Shadows and optical appearance of BH surrounded by dark fluid with Chaplygin-like equation of state

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Outlines

- Motivation-BHs surrounded by quintessence DE
- BH solutions with surrounding Chaplygin gas
- Phase structures of asymptotic AdS BHs & Shadow radius
- Optical appearance of asymptotic dS BHs
- Summary and discussion

Quintessence and black holes, V.V.Kiselev, arXiv:gr-qc/0210040. 523 citations.

- EoS of quintessence:

$$p_q = \omega_q \rho_q$$

- General static spherically symmetric spacetime:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2$$

- $T_t{}^t = \chi(r), \;\; T_t{}^i = 0, \;\; T_i{}^j = \xi(r) r_i r^j + \eta(r)\delta_i{}^j$
- Taking isotropic average over the angles:

$$\langle T_i{}^j \rangle = p_q(r)\delta_i{}^j$$

Motivation-BHs surrounded by quintessence DE

- Stress-energy tensor tensor:

$$T_t{}^t = T_r{}^r = \rho_q,\ T_\theta{}^\theta = T_\phi{}^\phi = -\frac{1}{2}\rho_q(3\omega_q + 1).$$

- Energy density:

$$\rho_q = -\frac{a}{2}\frac{3\omega}{r^{3(\omega+1)}}$$

- Spacetime solutions:

$$f(r) = 1 - \frac{\mu}{r} + \frac{a}{r^{3\omega_q+1}},$$

where μ and b are normalized factors.

Considering a dark fluid with Chaplygin-like EoS $\rho = -\frac{B}{p}$ (CDF), surrounding a BH, what would the situation be?

BH solutions with surrounding Chaplygin gas

- The CDF is anisotropic and its stress-energy tensor can be written as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p_r k_\mu k_\nu + p_t \Pi_{\mu\nu},$$

with $\Pi_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu - k_\mu k_\nu$.

- Working in the comoving frame of the fluid

$$u_\mu = (-\sqrt{f}, 0, 0, 0), k_\mu = (0, 1/\sqrt{f}, 0, 0),$$

one obtains

$$T_\mu{}^\nu = -(\rho + p_t)\delta_\mu{}^0 \delta^\nu{}_0 + p_t \delta_\mu{}^\nu + (p_r - p_t)\delta_\mu{}^1 \delta^\nu{}_1.$$

BH solutions with surrounding Chaplygin gas

- Taking isotropic average over the angles and requiring $\langle T_i^{\ j} \rangle = p(\rho)\delta_i^{\ j}$, we have

$$p_t + \frac{1}{3}(p_r - p_t) = p(\rho).$$

$$\text{quintessence:} \quad p(\rho) = \omega\rho \quad \Rightarrow \quad p_t = \frac{1}{2}(1+3\omega)\rho$$

$$\text{CDF:} \quad p(\rho) = -\frac{B}{\rho} \quad \Rightarrow \quad p_t = \frac{1}{2}\rho - \frac{3B}{2\rho}$$

$$\text{MCG:} \quad p(\rho) = A\rho - \frac{B}{\rho^\beta} \quad \Rightarrow \quad p_t = \frac{1+3A}{2}\rho - \frac{3B}{2\rho^\beta}$$

BH solutions with surrounding Chaplygin gas

- The gravitational equations

$$\frac{1}{r^2}(f + rf' - 1) + \Lambda = -\rho$$
$$\frac{1}{2r}(2f' + rf'') + \Lambda = \frac{1}{2}\rho - \frac{3B}{2\rho}$$

- The energy density of CDF

$$\rho(r) = \sqrt{B + \frac{q^2}{r^6}},$$

where $q > 0$ is a normalization factor.

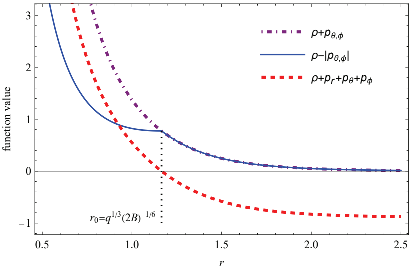


Figure: The variation of $\rho + p_{\theta,\phi}$, $\rho - |p_{\theta,\phi}|$ and $\rho + p_r + p_\theta + p_\phi$ versus r for the CDF taking $q = 1.0$ and $B = 0.2$.

