# NNNLO QCD predictions for Top-quark decay

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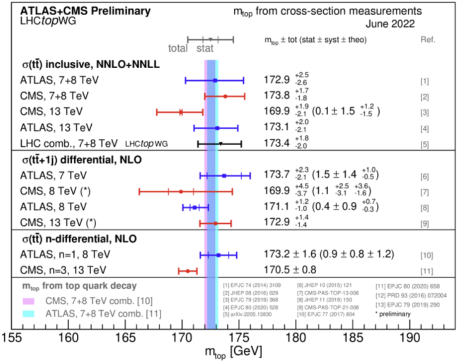
# Motivation

Top-quark mass is one of the fundamental parameters in the Standard Model.

Summary of the top-mass analyses at the LHC.

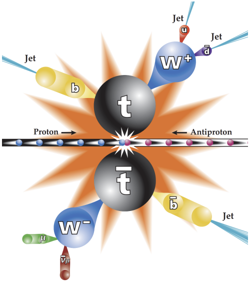


# Motivation

Top decay width $\Gamma_t$ is one of the fundamental properties of top-quark.

Due to its large mass, $\Gamma_t$ is expected to be very large.

The measurement of $\Gamma_t$ could hint new physics.




[Denisov, Vellidis 2015]

# Motivation

The top-quark decays almost exclusively to $Wb$. $\Gamma_t = \Gamma_t(t \to Wb)$.

$$\left|V_{\text{CKM}}\right| = \begin{pmatrix} 0.97435 \pm 0.00016 & 0.22500 \pm 0.00067 & 0.00369 \pm 0.00011 \\ 0.22486 \pm 0.00067 & 0.97349 \pm 0.00016 & 0.04182^{+0.00085}_{-0.00074} \\ 0.00857^{+0.00020}_{-0.00018} & 0.04110^{+0.00083}_{-0.00072} & 0.999118^{+0.000031}_{-0.000036} \end{pmatrix}$$

[PDG 2022]

The most precise measurement for $\Gamma_t$ by now is given by CMS

$\Gamma_t = 1.36 \pm 0.02\ (\text{stat.})^{+0.14}_{-0.11}\ (\text{syst.})$ GeV [CMS, 2014].

In the future $e^+e^-$ collider, $\Gamma_t$ can be measured with an uncertainty of 30 MeV [Martinez, Miquel 2019].

# Theoretical Development

NLO QCD corrections [Jezabek, Kuhn 1989, Czarnecki 1990, Li, Oakes, Yuan 1991]

NLO EW corrections [Denner, Sack 1991, Eilam, Mendel, Migneron, Soni 1991]

Numerical result of full NNLO QCD corrections [Gao, Li, Zhu 2013, Brucherseifer, Caola, Melnikov 2013]

Numerical result of full $N^3LO$ QCD corrections [Chen, Chen, Guan, Ma 2023]

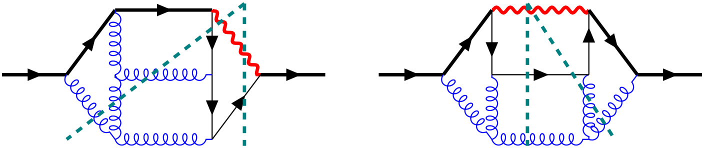
Analytic results of NNLO and $N^3LO$ QCD corrections [Chen, Li, Wang, Wang 2022, Chen, Li, Li, Wang, Wang, Wu 2023]

# Optical Theorem

The forward scattering amplitudes $\Sigma$ for $t \to Wb \to t$

$$\Gamma_t = \frac{\text{Im}(\Sigma)}{m_t} \quad (1)$$

For N$^3$LO QCD corrections, the self-energy four-loop diagrams are considered. For example,



The imaginary part comes from cut diagrams.

The complicated phase space integration can be avoided.

# Calculation Method

For $t \to Wb \to t$, b quark is considered as massless. We define $w = m_w^2/m_t^2$.

1. Generating diagrams and amplitudes.
2. Reducing amplitudes to master integrals by `FIRE` [Smirnov, Chuharev 2020].
3. Analytically calculating the master integrals.

