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Optimizing Entanglement and Bell Inequality Violation in $t\bar{t}$ Events

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arXiv:2311.09166 & work in progress

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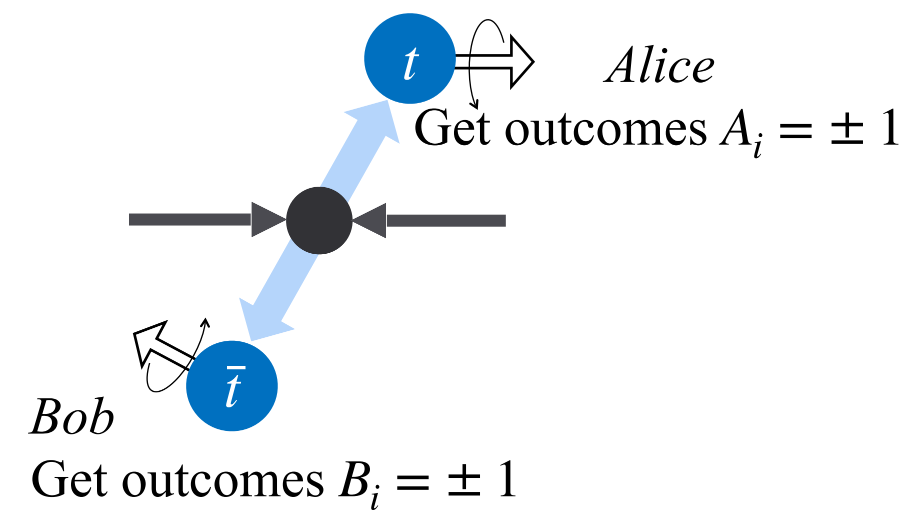
Introduction: Quantum state and Bell inequalities



Bell inequality: For a local theory, the results of two-outcome measurements $\hat{A}_{1,2}$ and $\hat{B}_{1,2}$ satisfy

$$\left|\langle \hat{A}_1 \hat{B}_1 \rangle + \langle \hat{A}_1 \hat{B}_2 \rangle + \langle \hat{A}_2 \hat{B}_1 \rangle - \langle \hat{A}_2 \hat{B}_2 \rangle\right| \leq 2$$

Next consider a quantum theory. The density matrix in $\mathscr{H}_A \otimes \mathscr{H}_B$ can be parametrized as

$$\rho_{t\bar{t}} = \frac{1}{4}\left(I_2 \otimes I_2 + B_i^+ \sigma_i \otimes I_2 + B_i^- I_2 \otimes \sigma_i + C_{ij} \sigma_i \otimes \sigma_j\right)$$

When choosing $\hat{A}_i$ and $\hat{B}_i$ as the angular momentum measurements along direction $\vec{a}_i$ and $\vec{b}_i$, $\hat{A}_i = \hat{\sigma} \cdot \vec{a}_i$, the Bell inequality is rewritten as:

$$\left|\vec{a}_1 \cdot C \cdot (\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot C \cdot (\vec{b}_1 + \vec{b}_2)\right| \leq 2$$

$$\mathscr{B}(\rho) = \max_{\vec{a}_i, \vec{b}_i} \left|\vec{a}_1 \cdot C \cdot (\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot C \cdot (\vec{b}_1 + \vec{b}_2)\right| = 2\sqrt{\mu_1^2 + \mu_2^2}$$

μ_1^2, μ_2^2 are the largest two eigenvalue of $C^T C$
When C_{ij} is symmetric, μ_i is its eigenvalue

Reconstruct density matrix produced at collider — quantum tomography

One qubit: $\rho^t = \frac{1}{2}(I_2 + B_i\sigma_i)$, $\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta_i} \approx \frac{1}{2}(1 + \vec{B}\cdot\vec{\ell}) \implies B_i = 3\langle \ell_i \rangle_{\rm av}$

$\ell_i = \cos\theta_i$: **cosine of the angle between $\vec{\ell}$ and axis $\hat{e}_i$**

Bi-qubit system, $\rho_{t\bar{t}} = \frac{1}{4}\left(I_2\otimes I_2 + B_i^+\sigma_i\otimes I_2 + B_i^- I_2\otimes\sigma_i + C_{ij}\,\sigma_i\,\sigma_j\right)$. **The density matrix constructed from** $t\to\ell^+\nu b,\ \bar{t}\to\ell^-\bar{\nu}\bar{b}$ **decay channel is**

$$B_i^+ = 3\langle \ell_i^+\rangle,\quad B_i^- = -3\langle \ell_i^-\rangle,\quad C_{ij} = -9\langle \ell_i^+\ell_j^-\rangle$$

Measure the momentum $\ell_i^\pm$ → $C_{ij} = -9\langle \ell_i^+\ell_j^-\rangle_{\rm av}$ → **Obtain a density matrix, and test entanglement…**

Quantum tomography: the processes to reconstruct a density matrix using measurements on an ensemble of events

$$\bar{\rho} = \frac{1}{N}\sum_{a=1}^{N}\rho_a,$$

$$\bar{\rho}^{\rm fic} = \frac{1}{N}\sum_a U_a^\dagger \rho_a U_a$$ (event-dependent basis choice $\implies$ fictitious state)

Current studies at the LHC are utilizing fictitious states, and the average $\langle \ell_i^+\ell_j^-\rangle_{\rm av}$ **is basis dependent.**

Event-by-event basis at collider

$$\rho_{t\bar{t}} = \frac{1}{4}\left(I_2 \otimes I_2 + B_i^+ \sigma_i \otimes I_2 + B_i^- I_2 \otimes \sigma_i + C_{ij}\, \sigma_i \sigma_j\right)$$
$$C_{ij} = -9\langle \ell_i^+ \ell_j^- \rangle_{\rm av}$$

Beam basis:
the spin basis $|\uparrow\rangle$ and $|\downarrow\rangle$ are define as spin eigenstates along $\hat{z}$-direction

Helicity basis:
the spin basis $|\uparrow\rangle$ and $|\downarrow\rangle$ are define as spin eigenstates along the moving direction of top quark.

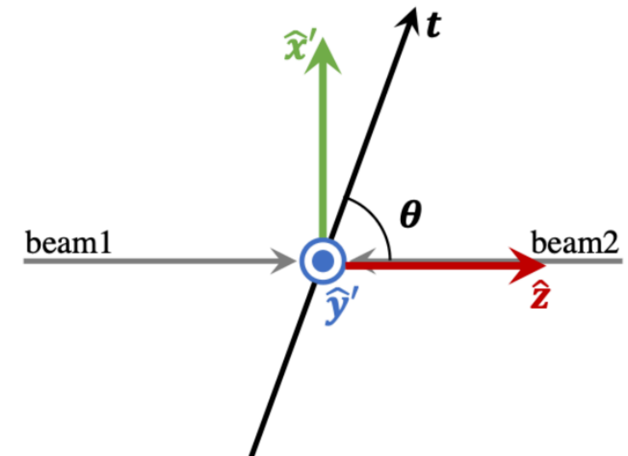
Example (fictitious state is basis-dependent):
Near threshold, the $q_R\bar{q}_L / e_R^+ e_L^- \to t\bar{t}$ processes produces a pure state $|\uparrow_z \uparrow_z\rangle$

$$\bar{\rho}^{\rm fixed} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \bar{\rho}^{\rm helicity} = \begin{pmatrix} \frac{8}{3} & -\frac{\pi}{2} & -\frac{\pi}{2} & \frac{4}{3} \\ -\frac{\pi}{2} & \frac{4}{3} & \frac{4}{3} & -\frac{\pi}{2} \\ -\frac{\pi}{2} & \frac{4}{3} & \frac{4}{3} & -\frac{\pi}{2} \\ \frac{4}{3} & -\frac{\pi}{2} & -\frac{\pi}{2} & \frac{8}{3} \end{pmatrix}$$

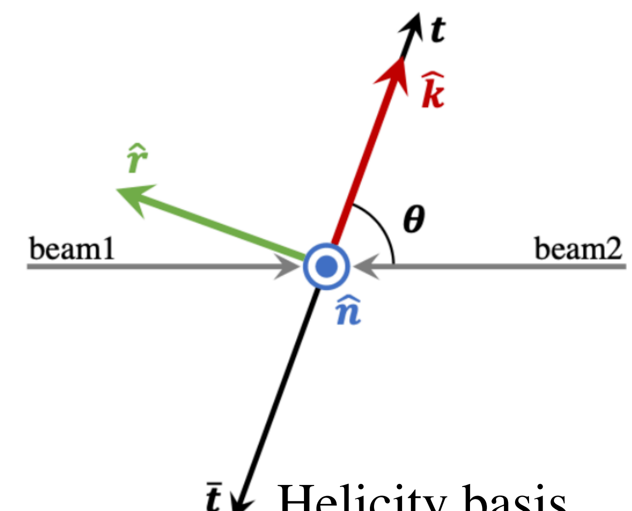
Physical state — $\mathrm{Tr}[(\bar{\rho}^{\rm fixed})^2] = 1$

Fictitious state — $\mathrm{Tr}[(\bar{\rho}^{\rm helicity})^2] \approx 0.7 < 1$

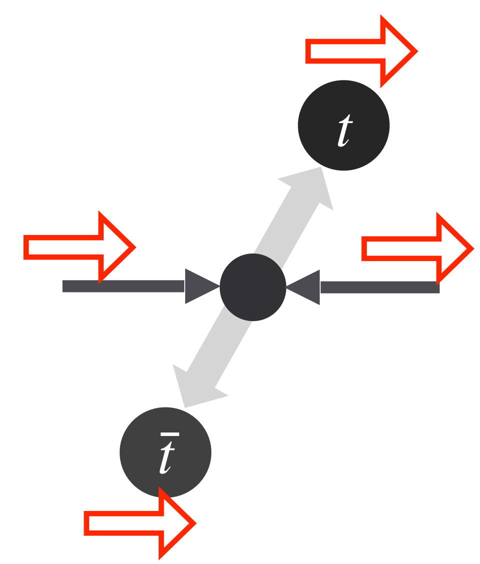
In the c.m. frame of $t\bar{t}$
$\mathbf{k} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$



