Quantum Hall Effect: Composite Fermions and Phase Transitions

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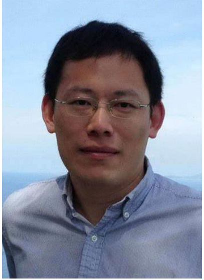

Hong-Hao Tu
Technische Universität Dresden

Meng Cheng
Yale University

Kai-Wen Huang
HUST

Y.-H. Wu, H.-H. Tu, and M. Cheng, arXiv: 2302.06501.

Some calculations in progress by graduate student Kai-Wen Huang.

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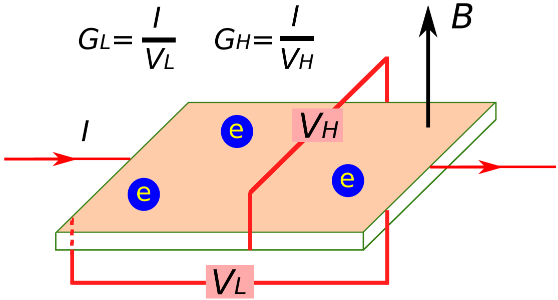
Introduction

longitudinal and Hall conductances



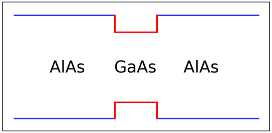
Classical eletromagnetism predicts the Hall conductance

$$G_{\mathrm{H}} = \frac{e\rho}{B} \qquad \rho \quad \text{is the electron density} \tag{1}$$

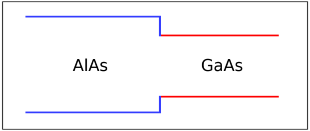
E. H. Hall, Am. J. Math. **2**, 287 (1879).

Technological progresses allow one to confine electrons in two dimensions.

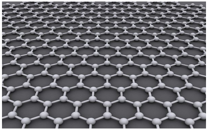
(a) semiconductor quantum well



(b) semiconductor heterostructure



(c) graphene



from wikipedia

The motion along the third direction is frozen.

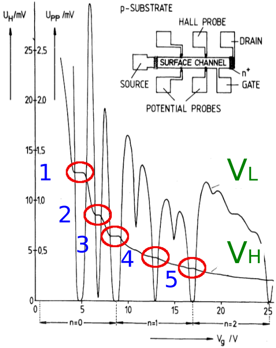
Experiments observed striking phenomena in 2DEG.

Quantum Hall Effect

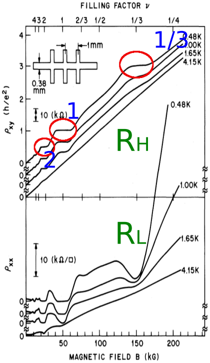
The Hall conductance is quantized at some plateaus.

$$G_{\mathrm{H}} = \nu(e^2/h) \quad \Longleftrightarrow \quad R_{\mathrm{H}} = \nu^{-1}(h/e^2) \tag{2}$$

integer quantum Hall states



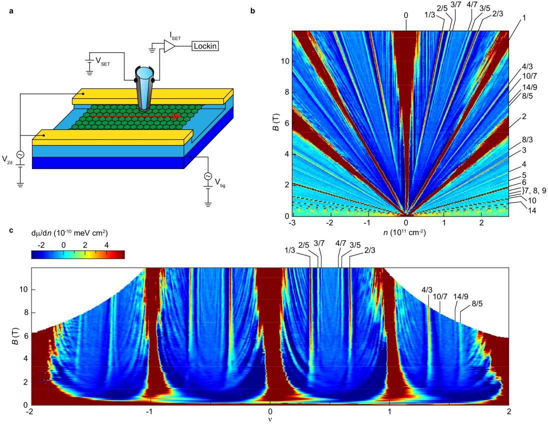
fractional quantum Hall states



K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
D. C. Tsui, H. L. Störmer, and A. C. Gossard, Phys. Rev. Lett. **48** 1559 (1982).

The electron plane in graphene and other 2D materials can be accessed more directly.

local compressibility measurement



B. E. Feldman, B. Krauss, J. H. Smet, and A. Yacoby, Science **337**, 1196 (2012).

Quantum Hall Effect

Many quantum Hall states have been observed.

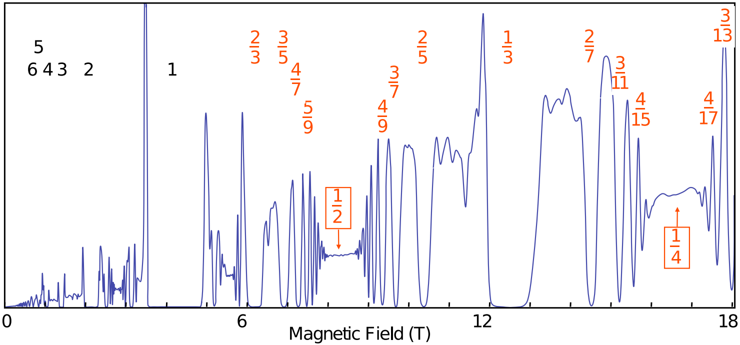
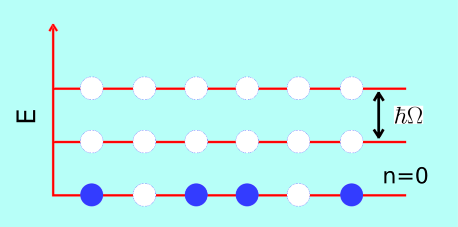


Figure made by Prof. H. L. Störmer.

W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **88**, 176802 (2002); Phys. Rev. Lett. **90**, 016801 (2003).

