R-ratio from Lattice QCD

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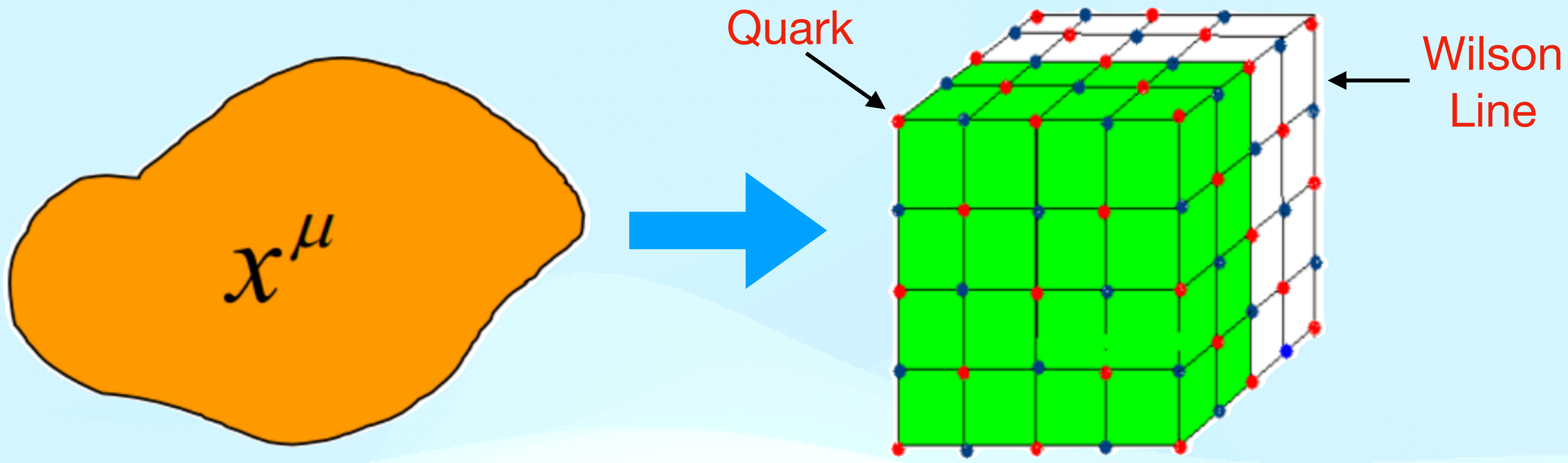
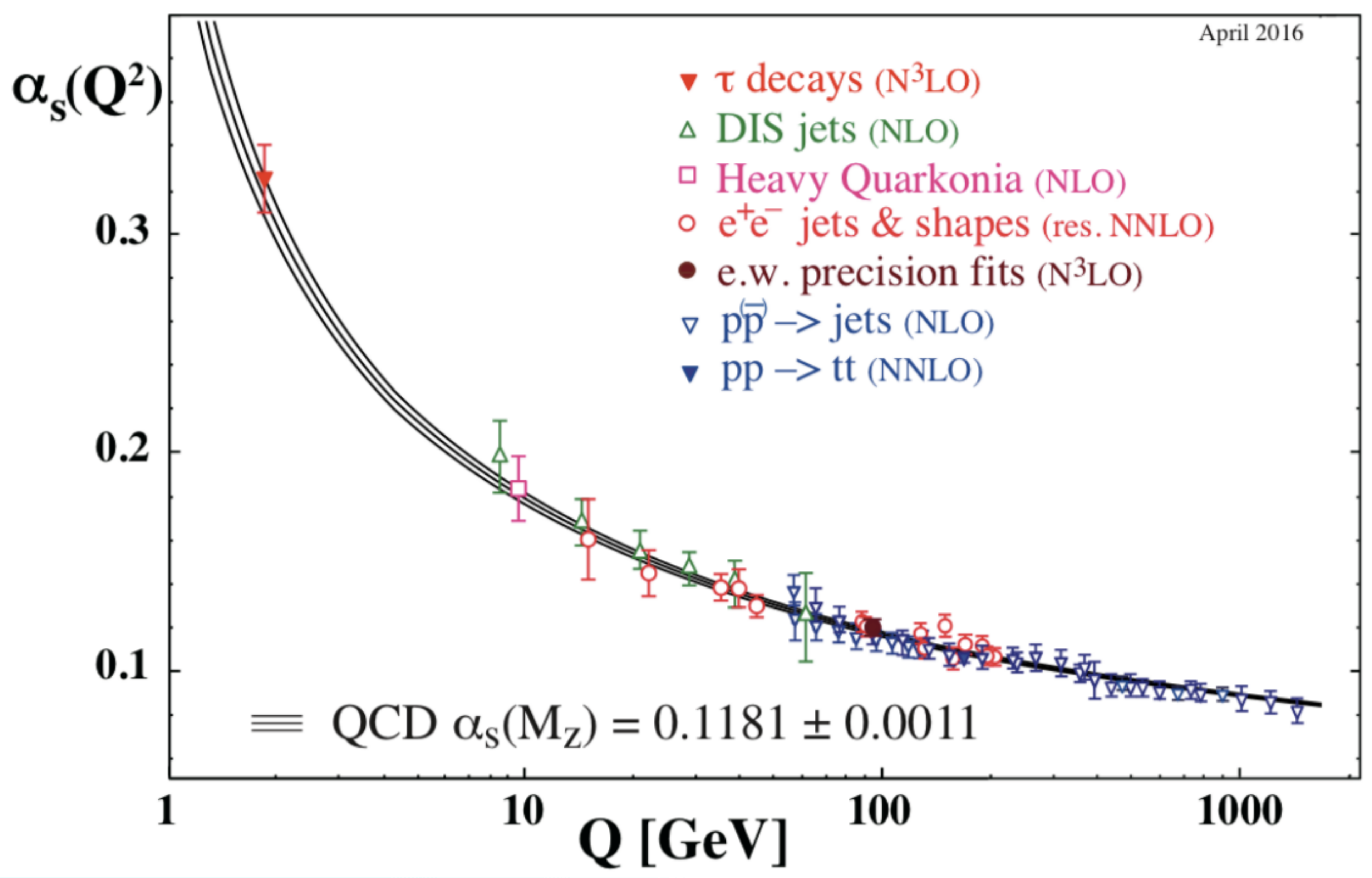
Outline

- Introduction to Lattice QCD
- Inverse Problem
- R-ratio on The Lattice
- Summary and Outlook

Introduction to Lattice QCD

The most systematic theoretical method for studying non-perturbative QCD is Lattice QCD.





The path integral on the lattice:

- Space-time discretization→Finite dimensional integral
- Euclidean space time→Importance sampling

$$\begin{cases} Z = \int D\phi e^{-S_E[\phi]} \\ \langle O[\phi] \rangle = \frac{1}{Z} \int D\phi O[\phi] e^{-S_E[\phi]} \end{cases}$$

- Monte Carlo simulations →Calculate path integral.

