Coefficient function for the double deeply-virtual Compton scattering

Hua-Yu Jiang (蒋华玉)

School of Physics, Nankai University

Nov 4, 2023 Lanzhou University

In collaboration with: Vladimir Braun, Alexander Manashov

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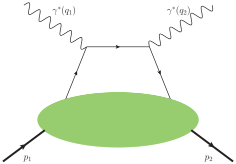
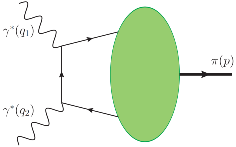
Motivation

- Double deeply-virtual Compton scattering (DDVCS) gives access to the generalized parton distributions (GPD) that encode the information on the transverse position of partons in the proton with dependence on their longitudinal momentum.





- The general framework for the QCD description of DVCS is based on the collinear factorization in terms of GPDs and is well understood at the leading-twist level.
- The NNLO analysis of DIS and DVCS has become the standard in this field, so that the NNLO precision for DDVCS is necessary as well.
- This implies that one needs to derive three-loop evolution equations for GPDs [Braun, Manashov, Moch and Strohmaier, 2017;2021] and calculate the two -loop corrections to the coefficient functions (CFs) of the DDVCS amplitude.

The spirit of conformal OPE

- The idea is to consider QCD in non-integer $d = 4 - 2\epsilon$ dimensions at the intermediate step, for the specially chosen (critical) value of the coupling α_s^* such that the $\beta(\alpha_s^*) = 0$ [Braun, Manashov, Moch and Strohmaier, 2018].

- This theory is conformally invariant and all anomalous dimensions for composite operators in a specified MS or $\overline{\mathrm{MS}}$ coincide with the anomalous dimensions of the corresponding operators for the QCD in $d = 4$.

- So that, the contributions of operators with total derivatives are related to the contributions without total derivatives by symmetry transformations.

- This symmetry is exact, however, the generators are modified by quantum corrections and differ from their canonical form.

The progress with this conformal method

- V. M. Braun and A. N. Manashov, Evolution equations beyond one loop from conformal symmetry, [Eur. Phys. J. C73 (2013) 2544].
- V. M. Braun, A. N. Manashov, S. Moch and M. Strohmaier, Conformal symmetry of QCD in d-dimensions, [Phys. Lett. B793 (2019) 78–84].
- V. M. Braun, A. N. Manashov, S. Moch and M. Strohmaier, Two-loop conformal generators for leading-twist operators in QCD, [JHEP 03 (2016) 142].
- V. M. Braun, A. N. Manashov, S. Moch and M. Strohmaier, Three-loop evolution equation for flavor-nonsinglet operators in off-forward kinematics, [JHEP 06 (2017) 037].
- V. M. Braun, A. N. Manashov, S. Moch and M. Strohmaier, Two-loop evolution equations for flavor-singlet light-ray operators, [JHEP 02 (2019) 191].
- V. M. Braun, A. N. Manashov, S. Moch and J. Schoenleber, Two-loop coefficient function for DVCS: vector contributions, [JHEP 09 (2020) 117].
- V. M. Braun, A. N. Manashov, S. Moch and J. Schoenleber, Axial-vector contributions in two-photon reactions: Pion transition form factor and deeply-virtual Compton scattering at NNLO in QCD, [Phys. Rev. D 104 (2021) 094007].
- V. M. Braun, Y. Ji and J. Schoenleber, Deeply Virtual Compton Scattering at Next-to-Next-to-Leading Order, [Phys. Rev. Lett. 129 (2022) 172001].

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Double deeply-virtual Compton scattering

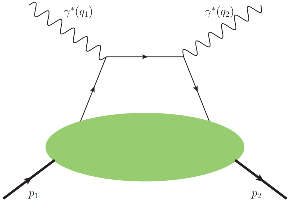
- The amplitude of the DDVCS process is given by the following matrix element

$$T_{\mu\nu}(q_1, q_2, p_1) = i \int d^4x\, e^{iq_1\cdot x} \langle p_2 | T\{j_\mu^{\rm em}(x)\, j_\nu^{\rm em}(0)\} | p_1 \rangle,$$
$$= -g_{\mu\nu}^{\perp} V + \varepsilon_{\mu\nu}^{\perp} A + \ldots$$