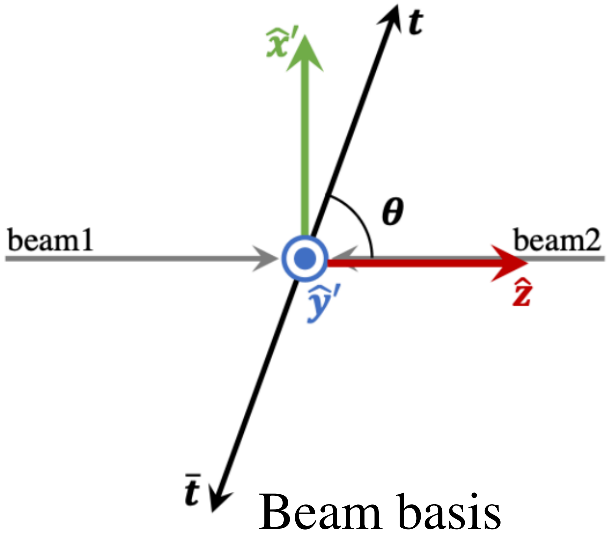
Beam basis



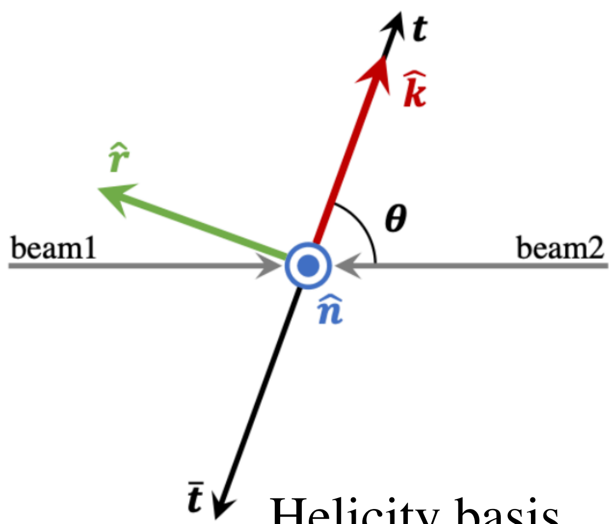
Helicity basis



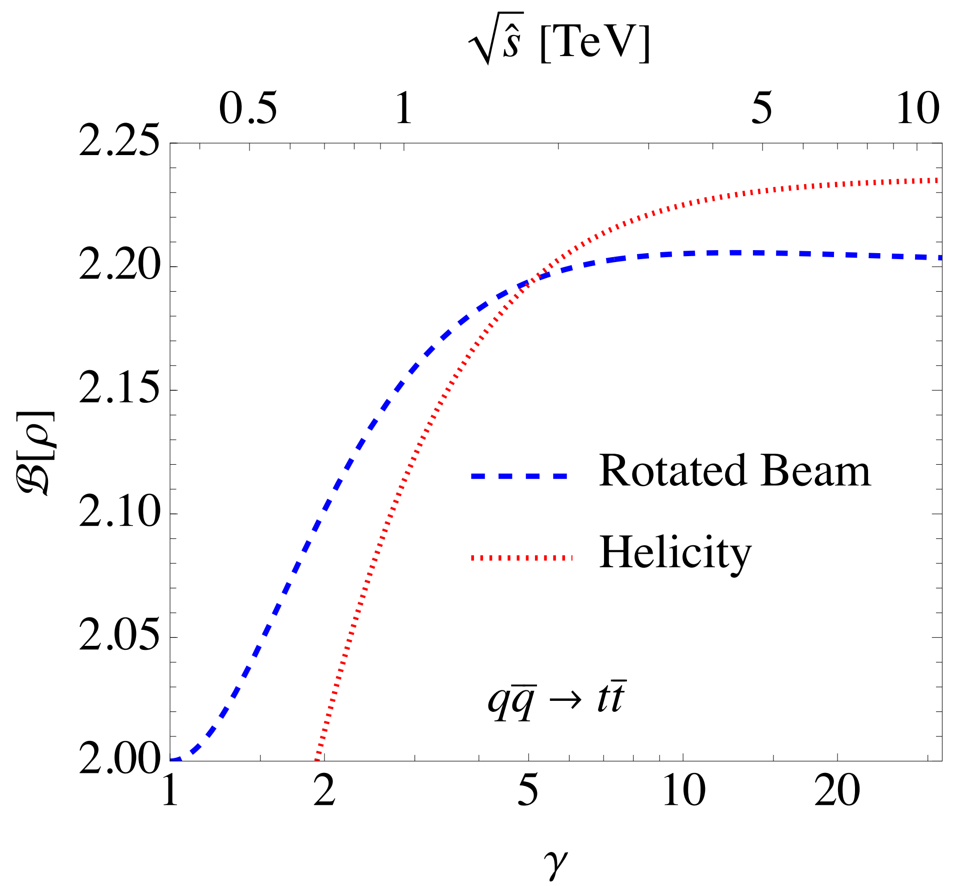
Basis dependence of Bell inequality violation



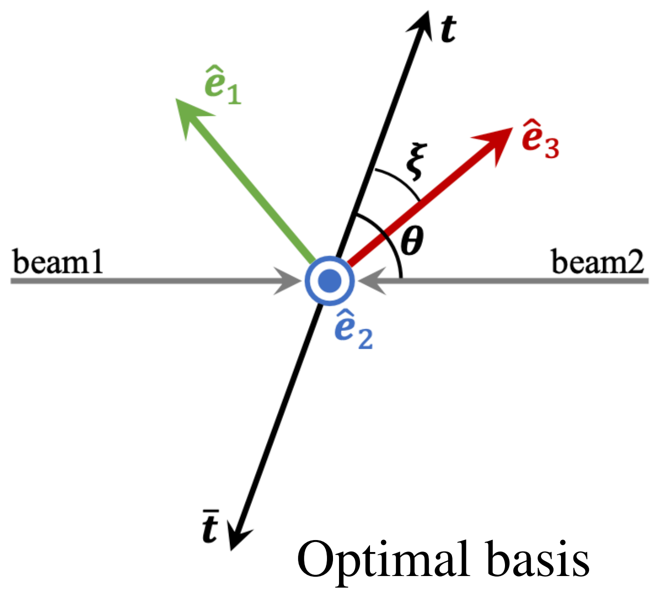
Beam basis



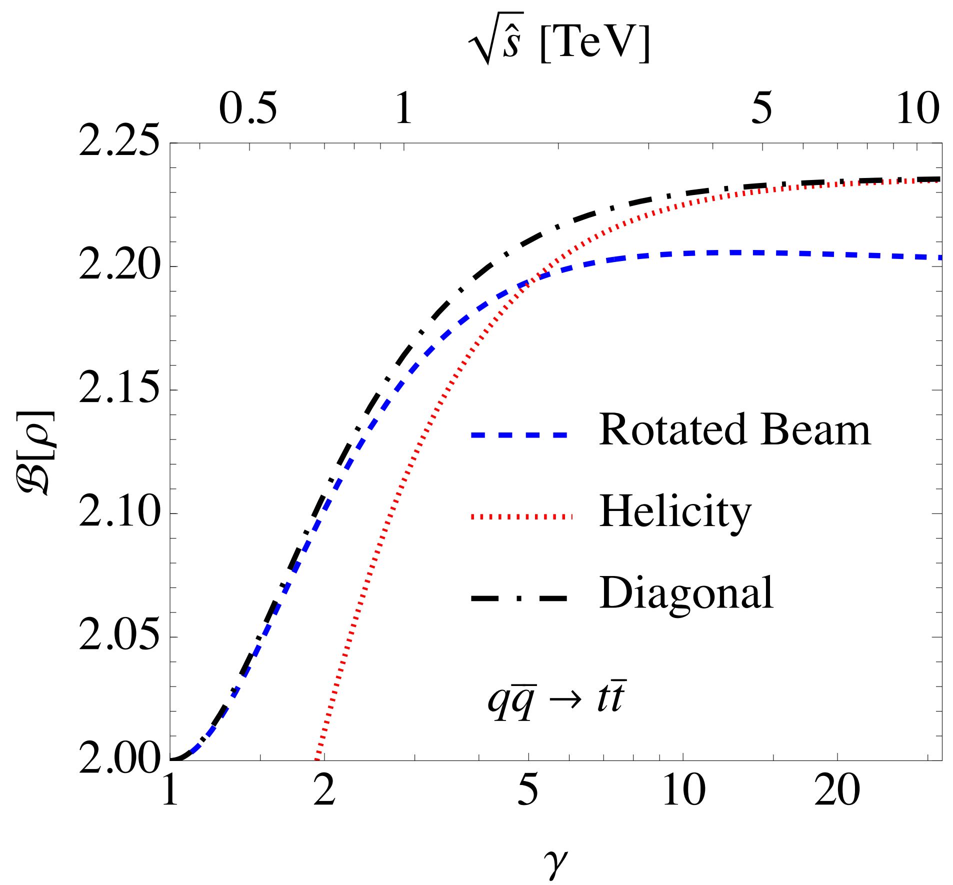
Helicity basis



Basis dependence of Bell inequality violation



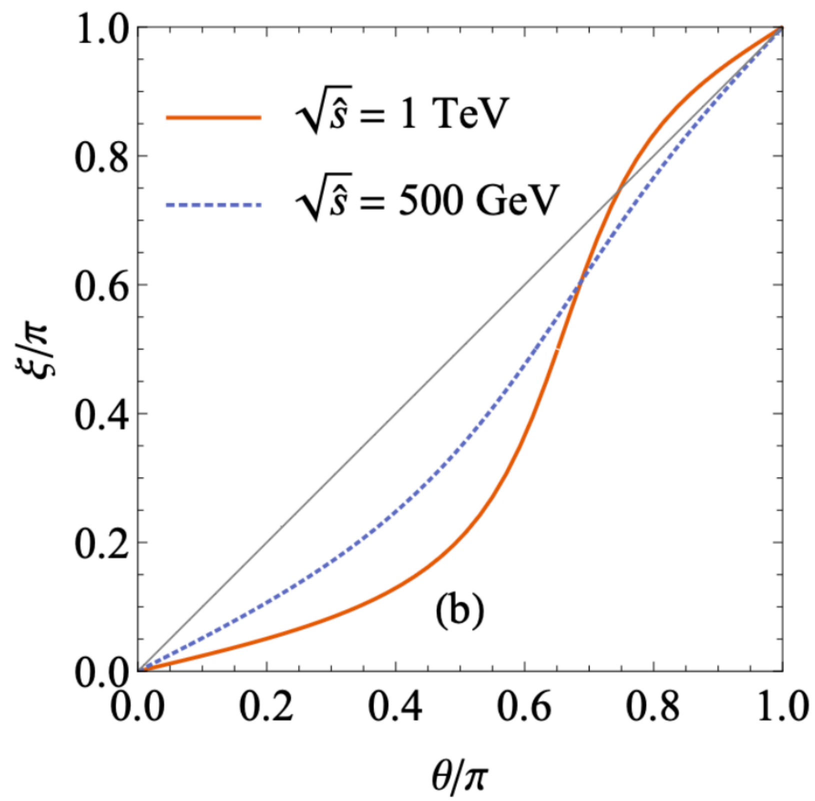
$$C^{\mathrm{diag}}(\mathbf{k}) = \begin{pmatrix} \mu_1(\mathbf{k}) & 0 & 0 \\ 0 & \mu_2(\mathbf{k}) & 0 \\ 0 & 0 & \mu_3(\mathbf{k}) \end{pmatrix}$$