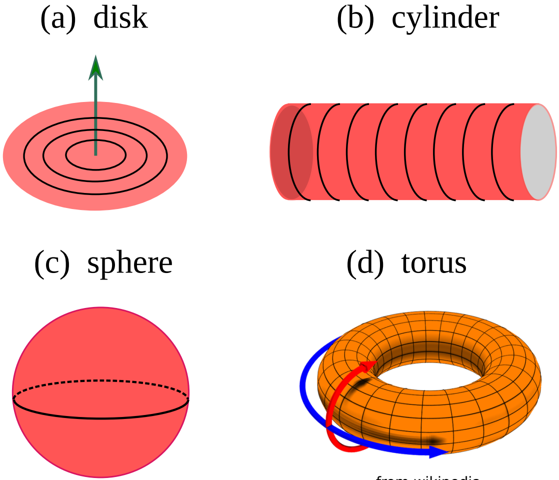
In two dimensions, charged particles in a magnetic field form Landau levels.

$$
\begin{aligned}
H_0 &= \frac{1}{2M}(\mathbf{p} - e\mathbf{A})^2 \\
&= \hbar\Omega\left(a^\dagger a + 1/2\right) \\
&= \hbar\Omega\,(n + 1/2)
\end{aligned}
$$



The single-particle eigenstates are highly degenerate.

The two-dimensional surface can have different shapes.



from wikipedia

Landau Levels

Explicit solutions on the disk with $\mathbf{A} = (-By/2, Bx/2, 0)$

$$\phi_{n,\alpha}(x,y) = \frac{1}{\sqrt{2\pi 2^{n+\alpha} n! \alpha!}} \left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z} \right)^n \left(\frac{z}{2} - 2\frac{\partial}{\partial \bar{z}} \right)^\alpha \exp\left(-\frac{|z|^2}{4} \right) \quad (3)$$

- n is the Landau level index
- α labels the states within a Landau level
- $\ell_B = \sqrt{\frac{\hbar}{eB}} = \frac{25.6 \text{ nm}}{\sqrt{B[\text{Tesla}]}} \qquad z = (x+iy)/\ell_B$

These states are localized due to the Gaussian factor.

For typical magnetic field, ℓ_B is much larger than lattice constants (~ 0.5 nm).

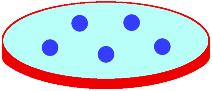
Lattice effects can be neglected in most cases. If they are important, there could be new physics.

IQH Effect

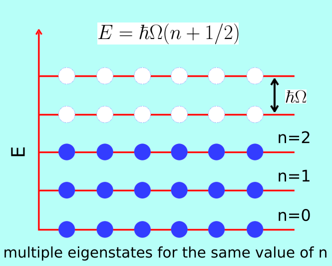
$$\text{filling factor } \nu = \frac{\text{number of electrons}}{\text{number of available states in one Landau level}} = \frac{N_e}{N_s} \qquad (4)$$

There is an energy gap when ν is an integer. If C Landau levels are fully occupied, the Hall conductance is Ce^2/h.

real space picture



energy level diagram



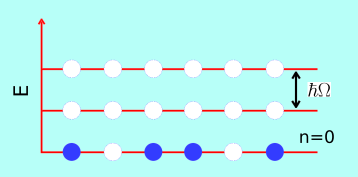
$\nu = 3$

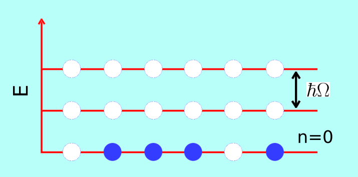
A complete theory of IQH states is much more complicated.

Composite Fermion Theory

FQH states occur at certain filling factors.

If free electrons partially occupy a Landau level, there would be no gap and no FQH state.





Interactions between electrons are necessary for generating FQH states.

What are special about these filling factors?

Many-Body Problem

The full Hamiltonian is very difficult

$$H = \frac{1}{2M} \sum_j [\mathbf{p}_j - e\mathbf{A}(\mathbf{r}_j)]^2 + \sum_{j<k} \frac{e^2}{4\pi\varepsilon|\mathbf{r}_j - \mathbf{r}_k|} + g\mu\mathbf{B}\cdot\mathbf{S} + \sum_j U(\mathbf{r}_j) \tag{5}$$

The four terms are

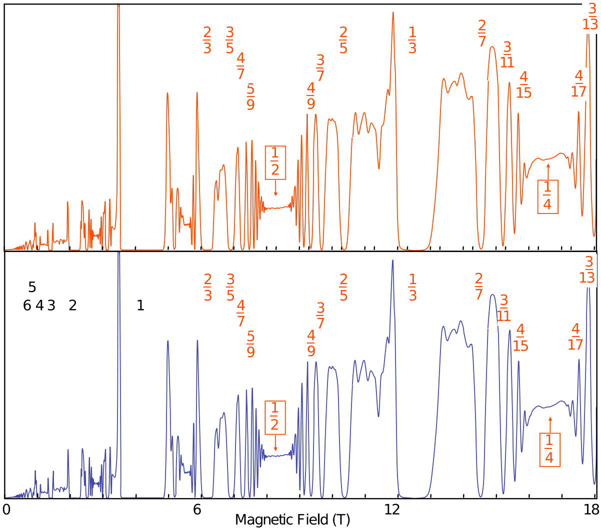
- kinetic energy with scale $\hbar\Omega \approx 20B[\text{Tesla}]\text{K}$
- Coulomb interaction with scale $\frac{e^2}{4\pi\varepsilon\ell_B} \approx 50\sqrt{B[\text{Tesla}]}\text{K}$
- Zeeman coupling with scale $\approx 0.3B[\text{Tesla}]\text{K}$
- background positive charges and disorder

The fourth term is neglected first. In the limit of large B and $\nu < 1$, all electrons reside in the lowest Landau level.