- Standard energy conditions

$$\text{NEC}: \quad \rho + p_i \geq 0 \, (i = r, \theta, \phi);$$

$$\text{WEC}: \quad \rho \geq 0 \ \ \& \ \ \rho + p_i \geq 0 \, (i = r, \theta, \phi);$$

$$\text{SEC}: \quad \rho + \sum_i p_i \geq 0 \ \ \& \ \ \rho + p_i \geq 0 \, (i = r, \theta, \phi);$$

$$\text{DEC}: \quad \rho \geq 0 \ \ \& \ \ |p_i| \leq \rho \, (i = r, \theta, \phi).$$

BH solutions with surrounding Chaplygin gas

- The analytical solution for $f(r)$[PRD 107, 104055 (2023)]

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{3}\sqrt{B + \frac{q^2}{r^6}} + \frac{q}{3r}\mathrm{ArcSinh}\frac{q}{\sqrt{B}r^3} - \frac{r^2}{3}\Lambda.$$

For MCG[Annals of Physics 446 (2022) 169125],

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{3}\left(\frac{B}{1+A}\right)^{\frac{1}{1+\beta}}\mathcal{F}(r) - \frac{r^2}{3}\Lambda,$$

with

$$\begin{aligned} \mathcal{F}(r) &= {}_2F_1\left(\left[-\frac{1}{1+\beta}, -\frac{1}{w}\right], 1 - \frac{1}{w}, -\frac{1}{B}\left(\frac{Q}{r^3}\right)^w\right), \\ w &= (1+A)(1+\beta). \end{aligned}$$

- The mass of the BH

$$M = \frac{r_h}{2} - \frac{r_h^3}{6}\Lambda - \frac{r_h^3}{6}\sqrt{B + \frac{q^2}{r_h^6}} + \frac{q}{6}\mathrm{ArcSinh}\frac{q}{\sqrt{B}r_h^3}$$

- The Hawking temperature

$$T = \frac{f'(r_h)}{4\pi} = \frac{1}{4\pi}\left(\frac{1}{r_h} - r_h\sqrt{B + \frac{q^2}{r_h^6}} - r_h\Lambda\right)$$

- The entropy

$$S = \int_0^{r_h} \frac{1}{T}\left(\frac{\partial M}{\partial r_h}\right) dr_h = \pi r_h^2$$

- In the extended phase space

$$P = -\Lambda,$$
$$V = \left(\frac{\partial M}{\partial P}\right)_{S,q} = \frac{r_h^3}{6}.$$

- Treating q as a variable, the conjugated physical quantity

$$\Phi = \left(\frac{\partial M}{\partial q}\right)_{S,P} = \frac{1}{6}\mathrm{ArcSinh}\frac{q}{\sqrt{B}r_h^3}$$

- The first law of BH thermodynamics

$$dM = TdS + VdP + \Phi dq$$

- The Smarr relation: $M = 2TS - 2VP + \Phi q.$

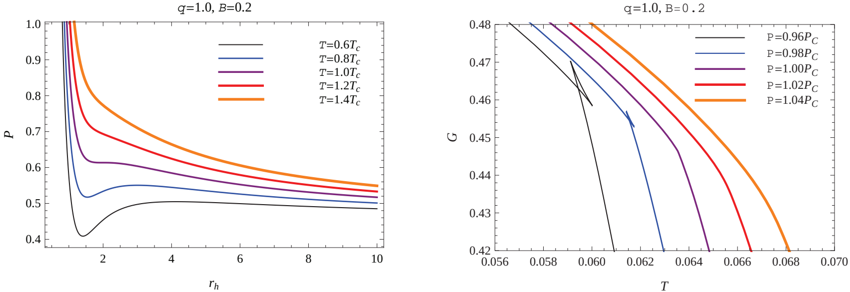


Figure: Left: The behavior of isothermal $P - V(r_h)$. Right: The behavior of isobaric $G - T$. The swallow-tail shape appears when $P < P_c$.

Table: Numerical solutions for the critical physical quantities in $P - V$ criticality.