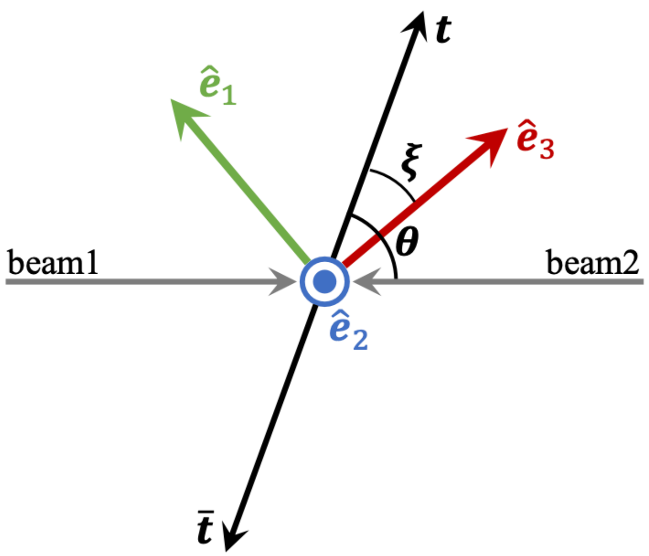


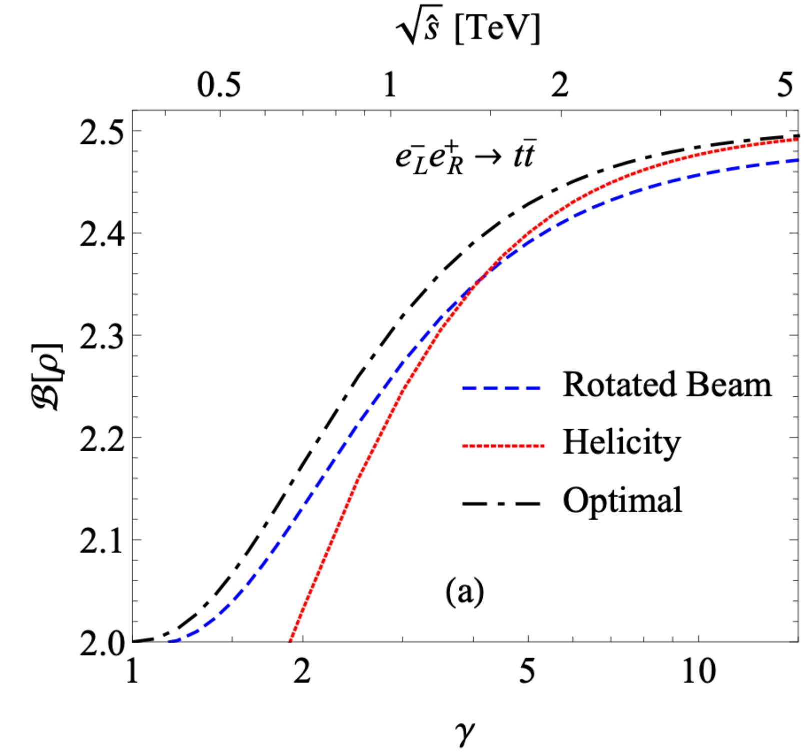
The basis that diagonalized the spin-spin correlation matrix C_{ij} maximize Bell inequality violation

arXiv: 2311.09166

Parton-level processes: $e^+e^- \to t\bar{t}$







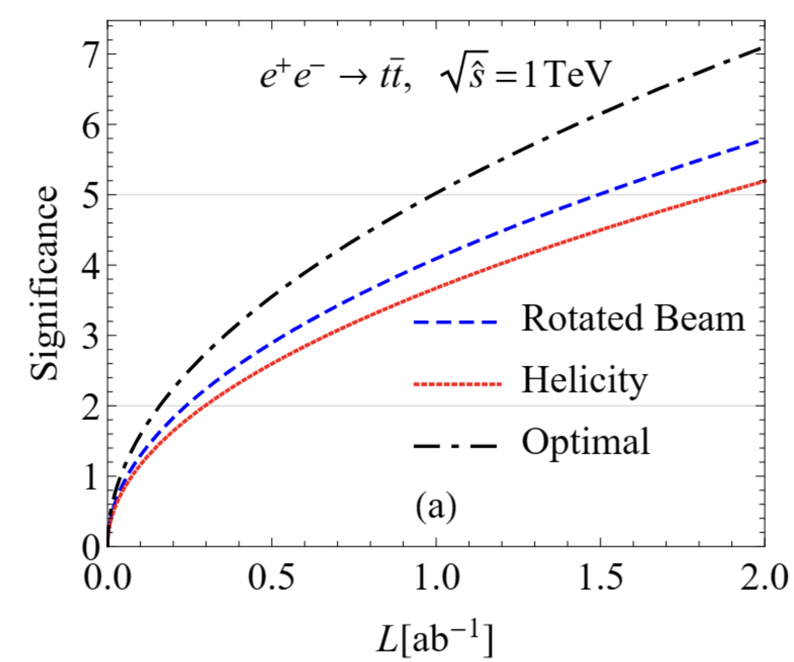
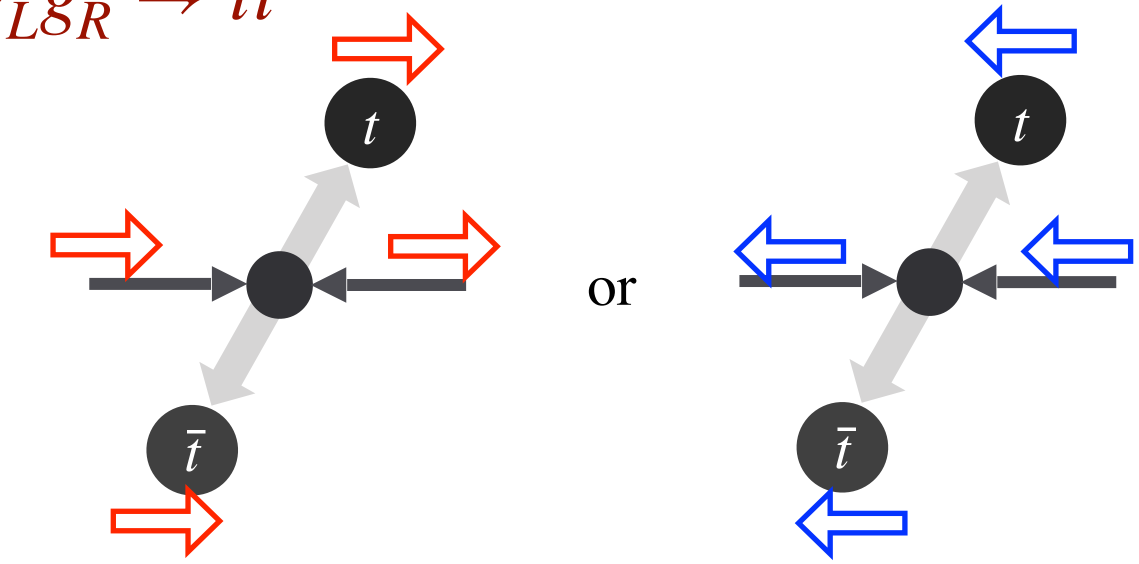


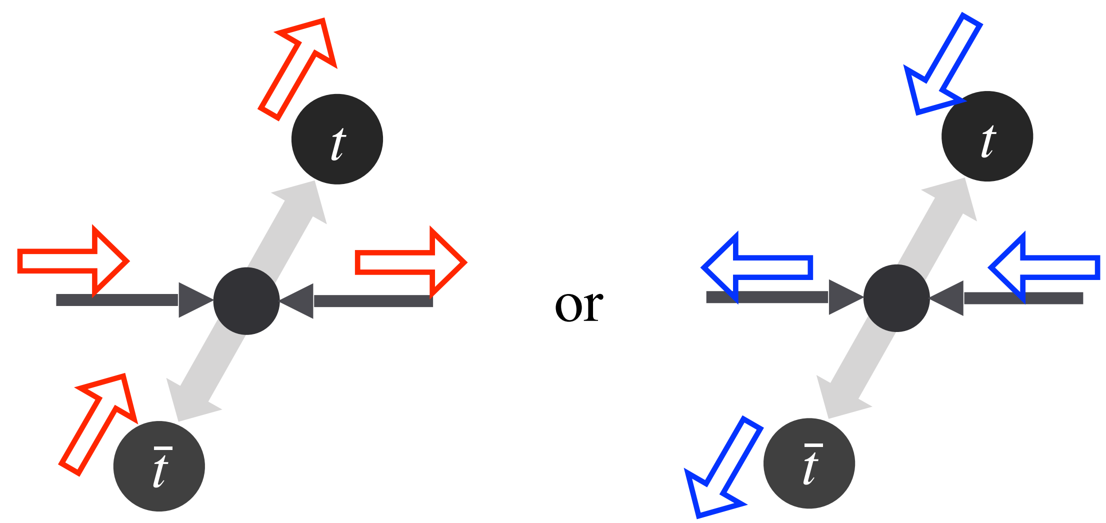
Fig. Optimal basis choice for $e^+e^- \to t\bar{t}$ processes.
work in progress

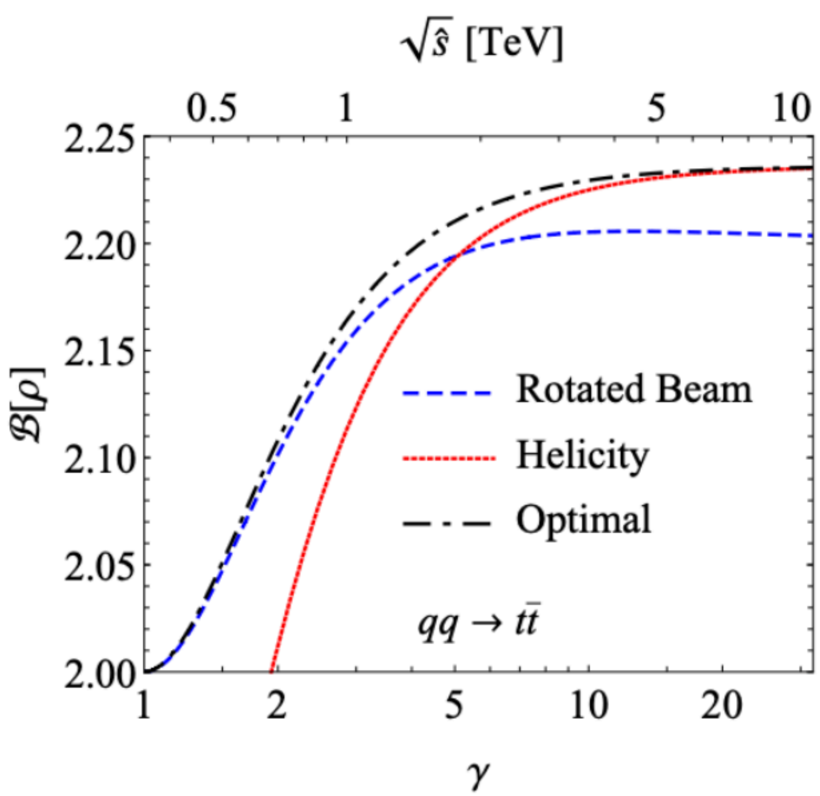
Parton-level processes: $q\bar{q} \to t\bar{t}, g_L g_R \to t\bar{t}$

