g_{D_s^* D_s \gamma}$ $\to$ non-perturbative approaches

	$g_{D^{*+}D^+\gamma}$ (GeV^{-1})	$g_{D^{*0}D^0\gamma}$ (GeV^{-1})	$g_{D_s^{*+}D_s^+\gamma}$ (GeV^{-1})
this work	$-0.15^{+0.11}_{-0.10}$	$1.48^{+0.29}_{-0.27}$	$-0.079^{+0.086}_{-0.078}$
HHχPT [24]	-0.27 ± 0.05	2.19 ± 0.11	0.041 ± 0.056
HQET+VMD [35]	$-0.29^{+0.19}_{-0.11}$	$1.60^{+0.35}_{-0.45}$	$-0.19^{+0.19}_{-0.08}$
HQET+CQM [71]	$-0.38^{+0.05}_{-0.06}$	1.91 ± 0.09	–
Lattice QCD [32]	-0.2 ± 0.3	2.0 ± 0.6	–
LCSR [21]	-0.50 ± 0.12	1.52 ± 0.25	–
QCDSR [20]	$-0.19^{+0.03}_{-0.02}$	0.62 ± 0.03	-0.20 ± 0.03
RQM [72]	-0.44 ± 0.06	2.15 ± 0.11	-0.19 ± 0.03
experiment [16–18]	-0.47 ± 0.06	1.77 ± 0.03	–

LCSRs, hadronic photon NLO[Li 2020]

LCSRs, LP NLO corrections[Pullin 2021]

$$g_{D_s^* D_s \gamma} = 0.60^{+0.19}_{-0.18}$$

very sensitive to different contributions
(radiative corrections, power corrections)
a benchmark to probe the involved dynamics

- impressive lattice QCD evaluation[HPQCD 2013] $\Gamma^{\mathrm{HPQCD}}_{D_s^{*+}} = 0.070(28)\,\mathrm{keV}$
  - △ the longest-lived charged vector meson △ encourage us to study the exclusive D_s^* weak decay

D_s^* weak decay

- D_s^* weak decay are highly anticipated to determine $|V_{cs}|$
- leptonic decays, helicity enhanced $D_s^* \to l\nu$, $|V_{cs}|f_{D_s^*}$

$$\Gamma_{D_s^* \to l\nu} = \frac{G_F^2}{12\pi}|V_{cs}|^2 f_{D_s^*}^2 m_{D_s^*}^3 \left(1 - \frac{m_l^2}{m_{D_{s*}^2}}\right)\left(1 + \frac{m_l^2}{m_{D_{s*}^2}}\right) = 2.44 \times 10^{-12}\,\mathrm{GeV}$$

△ $\mathcal{B}(D_s^* \to \mu\nu) = \frac{\Gamma_{D_s^* \to \mu\nu}}{\Gamma_{D_s^*}} \sim \frac{\Gamma_{D_s \to \mu\nu}}{\Gamma_{D_s^*}} \frac{2m_{D_s^*}^2}{3m_\mu^2} \sim 2 \times 10^{-5}$, close to the LQCD[HPQCD 2013]

△ the most favored modes, $\left(2.1^{+1.3}_{-0.9}\right) \times 10^{-5}$[arXiv:2304.12159 BESIII]

△ confirms the total width of D_s^* but need more precise lattice evaluation

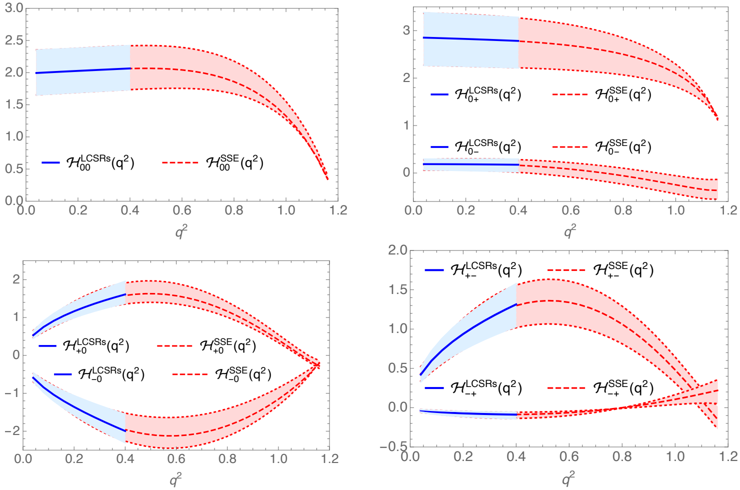
- semileptonic decays, $D_s^* \to \phi l\nu$, $|V_{cs}|$ and helicity form factors

△ heavy quark symmetry (HQS) has been examined in $\bar{B} \to D^*(D)l\bar{\nu}$, also in $D_s^*(D_s) \to \phi l^+\nu$?

