Heavy Quark Physics.

Oct. 4. 2022

- References

1. Manohar, Wise, " Heavy quark Physics ", Cam. Uni. Pres. 2000.

2. Neubert, " Heavy quark symmetry ", Phys. Rep. 1994.

3. Schwartz, " Quantum field theory & the SM ", Cam. Uni. Pres. 2014

4. Georgi, " Heavy quark effective field theory ", TASI, 1991.

Part I. Renormalization of the HQET lagrange & operators

Introducing the w.f. renormalization for the effective heavy quark

$$\underbrace{Q_v}_{\text{renormalized field.}} = Z_h^{-1/2} \cdot \overbrace{Q_v^{(0)}}^{\text{bare field.}} , \qquad \text{——— (1)}$$

We can express the "bare" HQET lagrange

$$\mathcal{L}_{eff} = i \cdot \bar{Q}_v^{(0)} \, v^\mu \, [\, \partial_\mu + i \, \underline{g^{(0)} \cdot A_\mu^{(0)}} \,] \cdot Q_v^{(0)} \qquad \text{——— (2).}$$

For the background gauge, $\boxed{g^{(0)} \cdot A_\mu^{(0)} = \mu^{\epsilon/2} \cdot g \cdot A_\mu}$ with $n = 4 - \epsilon$,

Comments: • The advantage of using the background gauge is that the form of the covariant derivative and the gluon field strength is preserved during renormalization. (see Ref.[2]).

• The only modification of the Feynman rules is the replacement of $g^{bare} \to \mu^{\epsilon/2} g$.

• More discussions on the background field method can be found in Ref. [3].

in the following form.

renormalized field. renormalized quantities!

$$\mathcal{L}_{eff} = i \cdot Z_h \cdot \bar{Q}_v \cdot v^\mu [\, \partial_\mu + i g \mu^{\epsilon/2} \cdot A_\mu \,] \, Q_v$$

$$= i \, \bar{Q}_v \, v^\mu [\, \partial_\mu + i \, \mu^{\epsilon/2} \cdot g \, A_\mu \,] \, Q_v \qquad \text{——— (3).}$$

$$\underbrace{\boxed{+ \, [\, Z_h - 1 \,] \cdot i \cdot \bar{Q}_v \, v^\mu \, [\, \partial_\mu + i \, \mu^{\epsilon/2} \, g \, A_\mu \,] \, Q_v}}_{\text{Counterterms!}}$$

Now we proceed to compute the renormalization constant "Z_h" by computing the quark self-energy diagram.

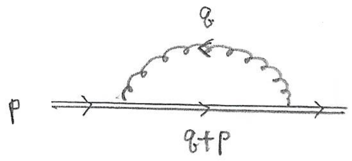


Fig 1. quark self-energy diagram @ 1-loop.

The resulting amplitude can be written as.

$$i\Sigma_2(v) = \int \overset{\text{loop momentum.}}{[d^n q]} \underbrace{\boxed{(-i\cdot g\, T^A \cdot \mu^{\epsilon/2})}}_{\text{due to the convention of } D_\mu\text{, see eq. (2)}} \frac{i\, \overset{\text{vertex}}{\boxed{v_\lambda}}}{\underbrace{v\cdot(q+p)+i0}_{\text{heavy-quark propagator}}} \boxed{(-ig\, T^A\, \mu^{\epsilon/2}) \cdot \overset{\text{vertex}}{v^\lambda}} \frac{-i}{q^2+i0}$$

$$= -g^2 \cdot \overset{=\frac{4}{3} \text{ for } SU(3)_c}{C_F} \cdot \mu^\epsilon \int [d^n q] \frac{1}{[v\cdot(q+p)+i0]\cdot \underbrace{[q^2+i0]}}$$

In order to seperate U.V. physics from IR physics, we introduce the IR regulator: gluon mass.

$$= -2g^2 \cdot C_F \cdot \mu^\epsilon \int_0^\infty d\lambda \int [d^n q] \frac{1}{[q^2 + 2\lambda v\cdot(q+p) - m^2 + i0]} \qquad (4)$$

Gergi parametrization of Feynman integrals

Georgi parametrization: [Exercise I].

$$\boxed{\frac{1}{A^m \cdot B^n} = 2^n \cdot \frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)} \cdot \int_0^\infty d\lambda \cdot \frac{\lambda^{n-1}}{[A + 2\lambda \cdot B]^{m+n}}} \qquad \text{——— (5)}$$

Then we have

$$i\,\Sigma_2(V) = -2 \cdot g^2 \cdot C_F \cdot \mu^\epsilon \cdot \int_0^\infty d\lambda \, \frac{i}{(4\pi)^{2-\epsilon/2}} \, \Gamma(\tfrac{\epsilon}{2}) \cdot \underline{\frac{1}{[\lambda^2 - 2\lambda V\cdot P + m^2 - i0]^{\epsilon/2}}}$$

Clearly, there will be IR div, when $\lambda \to 0$, $m \to 0$!