Near threshold



Boosted





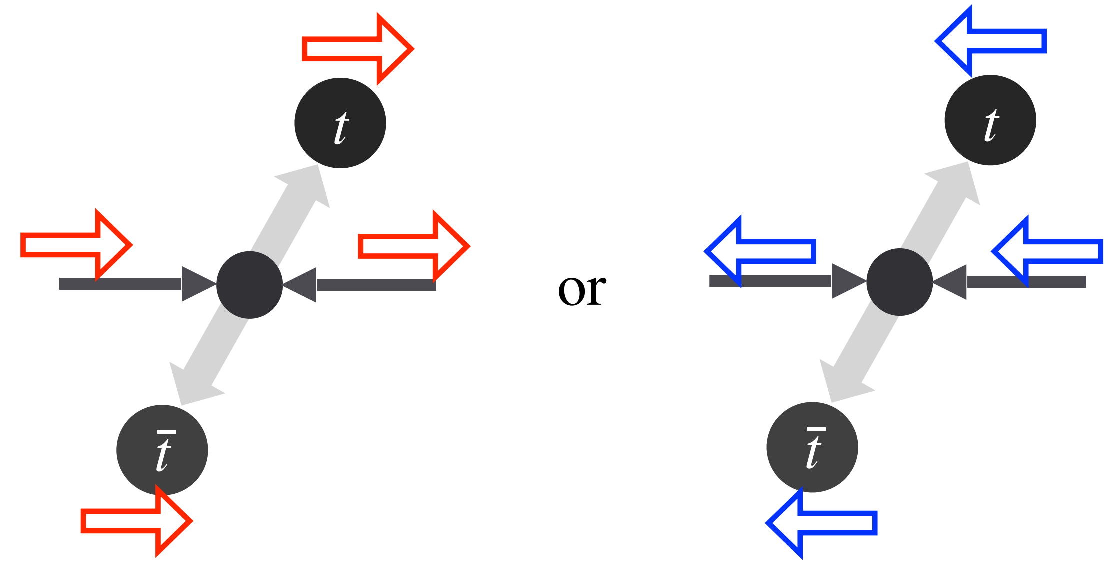
$q\bar{q} \to t\bar{t}$: **positive spin correlation,** $\xi = \dfrac{\tan\theta}{\gamma}$

Reproduces the basis in Phys. Rev. D 53, 4886 (1996)

LHC: $\rho^{t\bar{t}} = \omega^{q\bar{q}}\rho^{q\bar{q}\to t\bar{t}} + \omega^{gg}\rho^{gg\to t\bar{t}}$

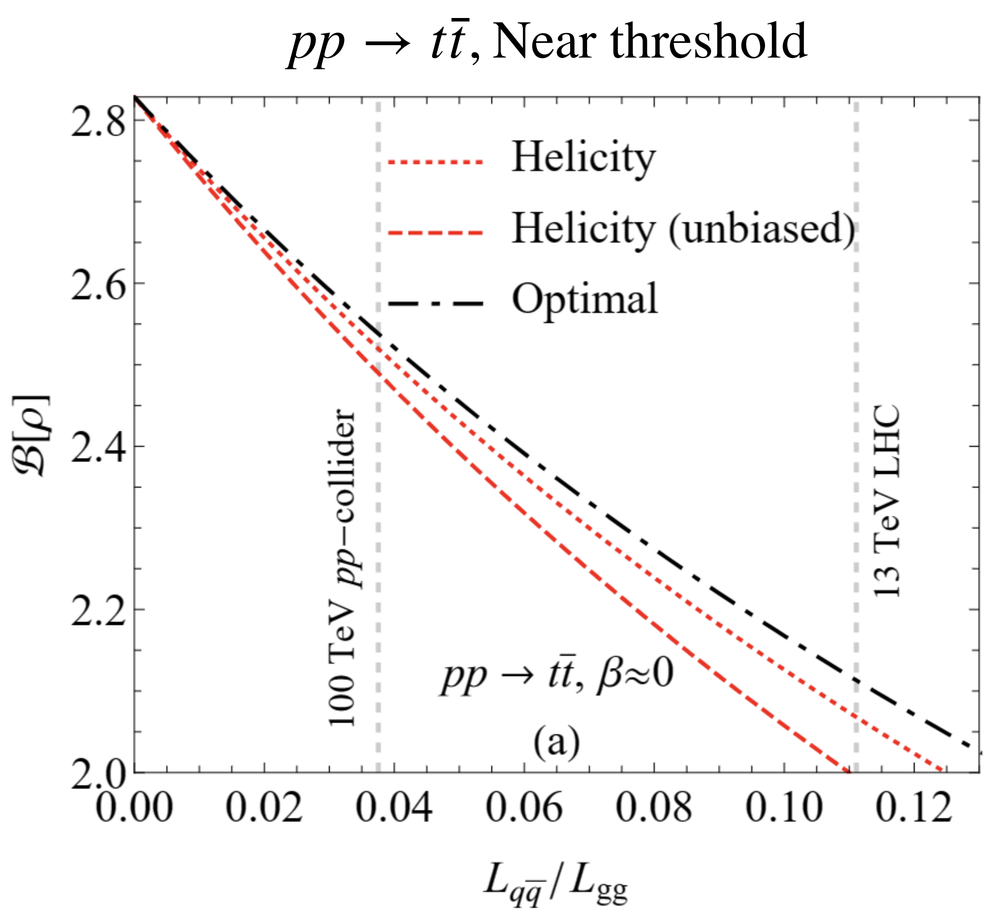
Boosted region: unlike-helicity gluon dominates, $gg \to t\bar{t}$ and $q\bar{q} \to t\bar{t}$ produce the same spin correlation.

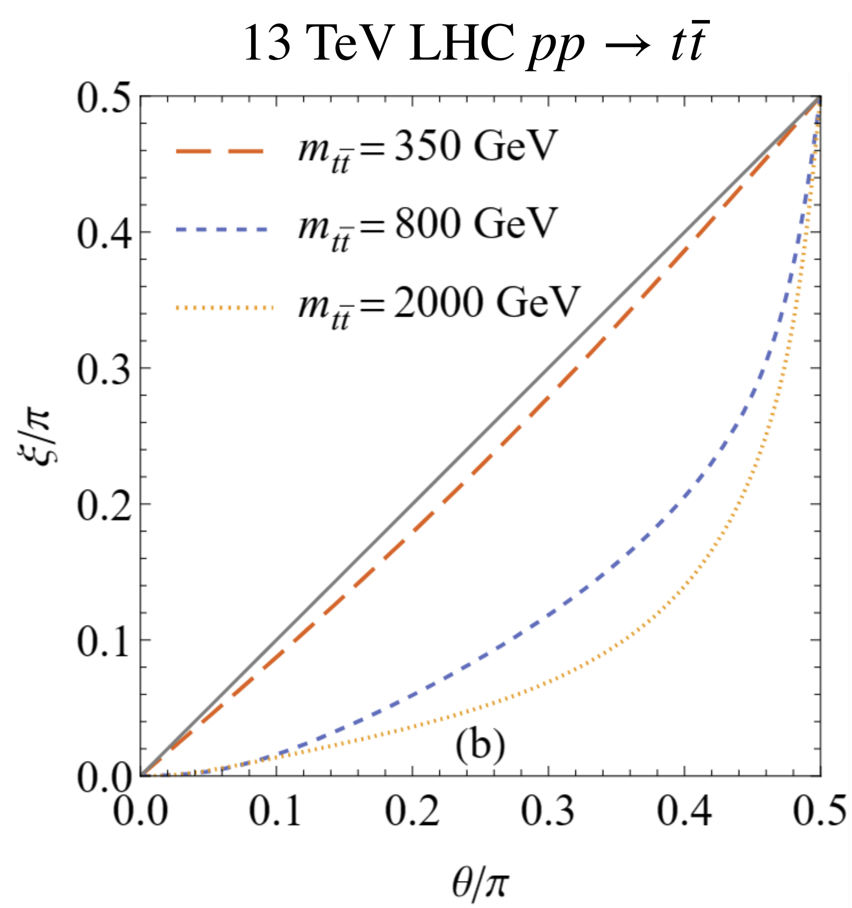
Near threshold: like-helicity gluon dominates, $gg \to t\bar{t}$ and $q\bar{q} \to t\bar{t}$ produce different spin correlation. The spin correlation from different initial state cancel with each other.



$q\bar{q} \to t\bar{t}$: positive spin correlation

$g_L g_L / g_R g_R \to t\bar{t}$: spin singlet $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$





Summary

By looking at the distribution of $t\bar{t}$ decay products, which quantum state we are studying?