B	q	r_c	P_c	T_c	$\frac{P_c r_c}{T_c}$
0.2	1.0	1.98018	0.61349	0.06353	19.1236
0.3	1.0	1.88657	0.73135	0.06676	20.6663
0.4	1.0	1.82257	0.82950	0.06916	21.8586
0.2	0.8	1.76045	0.65626	0.07121	16.2247
0.2	1.2	2.17761	0.58527	0.05789	22.0144

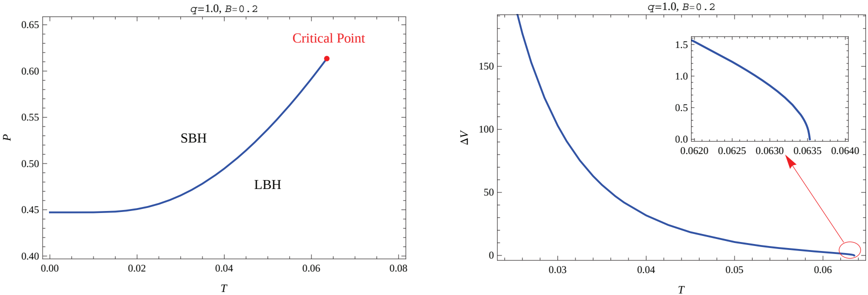


Figure: Left: Coexistence curve of small/large BH phase transition in the $P-T$ plane. Right: The ΔV as function of temperature T. The fitting $\Delta V - T$ curve near the critical temperature is magnified into view.

- Critical exponents for $P-V$ criticality

  Introduce the following notations

$$t=\frac{T}{T_c}-1,\quad \epsilon=\frac{V}{V_c}-1.$$

$$\begin{aligned}
C_V = T\left(\frac{\partial S}{\partial T}\right)_V &\propto |t|^{-\alpha}\\
\eta = V_l - V_s &\propto |t|^{\beta}\\
\kappa_{T@\langle P-V\rangle} = -\frac{1}{V}\frac{\partial V}{\partial P}\bigg|_{V_c} &\propto |t|^{-\gamma}\\
|P-P_c| &\propto |V-V_c|^{\delta}
\end{aligned}
\quad\Longrightarrow\quad
\begin{aligned}
\alpha &= 0\\
\beta &= 1/2\\
\gamma &= 1\\
\delta &= 3
\end{aligned}$$

- Rewrite the relation of temperature

$$T = \frac{1}{4\pi}\left(\frac{Pq^{1/3}}{\Xi(\Phi)} + \frac{\Xi(\Phi)}{q^{1/3}} - \frac{q^{1/3}\sqrt{B+\Xi(\Phi)^6}}{\Xi(\Phi)}\right),$$

with $\Xi(\Phi) \equiv B^{1/6}(\mathrm{Sinh}6\Phi)^{1/3}$.

- Solving q, we obtain two solutions for the BH equation of state

$$q_1 = \left(\frac{-\sqrt{64\pi^2T^2+8\Upsilon(\Phi)}-8\pi T}{2\Upsilon(\Phi)}\Xi(\Phi)\right)^3,$$

$$q_2 = \left(\frac{\sqrt{64\pi^2T^2+8\Upsilon(\Phi)}-8\pi T}{2\Upsilon(\Phi)}\Xi(\Phi)\right)^3,$$

with $\Upsilon(\Phi) \equiv -2P + 2\sqrt{B}\mathrm{Cosh}6\Phi$.

$q - \Phi$ criticality

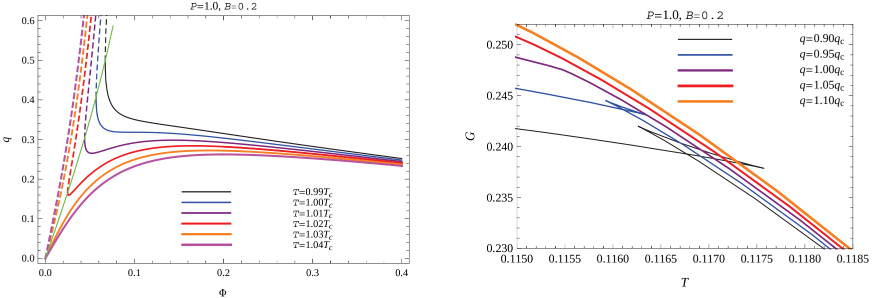


Figure: Left: The behavior of isothermal $q - \Phi$. The dashed and solid lines are for q_1 and q_2, respectively. The thin green line denotes the connection point (Φ_0, q_0) of the curves q_1 and q_2. Right: Gibbs free energy G as a function of temperature T for fixed q. The swallow-tail shape appears when $q < q_c$.

Table: Numerical solutions for the critical physical quantities in $P - V$ criticality.