$$= -i \cdot \frac{\alpha_s}{2\pi} \cdot C_F \cdot (4\pi\mu^2)^{\epsilon/2} \cdot \Gamma(\tfrac{\epsilon}{2}) \cdot \frac{1}{1-\epsilon} \cdot \Bigg\{ \frac{V\cdot P}{(m^2)^{\epsilon/2}}$$

$$\boxed{- \epsilon \cdot \underbrace{[m^2 - (V\cdot P)^2] \cdot \int_0^\infty d\lambda \cdot \frac{1}{\lambda^2 - 2\lambda V\cdot P + m^2}}_{\text{finite term.}}} \Bigg\}$$

↓

a $O(\epsilon)$ piece !

$$= -i \cdot \frac{\alpha_s}{\pi} \cdot C_F \cdot \frac{V\cdot P}{\epsilon} + (\text{finite terms})$$

Note that here "ϵ" is defined as $\boxed{n = 4 - \epsilon\,!}$

see eq. (3.59) of Ref. [1].

$$= -i \, \frac{\alpha_s C_F}{\pi} \cdot V\cdot P \, \Big[\frac{1}{\epsilon} + \underbrace{\frac{1}{2} \cdot \ln \frac{\mu^2}{(V\cdot P)^2}}_{\text{Keep the "}\ln\mu\text{" term here!}} \Big] + (\text{finite terms}). \qquad \text{——— (6)}$$

From eq. (3), we can write down the counterterm contribution.

$$[Z_h - 1] \quad \underbrace{i}_{\substack{\downarrow \\ \text{from the prefactor} \\ \text{"}i\text{" in "}i\mathcal{L}_{eff}\text{"}.}} \cdot \quad \underbrace{(v\cdot p)}_{\text{from "}iv\cdot D\text{" in } \mathcal{L}_{eff}} \qquad (7)$$

Adding eqs. (6) and (7) must lead to the finite amplitude, and we find (in the $\overline{MS}$ scheme).

$$Z_h = 1 + \frac{\alpha_s C_F}{\pi} \frac{1}{\epsilon} + O(\alpha_s^2). \qquad (8)$$

Which means that

$$\gamma_h = \frac{1}{2} \cdot \frac{d \ln Z_h}{d \ln \mu} = - \frac{\alpha_s C_F}{2\pi} \qquad (9)$$

$\downarrow$ recall that $Q_v^{(0)} = Z_h^{1/2} \cdot Q_v$ in eq. (1).

Comments: one can also determine the renormalization constant in the on-shell scheme.

$$Z_h^{-1} = 1 - \frac{\partial \Sigma(v\cdot p)}{\partial (v\cdot p)} \Big|_{v\cdot p = 0} \qquad (10)$$

Then we obtain

$$Z_{h,\,OS} = 1 + \frac{\alpha_s C_F}{\pi} \left[\frac{1}{\epsilon} + \frac{1}{2} \cdot \ln \frac{\mu^2}{m^2} \right], \qquad (11)$$

$\downarrow$ see Ref. [2], eq. (3.23).

- **Renormalization of the heavy-to-light current.**

Now we proceed to consider the bare heavy-to-light current

$$O_\Gamma^{(0)} = \bar{q}^{(0)} \Gamma \cdot Q_v^{(0)} = \underbrace{[Z_q \cdot Z_h]^{1/2}}_{\text{w.f. renormalization}} \cdot \bar{q} \Gamma \cdot Q_v$$

$$\equiv Z_O \cdot O_\Gamma \qquad (12).$$

Z_O: operator renormalization

O_Γ: renormalized operator.

Neither bare NOR renormalized quantity !

Namely. $$O_\Gamma = [Z_q \cdot Z_h]^{1/2} / Z_O \cdot \boxed{\bar{q} \Gamma Q_v}$$

$$= \bar{q} \Gamma Q_v + \underbrace{\left(\frac{\sqrt{Z_q \cdot Z_h}}{Z_O} - 1 \right) \cdot \bar{q} \Gamma \cdot Q_v}_{\text{counterm term}} \qquad (13)$$

We now compute the 1-loop correction to the heavy-to-light current.

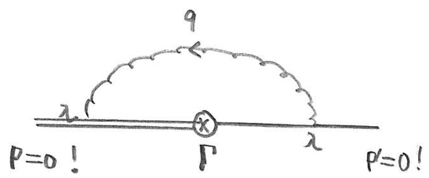


Fig 2. 1-loop correction to the $\bar{q} \Gamma h_v$ current

$$\langle O_\Gamma \rangle^{(1)} = \int [d^n q] \cdot \bar{q}\, (-ig\,\mu^{\epsilon/2}\, T^A) \cdot \gamma^\lambda \cdot \frac{i\,\not{q}}{q^2+i0}\, \Gamma \cdot (-ig \cdot \mu^{\epsilon/2}\, T^A) \cdot \frac{i V_\lambda}{V\cdot q + i0} \cdot \underbrace{\frac{-i}{q^2+i0}}_{\text{gluon propagator}}\, b$$

$$+ \text{(counterterm)}$$

can only generate $(\# \not{V})$ structure

$$= -i \cdot g^2 \cdot C_F \cdot \mu^\epsilon \cdot \int [d^n q]\ \bar{q} \cdot \frac{\not{V}\, \not{q}\, \Gamma}{\underbrace{(q^2+i0)^2}\cdot (V\cdot q + i0)}\, b \quad + \text{(counter term)}$$

$(q^2 - m^2 + i0)^2$, by introducing the I.R. regulator m^2.