- **Using angular-averaged state in event-by-event basis $\left(\hat{e}_1(\mathbf{k}), \hat{e}_2(\mathbf{k}), \hat{e}_3(\mathbf{k})\right)$:**
  - **Fictitious state instead of physical state.**
  - **Basis dependent**
  - **Optimal basis exists**

- **Current studies of $t\bar{t}$ at the LHC:**
  - **Entanglement (concurrence) is easier to test than Bell violation. (ATLAS-CONF-2023-069)**
  - **Helicity basis is mostly used.**
  - **At boosted region, e.g. $m_{t\bar{t}} > 1\,\mathrm{TeV}$, the optimal basis can give 20% improvement on the signal of Bell inequality violation. Near threshold, there is Bell inequality violation in the optimal basis but not in the helicity basis.**
  - **An improvement on testing the Bell inequality violation can be very useful.**

Backup

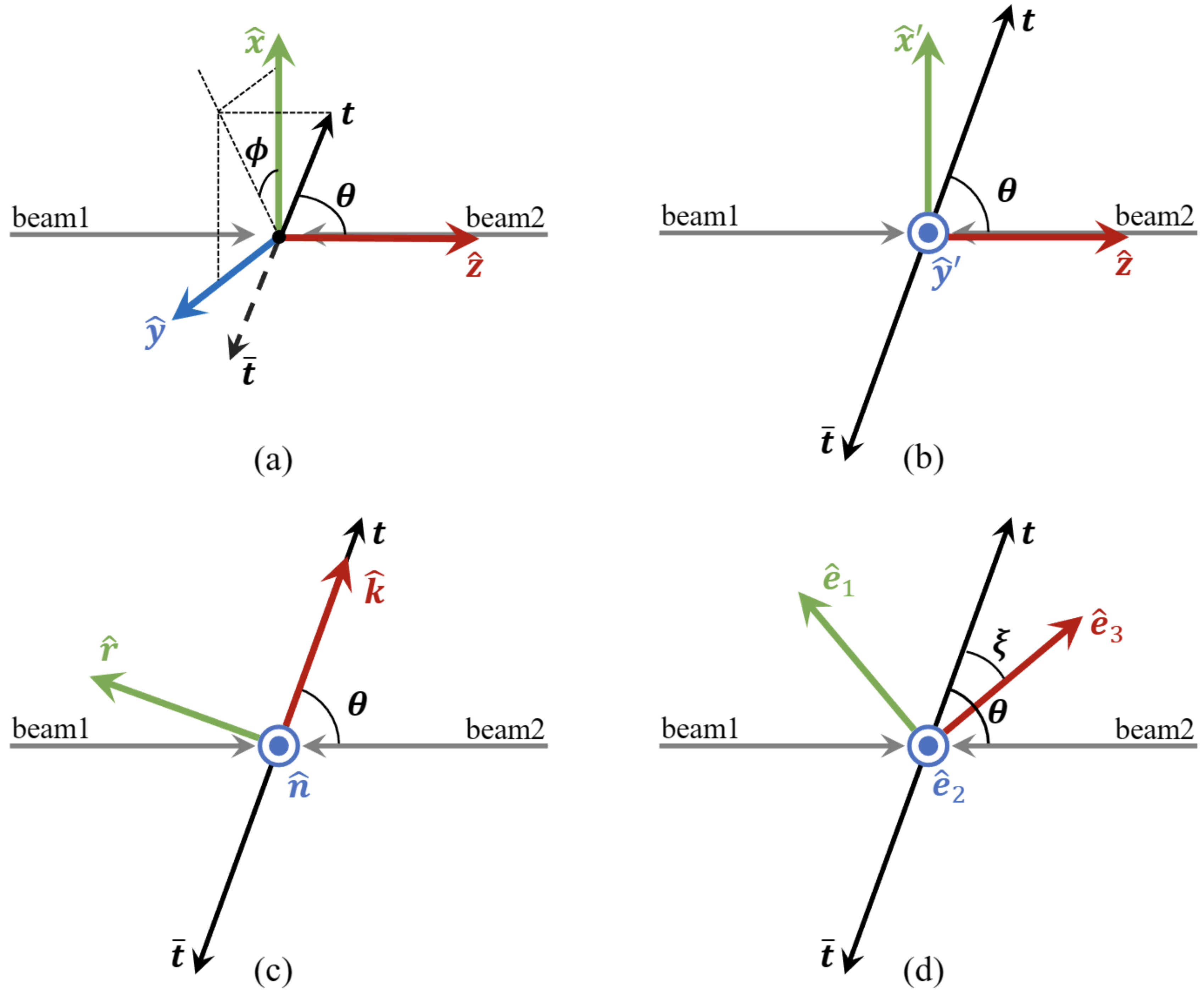
Choose a basis to maximize the entangle of angular-averaged state

(a) Fixed beam basis

(b) Rotated beam basis

(c) Helicity basis

(d) Optimal basis??

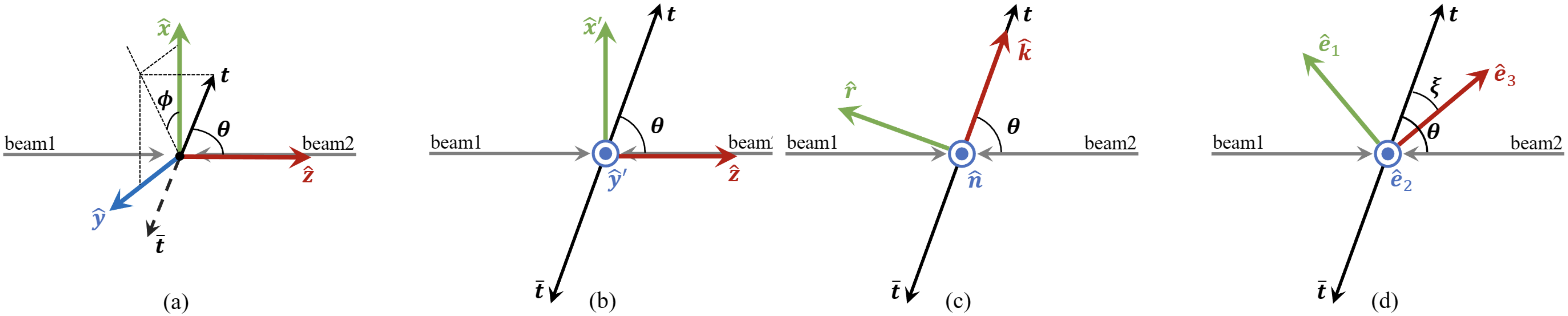


Physical states and fictitious states

$$\rho_{t\bar{t}} = \frac{1}{4}\left(I_2 \otimes I_2 + B_i^+ \sigma_i \otimes I_2 + B_i^- I_2 \otimes \sigma_i + C_{ij}\,\sigma_i \sigma_j\right)$$
$$C_{ij} = -9\langle \ell_i^+ \ell_j^- \rangle_{\text{av}}$$

Averaging $\langle \ell_i^+ \ell_j^- \rangle_{\text{av}}$ in fixed basis $\Longrightarrow$ physical state.

Averaging $\langle \ell_i^+ \ell_j^- \rangle_{\text{av}}$ in an event-by-event basis $\Longrightarrow$ fictitious state



1. **Can we use the angular-averaged states in event-by-event basis?** *Not ideal, but fine*

$$\bar{\rho}^{\text{fic}}_{\alpha\bar{\alpha},\beta\bar{\beta}} = \frac{1}{\sigma_\Pi}\int_{\Omega\in\Pi} \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\rho(\mathbf{k})_{\alpha\bar{\alpha},\beta\bar{\beta}}$$

2. **Is there an optimal basis to use?**

It is still fine to use angular-averaged state (fictitious state)

Assume $C(k)_{ij}$ is the correlation matrix written in a event-by-event basis, then the angular averaged state is

$$\bar{C}_{ij}^{\text{fic}} = \frac{1}{\sigma_\Pi} \int_{\Omega \in \Pi} \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} C(\mathbf{k})_{ij}.$$

If the Bell inequality is not violated for any quantum sub-states, then for any directions $(\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2)$

$$\vec{a}_1 \cdot C(\mathbf{k})(\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot C(\mathbf{k})(\vec{b}_1 + \vec{b}_2) \in [-2, 2]$$

Then the Bell inequality is also conserved for the angular averaged state.

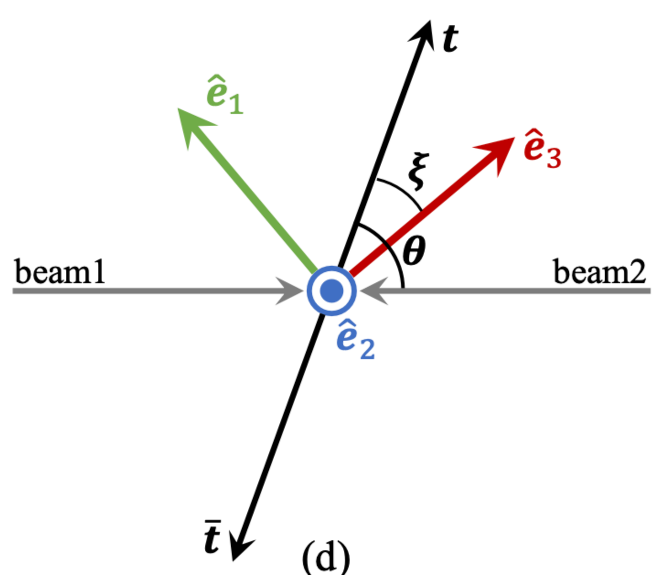
$$\begin{aligned} &\frac{1}{\sigma_\Pi} \int_{\Omega \in \Pi} \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \left(\vec{a}_1 \cdot C(\mathbf{k})(\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot C(\mathbf{k})(\vec{b}_1 + \vec{b}_2) \right) \\ &= \vec{a}_1 \cdot \bar{C} \cdot (\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot \bar{C} \cdot (\vec{b}_1 + \vec{b}_2) \\ &\in [-2, 2], \end{aligned}$$

The Bell inequality violation of the angular-averaged state implies the Bell inequality violation in some quantum sub-states

Parton-level processes: $q\bar{q} \to t\bar{t}$, $gg \to t\bar{t}$

initial state	$\sum\lvert\mathcal{M}\rvert^2$	Correlation matrix	ξ
$q\bar{q}$	$\kappa_q\left(2-\beta^2 s_\theta^2\right)$	$\begin{pmatrix} \frac{(2-\beta^2)s_\theta^2}{2-\beta^2 s_\theta^2} & 0 & -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2 s_\theta^2} \\ 0 & \frac{-\beta^2 s_\theta^2}{2-\beta^2 s_\theta^2} & 0 \\ -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2 s_\theta^2} & 0 & \frac{2c_\theta^2+\beta^2 s_\theta^2}{2-\beta^2 s_\theta^2} \end{pmatrix}$	$\tan\xi = \frac{1}{\gamma}\tan\theta$
$g_L g_R$	$\kappa_g\beta^2 s_\theta^2(2-\beta^2 s_\theta^2)$	$\begin{pmatrix} \frac{(2-\beta^2)s_\theta^2}{2-\beta^2 s_\theta^2} & 0 & -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2 s_\theta^2} \\ 0 & \frac{-\beta^2 s_\theta^2}{2-\beta^2 s_\theta^2} & 0 \\ -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2 s_\theta^2} & 0 & \frac{2c_\theta^2+\beta^2 s_\theta^2}{2-\beta^2 s_\theta^2} \end{pmatrix}$	$\tan\xi = \frac{1}{\gamma}\tan\theta$
$g_L g_L/g_R g_R$	$\kappa_g(1-\beta^4)$	$\begin{pmatrix} \frac{\beta^2-1}{\beta^2+1} & 0 & 0 \\ 0 & \frac{\beta^2-1}{\beta^2+1} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\xi = 0$

Same spin correlation



The third direction (with the largest eigenvalue of correlation matrix) is exactly the optimal basis of spin correlation found by Parke, Shadmi and Mahlon.