- The leading-twist vector amplitude can be factorized as

$$V(\xi, \eta, Q^2) = \sum_q e_q^2 \int_{-1}^{1} \frac{dx}{\eta} C\Big(\frac{x}{\eta}, \frac{\xi}{\eta}, Q^2, \mu\Big) F_q(x, \xi, t, \mu),$$

$$q = \frac{q_1 + q_2}{2}, \qquad p = \frac{p_1 + p_2}{2}, \qquad \Delta = p_2 - p_1$$
$$q^2 = -Q^2, \qquad \Delta^2 = t, \qquad \xi = -\frac{\Delta \cdot q}{2p \cdot q},$$
$$\eta = \frac{Q^2}{2p \cdot q}, \qquad \omega = \frac{q_2^2 - q_1^2}{q_2^2 + q_1^2} = \frac{\xi}{\eta}$$



The GPD and Coefficient function

- The GPD is defined by the appropriate matrix element

$$\langle p_2 | \mathcal{O}_q (z_1, z_2) | p_1 \rangle = 2P_+ \int_{-1}^{1} dx e^{-iP_+\xi(z_1+z_2)+iP_+x(z_1-z_2)} F_q(x,\xi),$$

of the light-ray operator

$$\mathcal{O}_q(z_1 n, z_2 n) = \bar{q}(z_1 n) \not{n} [z_1 n, z_2 n] q(z_2 n).$$

- The CF can be calculated in perturbation theory

$$C(x/\eta, \omega, Q^2, \mu) = C^{(0)}(x/\eta) + a_s C^{(1)}(x/\eta, \omega, Q^2/\mu^2) + a_s^2 C^{(2)}(x/\eta, \omega, Q^2/\mu^2) + \ldots,$$

where

$$C^{(0)}\left(\frac{x}{\eta}\right) = \frac{\eta}{\eta - x} - \frac{\eta}{\eta + x},$$

and so far knows up to NLO.

General framework

- Firstly, we consider the DDVCS process in a generic $d = 4 - 2\epsilon$ dimensional theory, then

$$C(x/\eta, \xi/\eta, a_s, \epsilon) = C_0(x/\eta) + a_s C^{(1)}(x/\eta, \xi/\eta, \epsilon) + a_s^2 C^{(2)}(x/\eta, \xi/\eta, \epsilon),$$
$$C^{(k)}(x/\eta, \xi/\eta, \epsilon) = C^{(k)}(x/\eta, \xi/\eta) + \epsilon C^{(k,1)}(x/\eta, \xi/\eta) + \epsilon^2 C^{(k,2)}(x/\eta, \xi/\eta),$$

- Secondly, the CF at the critical point $C_* = C(\alpha_s^*, \epsilon)$ can be expanded as

$$C_*(a_s) = C(a_s, \epsilon_*) = C^{(0)} + a_s C_*^{(1)} + a_s^2 C_*^{(2)} + \mathcal{O}(a_s^3),$$

with the condition $\beta(\alpha_s^*) = 0$ so that $\alpha_s^* = \alpha_s^*(\epsilon)$, that is

$$\epsilon_* = \epsilon(a_s^*) = -\left(\beta_0 a_s^* + \beta_1 (a_s^*)^2 + \dots\right), \quad \beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f,$$

- Then, we can obtain

$$C^{(1)} = C_*^{(1)}, \qquad C^{(2)} = C_*^{(2)} + \beta_0 C^{(1,1)}.$$

- The coefficients $C_*^{(k)}$ can be related to the known CFs for DIS by conformal invariance.

- The most general expression for the OPE of the product of two electromagnetic currents to the twist-two accuracy

$$\mathrm{T}\left\{j^{\mu}(x_1)\,j^{\nu}(x_2)\right\} = \sum_{N,\text{ even}} \frac{\mu^{\gamma_N}}{(-x_{12}^2)^{t_N}} \int_0^1 du \Big\{ -\frac{1}{2} A_N(u) \left(g^{\mu\nu} - \frac{2x_{12}^{\mu} x_{12}^{\nu}}{x_{12}^2} \right) + B_N(u) g^{\mu\nu} + C_N(u) x_{12}^{\nu} \partial_1^{\mu} - C_N(\bar{u}) x_{12}^{\mu} \partial_2^{\nu} + D_N(u) x_{12}^2 \partial_1^{\mu} \partial_2^{\nu} \Big\} \mathcal{O}_N^{x_{12}\ldots x_{12}}(x_{21}^u),$$

where $\mathcal{O}_N^{x\ldots x}(y) = x_{\mu_1}\ldots x_{\mu_N}\mathcal{O}_N^{\mu_1\ldots\mu_N}(y)$, $\mathcal{O}_N^{\mu_1\ldots\mu_N}(y)$ are the leading-twist conformal operators

$$\mathcal{O}_N^{\mu_1\ldots\mu_N}(0) = i^{N-1}\bar{q}(0)\gamma^{\{\mu_1}D^{\mu_2}\ldots D^{\mu_N\}}q(0).$$