# Master Integrals Calculations

Method: canonical differential equations. For example,

$$\frac{\partial F_4(w,\epsilon)}{\partial w} = \frac{\epsilon\,(F_5 - 2F_4)}{w-1} - \frac{\epsilon\,(F_4 + F_5)}{w}, \qquad w = \frac{m_W^2}{m_t^2}, \quad D = 4 - 2\epsilon \tag{2}$$

Two important ingredients:

1. Construct canonical form ($\epsilon$ form, $d\log$ form) – `Libra` [Lee 2021]

2. Boundary conditions – `AMFlow` [Liu, Ma 2022] and PSLQ method [Ferguson, Beiley, Arno 1992 1999]

Results: harmonic polylogarithms (HPLs).

Analytic calculations of four loop master integrals are non-trivial.

# Leading Color Contribution at NNLO

QCD corrections of $\Gamma_t$.

$$\Gamma(t \to Wb) = \Gamma_0 \left[ X_0 + \frac{\alpha_s}{\pi} X_1 + \left(\frac{\alpha_s}{\pi}\right)^2 X_2 + \left(\frac{\alpha_s}{\pi}\right)^3 X_3 \right], \tag{3}$$

$$\Gamma_0 = \frac{G_F m_t^3 |V_{tb}|^2}{8\sqrt{2}\pi}. \tag{4}$$

Leading Color Contribution at NNLO,

$$\begin{aligned} X_2 &= C_F \left[ C_F X_F + C_A X_A + T_R n_l X_l + T_R n_h X_h \right] \\ &= C_F \left[ N_c \left( X_A + \frac{X_F}{2} \right) + \frac{n_l}{2} X_l - \frac{1}{2N_c} X_F + \frac{n_h}{2} X_h \right] \end{aligned} \tag{5}$$

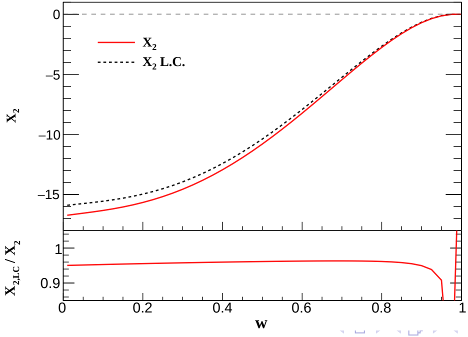
$$C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_A = N_c, \quad T_R = \frac{1}{2},$$

$n_l(n_h)$ = number of light (heavy) quark flavors.

# Leading Color Contribution at NNLO

$$
\begin{aligned}
X_2 &= C_F\left[C_F X_F + C_A X_A + T_R n_l X_l + T_R n_h X_h\right] \\
&= C_F\left[N_c\left(X_A + \frac{X_F}{2}\right) + \frac{n_l}{2}X_l - \frac{1}{2N_c}X_F + \frac{n_h}{2}X_h\right]
\end{aligned} \tag{6}
$$

In the top quark decay, $N_c = 3, n_l = 5, n_h = 1$, $w(m_w^2/m_t^2) \approx 0.22$, $X_{2,\text{L.C}}/X_2 > 95\%$.



## Analytic Calculations at N$^3$LO

According to NNLO, the leading color contributions are dominant.

The analytic calculation of full N$^3$LO is quite difficult. We concentrate on the leading color contributions at N$^3$LO.

$$X_3 = C_F\left[N_c^2 Y_A + \widetilde{Y}_A + \frac{\overline{Y}_A}{N_c^2} + n_l n_h Y_{lh} + n_l\left(N_c Y_l + \frac{\widetilde{Y}_l}{N_c}\right) + n_l^2 Y_{l2} + n_h\left(N_c Y_h + \frac{\widetilde{Y}_h}{N_c}\right) + n_h^2 Y_{h2}\right]. \tag{7}$$

$$X_{3,\text{L.C}} = C_F\left[N_c^2 Y_A + n_l N_c Y_l + n_l^2 Y_{l2}\right]. \tag{8}$$

In leading color approximation, there are still 408 Feynman diagrams and 185 master integrals need to be calculated.