B	P	Φ_c	q_c	T_c
0.2	1.0	0.09751	0.31829	0.11548
0.3	1.0	0.07481	0.41852	0.10462
0.4	1.0	0.06074	0.54486	0.09439
0.2	0.9	0.08513	0.38250	0.10461
0.2	1.1	0.11035	0.27410	0.12536

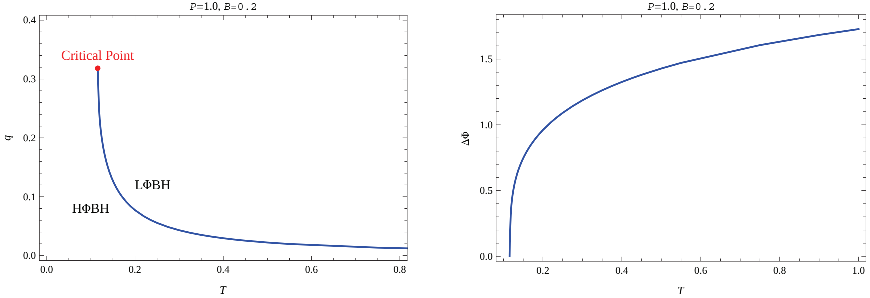


Figure: Left: Coexistence curve of low/high-Φ BH phase transition in the $q - T$ plane. Right: The $\Delta\Phi$ as function of temperature T.

- Critical exponents for $q-\Phi$ criticality

$$
\begin{aligned}
C_\Phi = T\frac{\partial S}{\partial T}\bigg|_\Phi &\propto |t|^{-\lambda} \\
\vartheta = \Phi_H - \Phi_L &\propto |t|^{\chi} \\
\kappa_{T@\langle q-\Phi\rangle} = -\frac{1}{\Phi}\frac{\partial \Phi}{\partial q_2}\bigg|_{\Phi_c} &\propto |t|^{-\sigma} \\
|q_2 - q_c| &\propto |\Phi - \Phi_c|^{\iota}
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\lambda &= 0 \\
\chi &= 1/2 \\
\sigma &= 1 \\
\iota &= 3
\end{aligned}
$$

- The photon sphere radius r_p satisfies

$$6M - 2r_p - q\mathrm{ArcSinh}\frac{q}{\sqrt{B r_p^3}} = 0$$

- The turning point of the photon orbit

$$\left.\frac{dr}{d\phi}\right|_{r=R} = 0$$

- The orbit equation for the photon

$$\frac{dr}{d\phi} = \pm r\sqrt{f(r)\left[\frac{r^2 f(R)}{R^2 f(r)} - 1\right]}$$

- Considering a light ray sending from a static observer placed at r_o and transmitting into the past with an angle α with respect to the radial direction

$$\cot\alpha = \frac{\sqrt{g_{rr}}}{g_{\phi\phi}} \frac{dr}{d\phi}\bigg|_{r=r_o}$$

- Shadow radius of the BH

$$r_s = r_o \tan\alpha \approx r_o \sin\alpha = R\sqrt{\frac{f(r_o)}{f(R)}}\bigg|_{R\to r_p}$$

The approximation is valid only for small value of α.

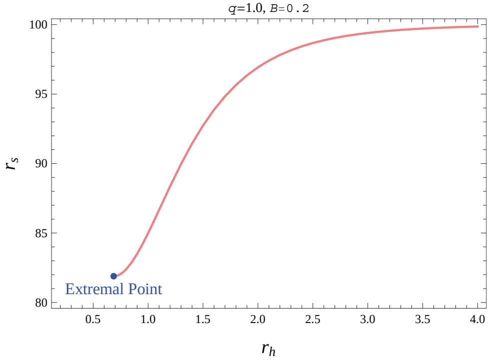


Figure: The variation of shadow radius r_s in terms of the event horizon radius r_h. The extremal point corresponds to the horizon radius at which the inner and outer horizons coincide.