$q' = q + \lambda V$

$$= -4i \cdot g^2 \cdot C_F \cdot \mu^\epsilon \cdot \int_0^\infty d\lambda \int [d^n q']\ \bar{q} \cdot \underbrace{\frac{\not{V}\,(\not{q}' - \lambda \not{V})\,\Gamma}{[q'^2 - \lambda^2 - m^2]^3}}\, b \quad + \text{(counterterm)}.$$

the odd term "$\not{q}'$" in the numerator does NOT contribute.

$$= -4i \cdot g^2 \cdot C_F \cdot \mu^\epsilon \int_0^\infty d\lambda \int [d^n q'] \cdot \frac{-\lambda}{(q'^2 - \lambda^2 - m^2)^3}\ \underbrace{\bar{q}\,\Gamma\, b} \quad + \text{(counterterm)}.$$

Soft interaction preserves the spin structure !

$$= \frac{\alpha_s C_F}{2\pi} \cdot \left[\frac{1}{\epsilon} + \frac{1}{2} \cdot \ln \frac{\mu^2}{m^2} + \text{(finite terms)} \right] \cdot \langle O_\Gamma \rangle^{(0)}$$

$$+ \underbrace{\text{(counterterm)}}_{\text{see eq. (13)}} \qquad\qquad (14)$$

Combining eqs. (13) & (14) leads to

$$\frac{\sqrt{Z_q \cdot Z_h}}{Z_0} = 1 - \frac{\alpha_s \cdot C_F}{2\pi} \cdot \frac{1}{\epsilon} + O(\alpha_s^2),$$

$Z_q^{1/2} = 1 - \frac{\alpha_s C_F}{4\pi} \cdot \frac{1}{\epsilon}$, see eq. (1.86) of Ref. [1]

$$\Rightarrow \quad Z_0 = \left(1 + \frac{\alpha_s C_F}{2\pi} \cdot \frac{1}{\epsilon}\right) \cdot (Z_q \cdot Z_h)^{1/2}.$$

eq. (8)

$$= 1 + \frac{3}{4} \cdot \frac{\alpha_s C_F}{\pi} \frac{1}{\epsilon} + O(\alpha_s^2). \qquad (15).$$

Therefore, the corresponding anomalous dimension reads

see eq. (1.129) of Ref. [1].

$$\boxed{\gamma_0 = \frac{d \ln Z_0}{d \ln \mu} = -\frac{3}{4} \cdot \frac{\alpha_s C_F}{\pi} + O(\alpha_s^2).} \qquad (16).$$

<u>Comments on the anomalous dimension</u> <u>(Ph.D thesis of Rohrwild).</u>

- In the $\overline{MS}$-like scheme, the renormalization constants <u>do NOT depend on "μ" EXPLICITLY</u>, but only through the indirect dependence on $\alpha_s(\mu)$.

- Defining $n = 4 - \epsilon$ (dimension of space-time), $\gamma_Q = \frac{d \ln Z_Q}{d \ln \mu}$ (such that $Q^{(0)} = Z_Q \cdot Q$) (renormalized).

  then γ_Q is equal to <u>the negative of the residue</u> of the corresponding renormalization constant. [Exercise II]. (hint: see also Collins, or Coleman).

- Renormalization of the heavy-to-heavy current.

We now continue to discuss the renormalization for the heavy-to-heavy current in HQET.

bare fields.

$$T_\Gamma^{(0)} \equiv \bar{Q}_{v'}^{(0)} \cdot \Gamma \cdot Q_v^{(0)} = Z_h \cdot \bar{Q}_{v'} \, \Gamma \cdot Q_v$$

$$= Z_\Gamma \cdot T_\Gamma \qquad (17)$$

renormalized current.

renormalization constant

Namely,

$$T_\Gamma = (Z_h / Z_\Gamma) \cdot \bar{Q}_{v'} \, \Gamma \cdot Q_v$$

$$= \bar{Q}_{v'} \, \Gamma \cdot Q_v + \underbrace{\left(\frac{Z_h}{Z_\Gamma} - 1\right) \cdot \bar{Q}_{v'} \, \Gamma \cdot Q_v}_{\text{Counter-terms.}} \qquad (18)$$

We now compute the 1-loop correction to the heavy-to-heavy current

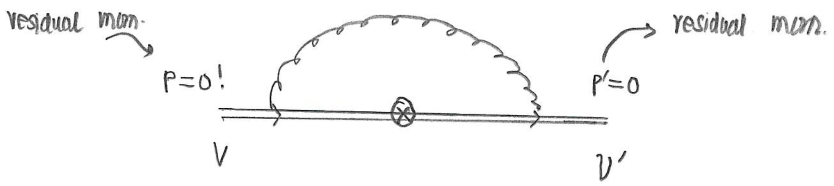


Fig 3. 1-loop correction to the heavy-to-heavy current.

$$\langle T_\Gamma \rangle^{(1)} = -i \cdot g^2 \cdot C_F \cdot \mu^\epsilon \cdot \underbrace{(v \cdot v')}_{\boxed{w \equiv v \cdot v'\,!}} \cdot \int [d^n q] \frac{1}{(q^2+i0)\underbrace{(v \cdot q + i0)(v' \cdot q + i0)}} \bar{Q}_{v'} \cdot \Gamma \cdot Q_v$$

eikonal propagators.