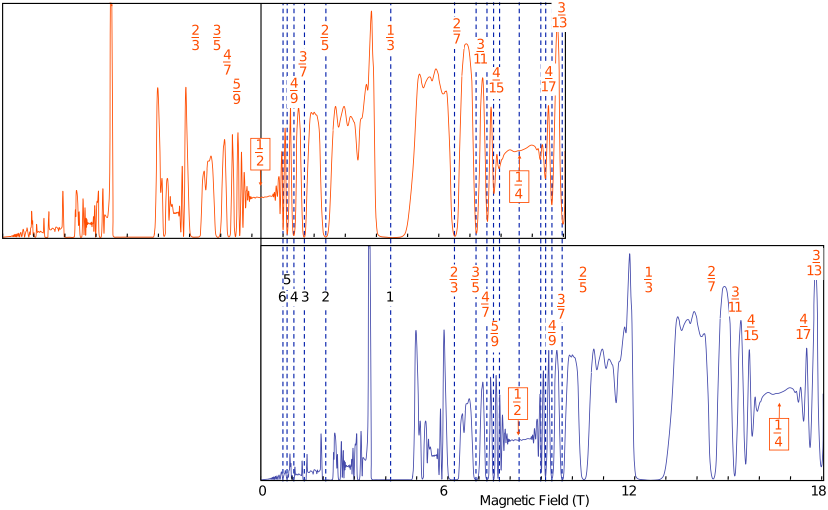
The simplified Hamiltonian is still difficult.

$$H = \mathcal{P}_{\text{LLL}} \sum_{j<k} \frac{e^2}{4\pi\varepsilon|\mathbf{r}_j - \mathbf{r}_k|} \mathcal{P}_{\text{LLL}} \tag{6}$$

A Nice Observation

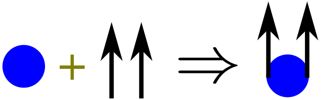


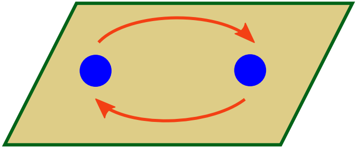
There seems to be a one-to-one correspondence between FQH and IQH states.



This correspondence can be understood using "composite fermions".

electron + p (even number) flux quanta $\Rightarrow$ composite fermion





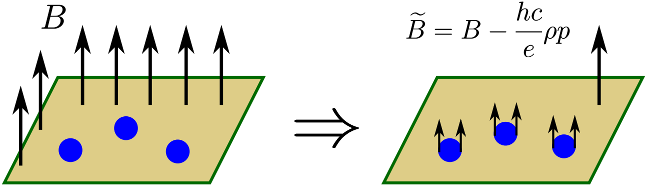
A minus sign appears if two fermions are exchanged. This sign is retained if p is even.

The formation of composite fermions is non-perturbative.

It is taken as a fundamental assumption that requires subsequent verifications.

J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).

The effective magnetic field for composite fermions is $\widetilde{B}$. It may have the same or opposite direction as the actual magnetic field B.

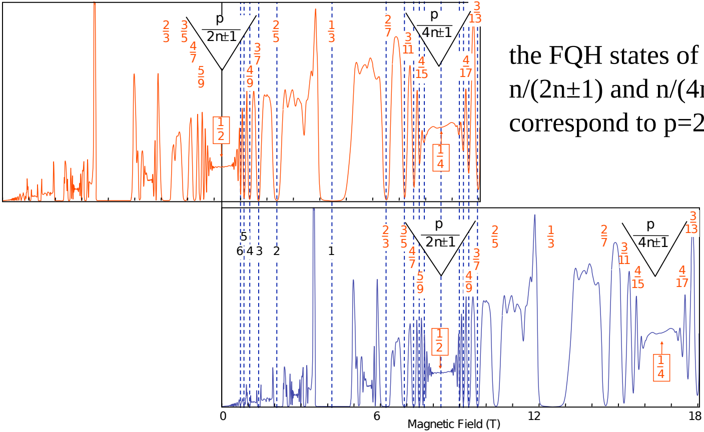


$$\text{composite fermion} \quad \widetilde{\nu} = \frac{\rho hc}{e|\widetilde{B}|} \qquad \text{electron} \quad \nu = \frac{\rho hc}{eB} = \frac{\widetilde{\nu}}{p\widetilde{\nu} \pm 1} \tag{7}$$

$+$ $(-)$ corresponds to positive (negative) $\widetilde{B}$.

An IQH state of composite fermions is realized when $\widetilde{\nu}$ is an integer.

FQH states of electrons ⇔ IQH states of composite fermions.



the FQH states of electrons at $n/(2n\pm1)$ and $n/(4n\pm1)$ correspond to p=2 and p=4

This analysis can be made more precise using wave functions and field theory.

If all magnetic fluxes are absorbed, the composite fermions would experience zero effective magnetic field and form a Fermi sea.

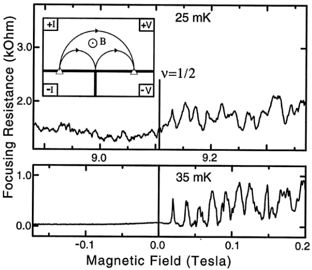


Fig. 3. Magnetic focusing of composite fermions near $\nu = \frac{1}{2}$ compared with focusing of electrons near $B = 0$. Composite fermions experience $B_{cf} = B - B(\nu = \frac{1}{2})$. The two B axes differ by $\sqrt{2}$ to account for spin polarization at high B. Inset shows the focusing sample geometry, where two possible focusing paths are shown by arrows [29].