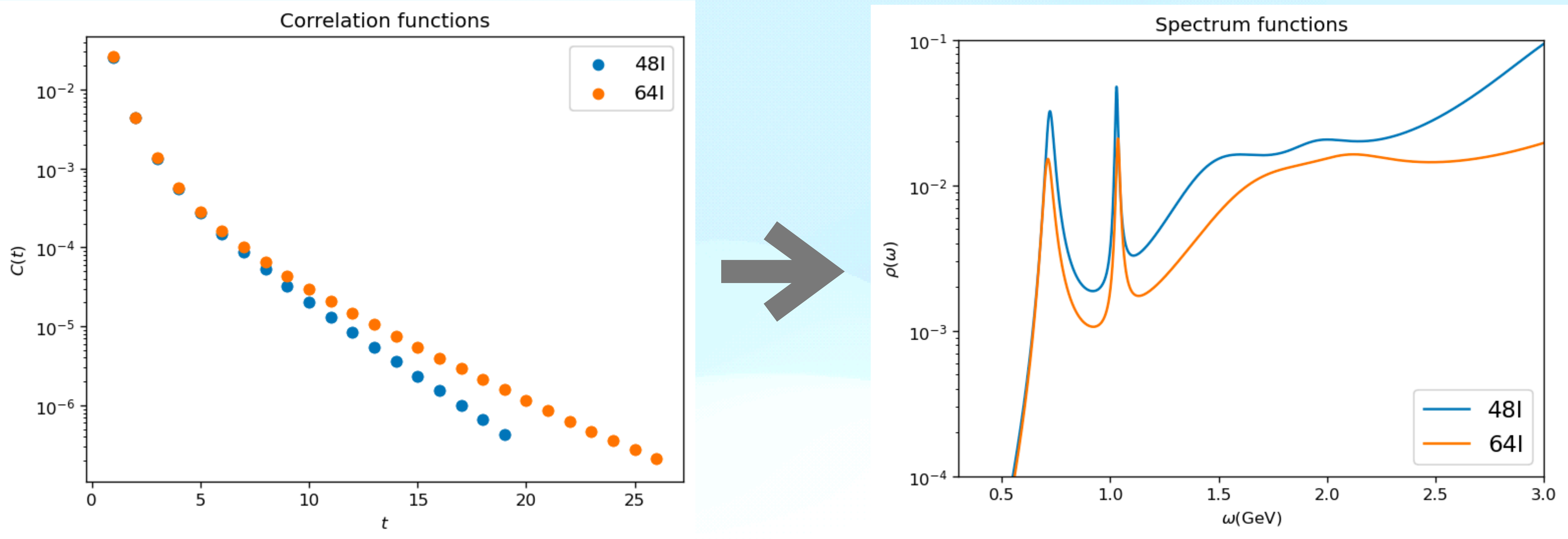
Inverse Problem

The two points correlation functions in finite volume can be expressed as:

$$C(t) = \int d\omega \rho_L(\omega) e^{-\omega t}$$

$\rho_L(\omega)$ is the spectrum function at finite volume, $e^{-\omega t}$ is the kernel function.

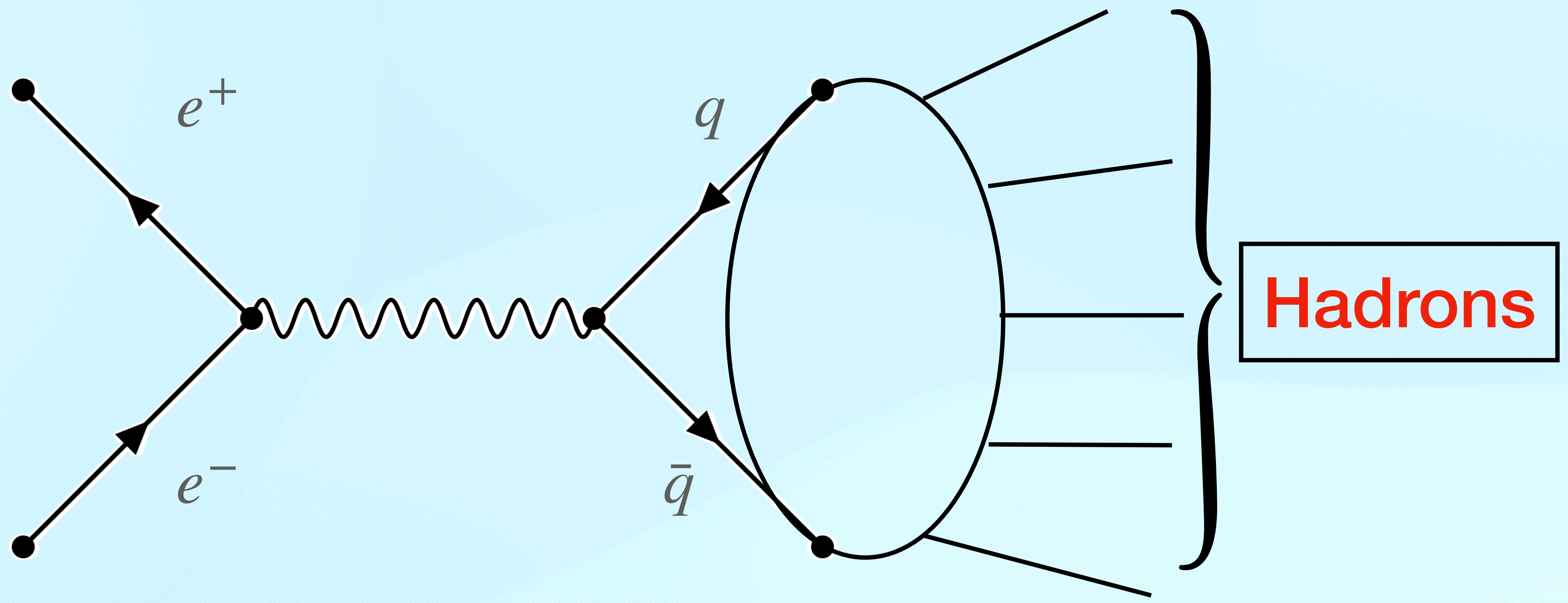
Generally, the number of lattice data is much less that the number of ω we want, so it called an ‘ill-posed’ problem (Inverse Problem).



- Studying Hadronic Spectroscopy (Calculate the spectral functions) Y. Burnier and A. Rothkopf, PRL. 111, 182003 (2013).
- Studying Deep Inelastic Scatterings (Calculate the hadronic tensor) J. Liang et al., PRD. 101, 114503 (2020)
- R-ratio、Hadronic decay widths、PDFs、etc.