↓

Combine these two terms together with the Feynman parametrization and then apply the Georgi parametrization!

$$+ [\text{counterterm}]$$

$$= -8i \cdot g^2 \cdot C_F \cdot \mu^\epsilon \cdot w \cdot \int_0^\infty d\lambda \int_0^1 dx \int [d^n q] \frac{\lambda}{\{q^2 + 2\lambda [x v + \bar{x} v'] \cdot q - m^2 + i0\}^3}$$

IR regulator to separate U.V. from I.R!

$$\cdot \bar{Q}_{v'} \cdot \Gamma \cdot Q_v + [\text{counterterm}]$$

$$= -\frac{\alpha_s C_F}{\pi} \mu^\epsilon \cdot w \cdot \underbrace{\int_0^\infty d\lambda}_{\downarrow} \int_0^1 dx \frac{\lambda}{\{\lambda^2 [1 + 2x \cdot \bar{x} \cdot \underbrace{(w-1)}_{w \geq 1.}] + m^2\}^{1+\epsilon/2}} \cdot \langle T_\Gamma \rangle^{(0)}$$

can be computed easily!

$$+ [\text{counterterm}]$$

$$= -\frac{\alpha_s C_F}{\pi} \cdot \left(\frac{\mu^2}{m^2}\right)^\epsilon \cdot w \cdot \int_0^1 dx \cdot \frac{1}{[1 + 2x \cdot \bar{x} (w-1)]} \cdot \langle T_\Gamma \rangle^{(0)} + [\text{counterterm}].$$

$$= -\frac{\alpha_s C_F}{\pi} \cdot w \cdot r(w) \cdot \left[\frac{1}{\epsilon} + \frac{1}{2} \cdot \ln \frac{\mu^2}{m^2} + (\text{finite terms}\right] \cdot \langle T_\Gamma \rangle^{(0)}$$

$$+ \underbrace{[\text{counter term}]}_{\text{see eq. (18)}}. \qquad\qquad (19)$$

Combining eqs. (18) & (19) leads to

$$\frac{Z_h}{Z_T} = 1 + \frac{\alpha_s C_F}{\pi} \omega \cdot r(\omega) \cdot \frac{1}{\epsilon} + O(\alpha_s^2).$$

$$\Rightarrow \quad Z_T = \left[1 - \frac{\alpha_s C_F}{\pi} \omega \cdot r(\omega) \cdot \frac{1}{\epsilon}\right] \cdot Z_h$$

eq. (8).

$$= 1 - \frac{\alpha_s C_F}{\pi} \cdot \underline{[\omega \cdot r(\omega) - 1]} \cdot \frac{1}{\epsilon} + O(\alpha_s^2). \qquad \text{—— (20)}$$

This form must be the case, due to the conservation of the HQET vector current!

The explicit form of the loop function "$r(\omega)$" is given by

$$r(\omega) = \frac{1}{\sqrt{\omega^2-1}} \cdot \ln\left(\omega + \sqrt{\omega^2-1}\right). \qquad \text{—— (21)}$$

[Exercise III]: Please compute the loop integral of eq. (19) by using the Georgi parametrization only!

[Hint: In this case you will encounter the 2-dimensional integrals like

$$\int_0^\infty d\lambda \int_0^\infty d\lambda' \frac{1}{[1+\lambda^2+\lambda'^2+2\lambda\cdot\lambda'\cdot\omega]^3}$$

$$= \frac{1}{2} \int_0^\infty \underbrace{d r^2}_{= 2\cdot r\cdot dr} \int_0^{\pi/2} d\theta \frac{1}{[1+r^2(1+\omega\cdot\sin 2\theta)]^3}$$

$$= \frac{1}{4} \cdot \int_0^{\pi/2} d\theta \frac{d\theta}{(1+\omega\cdot\sin 2\theta)}$$

$$= \frac{1}{4} \cdot \frac{1}{\sqrt{\omega^2-1}} \cdot \ln\left(\omega + \sqrt{\omega^2-1}\right).$$ (see, Ref. [4]).]

From eq. (20), it's then clear that the anomalous dimension γ_T is given by.

$$\gamma_T = \frac{\alpha_s C_F}{\pi} \cdot \underbrace{[\,w \cdot r(w) - 1\,]} + O(\alpha_s^2) \equiv \frac{d \ln Z_T}{d \ln \mu}. \qquad (22)$$

comment: expanding the terms in the square bracket leads to

" $\frac{2}{3}(w-1) - \frac{1}{5}(w-1)^2 + O((w-1)^3)$ " !!

which vanises at $w=1$ obviously.