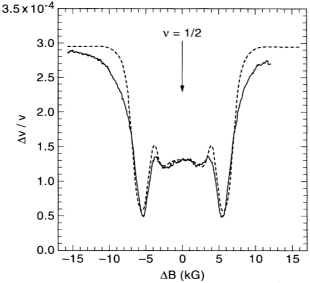


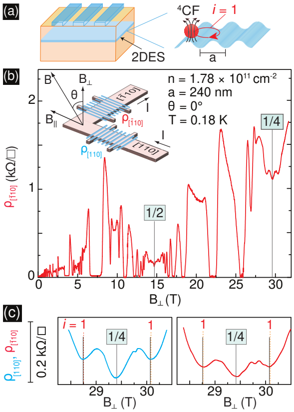
FIG. 1. Sound velocity shift versus magnetic field for 10.7 GHz surface acoustic waves near filling factor $\frac{1}{2}$. Both principle and secondary resonances are present. Temperature is ~ 130 mK. The dashed line shows the theoretical fit to the data using parameters defined in the text.

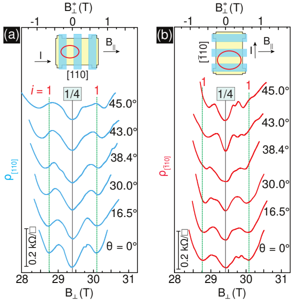
B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).
V. J. Goldman, B. Su, J.K. Jain, Phys. Rev. Lett. **72**, 2065 (1994).
R. L. Willett, K. W. West, L. N. Pfeiffer, Phys. Rev. Lett. **75**, 2989 (1995).

Composite Fermion

Geometric resonance of composite fermions in modulated potential.

2





Md. S. Hossain, M. K. Ma, M. A. Mueed, D. Kamburov, L. N. Pfeiffer, K. W. West, K. W. Baldwin, R. Winkler, and M. Shayegan, Phys. Rev. B **100**, 041112(R) (2019).

Boson vs. Fermion

If we have a system of charged bosons in two dimensions, there would be bosonic FQH states.

Bosonic FQH states are predicted to occur at filling factors $\nu = n/(n \pm 1)$ (and other values).

This can be also be understood using the composite fermion theory.

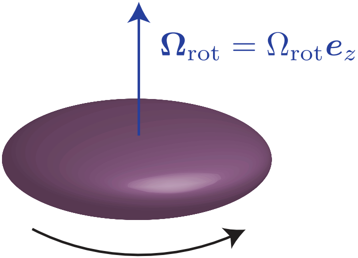
boson + p (odd number) flux quanta $\Rightarrow$ composite fermion

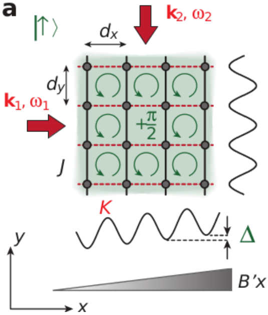
Bosonic FQH states may be studied using cold atoms or excitons in solid state systems.

N. R. Cooper, Adv. Phys. **57**, 539 (2008).
S. Viefers, J. Phys: Cond. Mat. **20**, 123202 (2008).

Synthetic Magnetic Field

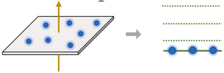
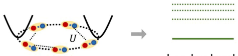
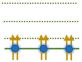
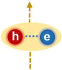
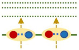
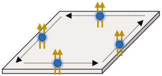
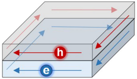
Atoms are charge neutral and require synthetic magnetic field.





N. Gemelke, E. Sarajlic, and S. Chu, arXiv:1007.2677.
M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Phys. Rev. Lett. **111**, 185301 (2013).
H. Miyake, G. A. Siviloglou, C. J. Kennedy, W. C. Burton, and W. Ketterle, Phys. Rev. Lett. **111**, 185302 (2013).
R. J. Fletcher, A. Shaffer, C. C. Wilson, P. B. Patel, Z. Yan, V. Crépel, B. Mukherjee, and M. W. Zwierlein, Science **372**, 1318 (2021).
J. Léonard, S. Kim, J. Kwan, P. Segura, F. Grusdt, C. Repellin, N. Goldman, and M. Greiner, Nature 619, 495 (2023),

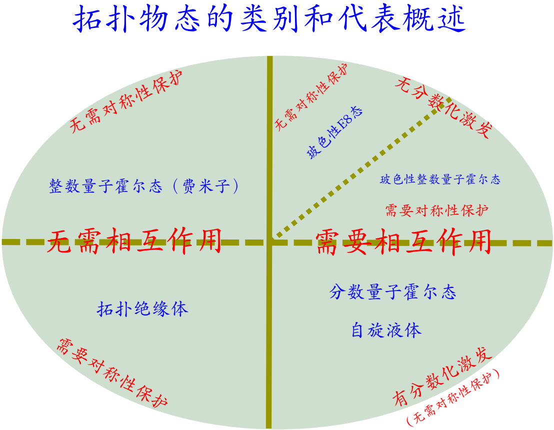
Electron-hole bound states with special band structure.

	Electronic FQH	Excitonic FQH (ETO)
Constituents	Electron	Exciton
Origin of Landau quantization	External field $B_\perp$ 	 “moat” + correlation U; Landau level at zero field $B_\perp = 0$
Bulk Low-energy excitation	Laughlin liquid (e.g. $m = 3$) Charge: $Q = e/3$ Statistics: anyons with $\phi = \pi/3$	Excitonic Laughlin liquid ($m = 2$) Charge: $Q = 0$ Statistics: anyons with $\phi = \pi/2$
Chern-Simons transmutation	 Electron bound to 2-flux quanta;  QH of CFs ($\nu = 1$)	 Exciton bound to 1-flux quanta;  Quantum anomalous Hall of CFs ($C = 1$)
Edge excitation and edge state conductance	 Chiral mode of CFs: $R_{xx} = 0$, $R_{xy} = 3h/e^2$	 Manifest as helical edge in conductance measurement ($B_\perp = 0$) $R_{xx} = h/2e^2$, $R_{xy} = 0$ (for small Hall bar)

R. Wang, T. A. Sedrakyan, B. Wang, L. Du, R.-R. Du, Excitonic topological order in imbalanced electron-hole bilayers, Nature **619**, 57 (2023).