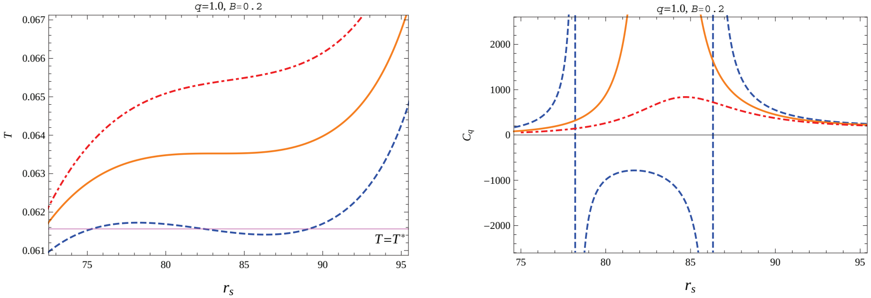


Figure: Hawking temperature (left panel) and the heat capacity (right panel) with respect to the BH shadow radius r_s for an observer at $r_o = 100$. The blue, orange and red curves correspond to the curves with $P = 0.98P_c$, $P = 1.00P_c$, and $P = 1.02P_c$. T^* represents the coexistence temperature.

- The geometry of the black hole

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\psi^2),$$

with

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{3}\sqrt{B + \frac{q^2}{r^6}} + \frac{q}{3r}\mathrm{ArcSinh}\frac{q}{\sqrt{B}r^3}.$$

- The time, azimuthal and radial four-velocity components

$$\dot{t} = \frac{1}{bf(r)},$$
$$\dot{\psi} = \pm\frac{1}{r^2},$$
$$\dot{r}^2 + V(r) = \frac{1}{b^2},$$

where $V(r) = \frac{1}{r^2}f(r)$ and $b = |L|/E$ is the impact parameter.

- The radius of photon sphere satisfies

$$V(r_{ph}) = \frac{1}{b^2}, \quad V'(r_{ph}) = 0.$$

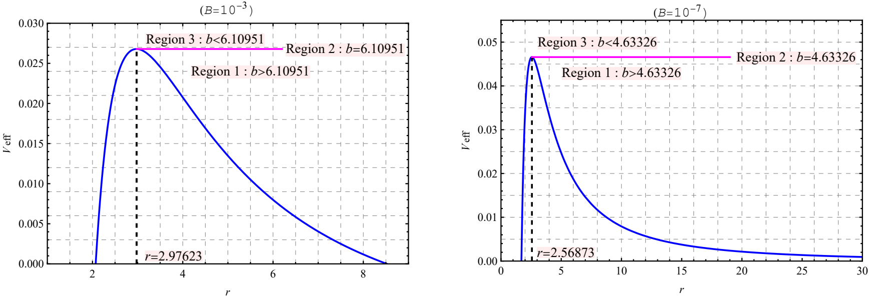


Figure: The profile of the effective potential (blue lines) for $B = 10^{-3}$ (left) and $B = 10^{-7}$ (right) with $M = 1$, $q = 0.2$.

Photon trajectory

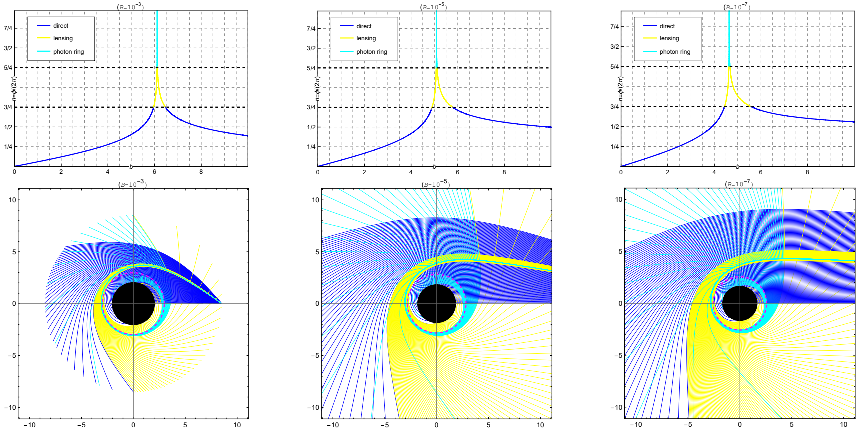


Figure: Behavior of photons as a function of impact parameter b with $M = 1$, $q = 0.2$.