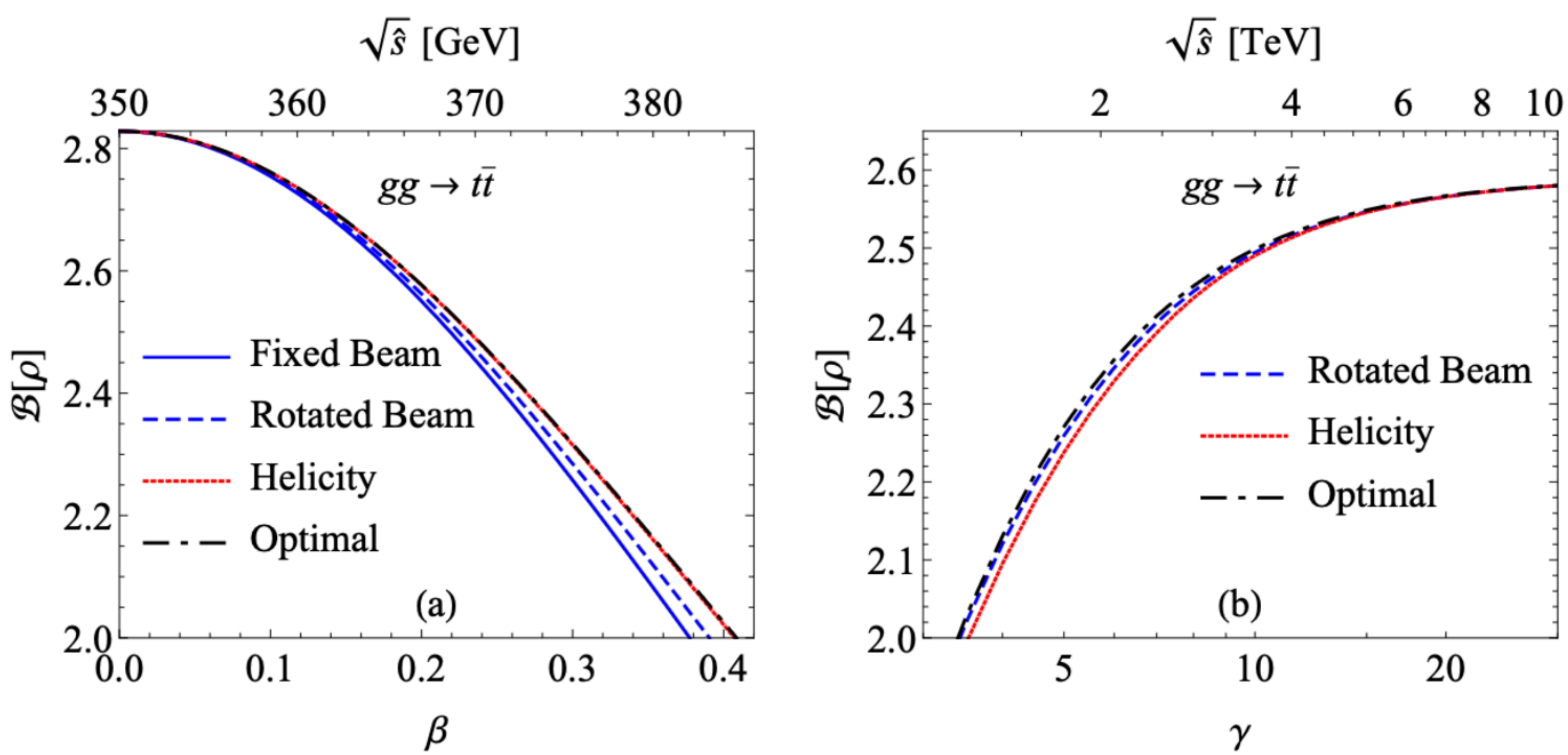
Parton-level processes: $gg \to t\bar{t}$

$t\bar{t}$ spin correlation from like-helicity gluon and unlike-helicity gluon cancel with each other

- Near threshold like-helicity gluon scattering $|S_z| = 0$ dominates, a spin singlet is produced and correlation matrix ~ diag(-1,-1,-1)

- High-pt region, unlike-helicity gluon scattering $|S_z = 2|$ dominates, a spin triplet $|\psi_2\rangle$ is produced and the correlation matrix ~ diag(1,-1,1)

- Other region: like- and unlike-helicity gluon comparable, no entanglement.



Backup: basis transformation

The spin density matrix of a spin-1/2 particle is a 2×2 trace-1 hermitian matrix, therefore can be always expanded as $\rho = \frac{1}{2}(I_2 + B_i \sigma_i)$. Likewise, the density matrix in $\mathscr{H}_A \otimes \mathscr{H}_B$ can be parametrized as

$$\rho_{t\bar{t}} = \frac{1}{4}\left(I_2 \otimes I_2 + B_i^+ \sigma_i \otimes I_2 + B_i^- I_2 \otimes \sigma_i + C_{ij} \sigma_i \otimes \sigma_j\right)$$

$B_i^{\pm}$ **parametrize the polarization of each particle;** $\langle \sigma_i^t \rangle = B_i^+, \qquad \langle \sigma_i^{\bar{t}} \rangle = B_i^-$

C_{ij} **parametrize their spin correlation** $\langle \sigma_i^t \sigma_j^{\bar{t}} \rangle = C_{ij}$

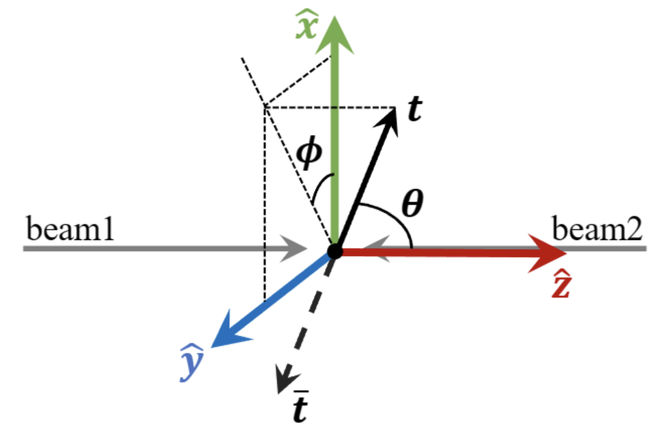
It is convenient to discuss different basis choices using this parametrization.

$$\sigma_1 = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|, \qquad \sigma_2 = -i\,|\uparrow\rangle\langle\downarrow| + i\,|\downarrow\rangle\langle\uparrow|, \qquad \sigma_3 = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$$

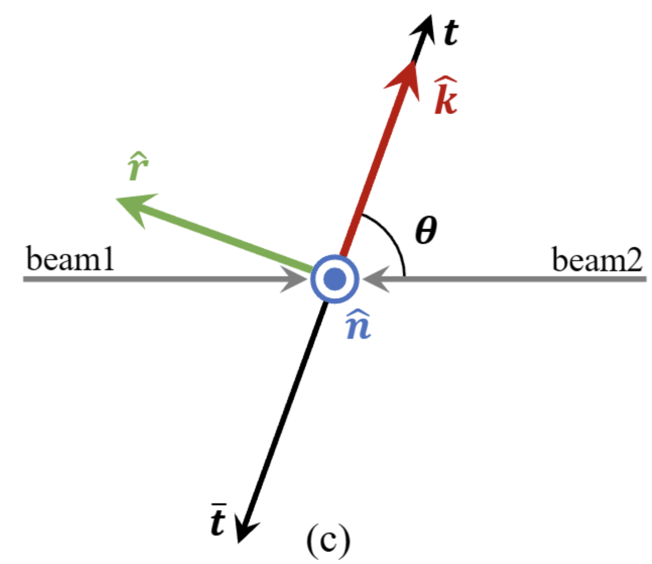
The basis transformation $U \otimes U$ on $\rho_{t\bar{t}}$ is now a simple rotation on C_{ij}……

Treating $t\bar{t}$ produce at colliders as quantum states

In the c.m. frame of $t\bar{t}$
$\mathbf{k} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$



$|\uparrow\rangle$: *defined along $\hat{e}_3$-direction*



The quantum state produced at collider is defined in $\mathscr{H}_k \otimes \mathscr{H}_{\rm spin} \otimes \mathscr{H}_{\rm color}$, we can expand it in terms of $|\mathbf{k}, \alpha\bar{\alpha}\rangle$

$$|t\bar{t}\rangle \propto \int \mathrm{d}\mathbf{k} \sum_{\alpha\bar{\alpha}} |\mathbf{k}, \alpha\bar{\alpha}\rangle \langle \mathbf{k}, \alpha\bar{\alpha}| T |I, \lambda\rangle = \int \mathrm{d}\mathbf{k} \sum_{\alpha\bar{\alpha}} \mathcal{M}^{\lambda}_{\alpha\bar{\alpha}}(\mathbf{k}) |\mathbf{k}, \alpha\bar{\alpha}\rangle$$

To obtain a physical density matrix in the spin space:

1) **Project the states to a momentum eigenstate**

$$\rho(\mathbf{k}) = \langle \mathbf{k}| \rho |\mathbf{k}\rangle$$
$$= \rho(\mathbf{k})_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'} |\alpha\bar{\alpha}\rangle \langle \alpha'\bar{\alpha}'|$$

Need infinitesimal bins
"Quantum sub-states"

2) **Trace in the momentum space.**

$$\rho_{\Pi} = \mathrm{Tr}_{\mathbf{k}\in\Pi}\left(\rho |\mathbf{k}\rangle \langle \mathbf{k}|\right)$$
$$= \frac{1}{\sigma_{\Pi}} \int_{\Omega\in\Pi} \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \rho(\mathbf{k})_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'} |\alpha\bar{\alpha}\rangle \langle \alpha'\bar{\alpha}'|$$

The basis $|\alpha\bar{\alpha}\rangle$ can be take out of the integral if it is defined in a fixed reference axis independent of $\mathbf{k}$

$$\rho^{\Pi}_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'} = \frac{1}{\sigma} \int_{\Omega\in\Pi} \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \rho(\mathbf{k})_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'}$$

However, it is a usual practice to average the density matrix in an event-by-event basis such as the helicity basis

$$\bar{\rho}^{\rm helicity}_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'} = \frac{1}{\sigma} \int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} U^{\dagger}_{\alpha\bar{\alpha},\beta\bar{\beta}} \rho(\mathbf{k})_{\beta\bar{\beta},\beta'\bar{\beta}'} U_{\beta'\bar{\beta}',\alpha'\bar{\alpha}'}$$