Topological States

IQH and FQH states are topological states.



X.-G. Wen, Rev. Mod. Phys. **89**, 041004 (2017).

Topological Zoo

General properties of topological states

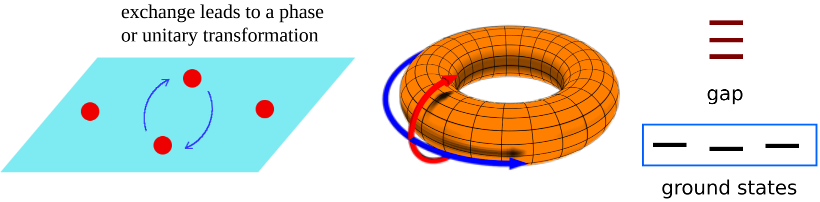
- quantized responses: the Hall conductance, the magnetoelectric effect, etc.
- fractionalized excitations: an excitation may carry fractional quantum numbers
- gapless edge states: localized at the boundary and stable under perturbations
- quantum entanglement: reduced density matrix of a subsystem has special properties
- symmetry protection: some states are nontrivial only if certain symmetries are retained

Nomenclature for different classes

- intrinsic topological order
- invertible topological order
- symmetry-protected topological (SPT) order
- ......

Topological Zoo

FQH states possess intrinsic topological order.



Two important features

- some excitations are anyons with fractional charge and/or nontrivial braid statistics.
- there are multiple (quasi-) degenerate ground states when defined on torus.

SPT states have no anyons and no GSD.

X. G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).

Field Theory

A (2+1)D system is in a topological phase if its low-energy effective field theory is a topological quantum field theory.

Lagrangian density of the Abelian Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi} \epsilon^{\tau\lambda\mu} K_{IJ} a_{I\tau} \partial_\lambda a_{J\mu} = \frac{1}{4\pi} K_{IJ} a_I d a_J \tag{8}$$

$\hbar = 1$ is assumed. K_{IJ} are integers and a_I are emergent U(1) gauge fields.

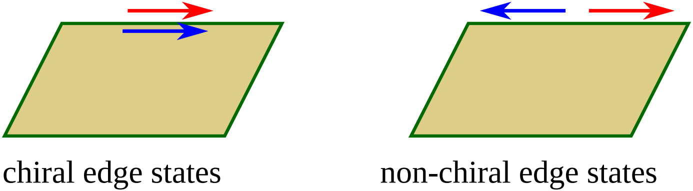
$$\begin{aligned} \nu = 1 \text{ IQH state} \qquad & K = \begin{bmatrix} 1 \end{bmatrix} \\ \nu = 1/3 \text{ FQH state} \qquad & K = \begin{bmatrix} 3 \end{bmatrix} \\ \nu = 2/5 \text{ FQH state} \qquad & K = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \end{aligned} \tag{9}$$

C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Rev. Mod. Phys. **80**, 1083 (2008).

- there are $|\det K|$ degenerate ground states on torus
- each excitation is labeled by an integer vector $\mathbf{l}$
- the self statistics of $\mathbf{l}$ is $\theta_{\mathbf{l},\mathbf{l}} = \pi \mathbf{l}^{\mathsf{T}} K^{-1} \mathbf{l}$
- the mutual statistics of $\mathbf{l}_1$ and $\mathbf{l}_2$ is $\theta_{\mathbf{l}_1,\mathbf{l}_1} = 2\pi \mathbf{l}_1^{\mathsf{T}} K^{-1} \mathbf{l}_2$

If the system is placed on a surface with boundary, there are gapless edge states.

- the chirality of edge states is determined by the sign of the eigenvalue of K.



M. Levin, Phys. Rev. X **3**, 021009 (2013).

In quantum Hall systems, microscopic Hamiltonians usually conserve the total number of particles, so the system has a U(1) symmetry.

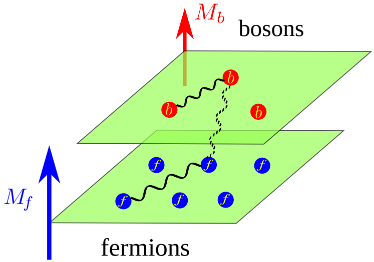
Each particle is assigned charge 1 and a probing field **A** is introduced to couple with this charge.

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_I d a_J + \frac{1}{2\pi} t_I A d a_I \tag{10}$$

t_I are integers.

- the charge of $\mathbf{l}$ is $\mathcal{Q} = \mathbf{t}^{\mathsf{T}} K^{-1} \mathbf{l}$
- the Hall conductance is $\sigma_{\mathrm{H}} = \mathbf{t}^{\mathsf{T}} K^{-1} \mathbf{t}$ (in units of e^2/h)

We study topological states in Bose-Fermi mixtures.



The particles couple to their respective synthetic magnetic fields.

The numbers of particles are N_b and N_f. The magnetic fluxes are M_b and M_f.

The system has a U(1)×U(1) symmetry associated with particle number conservations.

There are two probing fields $\mathbf{A}_b$ and $\mathbf{A}_f$.

each boson has unit charge with respect to $\mathbf{A}_b$ and zero charge with respect to $\mathbf{A}_f$.

each fermion has zero charge with respect to $\mathbf{A}_b$ and unit charge with respect to $\mathbf{A}_f$.