Part II. Matching of the heavy-to-light currents from QCD $\rightarrow$ HQET.

Step I) Consider the renormalization of the heavy-to-light currents in QCD

$$J^{\mu(0)}_{V(A)} = \bar{q}^{(0)} \gamma^\mu (\gamma_5)\, Q^{(0)}, \qquad \text{with} \quad m_q = 0. \qquad (23).$$

chiral symmetry

The renomalized QCD amplitude @ 1-loop is give by

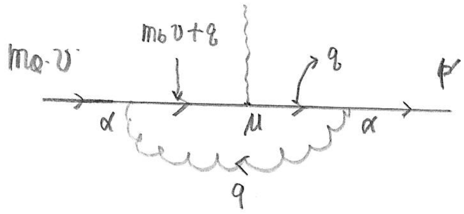
operator renormalization.

$$\langle J^\mu_V \rangle^{(1)} = Z_q^{1/2} \cdot Z_Q^{1/2} \cdot Z_{V,QCD}^{-1} \cdot \langle \bar{q}\, \gamma_\mu\, Q \rangle \big|_{1\text{-loop}}$$

vanishes for $J^\mu_{V(A)}$ in QCD

$$= \langle \bar{q}\, \gamma_\mu\, Q \rangle \big|_{1\text{-loop}} + \underbrace{[\, Z_q^{1/2} \cdot Z_Q^{1/2} / Z_{V,QCD} - 1\,] \cdot \langle \bar{q}\, \gamma_\mu\, Q \rangle \big|_{\text{tree}}}_{\text{counter term.}}$$

$$\Rightarrow (Z_q^{1/2} \cdot Z_Q^{1/2} - 1) \cdot \langle \bar{q}\, \gamma_\mu\, Q \rangle^{(0)} \qquad (24)$$



p with $p^2 = 0$! In fact, we can set $p_\mu = 0$.

This is because we are doing hard-soft factorization.

Fig 4. 1-loop QCD correction to the heavy-to-light current

$$\langle J_V^\mu \rangle^{(1)} = -i \cdot g^2 \cdot C_F \cdot \mu^{\epsilon} \int [d^n q] \cdot \frac{1}{(q^2 + i0)^2 \cdot [(q + m_b v)^2 - m_b^2 + i0]}$$

Introduce the gluon mass here.

$$\cdot\ \bar{q}\, \gamma_\alpha \cdot \not{q} \cdot \gamma^\mu\, (m_b \not{v} + \not{q} + m_Q) \cdot \gamma^\alpha \cdot b \quad + \quad [\text{counterterm}]$$

$$= \underline{\left\{ \frac{\alpha_s C_F}{2\pi} \cdot \left[\frac{1}{\epsilon} + \frac{1}{2} \ln \frac{\mu^2}{m^2} - \frac{1}{2} \right] \cdot \right\}}\ \bar{q}\, \gamma^\mu\, b \quad + \quad \left[\underline{\frac{\alpha_s C_F}{2\pi}} \cdot \right] \bar{q}\, v^\mu\, b$$

$$+ \ [\underbrace{Z_q^{1/2} \cdot Z_Q^{1/2}}_{\text{on-shell scheme!}} - 1] \cdot \ \bar{q}\, \gamma^\mu\, b \qquad \text{———————} \quad (25)$$

Using the renormalization constants of Z_q and Z_Q in the on-shell scheme (see eq. (3.23) of Ref. [2]).

gluon mass, IR regulator.

$$Z_Q = 1 - \frac{\alpha_s C_F}{2\pi} \left[\frac{1}{\epsilon} + \frac{1}{2} \cdot \ln \frac{\mu^2}{m_Q^2} + \ln \frac{m^2}{m_Q^2} + 2 \right] + O(\alpha_s^2),$$

$$Z_q = 1 - \frac{\alpha_s C_F}{2\pi} \left[\frac{1}{\epsilon} + \frac{1}{2} \cdot \ln \frac{\mu^2}{m^2} - \frac{1}{4} \right] + O(\alpha_s^2).$$

$$\text{———————} \quad (26)$$

[Exercise IV: Please verify the above results explicitly !]

we obtain

$$Z_q^{1/2} \cdot Z_Q^{1/2} - 1 = - \frac{\alpha_s C_F}{2\pi} \cdot \left[\frac{1}{\epsilon} + \frac{1}{2} \cdot \ln \frac{\mu^2}{m_Q^2} + \frac{1}{4} \ln \frac{m^2}{m_Q^2} + \frac{7}{8} \right] + O(\alpha_s^2). \qquad (27)$$

Inserting eq (27) into eq. (25) leads to.

$$\langle J_V^\mu \rangle^{(1)} = \underline{\left\{ \frac{\alpha_s C_F}{2\pi} \cdot \left[\frac{3}{4} \cdot \ln \frac{m_Q^2}{m^2} - \frac{11}{8} \right] \right\}} \bar{q} \gamma^\mu b + \left[\frac{\alpha_s C_F}{2\pi} \right] \cdot \bar{q} v^\mu b \qquad (28)$$

① No u.v. divergences, since the vector current is conserved in QCD ! (not in QED !!) see Collins, Manohar, Wise, hep-th/0512187.