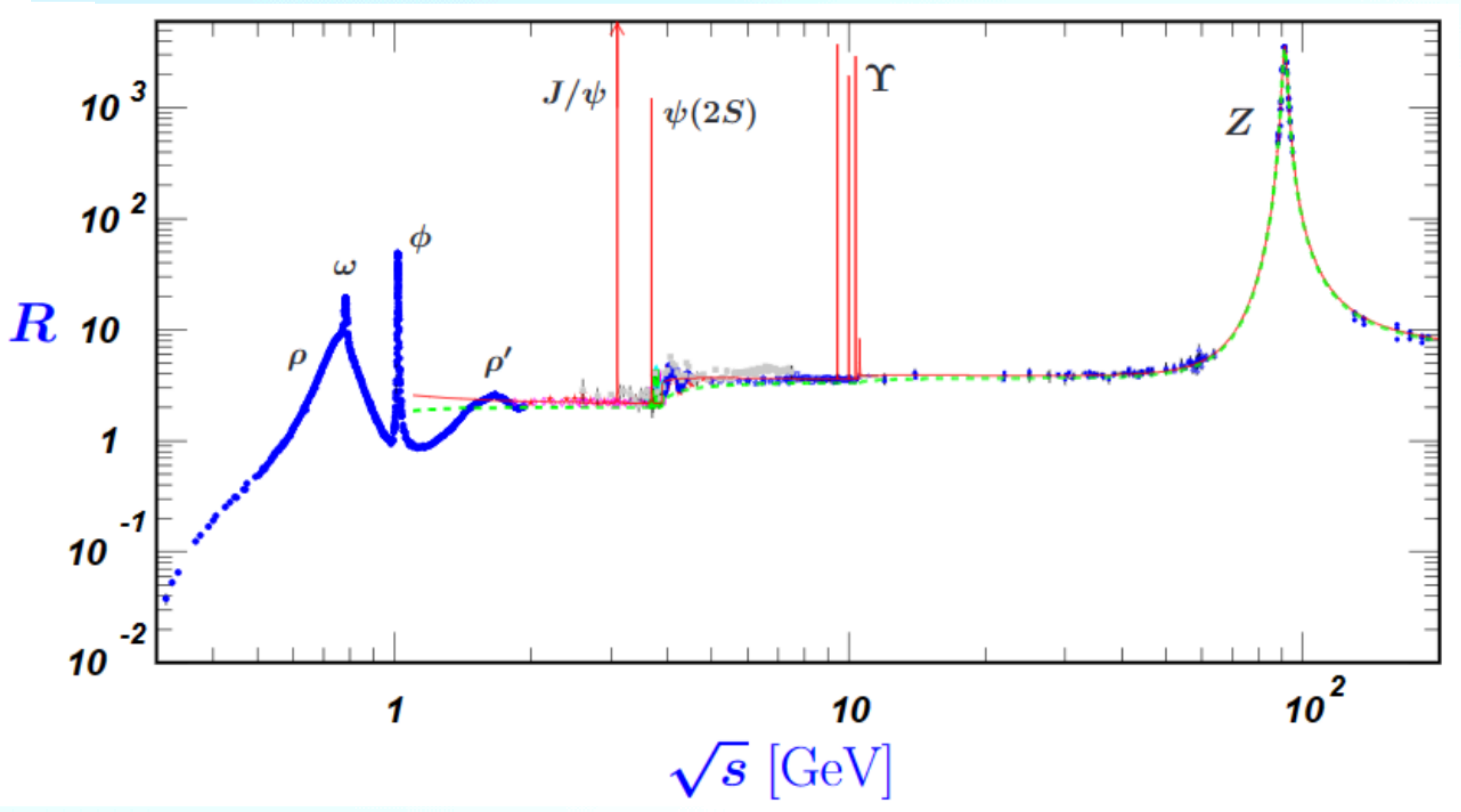
C. Alexandrou et al.(ETMC), PRL. 130, 241901 (2023).
M. T. Hansen et al., PRD. 96.094513(2017).
J. Karpie, et al., jhep. 04, 057 (2019).

Motivation



$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

$$R = N_c \sum_{i=1}^{n} Q_i^2 = \begin{cases} \frac{2}{3}N_c & \text{for } q = u, d, s; \\ \frac{10}{9}N_c & \text{for } q = u, d, s, c; \\ \frac{11}{9}N_c & \text{for } q = u, d, s, c, b; \end{cases}$$



V. V. Ezhela et al. arXiv:hep-ph/0312114 2004.

- R-ratio is a basic experimental measurement and has very important physical significance.
- Good playground to check the Bayesian Reconstruction (BR) algorithms solving the inverse problem.

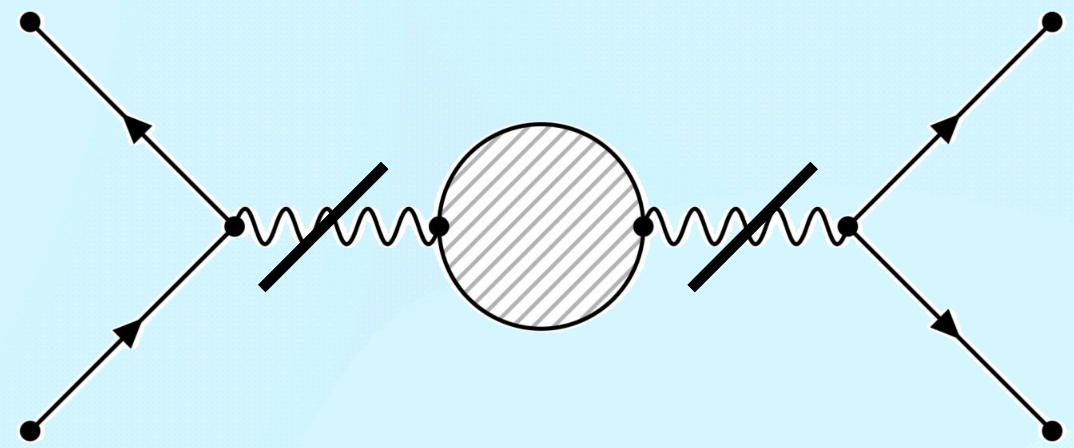
J. Liang et al., PRD101, 114503 (2020)
Y. Burnier and A. Rothkopf, PRL 111, 182003 (2013)

- Closely related to the HVP contribution to muon g - 2.

$$a_\mu^{\mathrm{Had}}[\mathrm{LO}] = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2}^{\infty} ds \frac{K(s)}{s} R^{(0)}(s)$$

P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2021)

R-ratio on The Lattice



The electromagnetic current：

$$J_\mu^{em} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s$$

By using the method based of BR that allows to extract spectral densities from current-current correlation functions.

$$\left\langle J_\mu^{em}(t) J_\mu^{em}(0) \right\rangle = \int d\omega \rho(\omega) e^{-\omega t}$$

$$\rho(\omega) = \sum_n A_n \delta(\omega, \omega_n)$$

$$R(\omega) = \frac{12\pi^2}{\omega^2}\rho(\omega)$$

Label	L/T	Mpi (MeV)	a (fm)	L (fm)
48I	48/96	139	0.11406	5.47
64I	64/128	139	0.08365	5.35
24D	24/64	141	0.1940	4.656
32D	32/64	141	0.1940	6.208
48D	48/96	141	0.1940	9.312

R. Arthur et al., PRD87, 094514 (2013)
T. Blum et al., PRD93, 074505 (2016)
P. Boyle et al., PRD 93, 054502 (2016)

- Overlap fermions on RBC/UKQCD domain wall gauge ensembles at the physical point with different lattice spacings and volumes.
- High-precision current-current correlation functions for both u/d and s, but no charm and no disconnected insertions for now.

Wang G et al. PRD, 2023, 107(3): 034513.

Numerical Calculation

○ Bayesian Reconstruction Method(BR)

$$P[\rho\,|\,D,\alpha,m] \propto e^{Q(\rho)}$$

$$Q = \alpha S - L - \gamma(L - N_\tau)^2$$

$$S = \sum_{\omega}\left[1 - \frac{\rho(\omega)}{m(\omega)} + \log\left(\frac{\rho(\omega)}{m(\omega)}\right)\right]\Delta\omega$$

$$P[\rho\,|\,D,m] = \frac{P[D\,|\,\rho,I]}{P[D\,|\,m]}\int d\alpha P[\alpha\,|\,D,m]$$