## Analytic Results of $N^3LO$

$$X_{3,\mathrm{L.C}} = C_F\left[N_c^2 Y_A + n_l N_c Y_l + n_l^2 Y_{l2}\right].$$

$$Y_l = (8w^3+12w^2+43w+8)\Big(H(0,0,0,0,1,w) - H(0,0,0,1,0,w)\Big)$$
$$+(-2w^3+51w^2+86w+7)\,H(0,0,0,1,1,w) + (2w^3-15w^2-20w-1)\,H(0,0,1,0,1,w)$$
$$+(-2w^3-45w^2-80w-9)\,H(0,0,1,1,0,w) - 2\,(6w^3-33w^2-46w-1)\,H(0,0,1,1,1,w)$$
$$+(6w^3+3w^2+14w+5)\,H(0,1,0,0,1,w) + 2\,(6w^3+21w^2+50w+8)\,H(0,1,0,1,1,w)$$
$$-2\,(2w^3+27w^2+56w+6)\,H(0,1,1,0,1,w) + 2\,(2w^3-27w^2-40w-3)\,H(0,1,1,1,0,w)$$
$$-(2w+1)(w-1)^2\Big(2H(0,1,0,1,0,w) - H(1,0,0,0,1,w) + H(1,0,0,1,0,w)$$
$$+4H(1,0,0,1,1,w) - 2H(1,0,1,0,1,w) - 2H(1,0,1,1,0,w)$$
$$-4H(1,1,0,0,1,w) + 4H(1,1,0,1,0,w)\Big) + \cdots$$

$$Y_{l2} = -\frac{(w-1)^2(2w+1)}{18}\Big(H(0,0,0,1,w) - H(0,0,1,0,w) + 2H(0,0,1,1,w)$$
$$-H(0,1,1,0,w) + 4H(0,1,1,1,w) + 2H(1,0,0,1,w) - 3H(1,0,1,0,w) - 4H(1,1,1,0,w)\Big) + \cdots$$

# Numerical Results

Input parameters from [P.D.G 2022]

$$
\begin{aligned}
m_t &= 172.69 \text{ GeV}, \quad m_b = 4.78 \text{ GeV}, \\
m_W &= 80.377 \text{ GeV}, \quad \Gamma_W = 2.085 \text{ GeV}, \\
m_Z &= 91.1876 \text{ GeV}, \quad G_F = 1.16638 \times 10^{-5} \text{ GeV}^{-2}, \\
|V_{tb}| &= 1, \quad \alpha_s(m_Z) = 0.1179.
\end{aligned} \tag{10}
$$

$\Gamma_t^{(0)} = 1.486$ GeV with $m_b = 0$ and on-shell $W$.

$$
\begin{aligned}
\Gamma_t = \Gamma_t^{(0)}[&(1 + \delta_b^{(0)} + \delta_W^{(0)}) \\
&+ (\delta_b^{(1)} + \delta_W^{(1)} + \delta_{\text{EW}}^{(1)} + \delta_{\text{QCD}}^{(1)}) \\
&+ (\delta_b^{(2)} + \delta_W^{(2)} + \delta_{\text{EW}}^{(2)} + \delta_{\text{QCD}}^{(2)} + \delta_{\text{EW}\times\text{QCD}}^{(2)}) \\
&+ (\delta_b^{(3)} + \delta_W^{(3)} + \delta_{\text{EW}}^{(3)} + \delta_{\text{QCD}}^{(3)} + \delta_{\text{EW}\times\text{QCD}}^{(3)})],
\end{aligned} \tag{11}
$$

# Numerical Results

Corrections in percentage ($\%$) normalized by the LO width $\Gamma_t^{(0)} = 1.486$ GeV with $m_b = 0$ and on-shell $W$.

| | $\delta_b^{(i)}$ | $\delta_W^{(i)}$ | $\delta_{\rm EW}^{(i)}$ | $\delta_{\rm QCD}^{(i)}$ | $\Gamma_t$ [GeV] |
|---|---|---|---|---|---|
| LO | -0.273 | -1.544 | — | — | 1.459 |
| NLO | 0.126 | 0.132 | 1.683 | -8.575 | $1.361^{+0.0091}_{-0.0130}$ |
| NNLO | $*$ | 0.030 | $*$ | -2.070 | $1.331^{+0.0055}_{-0.0051}$ |
| $\mathrm{N^3LO}$ | $*$ | 0.009 | $*$ | -0.667 | $1.321^{+0.0025}_{-0.0021}$ |

QCD corrections are dominant.