② IR. divergence appears, but it will be cancelled by the HQET matrix elements. physically, this is because of the unphysical external states ! (more discussions, see Ref. [2]).

Step II) Consider the renormalization of the heavy-to-light current in HQET.

$$J_{eff}^{(0)} = \bar{q}^{(0)} \cdot \Gamma_{eff} \cdot Q_v^{(0)} \qquad (29)$$

see eq. (12)

It's then clear that the renormalized effective current @ 1-loop is given by the following expression (see eq. (14)).

$$\langle J_{eff} \rangle^{(1)} = \underbrace{\left\{ \frac{\alpha_s C_F}{2\pi} \left[\frac{1}{\epsilon} + \frac{1}{2} \cdot \ln\frac{\mu^2}{m^2} + \frac{1}{2} \right] \cdot \right\}}_{\text{see eq. (14), OR eq. (3.58) of Ref. [2].}} \bar{q}\, \Gamma_{eff}\, Q_v$$

$$+ \underbrace{\boxed{\left[\frac{\sqrt{Z_q \cdot Z_h}}{Z_0} - 1 \right]} \cdot \bar{q}\, \Gamma_{eff} \cdot Q_v}_{\text{Counterterm!}} \qquad (30)$$

see eq. (13) !

on-shell scheme !

Using eqs. (26), eq. (11) and eq. (15), we have

$$Z_q^{1/2} \cdot Z_h^{1/2} / Z_0 - 1$$

$$= - \frac{\alpha_s C_F}{2\pi} \left[\frac{1}{\epsilon} - \frac{1}{4} \cdot \ln\frac{\mu^2}{m^2} - \frac{1}{8} \right] + O(\alpha_s^2). \qquad (31)$$

Inserting eq. (31) into eq. (30) leads to

$$\langle J_{eff} \rangle^{(1)} = \underline{\frac{\alpha_s C_F}{2\pi} \cdot \left[\frac{3}{4} \ln\frac{\mu^2}{m^2} + \frac{5}{8} \right]} \quad \bar{q}\, \Gamma_{eff} \cdot Q_v \qquad (32)$$

① No u.v. divergence, since the renormalization has been implemented.

② Must contain the same I.R. physics as eq. (28)

③ It's clear that the u.v. renormalization of the dimension-3 HQET operator is free of the operator-mixing !

Step-III) Applying the matching condition in perturbative QFT.

$$J^{\mu}_{V,\,QCD} = \sum_i C_{V,i} \cdot J_{\text{eff},\,i}.$$

renormalized QCD current

matching coefficient

renormalized HQET current

$$\equiv C_{V,1} \cdot \bar{q}\, \gamma^{\mu} \cdot Q_v + C_{V,2}\, \bar{q}\, v^{\mu} \cdot Q_v \qquad (33)$$

We can readily determine the tree-level matching coefficients

$$\boxed{C^{(0)}_{V,1} = 1, \qquad C^{(0)}_{V,2} = 0.} \qquad (34)$$

Comparing eqs. (28) and (30), we can proceed to determine the 1-loop matching coefficients.

$$\boxed{\begin{aligned} C^{(1)}_{V,1} &= \frac{\alpha_s C_F}{2\pi} \left[\frac{3}{4} \ln \frac{m_Q^2}{\mu^2} - 2 \right], \\ C^{(1)}_{V,2} &= \frac{\alpha_s C_F}{2\pi}. \end{aligned}} \qquad (35)$$

Comments:

① It's clear that the Wilson Coefficients are independent of the I.R. regulator.

② Actually, these matching coefficients must be also independent of the external partonic states. [Exercise V: Compute the hard function of $B \to \gamma e \nu$ with a soft photon radiation] (hint: The soft dynamics is determined by the HQET correlation function here)

☆ Perturbative matching with the dim. reg. for both. U.V & IR divergences.

This is clear the best strategy to perform the matching calculation in perturbative QFT.

Step I: 1-loop correction to the QCD current:

We first list the renormalization constants (see eq. (3.63) of Ref. [2]).

both U.V & I.R.

$$Z_Q = 1 - \frac{\alpha_s C_F}{2\pi} \cdot 3 \cdot \left[\frac{1}{\epsilon} - \frac{1}{2} \cdot \ln \frac{m_Q^2}{\mu^2} + \frac{2}{3} \right] + O(\alpha_s^2).$$

compare eq. (26)!

$$Z_q = 1. \qquad \text{——— (36)}$$

The yielding 1-loop correction to the heavy-to-light current in QCD reads: (see eq. (3.64) of Ref. [2])

$$\langle J_V^\mu \rangle^{(1)} = \left\{ -\frac{\alpha_s \cdot C_F}{\pi} \cdot \frac{3}{4} \cdot \left[\frac{1}{\epsilon} - \frac{1}{2} \cdot \ln \frac{m_Q^2}{\mu^2} + \frac{4}{3} \right] \right\} \cdot \bar{q} \gamma^\mu b + \left[\frac{\alpha_s C_F}{2\pi} \right] \bar{q} \gamma^\mu b \qquad \text{——— (37)}$$

only IR pole here!