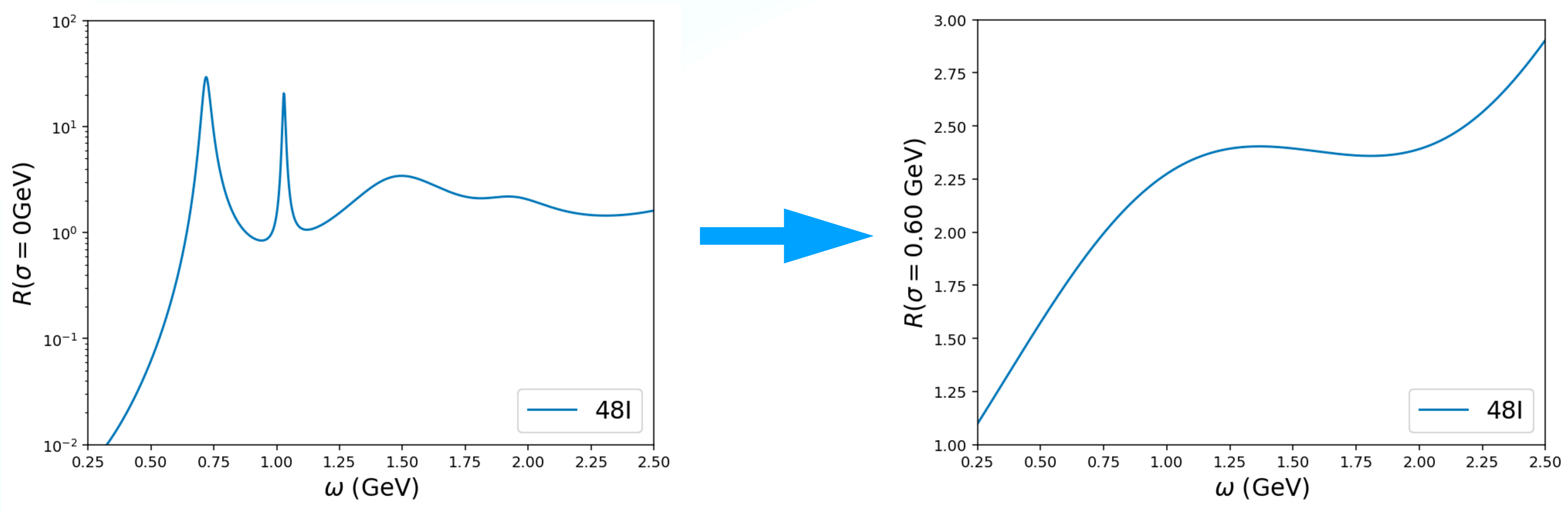
- Hyper parameter α is integrated over;
- Maximum search is in the entire parameter space($O(10^3)$)
- High precision architecture is needed(e.g.,512-bit floating point number).

J. Liang et al., PRD101, 114503 (2020)
Y. Burnier and A. Rothkopf, PRL 111, 182003 (2013)

Numerical Calculation

○ Smearing Method

- For finite lattice size the spectrum is discrete, in principle, lattice spectrum functions are delta functions.
- Experimentally, what we have a continuous result.
- We can compare the experimental results with the smeared lattice spectrum function.



$$C(t) = \sum_n A_n e^{-\omega_n t} = \int d\omega \rho_L(\omega) e^{-\omega t}$$