Optical appearance of asymptotic dS BHs

Photon trajectory

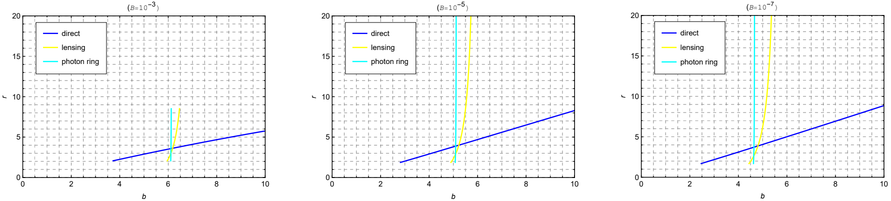


Figure: The first three transfer functions $r_m(b)$ for the thin disk with $M = 1$, $q = 0.2$.

- The effective potential of the timelike particles

$$U_{eff} = f(r)\left(1+\frac{L^2}{r^2}\right)$$

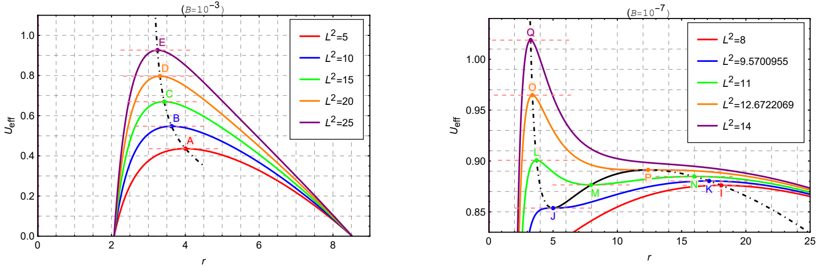


Figure: The profile of the effective potential curves of timelike particles with $M=1$, $q=0.2$.

Optical appearance of asymptotic dS BHs

Table: The values of radius $r_{\rm ph}$ and impact parameter $b_{\rm ph}$ of the photon sphere, the event horizon $r_{\rm h}$ and cosmological horizon $r_{\rm c}$, the radius $r_{\rm ISCO}$ of the innermost stable circular orbit, as well as the rays classification parameters $b_m^{\pm}$ with varying B and q for $M = 1.0$.

B	q	$r_{\rm h}$	$r_{\rm c}$	$r_{\rm ph}$	$b_{\rm ph}$	$r_{\rm ISCO}$	b_1^-	b_2^-	b_2^+	b_3^-	b_3^+
10^{-3}	10^{-3}	2.09723	8.52058	3.00000	6.14340	–	3.77645	6.00058	6.48810	6.13704	6.15765
	10^{-2}	2.09717	8.52058	2.99994	6.14332	–	3.77632	6.00048	6.48803	6.13696	6.15757
	0.2	2.07073	8.52084	2.97623	6.10951	–	3.72433	5.95982	6.46324	6.10259	6.12466
	0.4	1.99026	8.52163	2.90048	6.00309	–	3.56588	5.83074	6.38686	5.99411	6.02145
10^{-4}	0.2	1.95737	16.21750	2.92692	5.37247	–	3.09929	5.20594	5.88608	5.36413	5.39462
10^{-5}		1.86928	29.74730	2.82444	5.08701	–	2.81652	4.90003	5.79038	5.07696	5.11743
10^{-6}		1.78290	53.74350	2.69841	4.85819	5.84933	2.62476	4.66285	5.71446	4.84741	4.89368
10^{-7}	10^{-3}	2.00078	96.38450	2.99994	5.20351	6.25941	2.89373	5.03064	6.03561	5.19558	5.23211
	10^{-2}	1.99648	96.38450	2.99499	5.19888	6.25026	2.88805	5.02500	6.03334	5.19085	5.22773
	0.2	1.69616	96.38460	2.56873	4.63326	5.00492	2.46830	4.43750	5.57041	4.62230	4.67088
	0.4	1.14147	96.38460	1.77817	3.49840	3.30513	1.68256	3.25723	4.60648	3.47784	3.55956

- The observed intensity

$$I(r) = \sum_m f(r)^2 I_{em}(r) \,|_{r=r_m(b)},$$

with $I_{em}(r)$ is the total emitted specific intensity near the accretion.