Fictitious state

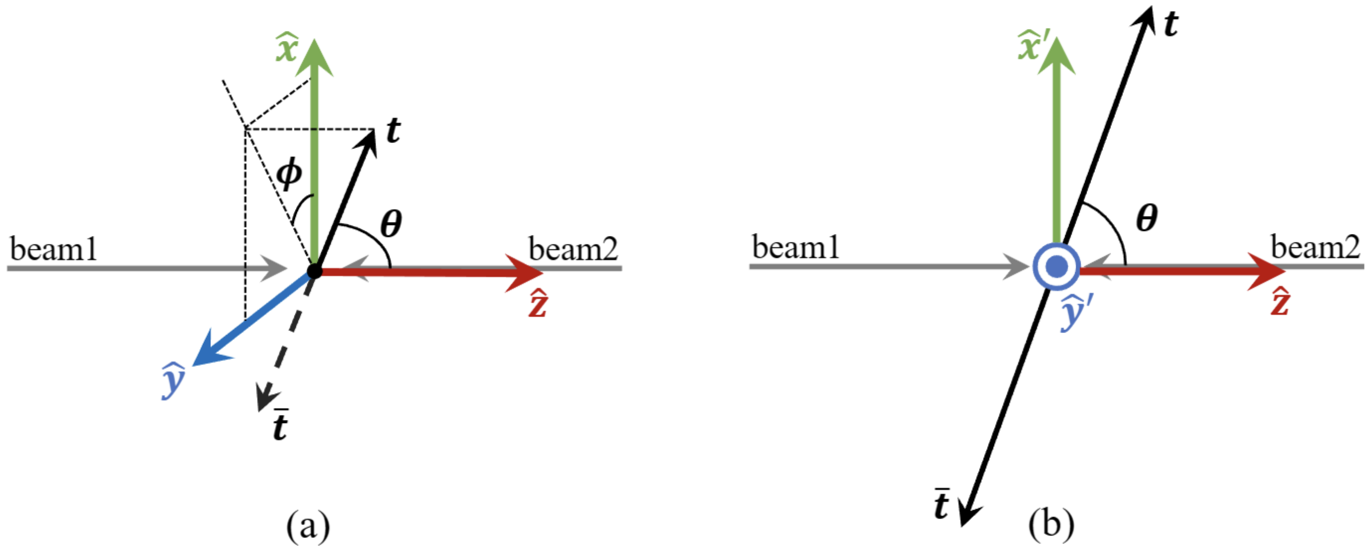
From angular momentum conservation, the $t\bar{t}$ quantum state can be written as

$$e^{iS_z^I\phi}\left(e^{-i\phi}\mathcal{M}_{\uparrow\uparrow}\left|\uparrow\uparrow\right\rangle+\mathcal{M}_{\uparrow\downarrow}\left|\uparrow\downarrow\right\rangle+\mathcal{M}_{\downarrow\uparrow}\left|\downarrow\uparrow\right\rangle+e^{i\phi}\mathcal{M}_{\downarrow\downarrow}\left|\downarrow\downarrow\right\rangle\right)$$

At leading order, the helicity amplitudes are real. After rotating the azimuthal angle ϕ to zero, the correlation matrix C_{ij} is diagonal in the second direction.

$$C^{\text{fixed}}\propto\begin{pmatrix}2\left(\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\uparrow\uparrow}\cos(2\phi)+\mathcal{M}_{\downarrow\uparrow}\mathcal{M}_{\uparrow\downarrow}\right) & 2\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\uparrow\uparrow}\sin(2\phi) & 2\cos(\phi)\left(\mathcal{M}_{\downarrow\uparrow}\mathcal{M}_{\uparrow\uparrow}-\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\uparrow\downarrow}\right)\\ 2\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\uparrow\uparrow}\sin(2\phi) & 2\mathcal{M}_{\downarrow\uparrow}\mathcal{M}_{\uparrow\downarrow}-2\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\uparrow\uparrow}\cos(2\phi) & 2\sin(\phi)\left(\mathcal{M}_{\downarrow\uparrow}\mathcal{M}_{\uparrow\uparrow}-\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\uparrow\downarrow}\right)\\ 2\cos(\phi)\left(\mathcal{M}_{\uparrow\downarrow}\mathcal{M}_{\uparrow\uparrow}-\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\downarrow\uparrow}\right) & 2\sin(\phi)\left(\mathcal{M}_{\uparrow\downarrow}\mathcal{M}_{\uparrow\uparrow}-\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\downarrow\uparrow}\right) & \mathcal{M}_{\downarrow\downarrow}^2-\mathcal{M}_{\downarrow\uparrow}^2-\mathcal{M}_{\uparrow\downarrow}^2+\mathcal{M}_{\uparrow\uparrow}^2\end{pmatrix}$$

$$C^{\text{rotated}}\propto\begin{pmatrix}2\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\uparrow\uparrow}+2\mathcal{M}_{\downarrow\uparrow}\mathcal{M}_{\uparrow\downarrow} & 0 & 2\mathcal{M}_{\downarrow\uparrow}\mathcal{M}_{\uparrow\uparrow}-2\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\uparrow\downarrow}\\ 0 & 2\mathcal{M}_{\downarrow\uparrow}\mathcal{M}_{\uparrow\downarrow}-2\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\uparrow\uparrow} & 0\\ 2\mathcal{M}_{\uparrow\downarrow}\mathcal{M}_{\uparrow\uparrow}-2\mathcal{M}_{\downarrow\downarrow}\mathcal{M}_{\downarrow\uparrow} & 0 & \mathcal{M}_{\downarrow\downarrow}^2-\mathcal{M}_{\downarrow\uparrow}^2-\mathcal{M}_{\uparrow\downarrow}^2+\mathcal{M}_{\uparrow\uparrow}^2\end{pmatrix}$$



Maximizing spin correlation vs. Maximizing entanglement of angular averaged state

The coordinate $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ that diagonalizes the correlation matrix maximizes the Bell inequality violation of angular averaged states.

$$C^{\text{diag}}(\mathbf{k}) = \begin{pmatrix} \mu_1(\mathbf{k}) & 0 & 0 \\ 0 & \mu_2(\mathbf{k}) & 0 \\ 0 & 0 & \mu_3(\mathbf{k}) \end{pmatrix}$$

Choosing a basis to maximize the entanglement of angular-averaged state is different from choosing a basis to maximize spin correlation

The spin correlation $\langle S_3^t \otimes S_3^{\bar{t}} \rangle$ is simply a function of reference axis $\hat{e}_3$, while the basis dependence of entanglement is introduced from angular averaging.
When using event-by-event basis $\hat{e}_i(\mathbf{k})$, the entanglement of angular averaged state is basis dependent is because the angular averaged state is a functional of $\hat{e}_i(\mathbf{k})$

$$\bar{C}_{ij} \propto \int \mathrm{d}\mathbf{k}\, |\mathcal{M}|^2\, \hat{e}_i(\mathbf{k}) \cdot C(\mathbf{k}) \cdot \hat{e}_j(\mathbf{k})$$

Maximizing spin correlation vs. Maximizing entanglement of angular averaged state

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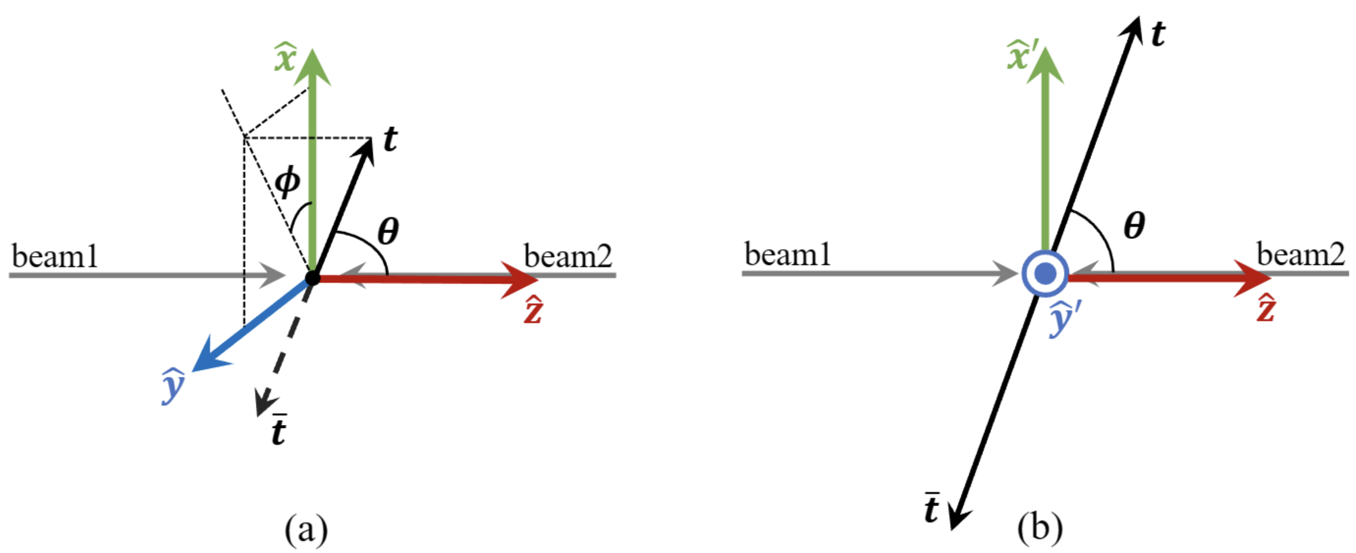


Maximizing the spin correlation only need to find a proper z direction to define $|\uparrow\rangle$ and $|\downarrow\rangle$, and the phase of $|\uparrow\rangle$ and $|\downarrow\rangle$ (the direction of $\hat{x}$ and $\hat{y}$) is irrelevant.

At high-p_T region, $q\bar{q}/gg$ scattering produce a triplet Bell state, $|\Psi_\phi\rangle = i\dfrac{e^{-i\phi}|\uparrow\uparrow\rangle + e^{i\phi}|\downarrow\downarrow\rangle}{\sqrt{2}}$

Rotated beam basis: $\bar{\rho}^{\text{rotated}}_{triplet} = \dfrac{1}{2\pi}\displaystyle\int \mathrm{d}\phi\, |\Psi_0\rangle\langle\Psi_0| = |\Psi_0\rangle\langle\Psi_0|$

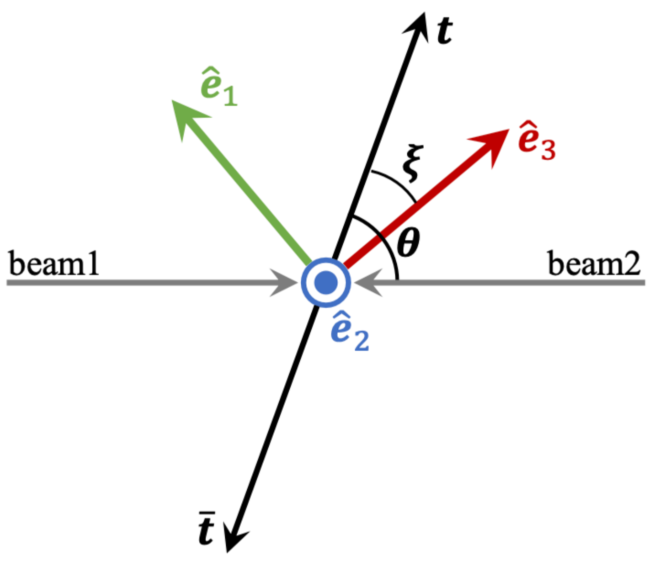
Fixed beam basis $\bar{\rho}^{\text{fixed}}_{triplet} = \dfrac{1}{2\pi}\displaystyle\int \mathrm{d}\phi\, |\Psi_\phi\rangle\langle\Psi_\phi| = \dfrac{|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|}{2}$

Maximum spin correlated
Separable, entangle…….