BR

$$\rho_L(\omega) = \sum_n A_n \delta(\omega, \omega_n)$$

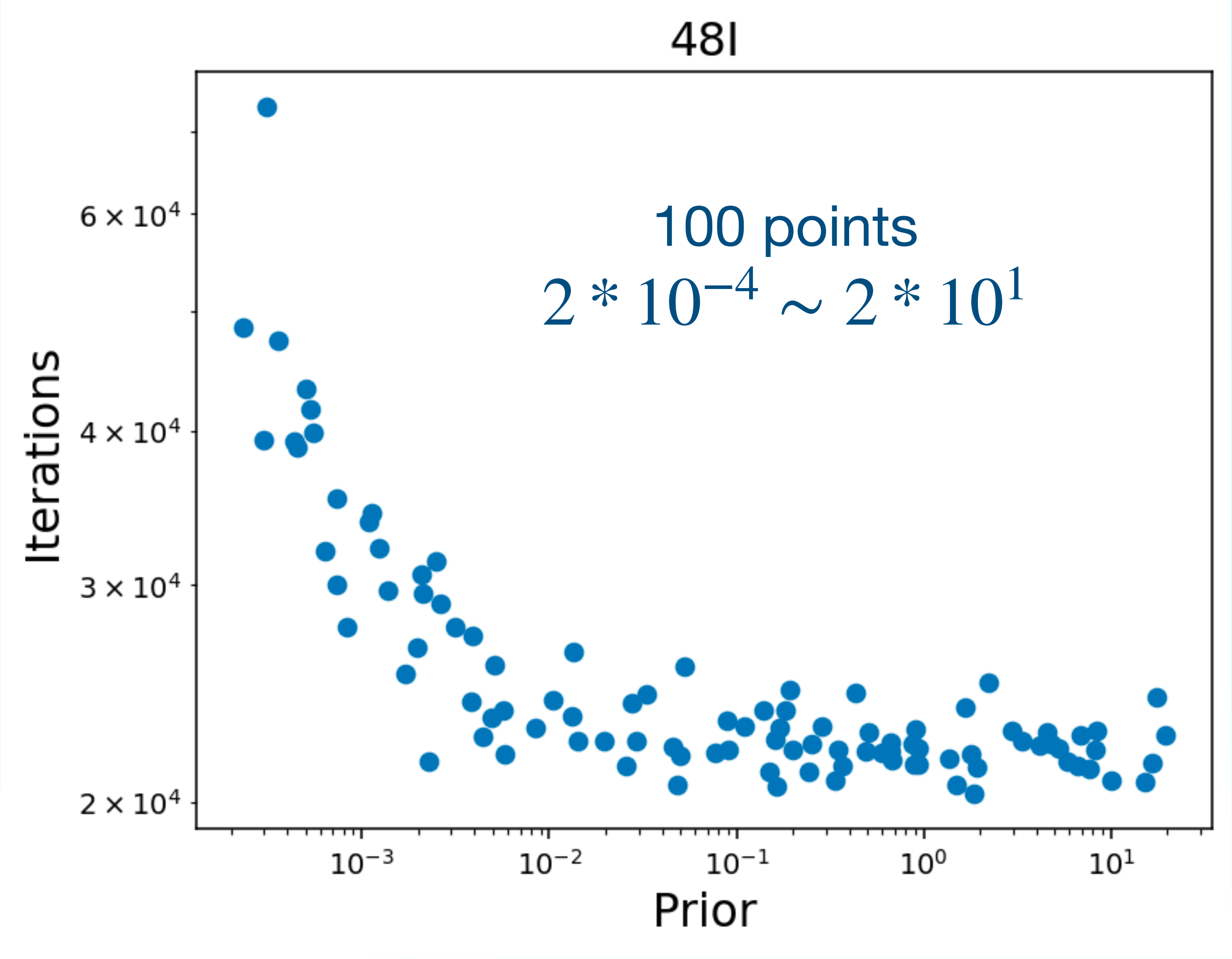
Smearing

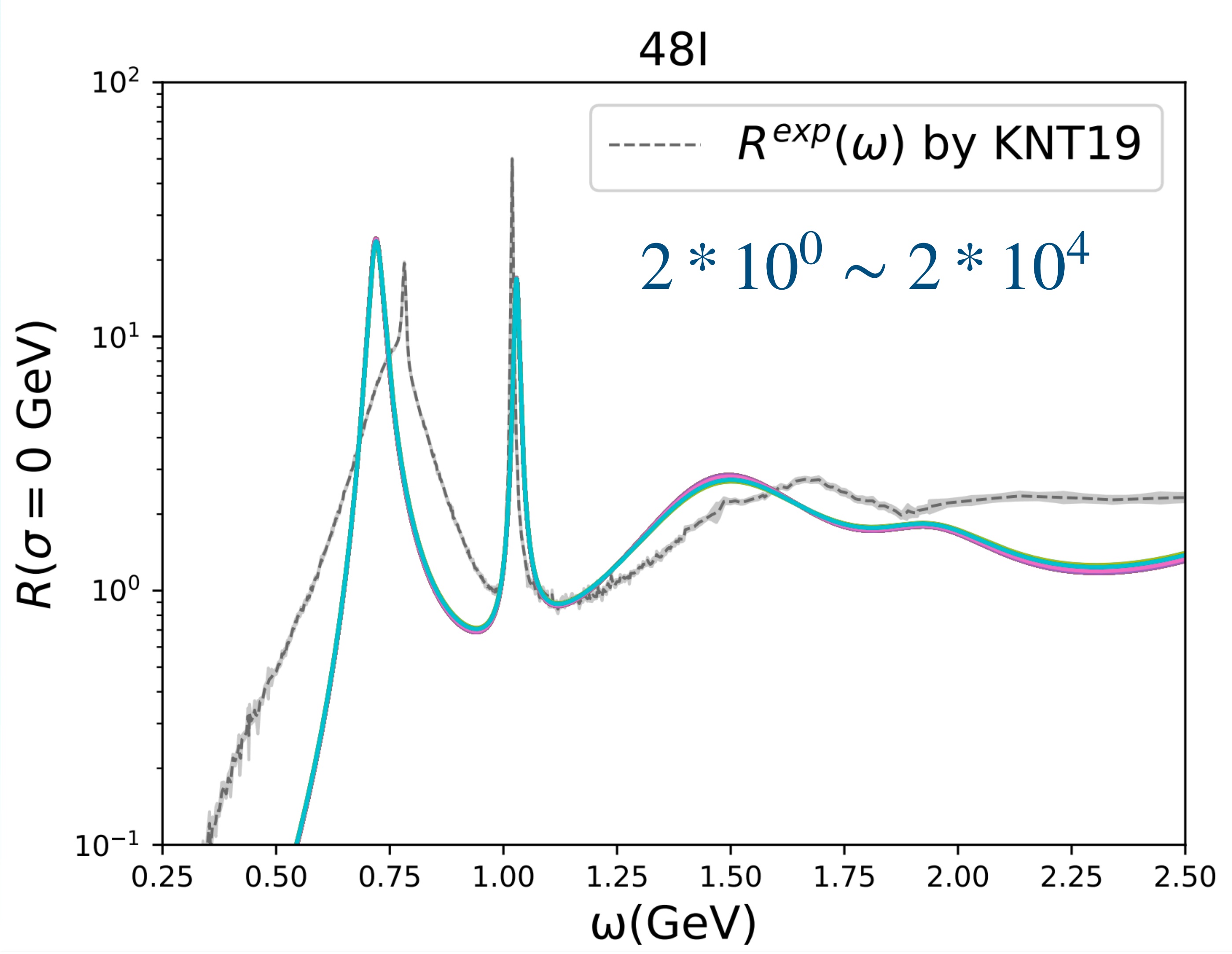
$$\rho_{\sigma,L}(E) = \int_0^\infty d\omega \Delta_\sigma(E,\omega)\rho_L(\omega)$$

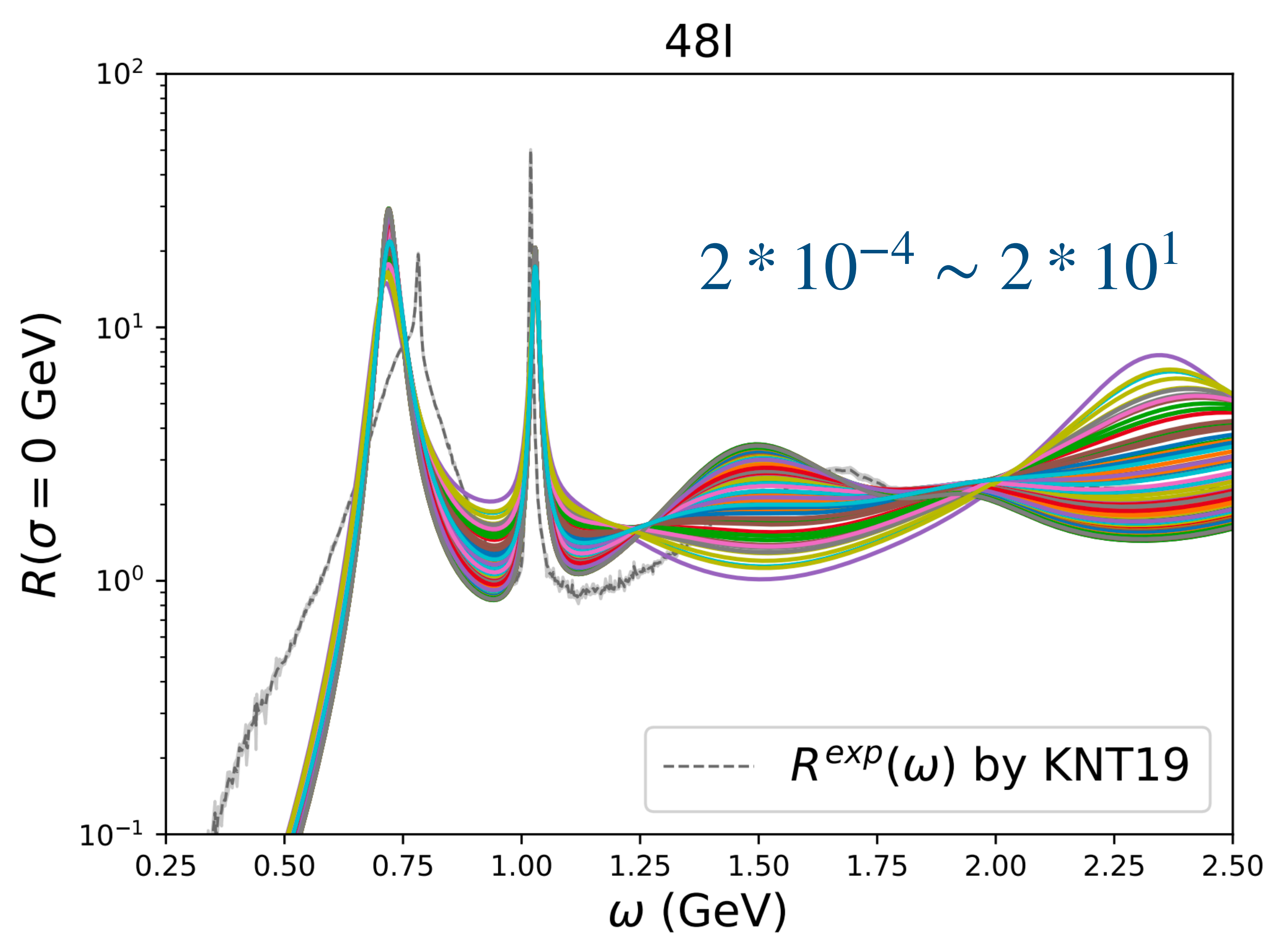
$$\Delta_\sigma(\omega, E) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(\mathrm{E}-\omega)^2}{2\sigma^2}\right)$$

Numerical Calculation

$$P[\rho|D,m] = \frac{P[D|\rho,I]}{P[D|m]}\int d\alpha P[\alpha|D,m]$$







- We tested the number of iterations of different prior in the BR program.
- With the prior decreases, the number of iterations increases exponentially.