Bose-Fermi SPT State

A SPT state is constructed using the composite fermion theory.

$$\Psi_{\rm SPT} \sim \left[\Phi_1^*(\{z_j^b\})\Phi_1^*(\{z_j^f\})\right] \prod_{j<k}^{N_b}(z_j^b - z_k^b) \prod_{j<k}^{N_f}(z_j^f - z_k^f)^2 \prod_{j}^{N_b}\prod_{k}^{N_f}(z_j^b - z_k^f) \tag{11}$$

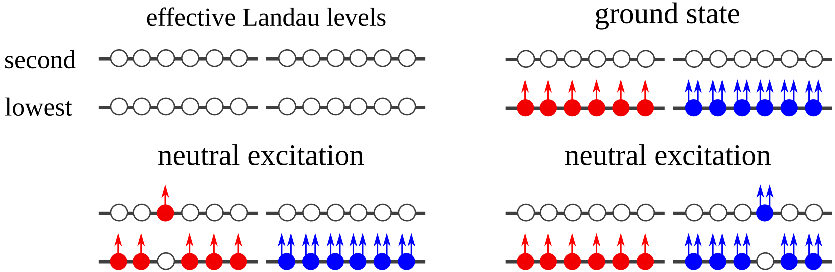
- $M_b = N_f, M_f = N_b + N_f$ (in the thermodynamic limit).
- each boson is attached with one flux from other bosons
- each fermion is attached with two fluxes from other fermions
- boson fermion correlation
- composite fermions form two $\nu = -1$ IQH states

Its Chern-Simons theory has

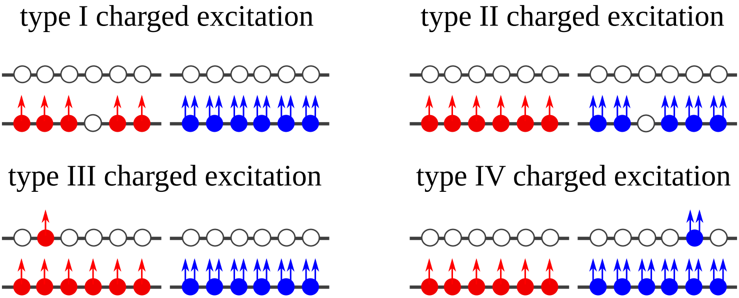
$$K = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad |\det K| = 1 \tag{12}$$

K has one positive and one negative eigenvalue, so the edge states are non-chiral and can be gapped out.

Neutral excitations are excitons of composite fermions.



Charged excitations are created by adding or removing composite fermions.



They do not carry fractional charges with respect to $\mathbf{A}_b$ or $\mathbf{A}_f$.

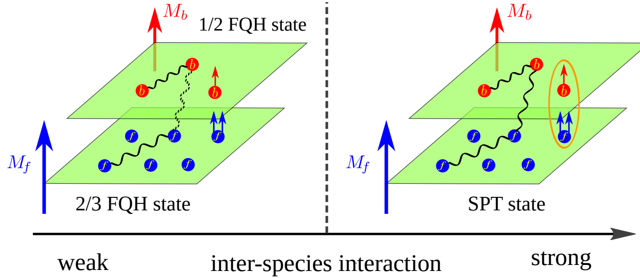
This is consistent with the fact that $|\det K| = 1$.

Quantum Phase Transitions

Continuum Model

$$H_{\text{mix}} = \underbrace{\sum_{j<k} 4\pi\ell_b^2 \, \delta(\mathbf{r}_j^b - \mathbf{r}_k^b)}_{\text{boson-boson}} + \underbrace{\sum_{j<k} 4\pi\ell_f^4 \, \nabla^2\delta(\mathbf{r}_j^f - \mathbf{r}_k^f)}_{\text{fermion-fermion}} + \underbrace{g_m \sum_{j,k} 4\pi\ell_b\ell_f \, \delta(\mathbf{r}_j^b - \mathbf{r}_k^f)}_{\text{boson-fermion}} \quad (13)$$

$$\nu_b = \frac{N_b}{M_b} \qquad \nu_f = \frac{N_f}{M_f} \qquad M_b = N_f \quad M_f = N_b + N_f$$



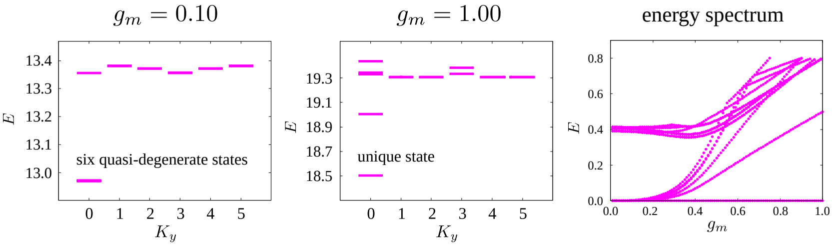
The individual filling factors ν_b and ν_f can be adjusted.

Numerical Results

If we choose $\nu_b = 1/2$ and $\nu_f = 2/3$, the system would realize two decoupled FQH states when $g_m = 0$.

Exact diagonalization is performed on the torus for many different $g_m \in [0, 1]$.

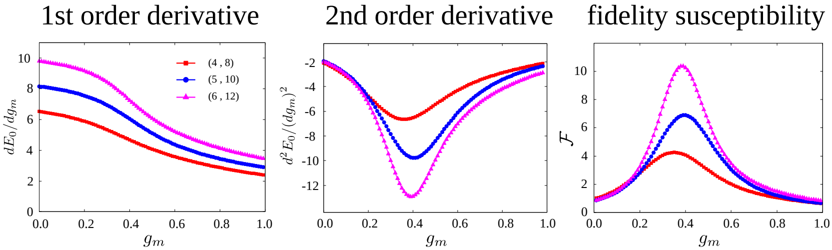
A finite small g_m does not destroy the FQH states.



The system turns into the SPT state when g_m becomes sufficiently large.