- It transforms in the proper way under conformal transformations

$$[\mathbb{K}_\mu, \mathcal{O}_N^{x\ldots x}(y)] = \left(2y_\mu y^\nu \frac{\partial}{\partial y^\nu} - y^2 \frac{\partial}{\partial y^\mu} + 2\Delta_N y_\mu + 2y^\nu \left(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} \right)\right) \mathcal{O}_N^{x\ldots x}(y).$$

Conformal operator product expansion

- By using the conditions of conformal invariance and current conservation $\partial^\mu j_\mu = 0$ lead to constraints on the functional form and also certain relations between the invariant functions $A_N(u), \ldots, D_N(u)$

$$
\begin{aligned}
A_N(u) &= a_N u^{j_N - 1} \bar{u}^{j_N - 1}, \\
B_N(u) &= b_N u^{j_N - 1} \bar{u}^{j_N - 1}, \\
C_N(u) &= u^{N-1} \int_u^1 \frac{dv}{v^N} v^{j_N} \bar{v}^{j_N - 2} \left(c_N - \frac{b_N}{v} \right), \\
D_N(u) &= -\frac{1}{N-1} \int_0^1 dv (v\bar{v})^{j_N - 1} \\
&\quad \times \left[\theta(v-u) \left(\frac{u}{v} \right)^{N-1} + \theta(\bar{v} - \bar{u}) \left(\frac{\bar{u}}{\bar{v}} \right)^{N-1} \right] \left(d_N - \frac{c_N - b_N}{2 v \bar{v}} \right).
\end{aligned}
$$

- The coefficients c_N and d_N are not independent and are given in terms of a_N and b_N

$$
\begin{aligned}
(j_N - 1) a_N &= 2 t_N (c_N - b_N), \\
2(j_N - 1) d_N &= -\frac{1}{2} a_N (N - j_N) - \gamma_N b_N + (j_N - 2 + 2 t_N)(c_N - b_N).
\end{aligned}
$$

Relating DIS and DDVCS

- The general matrix element of the conformal operator is parameterized as

$$\langle p_2|x_{\mu_1}\dots x_{\mu_N}O_N^{\mu_1\dots\mu_N}(ux)|p_1\rangle = e^{iu\Delta\cdot x}\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k f_N^{(k)}\xi^k(x\cdot p)^{N-k}(x\cdot\Delta)^k.$$

- The conformal OPE for the forward matrix element

$$\begin{aligned} T_{\mu\nu}^{\text{DIS}}(p,q) &\equiv i\int d^dx e^{-iqx}\langle p|T(j_\mu(x)j_\nu(0)|p\rangle \\ &= \sum_{N,\text{ even}} f_N\left(\frac{2p\cdot q}{Q^2}\right)^N\Big[\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)C_1\left(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*\right) \\ &+\frac{(q_\mu+2x_Bp_\mu)(q_\nu+2x_Bp_\nu)}{Q^2}C_2\left(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*\right)\Big]. \end{aligned}$$

The general OPE form of the two-point correlator

- For the DDVCS, the conformal OPE of the matrix element is changed as

$$T_{\mu\nu}^{\mathrm{DDVCS}}(p_1,p_2,q_1) \equiv i\int d^dx e^{-iq_1x}\langle p_2|T(j_\mu(x)j_\nu(0))|p_1\rangle$$
$$= \sum_N f_N(\xi)\Big(\frac{1}{(1+\omega)\eta}\Big)^N\Big(\frac{1}{1+\omega}\Big)^{\frac{1}{2}\gamma_N}{}_2F_1\big(N+\frac{1}{2}\gamma_N, j_N; 2j_N; \frac{2\omega}{1+\omega}\big)$$
$$\times\Big[C_1\Big(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*\Big)(-g_{\mu\nu})+\ldots\Big],$$

- this leads to

$$V(\xi,\eta,Q^2) = \sum_N f_N(\xi)\Big(\frac{1}{(1+\omega)\eta}\Big)^N\Big(\frac{1}{1+\omega}\Big)^{\frac{1}{2}\gamma_N}C_1\Big(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*\Big){}_2F_1\big(N+\frac{1}{2}\gamma_N, j_N; 2j_N; \frac{2\omega}{1+\omega}\big).$$