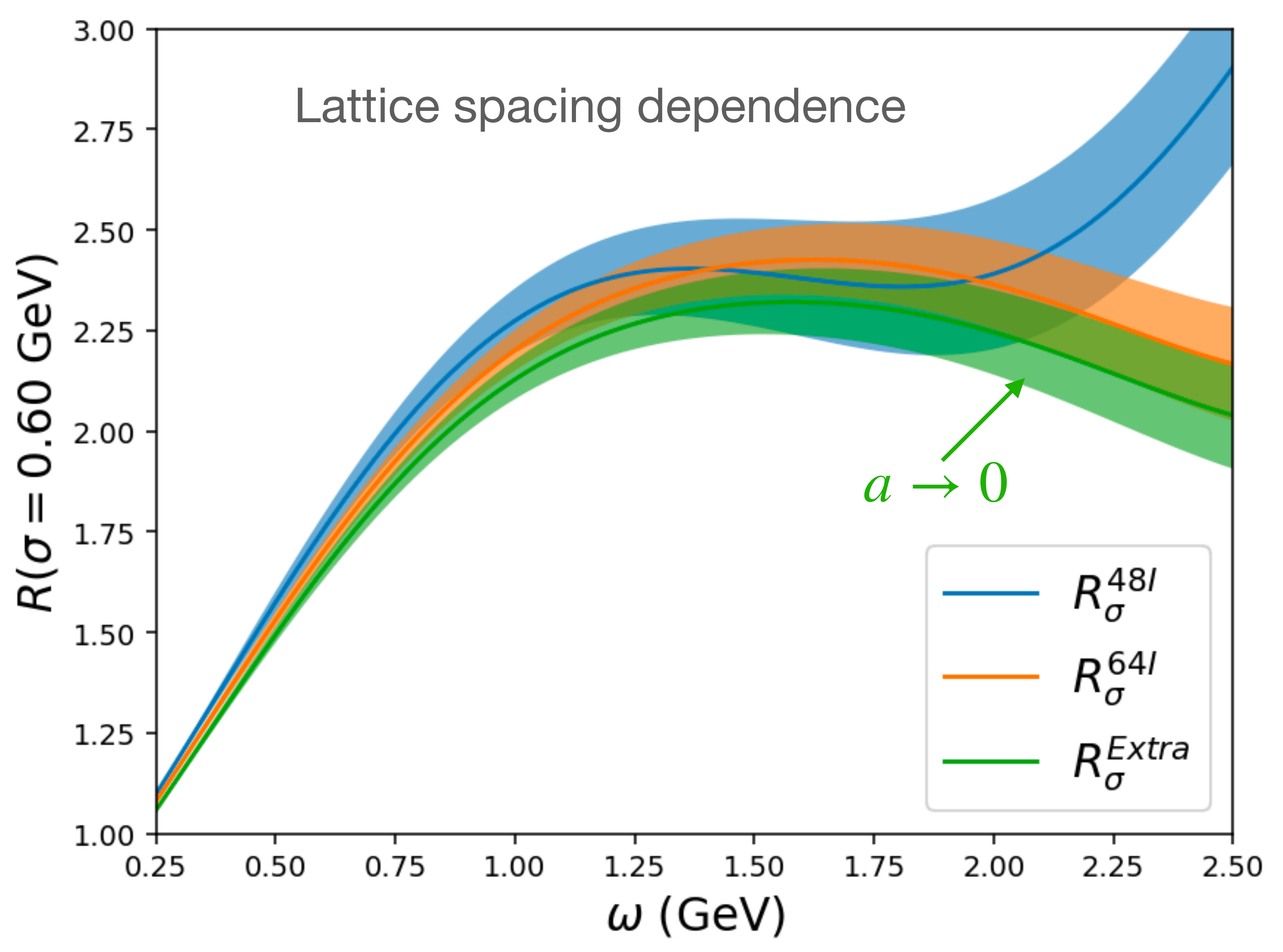
- BR is stable so long as the prior value is more than 2.

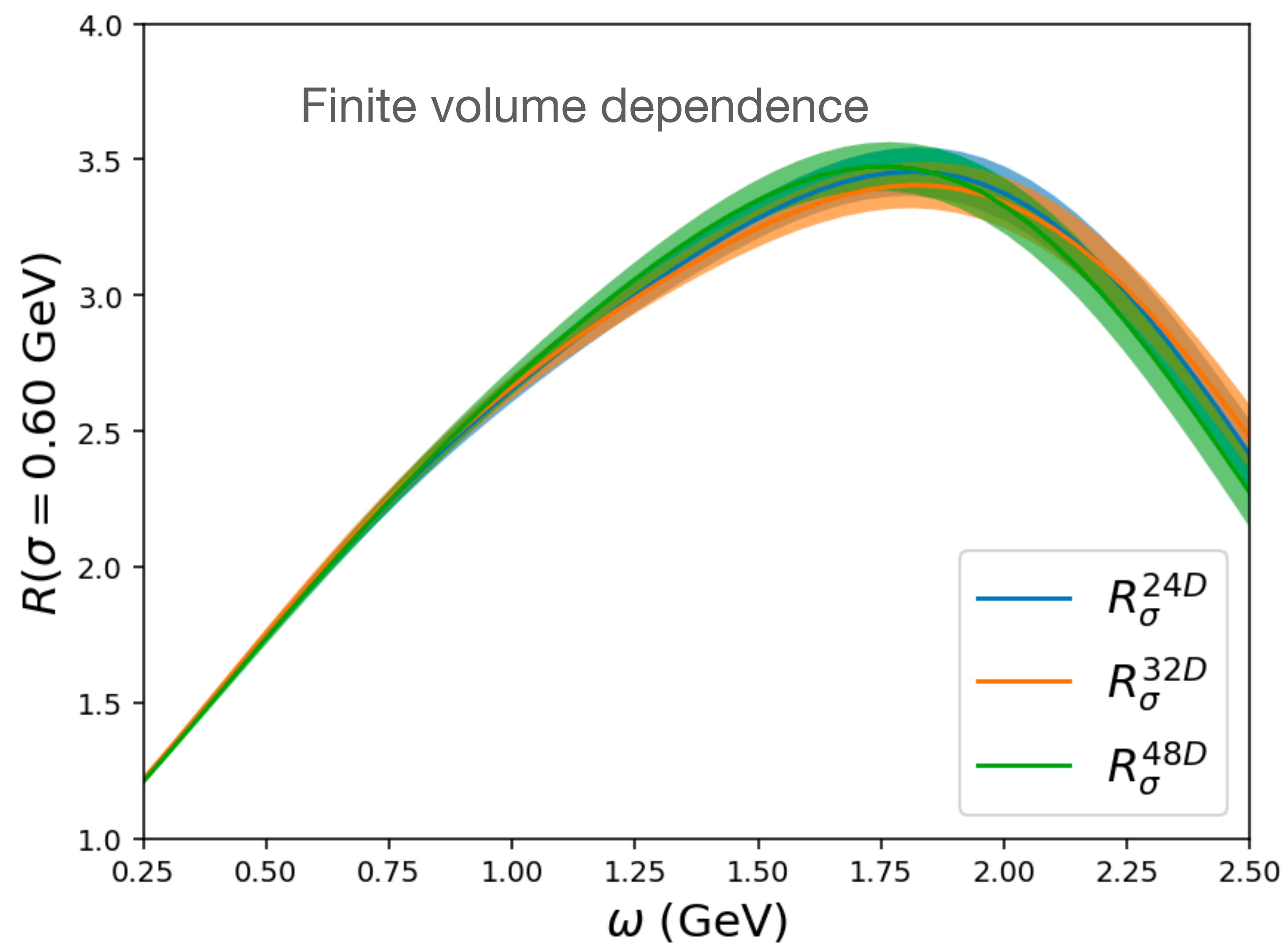
- When the prior of BR is less than 1, the lattice results change as the prior decreases.

The input variable function ‘prior’ of the BR program has a significant impact on the result of the R-ratio.