This is because the vector QCD current is NOT renormalized. (see also eq. (28))

Step II. 1-loop correction to the effective current.

only scaleless integrals appear. OR U.V + I.R = 0 !

$$\langle J_{eff} \rangle^{(1)} = 0 \; ! \qquad \text{——— (38)}$$

Step III. Implemeting the matching condition as shown in eq (33) leads to

$$C_{V,1}^{(1)} = -\frac{\alpha_s C_F}{\pi} \cdot \frac{3}{4} \left[\frac{1}{\epsilon} - \frac{1}{2} \cdot \ln \frac{m_Q^2}{\mu^2} + \frac{4}{3} \right]$$

$$+ \underbrace{\left[Z_{V,1}^{-1} - 1 \right]} \cdot \overbrace{C_{V,1}^{(0)}}^{=1}$$

$$= Z_0 - 1 \quad (\text{see eq. (15) for } Z_0)$$

$$= \frac{3}{4} \frac{\alpha_s C_F}{\pi} \cdot \frac{1}{\epsilon} \Leftarrow$$

(This must be the case, since the I.R. pole of the wilson coefficent is the negative of the U.V. pole of the effective operator)

$$= -\frac{\alpha_s C_F}{\pi} \cdot \frac{3}{4} \left[-\frac{1}{2} \cdot \ln \frac{m_Q^2}{\mu^2} + \frac{4}{3} \right]$$

$$= \frac{\alpha_s C_F}{2\pi} \cdot \left[\frac{3}{4} \cdot \ln \frac{m_Q^2}{\mu^2} - 2 \right], \qquad (39)$$

This is exactly the same as eq. (35).

Clearly. $$C_{V,2}^{(1)} = \frac{\alpha_s C_F}{2\pi}. \qquad (40)$$

Exactly the same as eq. (35).

Comment: Here, we have used an important relation in effective field theory generally

$$C^{(bare)} \cdot Q_{eff}^{(bare)} \overset{!}{=} C^{(ren)}(\mu) \cdot Q_{eff}^{(ren)}(\mu)$$

$$\Longleftrightarrow \quad Z_C \cdot Z_Q = 1 \quad (\text{by definition}).$$

This is precisely the argument used in the above.

This can be further generalized to the case with the nontrivial operator mixing.

Part III: Matching of the heavy-to-heavy currents from QCD → HQET.

The general discussion of matching the heavy-to-heavy currents from QCD → HQET is rather tedious, because the 1-loop QCD correction will depend on ① the kinematic variable $w = v \cdot v'$ and on ② the two heavy quark masses $(m_b\ m_c)$. To illustrate the strategy clearly we will seperate these two difficulties by either taking the $m_c \to m_b$ limit (flavour-conserving) as adopted by Ref [2]. or taking the zero-recoil $w \to 1$ limit. (as adopted by Ref. [1]). Since the conceptual fundamendation is rather standard in both cases, we will focus on the $w \to 1$ limit here. The great adlvantage is that the yielding matching condition is rather simple. More specially.

$$\underbrace{H^{\mu}_{V,\,QCD}}_{\substack{\text{renormalized QCD current} \\ \bar{c}\,\gamma^\mu b}} = f_V(m_b, m_c) \cdot \underbrace{H^{\mu}_{V,\,eff}}_{\substack{\text{renormalized effective current} \\ \bar{c}_v \cdot \gamma^\mu b_v}}, \qquad (41)$$

To achieve the matching condition (41) we have to proof the following equations.

$$\boxed{① \quad \bar{c}_v \cdot \gamma_5\, b_v = 0, \qquad ② \quad \bar{c}_v \gamma_\mu\, b_v = \bar{c}\, v_\mu\, b_v} \qquad (42)$$

Proof: ① $\bar{c}_v\, \gamma_5\, b_v = \bar{c}_v\, \gamma_5\, \not{v}\, b_v = -\bar{c}_v\, \not{v}\, \gamma_5\, b_v$

$$= -\bar{c}_v\, \gamma_5\, b_v = 0\,!$$

Another way to see this result is as follows.

$$\bar{c}_v \gamma_5 b_v = \bar{c}_v \frac{1+\slashed{v}}{2}\cdot \gamma_5 \frac{1+\slashed{v}}{2} b_v = \bar{c}_v \gamma_5 \underbrace{\frac{1-\slashed{v}}{2}\cdot \frac{1+\slashed{v}}{2}}_{=0} b_v = 0\,!$$

② $$\bar{c}_v \gamma_\mu b_v = \bar{c}_v \gamma_\mu \frac{1+\slashed{v}}{2} b_v = \underbrace{\bar{c}_v \frac{1-\slashed{v}}{2}}_{=0} \gamma_\mu b_v + \bar{c}_v v_\mu b_v$$

$$= \bar{c}_v v_\mu b_v.$$

Now, we proceed to compute the matching coefficient following the standard procedure.