- Three toy-model functions which may emit by some thin matters

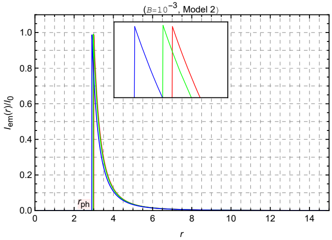
$$\text{Model 1:}\quad I_{\rm em}(r)=\begin{cases} I_0\left(\frac{1}{r-(r_{\rm ISCO}-1)}\right)^2, & r>r_{\rm ISCO}\\ 0, & r\leq r_{\rm ISCO}\end{cases},$$

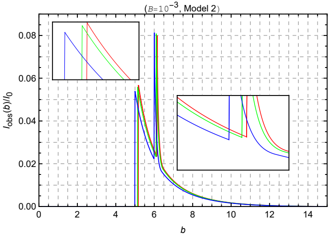
$$\text{Model 2:}\quad I_{\rm em}(r)=\begin{cases} I_0\left[\frac{1}{r-(r_{\rm ph}-1)}\right]^3, & r>r_{\rm ph}\\ 0, & r\leq r_{\rm ph}\end{cases},$$

$$\text{Model 3:}\quad I_{\rm em}(r)=\begin{cases} I_0\frac{\frac{\pi}{2}-\tan^{-1}[r-(r_{\rm ISCO}-1)]}{\frac{\pi}{2}-\tan^{-1}[r_{\rm h}-(r_{\rm ISCO}-1)]}, & r>r_{\rm h}\\ 0, & r\leq r_{\rm h}\end{cases}.$$

Optical appearance of asymptotic dS BHs

Thin disk accretion





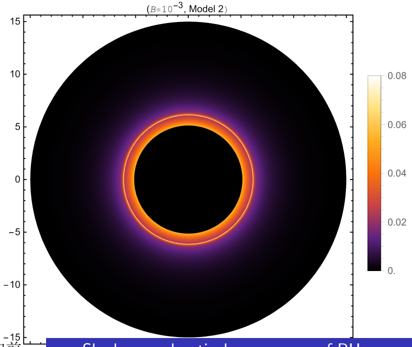
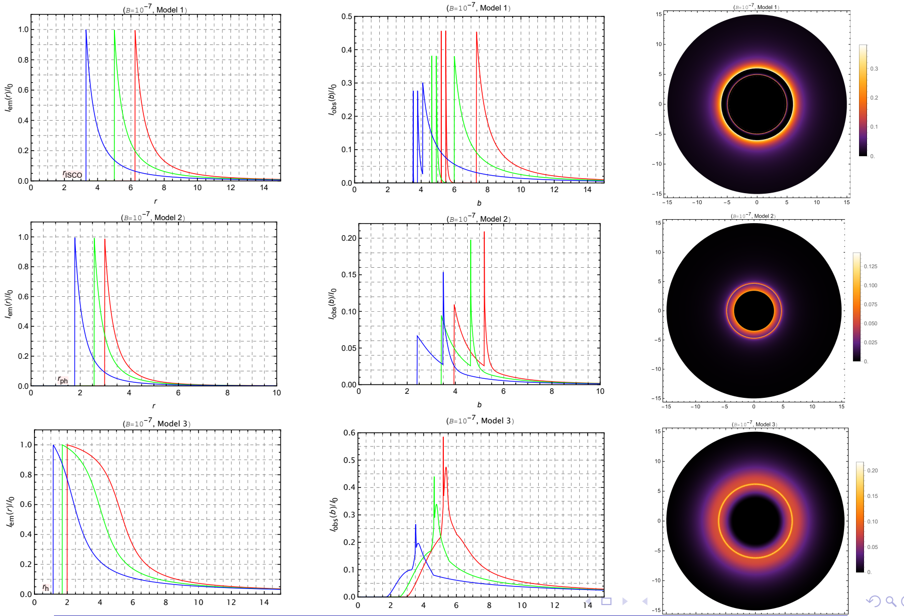


Figure: Observational appearances of a geometrically and optically thin disk with Model 2 profile, $M = 1$, $B = 10^{-3}$.

Optical appearance of asymptotic dS BHs



- Static observer

$$I_{obs} = \int_{\gamma} \frac{f(r)^{3/2}}{r^2} \sqrt{f(r)^{-1} + r^2 \left(\frac{d\psi}{dr}\right)^2} \, dr$$