- For $\omega = 1$, it will be reduced to the well known DVCS case

$$V(\xi,Q^2) = \sum_N f_N(\xi)\Big(\frac{1}{2\xi}\Big)^N C_1\Big(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*\Big)\frac{\Gamma(\frac{d}{2}-1)\Gamma(2j_N)}{\Gamma(j_N)\Gamma(j_N+\frac{d}{2}-1)}$$

Coefficient function in momentum fraction space

- The factorization formula

$$V(\xi = 1, Q^2) = \int_{-1}^{1} dx\, C(x, Q^2) F_q(x, \xi = 1) = \int_{-1}^{1} dx\, \mathbf{C}(x, Q^2) \mathbf{F}_q(x, \xi = 1).$$

- The CF and GPD in conformal scheme

$$C(x/\xi, \mu^2/Q^2) = \int_{-1}^{1} \frac{dx'}{\xi} \mathbf{C}(x'/\xi, \mu^2/Q^2) \mathrm{U}(x', x, \xi),$$

$$\mathbf{F}_q(x, \xi) = [\mathrm{U} F_q](x, \xi) \equiv \int_{-1}^{1} \frac{dx'}{\xi} \mathrm{U}(x, x', \xi) F_q(x', \xi).$$

- The constraints from conformal invariance for light-ray operator

$$\mathbf{O}(z_1, z_2) = \sum_{Nk} \frac{i^{N-1}}{(N-1)!} \sigma_N a_{Nk} (S_+(\gamma_N))^k z_{12}^{N-1} \partial_+^k \mathcal{O}_N(0),$$

where the conformal generators $S_+(\gamma_N) \equiv S_+^{(0)} + (z_1 + z_2)\left(-\epsilon_* + \frac{1}{2}\gamma_N\right)$ and satisfy the RGE $(\mu\partial_\mu + \mathbf{H}(a_s))\,[\mathbf{O}(z_1, z_2)] = 0.$

Coefficient function in momentum fraction space

- The GPD in conformal scheme

$$\langle p'|\mathbf{O}(z_1, z_2)|p\rangle = P_+ \sum_N \frac{\sigma_N f_N(\xi)}{(N-1!} \left(\frac{1}{2\xi}\right)^{N-1} \frac{1}{2}\omega_N \int_{-1}^{1} dx\, e^{-i\xi P_+(z_1+z_2-xz_{12})} P_{N-1}^{(\lambda_N)}(x),$$
$$\equiv 2P_+ \int_{-1}^{1} dx\, e^{-iP_+[z_1(\xi-x)+z_2(x+\xi)]}\mathbf{F}(x,\xi,t).$$

with $P_{N-1}^{(\lambda_N)}(x) = \left(\frac{1-x^2}{4}\right)^{\lambda_N-\frac{1}{2}} C_{N-1}^{\lambda_N}(x)$.

- Comparing the two sides of the this equation

$$\mathbf{F}(x,\xi=1) = \frac{1}{4}\sum_N \frac{\sigma_N\omega_N}{2^{N-1}(N-1)!} f_N(\xi=1) P_{N-1}^{(\lambda_N)}(x).$$

- It means that

$$V(\xi=1,\omega,Q^2) = \frac{1}{4}\sum_N \frac{\sigma_N\omega_N}{2^{N-1}(N-1)!} f_N(\xi=1)\int_{-1}^{1} dx\,\mathbf{C}(x,\omega,Q^2) P_{N-1}^{(\lambda_N)}(x).$$

The master formula

- We have obtained the vector amplitude from different side at $\xi = 1$

$$
\begin{aligned}
V(\xi=1,\omega,Q^2) = &\sum_N f_N(\xi=1)\left(\frac{\omega}{1+\omega}\right)^N \left(\frac{1}{1+\omega}\right)^{\frac{1}{2}\gamma_N} C_1\left(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*\right) \\
&\times {}_2F_1\left(N+\frac{1}{2}\gamma_N, j_N; 2j_N; \frac{2\omega}{1+\omega}\right), \\
&= \frac{1}{4}\sum_N \frac{\sigma_N}{2^{N-1}} \frac{\Gamma(2j_N)\Gamma(2\lambda_N) f_N(\xi=1)}{\Gamma(\lambda_N+\frac{1}{2})\Gamma(j_N)\Gamma(N-1+2\lambda_N)} \int_{-1}^{1} dx\, \mathbf{C}\left(x,\omega,Q^2,a_s\right) P_{N-1}^{(\lambda_N)}(x).
\end{aligned}
$$