The optimal basis for angular averaged state

The correlation matrix C_{ij} is symmetric for unpolarized final states,

$$\text{eigenvalues}\left(C_{ij}(\mathbf{k})\right) = (\ \mu_1(\mathbf{k}), \mu_2(\mathbf{k}), \mu_3(\mathbf{k})\ )$$



Diagonal basis:

$$C^{\rm diag}(\mathbf{k}) = \begin{pmatrix} \mu_1(\mathbf{k}) & 0 & 0 \\ 0 & \mu_2(\mathbf{k}) & 0 \\ 0 & 0 & \mu_3(\mathbf{k}) \end{pmatrix}$$

The diagonal basis maximizes the signal of entanglement of angular averaged states.

- The angular averaged state in the diagonal basis:

$$C^{\rm basis1}(\mathbf{k}) = R^T(\mathbf{k}) C^{\rm basis2}(\mathbf{k}) R(\mathbf{k})$$

$$\bar{C}^{\rm diag} = \left\langle C^{\rm diag}(\mathbf{k}) \right\rangle_{\mathbf{k}\in\Pi} = \begin{pmatrix} \bar{\mu}_1 & 0 & 0 \\ 0 & \bar{\mu}_2 & 0 \\ 0 & 0 & \bar{\mu}_3 \end{pmatrix} \qquad \bar{\mu}_i = \langle \mu_i(\mathbf{k}) \rangle_{\mathbf{k}\in\Pi}$$

- The angular averaged state in any other basis: (denote the eigenvalues of $\bar{C}^{\rm basis}$ as $\bar{c}_i$)

$$\bar{C}^{\rm basis} = \left\langle C^{\rm basis}(\mathbf{k}) \right\rangle_{\mathbf{k}\in\Pi} \qquad \bar{c}_1 + \bar{c}_2 + \bar{c}_3 = \bar{\mu}_1 + \bar{\mu}_2 + \bar{\mu}_3 = \mathrm{Tr}(\bar{C}),$$

The diagonal terms of a matrix are always bounded by its eigenvalues

$$\bar{\mu}_1 \geq \bar{c}_i \geq \bar{\mu}_3.$$

To show that the diagonal basis maximize the violation of Bell inequalities, we need to prove that for any $i \neq j$,

$$\bar{c}_i^2 + \bar{c}_j^2 \ \leq \ \max_{k\neq\ell} \left[\bar{\mu}_k^2 + \bar{\mu}_\ell^2\right]$$

$$\bar{c}_1 + \bar{c}_2 + \bar{c}_3 = \bar{\mu}_1 + \bar{\mu}_2 + \bar{\mu}_3 = \mathrm{Tr}(\bar{C}), \qquad \bar{\mu}_1 \geq \bar{c}_i \geq \bar{\mu}_3. \quad \overset{i \neq j}{\Longrightarrow} \quad \bar{c}_i^2 + \bar{c}_j^2 \leq \max_{k \neq \ell} \left[\bar{\mu}_k^2 + \bar{\mu}_\ell^2\right]$$

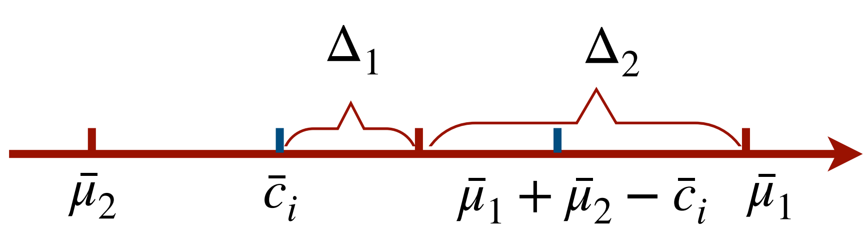
Case (a) $\bar{\mu}_1 \geq \bar{\mu}_2 \geq \bar{\mu}_3 \geq 0$

- **If:** $0 \leq \bar{c}_i \leq \bar{\mu}_2 \quad \bar{c}_j \leq \bar{\mu}_1 \quad \Longrightarrow \quad \bar{c}_i^2 + \bar{c}_j^2 \leq \bar{\mu}_1^2 + \bar{\mu}_2^2$
- **Else:** $\bar{\mu}_2 \leq \bar{c}_i \leq \bar{\mu}_1$

  $\bar{c}_i + \bar{c}_j \leq \bar{\mu}_1 + \bar{\mu}_2 \quad \Longrightarrow \quad \bar{c}_i^2 + \bar{c}_j^2 \leq \bar{c}_i^2 + (\bar{\mu}_1 + \bar{\mu}_2 - \bar{c}_i)^2$

We need to prove:

$$\bar{c}_i^2 + \bar{c}_j^2 \leq \underbrace{\bar{c}_i^2 + (\bar{\mu}_1 + \bar{\mu}_2 - \bar{c}_i)^2}_{f(\Delta_1)} \leq \underbrace{\bar{\mu}_1^2 + \bar{\mu}_2^2}_{f(\Delta_2)}$$



Define $f(\Delta) = \dfrac{(\bar{\mu}_1 + \bar{\mu}_2)^2}{2} + 2\Delta^2$

$f(\Delta_1) < f(\Delta_2)$ **when** $|\Delta_1| < |\Delta_2|$

$$\Delta_1 = \frac{\bar{\mu}_1 + \bar{\mu}_2}{2} - \bar{c}_i$$

$$\Delta_2 = \frac{\mu_1 - \mu_2}{2}$$

Case (b) $0 \geq \bar{\mu}_1 \geq \bar{\mu}_2 \geq \bar{\mu}_3$

Case (c) $\bar{\mu}_1 \geq 0 \geq \bar{\mu}_3$

Spin correlation matrix of different processes

$$\rho^{t\bar{t}} = \omega^{q\bar{q}}\rho^{q\bar{q}\to t\bar{t}} + \omega^{gg}\rho^{gg\to t\bar{t}},$$

$$\omega^{I} = \frac{L_I|\mathcal{M}_{I\to t\bar{t}}|^2}{L_{q\bar{q}}|\mathcal{M}_{q\bar{q}\to t\bar{t}}|^2 + L_{gg}|\mathcal{M}_{gg\to t\bar{t}}|^2}, \quad I = q\bar{q}, gg,$$

TABLE I. Here, the correlation matrix is expressed in the helicity basis (r, n, k). The QCD color factor $\kappa_q = g_s^2\frac{N^2-1}{N^2}$, $\kappa_g = \frac{2g_s^2}{(\beta^2c_\theta^2-1)^2}\frac{N^2(\beta^2c_\theta^2+1)-2}{N(N^2-1)}$, and $N=3$ is the number of colors.

initial state	$\overline{\sum}\|\mathcal{M}\|^2$	Correlation matrix	ξ
$q\bar{q}$	$\kappa_q\left(2-\beta^2s_\theta^2\right)$	$\begin{pmatrix} \frac{(2-\beta^2)s_\theta^2}{2-\beta^2s_\theta^2} & 0 & -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2s_\theta^2} \\ 0 & \frac{-\beta^2s_\theta^2}{2-\beta^2s_\theta^2} & 0 \\ -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2s_\theta^2} & 0 & \frac{2c_\theta^2+\beta^2s_\theta^2}{2-\beta^2s_\theta^2} \end{pmatrix}$	$\tan\xi = \frac{1}{\gamma}\tan\theta$
g_Lg_R	$\kappa_g\beta^2s_\theta^2(2-\beta^2s_\theta^2)$	$\begin{pmatrix} \frac{(2-\beta^2)s_\theta^2}{2-\beta^2s_\theta^2} & 0 & -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2s_\theta^2} \\ 0 & \frac{-\beta^2s_\theta^2}{2-\beta^2s_\theta^2} & 0 \\ -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2s_\theta^2} & 0 & \frac{2c_\theta^2+\beta^2s_\theta^2}{2-\beta^2s_\theta^2} \end{pmatrix}$	$\tan\xi = \frac{1}{\gamma}\tan\theta$
g_Lg_L/g_Rg_R	$\kappa_g(1-\beta^4)$	$\begin{pmatrix} \frac{\beta^2-1}{\beta^2+1} & 0 & 0 \\ 0 & \frac{\beta^2-1}{\beta^2+1} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\xi = 0$

$$\tan 2\xi = \frac{(L_{qq}\kappa_q + L_{gg}\kappa_g\beta^2s_\theta^2)s_{2\theta}\sqrt{1-\beta^2}}{(L_{qq}\kappa_q + L_{gg}\kappa_g\beta^2s_\theta^2)(c_{2\theta}+\beta^2s_\theta^2) + L_{gg}\kappa_g\beta^2(1-\beta^2)}.$$

Reconstruction

arXiv: 2310.17696

Writing the truth distribution as a function of Θ we have

$$\vec{x}_{\text{truth}}(\Theta) \xrightarrow{\text{folding}} \vec{x}_{\text{predicted}}(\Theta) = R \cdot \vec{x}_{\text{truth}}(\Theta). \tag{A.3}$$

The parameter Θ is now extracted by fitting $\vec{x}_{\text{predicted}}(\Theta)$ to $\vec{x}_{\text{detected}}$.

We perform this parameter extraction by a binned maximum likelihood fit where the likelihood function is

$$L(\Theta) = \prod_{\alpha=1}^{\mathrm{n_{bins}}} \text{Poisson}\left(x_{\text{detected},\alpha}, x_{\text{predicted},\alpha}(\Theta)\right), \tag{A.4}$$

$$= \prod_{\alpha=1}^{\mathrm{n_{bins}}} \text{Poisson}\left(x_{\text{detected},\alpha}, \sum_{\beta} R_{\alpha\beta} x_{\text{truth},\beta}(\Theta)\right), \tag{A.5}$$

where Poisson(x, λ) is the Poisson distribution for random variable x with mean λ. The response matrix R is calculated from simulation, the distribution $x_{\text{truth}}(\Theta)$ as a function of Θ is known analytically in all cases that we study. For example, for $\Theta = C_{ij}$, the truth distribution is given by Eq. (3.10). To obtain Θ we maximize the logarithm of the likelihood function.