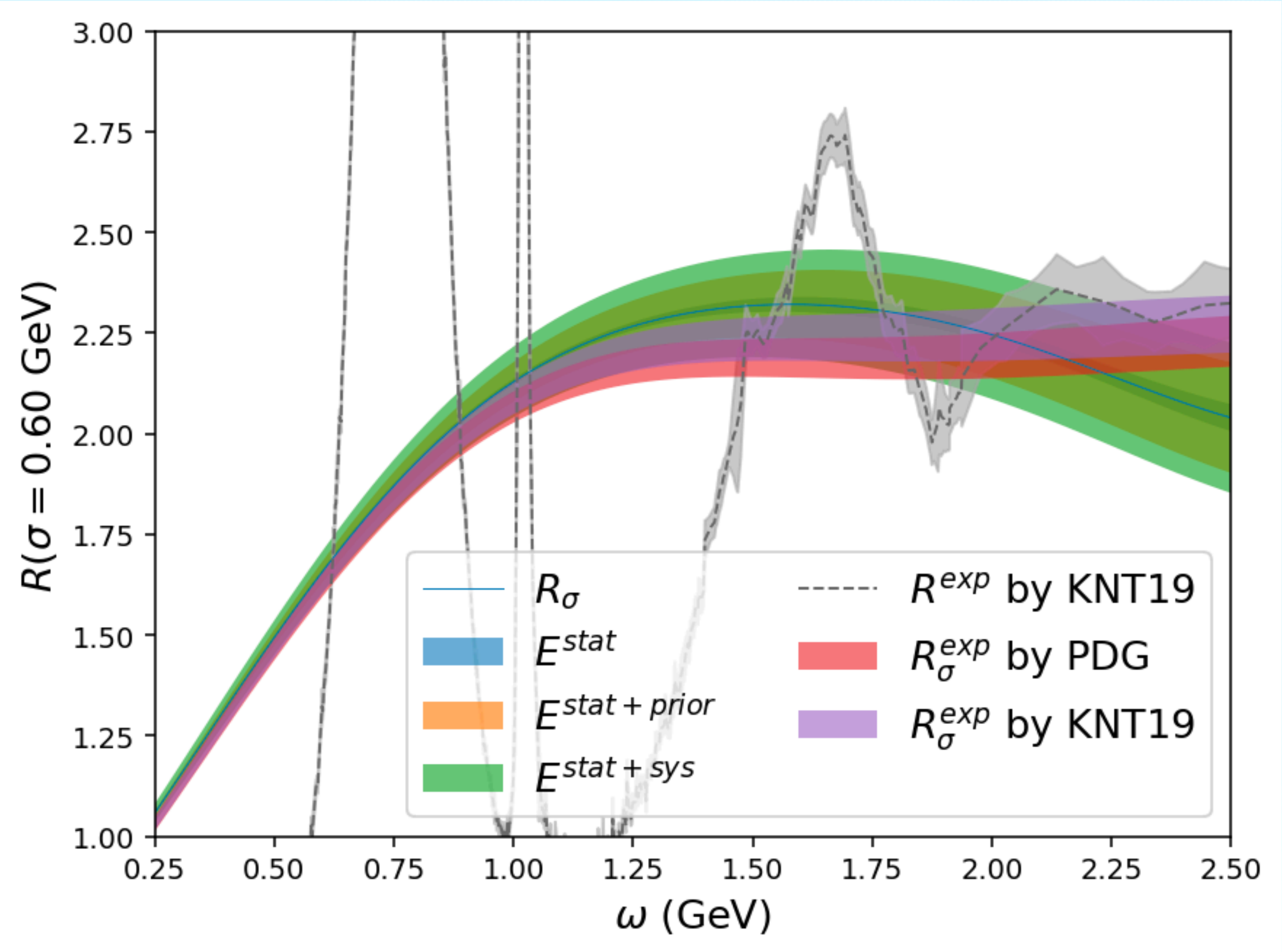
Numerical Calculation

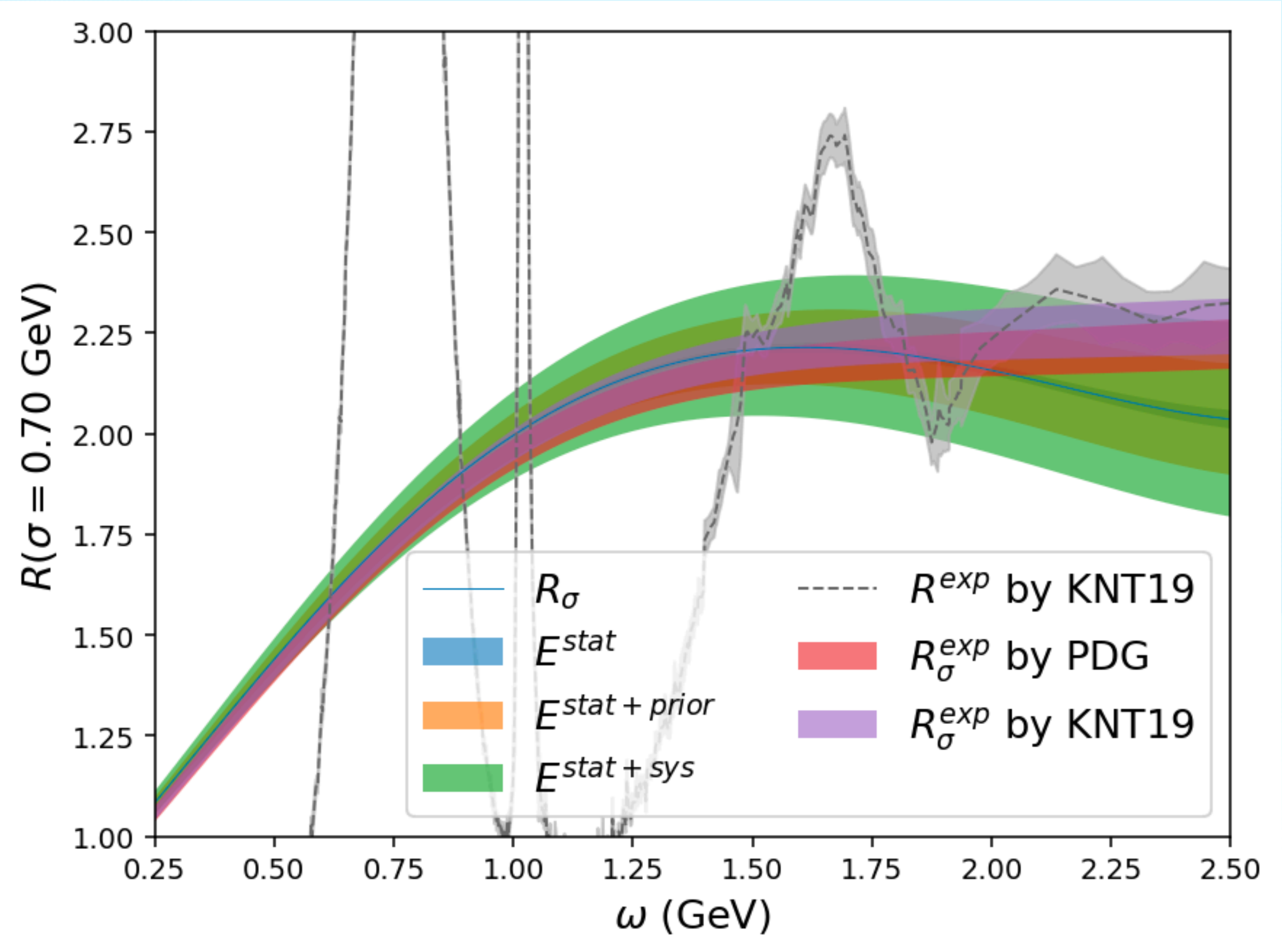
- We use the difference of the R-ratio with prior in range $2 \times 10^{-4} \sim 2 \times 10^{1}$ as a part of the systematic uncertainty.
- The impact of lattice spacing on the results is significant. We extrapolate the lattice data to a lattice spacing of 0.
- The effect of finite volume on the results is negligible.