We study the lowest energy $E_0(g_m)$ and the associated state $|\Psi_0(g_m)\rangle$.

$$\text{fidelity susceptibility} \quad \mathcal{F}(g_m) = \frac{2}{(\delta g_m)^2}\left[1 - \left|\langle\Psi_0(g_m)|\Psi_0(g_m + \delta g_m)\rangle\right|\right] \tag{14}$$



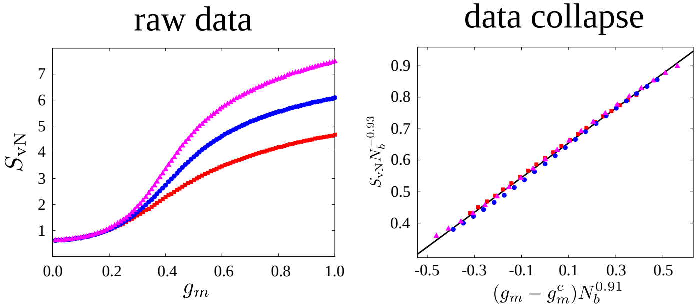
These results suggest that there is a continuous phase transition.

M. Cozzini, R. Ionicioiu, and P. Zanardi, Phys. Rev. B **76**, 104420 (2007).
W.-L. You, Y.-W. Li, and S.-J. Gu, Phys. Rev. E **76**, 022101 (2007).

Critical scaling plays a prominent role in continous phase transitions.

We study the von Neumann entanglement entropy between bosons and fermions.

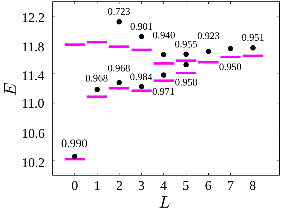
$$\rho_b = \mathrm{Tr}_f \, |\Psi_0(g_m)\rangle\langle\Psi_0(g_m)| \qquad S_{\mathrm{vN}} = -\mathrm{Tr}\, \rho_b \ln \rho_b \tag{15}$$



Data collapse is achieved using a bold conjecture: $S_{\mathrm{vN}}(g_m) N_b^{\alpha} = f[(g_m - g_m^c) N_b^{\beta}]$.

Exact diagonalization is performed on the sphere for $g_m = 1$.

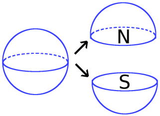
energy spectrum



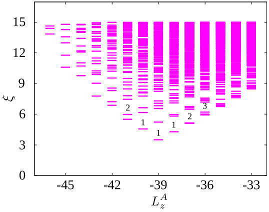
$$\rho = |\Psi_0\rangle\langle\Psi_0|$$

$$\rho_{\mathrm{S}} = \mathrm{Tr}_N \rho$$

$$\rho_{\mathrm{S}} = \exp(-\xi)$$



entanglement spectrum



The composite fermion wave functions are quite accurate.

There are two edge modes with opposite chiralities.

H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).
J. Dubail, N. Read, and E. H. Rezayi, Phys. Rev. B **85**, 115321 (2012).
A. Sterdyniak, A. Chandran, N. Regnault, B. A. Bernevig, and P. Bonderson, Phys. Rev. B **85**, 125308 (2012).
I. D. Rodríguez, S. H. Simon, and J. K. Slingerland, Phys. Rev. Lett. **108**, 256806 (2012).

Chern-Simons theory for the decoupled FQH states are

$$\text{boson} \quad \nu_b = 1/2 \quad K_b = [2] \quad \mathbf{t}_b = [1]$$
$$\mathcal{L}_b = \frac{1}{4\pi} 2 a_1 d a_1 + \frac{1}{2\pi} A_b d a_1 \tag{16}$$

$$\text{fermion} \quad \nu_f = 2/3 \quad K_f = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \quad \mathbf{t}_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{the common version}$$
$$\mathcal{L}_f = \frac{1}{4\pi} a_2 d a_2 - \frac{1}{4\pi} 3 a_3 d a_3 + \frac{1}{2\pi} A_f d a_2 + \frac{1}{2\pi} A_f d a_3 \tag{17}$$

It is more convenient to use another version for the fermions

$$K_f = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{t}_f = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{18}$$

They can be obtained from previous ones using $\mathrm{GL}(2, \mathbb{Z})$ transformations.

An intuitive picture

$$K_b \oplus K_f = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{folding} \Rightarrow \quad \widetilde{K} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{19}$$

Formally, we introduce a scalar boson field ϕ to construct the critical theory

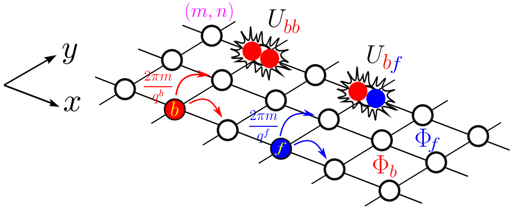
$$\mathcal{L}_{\text{mix}} = \mathcal{L}_b + \mathcal{L}_f + \left|(\partial - ia_1 + ia_2)\,\phi\right|^2 + r|\phi|^2 + u|\phi|^4 + \cdots \tag{20}$$

- $r > 0$, ϕ is gapped and can be integrated out. It reduces to $\mathcal{L}_b + \mathcal{L}_f$.
- $r < 0$, ϕ condenses to generate the Higgs phase. a_1 can be eliminated by setting it to a_2. This reproduces the SPT theory

$$\widetilde{\mathcal{L}} = \frac{1}{4\pi} 2a_2 da_3 + \frac{1}{4\pi} a_3 da_3 + \frac{1}{2\pi} A_b da_2 + \frac{1}{2\pi} A_f da_3 \tag{21}$$

$$\widetilde{K} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \widetilde{\mathbf{t}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In the Harper-Hofstadter model, the effects of magnetic field is manifested via phases in hopping terms.