- Infalling observer

$$I_{obs} \propto \int_{\gamma} \frac{g^3 k_t dr}{r^2 |k_r|},$$

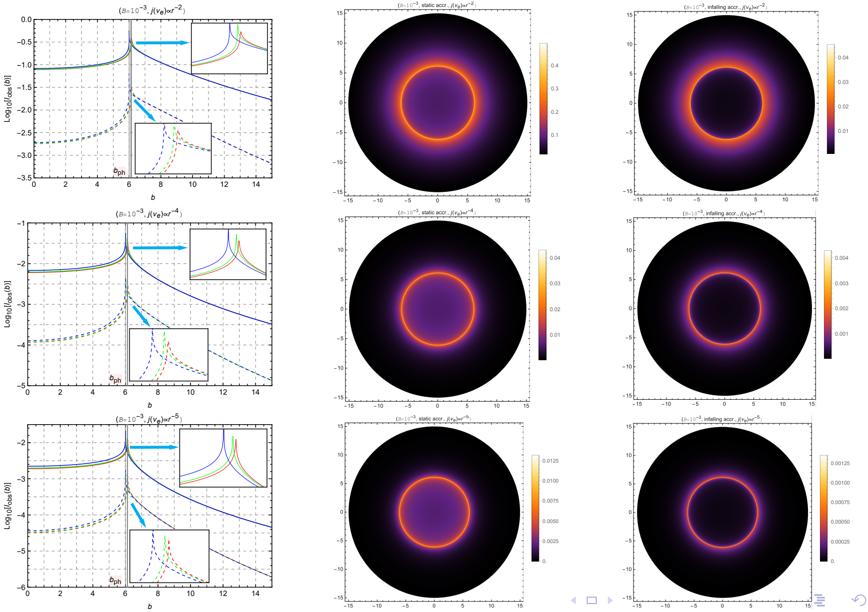
with

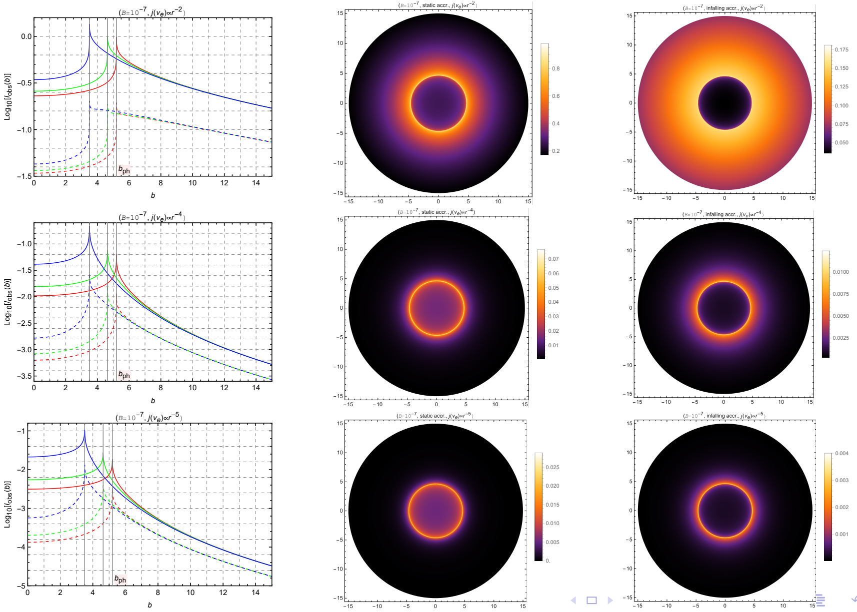
$$g = \frac{1}{u_e^t + k_r/k_t u_e^r},$$

and

$$\frac{k_r}{k_t} = \pm \frac{1}{f(r)} \sqrt{1 - \frac{b^2 f(r)}{r^2}}.$$

Optical appearance of asymptotic dS BHs




(B=10^-7, j(νe)∝r^-2)
(B=10^-7, static accr., j(νe)∝r^-2)
(B=10^-7, infalling accr., j(νe)∝r^-2)
(B=10^-7, j(νe)∝r^-4)
(B=10^-7, static accr., j(νe)∝r^-4)
(B=10^-7, infalling accr., j(νe)∝r^-4)
(B=10^-7, j(νe)∝r^-5)
(B=10^-7, static accr., j(νe)∝r^-5)
(B=10^-7, infalling accr., j(νe)∝r^-5)
Log10[Iobs(b)]
b
bph

Summary and discussion

- BH solutions with surrounding Chaplygin gas
- Phase structures of asymptotic AdS BHs & Shadow radius
  - $P-V$ criticality
  - $q-\Phi$ criticality
  - Phase structures using shadow analysis
- Optical appearance of asymptotic dS BHs
  - Thin disk accretion
  - Spherical accretions
- Rotating BH solutions?