- Then we can obtain the master formula

$$
\begin{aligned}
\int_{-1}^{1} dx\, \mathbf{C}\left(x,\omega,Q^2,a_s\right) P_{N-1}^{(\lambda_N)}(x) = &\, C_1\left(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*\right) \frac{2\Gamma(\lambda_N+\frac{1}{2})\Gamma(N-1+2\lambda_N)\Gamma(j_N)}{\sigma_N \Gamma(2\lambda_N)\Gamma(2j_N)} \\
&\times \left(\frac{2\omega}{1+\omega}\right)^N \left(\frac{1}{1+\omega}\right)^{\frac{1}{2}\gamma_N} {}_2F_1\left(N+\frac{1}{2}\gamma_N, j_N; 2j_N; \frac{2\omega}{1+\omega}\right).
\end{aligned}
$$

Reduced to the DVCS case

- The master formula for DDVCS

$$\int_{-1}^{1} dx\,\mathbf{C}(x,\omega,Q^2,a_s)\,P_{N-1}^{(\lambda_N)}(x) = C_1\left(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*\right)\frac{2\Gamma(\lambda_N+\frac{1}{2})\Gamma(N-1+2\lambda_N)\Gamma(j_N)}{\sigma_N\Gamma(2\lambda_N)\Gamma(2j_N)}$$
$$\times\left(\frac{2\omega}{1+\omega}\right)^N\left(\frac{1}{1+\omega}\right)^{\frac{1}{2}\gamma_N}{}_2F_1\left(N+\frac{1}{2}\gamma_N,j_N;2j_N;\frac{2\omega}{1+\omega}\right).$$

- For $\omega=1$, it is reduced to the master formula for DVCS

$$\int_{-1}^{1} dx\,\mathbf{C}(x,Q^2,a_s)\,P_{N-1}^{(\lambda_N)}(x) = C_1\left(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*\right)\frac{2\Gamma(\frac{d}{2}-1)\Gamma(\lambda_N+\frac{1}{2})\Gamma(N-1+2\lambda_N)}{\sigma_N\Gamma(2\lambda_N)\Gamma(j_N+\frac{d}{2}-1)}.$$

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The general solution of the master formula

- To leading order, $\gamma_N = 0$, $\lambda_N = \frac{3}{2}$, $C_1(N) = 1$, $\sigma_N = 1$, the CF is

$$\mathbf{C}^{(0)}(\xi = 1, x) = \frac{1}{1-x} - \frac{1}{1+x},$$

- The following step is to construct the full CF by the convolution of the LO CF $\mathbf{C}^{(0)}(x)$ with a specific kernel $K(x, x')$

$$\mathbf{C}(x) = \int_{-1}^{1} dx'\, C^{(0)}(x') K(x', x),$$

this kernel is defined as

$$\int_{-1}^{1} dx'\, K(x', x) P_{N-1}^{(\lambda_N)}(x') = K(N) P_{N-1}^{(\lambda_N)}(x).$$

- This leads to

$$\int_{-1}^{1} dx\, \mathbf{C}(x) P_{N-1}^{(\lambda_N)}(x) = K(N) \int_{-1}^{1} dx'\, C^{(0)}(x') P_{N-1}^{(\lambda_N)}(x') = 2K(N) B(\lambda_N + \frac{1}{2}, \lambda_N - \frac{1}{2}).$$

The general solution of the master formula

- Comparing with the master formula, we can extract out $K(N)$

$$K(N)=\frac{C_1(N,\frac{Q^2}{\mu^2},a_s,\epsilon_*)}{\sigma_N}\frac{\Gamma(\frac{d}{2}-1)\Gamma(j_N+\lambda_N-\frac{1}{2})}{\Gamma(\lambda_N-\frac{1}{2})\Gamma(j_N+\frac{d}{2}-1)}.$$

- This kernel has very perfect form in the expansion of a_s

$$K^{(1)}(N)=2C_F\left\{\left(\bar{\gamma}_N^{(1)}+\frac{3}{2}\right)^2+\frac{5}{2}\frac{1}{N(N+1)}-\frac{9}{2}\right\},$$

with $\bar{\mathbb{H}}^{(1)}z_{12}^{N-1}=\bar{\gamma}_N^{(1)}z_{12}^{N-1}$, $\mathcal{H}_+z_{12}^{N-1}=\frac{1}{N(N+1)}z_{12}^{N-1}$.