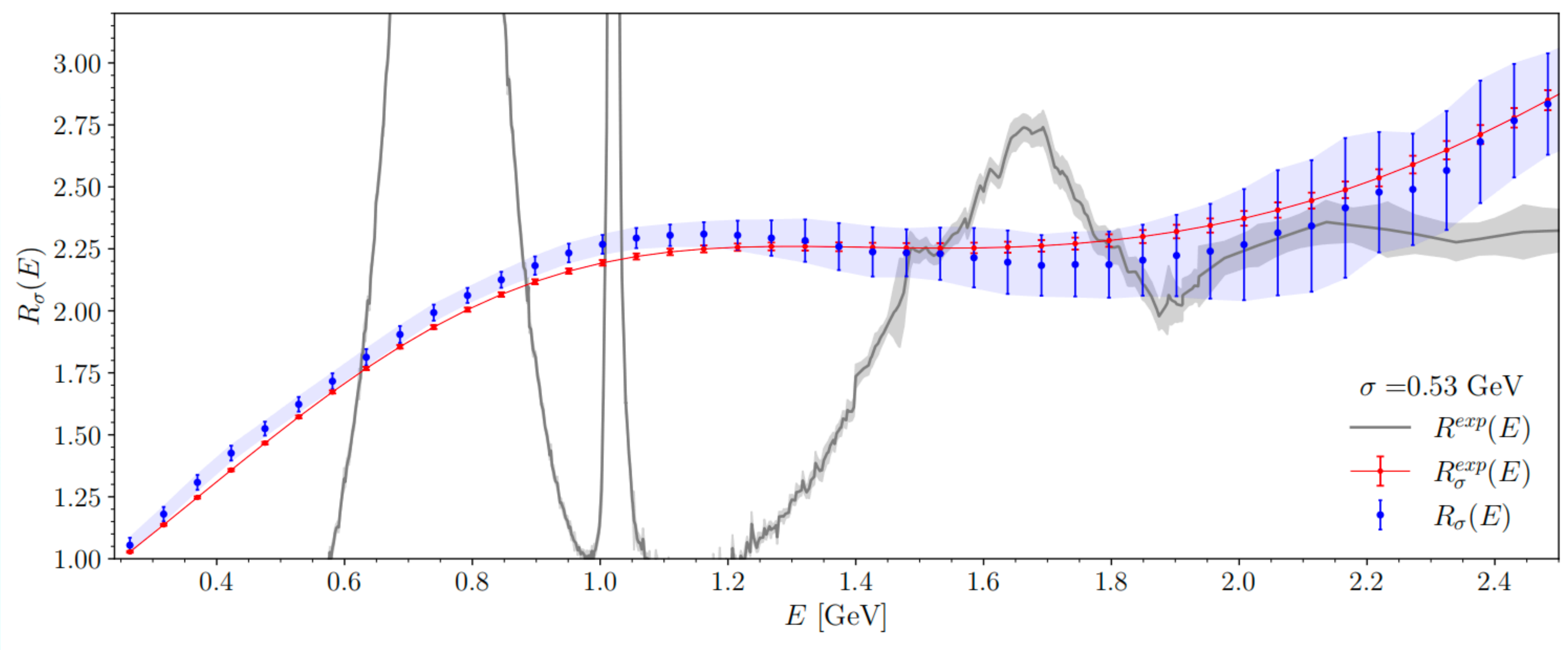


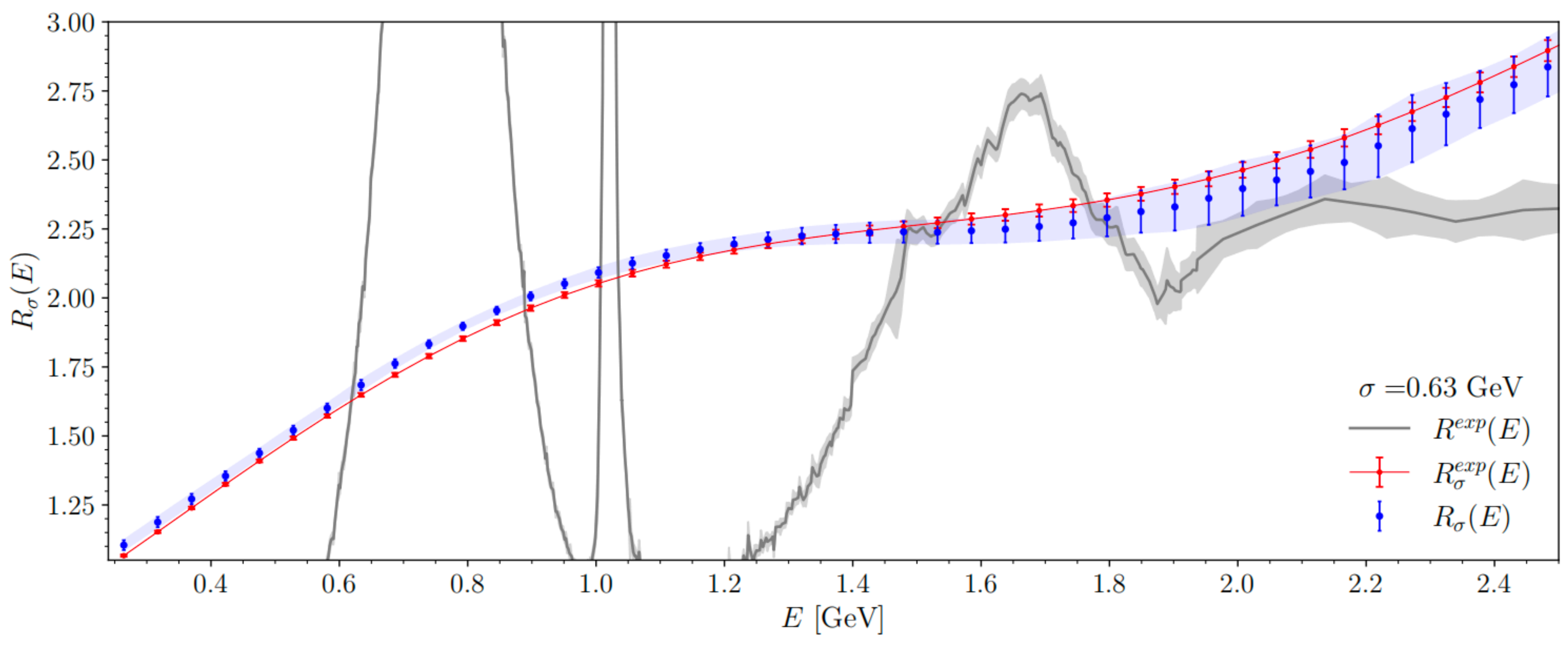
Results





- We considered the statistical error and system error caused by the prior、lattice spacing and finite volume, and obtained results for two smearing parameters.
- We compute the R-ratio with energy up to 2.5 GeV.





C. Alexandrou et al.(ETMC), PRL. 130, 241901 (2023).

Summary and Outlook

- We present a new method based on BR for addressing inverse problems and calculate the R-ratio through spectral function.
- This study provides a new possibility to support the lattice calculation for muon g - 2.
- We demonstrate a systematic approach based on BR to tackle the inverse problem with sophisticated error control.
- In the future, we hope this new approach can be applied to address the inverse problems arising in many aspects of lattice QCD.

Summary and Outlook

- We present a new method based on BR for addressing inverse problems and calculate the R-ratio through spectral function.
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Thanks for your attention!

Back up

$$\sigma(e^+e^- \to hadrons) \propto \sum_i Q_i^2 * 3 * \frac{4\pi\alpha^2}{3E_{cm}^2}$$

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3E_{cm}^2}$$

- **Smearing Method**

$$C(t) = \sum_n A_n e^{-\omega_n t} = \int d\omega \rho_L(\omega) e^{-\omega t}$$

$$\rho_L(\omega) = \sum_n A_n \delta(\omega, \omega_n)$$

- For finite lattice size the spectrum is discrete, in principle, lattice spectrum functions are delta functions.
- Experimentally, what we have is a continuous spectrum function.
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