△ lepton flavour university (LFU) in vector charm sector

- hadronic decays $D_s^* \to \phi\rho, \phi\pi$, factorisation theo. or topological analysis
- inclusive decays, $D_s^* \to X_s l\nu$, HQET and reliability of power expansion

D_s^* weak decay [arXiv:2203.06797]



- LCSRs parameters $s_0 = 6.8 \pm 1.0\ \mathrm{GeV}^2$, $M^2 = 4.50 \pm 1.0\ \mathrm{GeV}^2$
- Wigner-Eckart theorem: the helicity information at endpoint is only governed by the Clebsch-Gordan coefficients [Hiller 2014, Grattrex 2016, Hiller 2021]

D_s^* weak decay [arXiv:2203.06797]

- semileptonic decays $D_s^* \to \phi l \nu_l$

$$\frac{d\Gamma_{ij}(q^2)}{dq^2} = \frac{G_F^2 |V_{cs}|^2}{192\pi^3 m_{D_s^*}^3} \lambda^{1/2}(m_{D_s^*}^2, m_\phi^2, q^2)\, q^2\, |H_{ij}(q^2)|^2$$

$$\Gamma_{D_s^* \to \phi l \nu_l} = \frac{1}{3} \int_0^{q_0^2} dq^2 \sum_{i,j=0,\pm} \frac{d\Gamma_{ij}(q^2)}{dq^2} = \left(3.28^{+0.82}_{-0.71}\right) \times 10^{-14}\,\mathrm{GeV}$$

- hadronic decays (naive factorisation)

$$\mathcal{A}(D_s^{*+} \to \phi\pi^+) = (-i)\frac{G_F}{\sqrt{2}} V_{cs}\, a_1\, m_\pi f_\pi \sum_{i=0,\pm} H_{0j}(m_\pi^2)$$

$$\mathcal{A}(D_s^{*+} \to \phi\rho^+) = \frac{G_F}{\sqrt{2}} V_{cs}\, a_1\, m_\rho f_\rho^{\|(\perp)} \sum_{i,j} H_{ij}(m_\rho^2)$$

△ $a_1(\mu) = 0.999$, $f_\pi = 0.130$ GeV, $f_\rho^{\|} = 0.210$ GeV

△ $\Gamma_{D_s^{*+} \to \phi\pi^+} = \left(3.81^{+1.52}_{-1.33}\right) \times 10^{-14}\,\mathrm{GeV}$, $\Gamma_{D_s^{*+} \to \phi\rho^+} = \left(1.16^{+0.42}_{-0.39}\right) \times 10^{-13}\,\mathrm{GeV}$

△ the result of $\phi\pi$ channel is marginally consistent with the PQCD[Yang 2022]

- with the lattice evaluation of $\Gamma_{D_s^*} = (0.70 \pm 0.28) \times 10^{-8}$ GeV[HPQCD 2013]

$$\mathcal{B}(D_s^* \to l\nu) = (3.49 \pm 1.40) \times 10^{-5}, \quad \mathcal{B}(D_s^* \to \phi l \nu) = (0.47\,^{+0.12}_{-0.10} \pm 0.19) \times 10^{-6}$$

$$\mathcal{B}(D_s^{*+} \to \phi\pi^+) = (0.54\,^{+0.22}_{-0.19} \pm 0.22) \times 10^{-6}, \quad \mathcal{B}(D_s^{*+} \to \phi\rho^+) = (1.65\,^{+0.61}_{-0.56} \pm 0.66) \times 10^{-6}$$

D_s^* weak decay   without the background analysis

- Belle II   clear background

  △ 2022, 400 fb^{-1}, reconstruct 2×10^5 data samples of $D_s^*(D_s)$ from $\phi\pi$ channel

  △ phase 3 running (2024-2026), 10 ab^{-1}, $\mathcal{O}(1 \times 10^7)$ data sample of $D_s^*(D_s)$

  △ the number of D_s^* production is $\mathcal{O}(10^9)$ $\Leftarrow \mathcal{B}(D_s \to \phi\pi) = (4.5 \pm 0.4)\,\%$

  △ excellent potential to study the D_s^* weak decays, 50 ab^{-1} is hottest expected

- LHCb   excellent particle identification to distinguish K, π and μ

  △ the channel $D_s^* \to \phi\pi$ with the D_s^* producing by $B_s \to D_s^* \mu\nu$

- BESIII   low background

  △ directly produced from e^+e^- collision at the $D_s D_s^*$ threshold

  △ have collected $\sim 6 \times 10^6$ D_s^* mesons with the 3.2 fb^{-1} data at 4.178 GeV

  △ provides the good chance for the leptonic decay $D_s^* \to l\nu$, Statistical error

  △ first $\mathcal{B}$ measurement [2304.12159], determination of spin and parity [2305.14631]

- CEPC(4×Tere-Z) 8.8×10^{10} D_s/D_s^* production
- STCF 1.0×10^9 D_s/D_s^* production/year

Conclusion

- Pion structure in LCDAs

  △ F_π obtained from the modulus dispersion relation reveals more structure in Pion △ has a slightly derivation with the spacelike data △ $F_{\pi\gamma\gamma^*}$ is the most clean observable △ $F_K, F_{\eta^{(\prime)}}$ and et.al,

- DiPion (DiKaon) structure in LCDAs

  △ width effect of ρ, ϕ, f_0 in CKM determinations and anomalies study

  △ subleading twist LCDAs is still not known, as well as the k^2 evolution

- $D_s^{(*)}$ weak decay

  △ semileptonic D_s decay provides clean environment to study scale meson and (subleading twist) DiPion LCDAs △ D_s^* provides the opportunity of first measurement of weak decay of vector meson and further more physics

Thank you for your patience.