- The kernel in the momentum fraction space

$$K^{(1)}_{\rm DVCS}(x',x)=2C_F\left[\left(\bar{\mathbb{H}}^{(1)}(x',x)+\frac{3}{2}\delta(x'-x)\right)^2+\frac{5}{2}\mathcal{H}_+(x',x)-\frac{9}{2}\delta(x'-x)\right]$$

The general solution of the master formula

- Where the kernel has the form

$$\mathcal{H}_+(z', z) = \theta(z - z')\frac{z'}{z} + \theta(z' - z)\frac{1 - z'}{1 - z},$$
$$\widehat{\mathcal{H}}(z', z) = -\theta(z - z')\frac{z'}{z}\left[\frac{1}{z - z'}\right]_+ + \theta(z' - z)\frac{1 - z'}{1 - z}\left[\frac{1}{z - z'}\right]_+ - \delta(z - z')\left(\ln z + \ln \bar{z}\right),$$

with $z = (1 - x)/2$, $\bar{\mathbb{H}} = \widehat{\mathcal{H}} - \mathcal{H}_+ - \frac{3}{2}$.

- The NLO coefficient function in conformal scheme

$$\mathbf{C}^{(1)}(x) = \int_0^1 dz' \left(\frac{1}{z} - \frac{1}{\bar{z}}\right) K^{(1)}(x', x),$$

- we need to transform it back to the $\overline{\mathrm{MS}}$ scheme

$$C(x) = \int_{-1}^{1} dx'\, \mathbf{C}(x') U(x', x).$$

Content

Summary

- Based on a non-traditional approach, we are trying to calculate **the coefficient functions for double deeply-virtual Compton scattering up to NNLO**.
- The general framework for the calculation of CFs is **in the OPE of two electromagnetic currents using conformal symmetry of QCD at the Wilson-Fischer fixed point in non-integer dimensions.**
- **This approach avoids the computation of complicated mater integrals in traditional loop calculations.**

Summary

- Based on a non-traditional approach, we are trying to calculate **the coefficient functions for double deeply-virtual Compton scattering up to NNLO**.
- The general framework for the calculation of CFs is **in the OPE of two electromagnetic currents using conformal symmetry of QCD at the Wilson-Fischer fixed point in non-integer dimensions.**
- **This approach avoids the computation of complicated mater integrals in traditional loop calculations.**

Thank you for your attention!

Back up: The conformal symmetry and generators

- The dilatation (global scale transformation) and special conformal transformation [Braun, Korchemsky and Muller, 2003]

$$x^\mu \to x'^\mu = \lambda x^\mu, \quad x^\mu \to x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2a \cdot x + a^2 x^2}.$$

- The full conformal algebra in 4 dimensions includes fifteen generators
  $\mathbf{P}_\mu$ (4 translations)
  $\mathbf{M}_{\mu\nu}$ (6 Lorentz rotations)
  $\mathbf{D}$ (dilatation)
  $\mathbf{K}_\mu$ (4 special conformal transformations)
- The commutation relations that specify the conformal algebra

$$\begin{aligned}
&i[\mathbf{P}_\mu, \mathbf{P}_\nu] = 0, \quad i[\mathbf{M}_{\alpha\beta}, \mathbf{P}_\mu] = g_{\alpha\mu}\mathbf{P}_\beta - g_{\beta\mu}\mathbf{P}_\alpha, \\
&i[\mathbf{M}_{\alpha\beta}, \mathbf{M}_{\mu\nu}] = g_{\alpha\mu}\mathbf{M}_{\beta\nu} - g_{\beta\mu}\mathbf{M}_{\alpha\nu} - g_{\alpha\nu}\mathbf{M}_{\beta\mu} + g_{\beta\nu}\mathbf{M}_{\alpha\mu}, \\
&i[\mathbf{D}, \mathbf{P}_\mu] = \mathbf{P}_\mu, \quad i[\mathbf{D}, \mathbf{K}_\mu] = -\mathbf{K}_\mu, \\
&i[\mathbf{M}_{\alpha\beta}, \mathbf{K}_\mu] = g_{\alpha\mu}\mathbf{K}_\beta - g_{\beta\mu}\mathbf{K}_\alpha, \quad i[\mathbf{P}_\mu, \mathbf{K}_\nu] = -2g_{\mu\nu}\mathbf{D} + 2\mathbf{M}_{\mu\nu}, \\
&i[\mathbf{D}, \mathbf{M}_{\mu\nu}] = i[\mathbf{K}_\mu, \mathbf{K}_\nu] = 0.
\end{aligned}$$