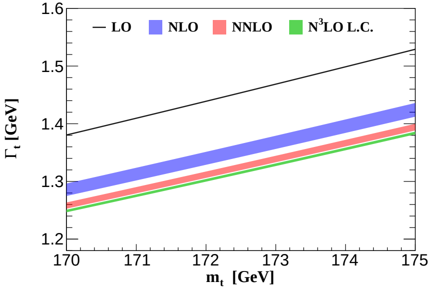
$X_{3,\mathrm{L.C}}/X_3 \approx 95\%$ compared with full numerical result. [Chen, Chen, Guan, Ma 2023]

NLO EW correction is 1.683%.

The off-shell W boson effect at NNLO and N3LO are further suppressed.

# Numerical Results

$m_t$ varies from 170 GeV to 175 GeV. $\mu$ varies from $\frac{m_t}{2}$ to $2m_t$.




The scale uncertainty at N$^3$LO is reduced to $\pm 0.2\%$, only half of that at NNLO.

It is very convenient to use fitted function within this range

$$\Gamma_t(m_t) = 0.027037 \times m_t - 3.34801 \text{ GeV} \tag{12}$$

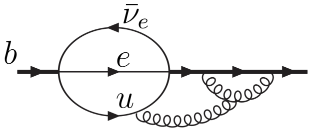
# Mathematica program

TopWidth.m can be downloaded from https://github.com/haitaoli1/TopWidth. The package HPL is required [Maitre 2006].

```
<< TopWidth`

        (****** TopWidth-1.0 ******)
    Authors: Long-Bin Chen, Hai Tao Li, Jian Wang, YeFan Wang
    TopWidth[QCDorder, mbCorr, WwidthCorr, EWcorr, mu] is provided for top width calculations
    Please cite the paper for reference: arXiv:2212.06341

*-*-*-*-*-* HPL 2.0 *-*-*-*-*-*

Author: Daniel Maitre, University of Zurich

(* SetParameters[mt,mb,mw,Wwidth, GF] *)
SetParameters[17269/100, 478/100, 80377/1000, 2085/1000, 911876/10000, 11663788 × 10^-12];
TopWidth[3, 1, 1, 1, 17269/100]

1.32073
```

## Relations With Other Process

Integrating over $w$ $(w = m_W^2/m_t^2)$ from 0 to 1, we obtain N$^3$LO QCD corrections in semileptonic decay $\Gamma(b \to X_u e\bar{\nu}_e)$ [Ritbergen 1999].




$$\Gamma(b \to X_u e\bar{\nu}_e) = \frac{G_F^2 |V_{ub}|^2 m_b^5}{192\pi^3} \left[1 + \sum_{i=1} \left(\frac{\alpha_s}{\pi}\right)^i b_i\right]. \tag{13}$$

$$\begin{aligned} b_3 = C_F \Bigg[ & N_c^2 \left(\frac{9651283}{82944} - \frac{1051339\pi^2}{62208} - \frac{67189\zeta(3)}{864} + \frac{4363\pi^4}{6480} + \frac{59\pi^2\zeta(3)}{32} + \frac{3655\zeta(5)}{96} \right. \\ & \left. - \frac{109\pi^6}{3780}\right) + n_l N_c \left(-\frac{729695}{27648} + \frac{48403\pi^2}{15552} + \frac{1373\zeta(3)}{108} + \frac{133\pi^4}{1728} - \frac{13\pi^2\zeta(3)}{72} - \frac{125\zeta(5)}{24}\right) \\ & + n_l^2 \left(\frac{24763}{20736} - \frac{1417\pi^2}{15552} - \frac{37\zeta(3)}{216} - \frac{121\pi^4}{6480}\right) + \text{subleading color} \Bigg] \\ = & (-195.3 \pm 9.8) C_F \,. \end{aligned} \tag{14}$$

# Summary

We provide the first leading color QCD correction at N$^3$LO analytically, which can be used to perform fast numerical evaluations.