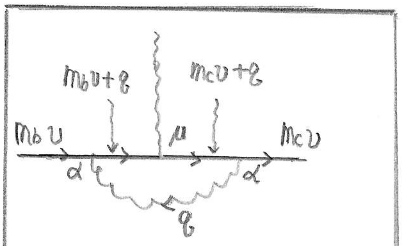


Fig. 5 1-loop QCD correction for the heavy-to-heavy current!

step I.) 1-loop correction to the heavy-to-heavy current in QCD.

$$\langle H_V^\mu \rangle^{(1)} = -i\cdot g^2\cdot C_F\cdot \mu^\epsilon \cdot \int [d^n q] \frac{1}{[(q+m_b v)^2 - m_b^2]\,[(q+m_c v)^2 - m_c^2]\cdot q^2}$$

$$\cdot\, \bar{c}\cdot \gamma_\alpha (\slashed{q} + m_c \slashed{v} + m_c)\, \gamma^\mu \cdot (\slashed{q} + m_b \slashed{v} + m_b)\, \gamma^\alpha \cdot b$$

$$+ \underbrace{[\, Z_b^{1/2}\cdot Z_c^{1/2} - 1\,]\cdot \bar{c}\, \gamma^\mu b}_{\text{Counter term}}$$

Counter term, (see eq. (25) for the heavy-to-light case)

both U.V & IR poles here. Note we do NOT use the gluon-mass regulator here.

$$= \left\{ + \frac{\alpha_s C_F}{2\pi}\cdot \left[\frac{3}{\epsilon} - 3\cdot \frac{m_b \cdot \ln\frac{m_c}{\mu} - m_c\cdot \ln\frac{m_b}{\mu}}{m_b - m_c} - 1 \right] \right\} \bar{c}\, \gamma^\mu b$$

from eq. (3.97) of Ref. [1].

$$+ [\, Z_b^{1/2}\cdot Z_c^{1/2} - 1\,]\cdot \bar{c}\, \gamma^\mu b \qquad \text{——— (43)}$$

Using the renormalization constants Z_Q in eq. (36) yields.

$$Z_b^{1/2} \cdot Z_c^{1/2} - 1 = - \frac{\alpha_s C_F}{2\pi} \cdot 3 \cdot \left[\frac{1}{\epsilon} - \frac{1}{2} \left(\ln \frac{m_b}{\mu} + \ln \frac{m_c}{\mu} \right) + \frac{2}{3} \right] \quad \text{——— (44)}$$

Inserting eq. (44) into eq. (43) leads to

$$\langle H_V^\mu \rangle^{(1)} = \frac{\alpha_s \cdot C_F}{2\pi} \left[\frac{3}{2} \cdot \frac{m_b + m_c}{m_b - m_c} \cdot \ln \frac{m_b}{m_c} - 3 \right]$$

$$= \frac{3}{4} \cdot \frac{\alpha_s C_F}{\pi} \left[\frac{m_b + m_c}{m_b - m_c} \cdot \ln \frac{m_b}{m_c} - 2 \right]. \quad \text{——— (45)}$$

① This QCD matrix element must be independent of the U.V. renormalization scale, since the vector current is conserved in QCD !

② This QCD matrix element must be independent of the I.R. regularization parameter, since the effective matrix element vanishes in dim. reg..

Of course, this QCD matrix element will dependent on the regulator, if we use the fictious gluon-mass as the I.R. regulator ! (see eq. (28) for the heavy-to-light case).

Step II) 1-loop correction to the effective matrix element.

At the first sight, one may argue that the effective matrix element may NOT vanish, since the relevant integrals in HQET can dependent on the variable $w = v \cdot v'$. But we only consider the zero-recoil limit $w \to 1$, where the effective current $\bar{c}_v \gamma_\mu b_v = c_v^\dagger b_v = h_v^\dagger h_v$ corresponds to the Noether charge of the heavy-quark flavour, the resulting effective matrix element therefore is NOT renormalized, i.e.,

$$\langle H^{\mu}_{V,\mathrm{eff}} \rangle^{(1)} \overset{w = v\cdot v' = 1}{=} \bar{c}_v \cdot \gamma^\mu b_v \; ! \qquad (46)$$

Step III) Implemeting the mactching condition shown in eq. (41) gives rise to

$$f_V(m_b, m_c) = \underbrace{1}_{\text{tree-level coefficent}} + \frac{3}{4}\frac{\alpha_s \cdot C_F}{\pi}\left[\frac{m_b + m_c}{m_b - m_c}\cdot \ln\frac{m_b}{m_c} - 2\right], \qquad (47).$$

Comments: ① In the limit $m_c \to m_b$, the QCD vector current is exactly conserved, namely, the perturbative matching coefficent must be equal to "1" to all orders in perturbation theory. This can be verified explicitly that

$$\lim_{m_c \to m_b} f_V(m_b, m_c) = 1. \qquad (48).$$

② More discussions on the flavour-conserving heavy-to-heavy current at $w \neq 1$ and on the flavour-changing heavy-to-heavy current case at $w \neq 1$ can be found in Sections 3.7 & 3.8 of Ref. [2].