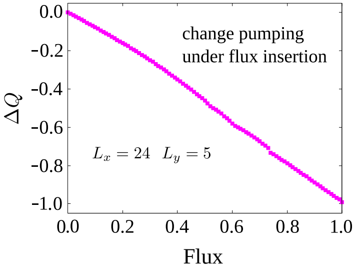


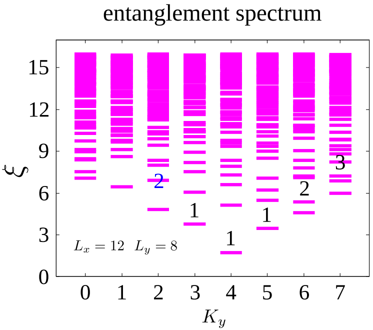
onsite boson-boson repulsion, nearest neighbor fermion-fermion repulsion, onsite boson-fermion repulsion

$$\begin{aligned} H \quad &= \quad -t \sum_{\sigma} \sum_{\langle jk \rangle_x} \left(\widehat{c}^\dagger_{\sigma,j} \widehat{c}_{\sigma,k} + \text{H.c.} \right) - t \sum_{\sigma} \sum_{\langle jk \rangle_y} \left(e^{i\phi^\sigma_{jk}} \widehat{c}^\dagger_{\sigma,j} \widehat{c}_{\sigma,k} + \text{H.c.} \right) \\ & \quad + U_{bb} \sum_{j} \widehat{n}_{b,j} \widehat{n}_{b,j} + U_{ff} \sum_{\langle jk \rangle} \widehat{n}_{f,j} \widehat{n}_{f,k} + U_{bf} \sum_{j} \widehat{n}_{b,j} \widehat{n}_{f,j} \end{aligned} \tag{22}$$

P. G. Harper, Proc. Phys. Soc. Lond. A **68**, 879 (1955).
D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

The lattice model has been studied using density matrix renormalization group (DMRG) on finite cylinders.





There is good evidence for the SPT state.

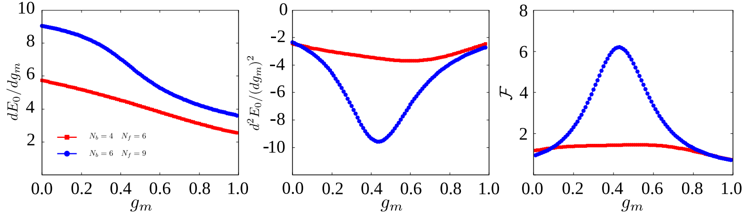
In general, we may choose $\nu_b = n/(n+1)$ and $\nu_f = (n+1)/(2n+1)$ $(n \in \mathbb{N}^+)$. The decoupled FQH states are also composite fermion states with

$$K_b = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} + \mathbb{I}_{n\times n} \qquad K_f = 2\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} - \mathbb{I}_{(n+1)\times(n+1)} \tag{23}$$

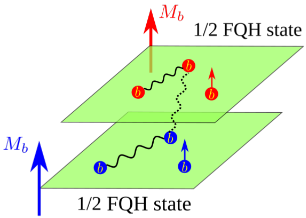
The critical theory is

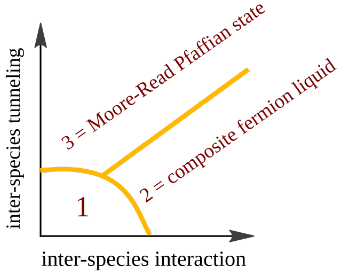
$$\mathcal{L}_{\rm mix} = \mathcal{L}_b + \mathcal{L}_f + |(\partial - ia_n + ia_{n+1})\,\phi|^2 + r|\phi|^2 + u|\phi|^4 + \cdots \tag{24}$$

Numerical results for the n=2 case



Bose-Bose mixture with inter-species interaction and tunneling.





1 = two decoupled Laughlin states

Both 1⇒2 and 1⇒3 seem to be continuous. Is there a tricritical point?

Y.-H. Wu and J. K. Jain, Phys. Rev. A **91**, 063623 (2015).
W. Zhu, S. S. Gong, D. N. Sheng1, L Sheng, Phys. Rev. B. **91**, 245126 (2015).
Z. Liu, A. Vaezi, C. Repellin, and N. Regnault, Phys. Rev. B **93**, 085115 (2016).
S. D. Geraedts, C. Repellin, C. Wang, R. S. K. Mong, T. Senthil, and N. Regnault, Phys. Rev. B **96**, 075148 (2017).
V. Crépel, B. Estienne, and N. Regnault, Phys. Rev. Lett. **123**, 126804 (2019).

The Laughlin state is described by $\mathrm{U}(1)_2$ Abelian Chern Simons theory. It is dual to $\mathrm{SU}(2)_1$ non-Abelian Chern-Simons theory.

$$\mathcal{L}_u = \frac{1}{4\pi}\mathrm{Tr}\left[a_u \wedge da_u + \frac{2}{3} a_u \wedge a_u \wedge a_u\right] \qquad \mathcal{L}_d = \frac{1}{4\pi}\mathrm{Tr}\left[a_d \wedge da_d + \frac{2}{3} a_d \wedge a_d \wedge a_d\right] \tag{25}$$

The Moore-Read state is described by $\mathrm{SU}(2)_2$ non-Abelian Chern-Simons theory

$$\mathcal{L}_{\mathrm{MR}} = \frac{2}{4\pi}\mathrm{Tr}\left[a \wedge da + \frac{2}{3} a \wedge a \wedge a\right] \tag{26}$$

The transition is described by a non-Abelian Chern-Simons-Higgs theory

$$\mathcal{L}_u + \mathcal{L}_d + \left|\left(\partial - i a_u + i a_d\right)\phi\right|^2 + \ldots . \tag{27}$$

Higgs condensation locks a_u with a_d.

Summary

Summary

A SPT state for Bose-Fermi mixtures can be realized using simple Hamiltonians.

Continuous phase transitions between FQH states and SPT state can be induced.

The transitions are described by Chern-Simons-Higgs theory in which Wilson-Fisher bosons couple to emergent gauge fields.

The condensation of bosons locks two gauge fields with each other. This is a quite general mechanism that also works in many other cases.

The critical theory is strongly coupled and very difficult to study. What else can be done?

Thanks!