The NNLO and N$^3$LO QCD corrections decrease the LO result by -2.07% and -0.667% with $m_t = 172.69$ GeV and $\mu = m_t$

We derive the analytic N$^3$LO QCD leading color predictions for the semileptonic $b \to u$ decay width.

Thanks !

## HPLs

The analytical results of master integrals can be written as multiple polylogarithms (GPLs)

$$G_{a_1,a_2,\ldots,a_n}(x) \equiv \int_0^x \frac{\mathrm{d}t}{t-a_1} G_{a_2,\ldots,a_n}(t)\,, \tag{15}$$

$$G_{\vec{0}_n}(x) \equiv \frac{1}{n!} \ln^n x\,. \tag{16}$$

In our problem, we only need harmonic polylogarithms (HPLs).

$$H_{a_1,a_2,\ldots,a_n}(x) = G_{a_1,a_2,\ldots,a_n}(x)|_{a_i\in\{-1,0,1\}}. \tag{17}$$

For example,

$$H_0(x) = \ln x, \quad H_{1,0}(x) = \int_0^x \frac{\mathrm{d}t}{t-1} \ln t, \quad H_{-1,1,0}(x) = \int_0^x \frac{\mathrm{d}t}{t+1} H_{1,0}(t). \tag{18}$$

HPLs have good mathematical properties.

# Off-Shell W Boson

Including the W boson width $\Gamma_W = 2.085$ GeV, the $\Gamma_t$ becomes [Jezabek, Kuhn 1989]

$$\tilde{\Gamma}_t \equiv \Gamma(t \to W^* b) = \frac{1}{\pi} \int_0^{m_t^2} dq^2 \frac{m_W \Gamma_W}{(q^2 - m_W^2)^2 + m_W^2 \Gamma_W^2} \Gamma_t(q^2/m_t^2), \tag{19}$$

# Asymptotic Behaviors

$$X_{3,\text{L.C}} = C_F\left[N_c^2 Y_A + n_l N_c Y_l + n_l^2 Y_{l2}\right].$$

The expansion series near the boundary $w = 0$ can be obtained easily.

$$\begin{aligned}
Y_A =& \left[\frac{203185}{41472} - \frac{12695\pi^2}{1944} - \frac{4525\zeta(3)}{576} - \frac{1109\pi^4}{25920} + \frac{37\pi^2\zeta(3)}{36} + \frac{1145\zeta(5)}{96} + \frac{47\pi^6}{2835} - \frac{3\zeta(3)^2}{4}\right] \\
&+ w\left[-\frac{157939}{2304} + \frac{140863\pi^2}{20736} + \frac{5073\zeta(3)}{64} - \frac{14743\pi^4}{6480} - \frac{169\pi^2\zeta(3)}{72} - \frac{45\zeta(5)}{16} + \frac{3953\pi^6}{22680} \right. \\
&\left. - \frac{15\zeta(3)^2}{4}\right] + \mathcal{O}(w^2), \\
Y_l =& \left[\frac{18209}{20736} + \frac{60025\pi^2}{31104} - \frac{197\zeta(3)}{288} - \frac{14\pi^4}{405} + \frac{5\pi^2\zeta(3)}{36} - \frac{25\zeta(5)}{12}\right] \\
&+ w\left[-\frac{179}{1152} - \frac{3709\pi^2}{2592} - \frac{73\zeta(3)}{6} + \frac{46\pi^4}{405} + \frac{19\pi^2\zeta(3)}{18} + \frac{5\zeta(5)}{2}\right] + \mathcal{O}(w^2), \\
Y_{l2} =& \left[-\frac{695}{2592} - \frac{91\pi^2}{972} + \frac{11\zeta(3)}{36} - \frac{2\pi^4}{405}\right] + w\left[\frac{245}{144} - \frac{73\pi^2}{648} - \frac{\zeta(3)}{3}\right] + \mathcal{O}(w^2).
\end{aligned} \tag{20}$$