Two-body scattering on the lattice in the presence of a long-range force

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FB23, Beijing, 24 September 2024

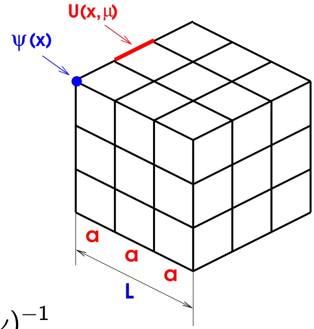
Plan

- Determining S-matrix elements: the place where lattice meets few-body physics
- The finite-volume formalism
- An example: inclusion of the long-range forces
- Modified effective range expansion and modified Lüscher equation
- Conclusion, outlook

QCD on the lattice

- In QCD, the structure of hadrons and their interactions at low energies cannot be studied in perturbation theory $\rightarrow$ QCD on the lattice

$$
\begin{aligned}
S &= \frac{1}{g^2}\sum_{x\mu\nu} \mathrm{Re\,tr}(1 - P_{\mu\nu}(x)) \\
&+ \sum_{x\mu} \bar{\psi}\left(\frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{a}{2}\nabla_\mu\nabla^*_\mu\right)\psi + \sum_x \bar{\psi}\, m\, \psi
\end{aligned}
$$



- The covariant derivative and the plaquette:

$$
\begin{aligned}
\nabla_\mu\psi(x) &= \frac{1}{a}\left(U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x)\right) \\
P_{\mu\nu}(x) &= U(x,\mu)U(x+a\hat{\mu},\nu)U(x+a\hat{\nu},\mu)^{-1}U(x,\nu)^{-1} \\
\mathrm{tr}(P_{\mu\nu}(x)) &= N_c - \frac{1}{2}a^4\,\mathrm{tr}(G_{\mu\nu}(x)G_{\mu\nu}(x)) + O(a^5)
\end{aligned}
$$

Evaluation of the spectrum

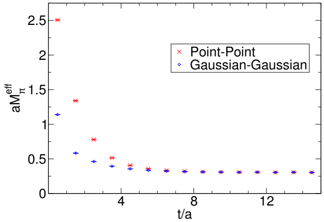
- The Euclidean path integral

$$D(t) = \sum_{\boldsymbol{x}} \langle 0 | T \mathcal{O}(t, \boldsymbol{x}) \mathcal{O}^\dagger(0, \boldsymbol{0}) | 0 \rangle = \frac{\int \mathscr{D}U \mathscr{D}\psi \mathscr{D}\bar{\psi} e^{-S} \sum_{\boldsymbol{x}} \mathcal{O}(t, \boldsymbol{x}) \mathcal{O}^\dagger(0, \boldsymbol{0})}{\int \mathscr{D}U \mathscr{D}\psi \mathscr{D}\bar{\psi} e^{-S}}$$

- If $t \to \infty$, then

$$D(t) \quad \to \quad |\langle 0 | \mathcal{O}(0, \boldsymbol{0}) | n \rangle|^2 e^{-E_n t} + \cdots$$

$$aM^{\text{eff}}(t) \quad = \quad \ln \frac{D(t)}{D(t+a)} \to aE_n + \cdots$$



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The no-go theorem (Maiani & Testa, 1990)

- The scattering S-matrix elements cannot be directly extracted from the amplitudes calculated on the lattice
- Example: the timelike form factor of the pion, $t', t \to \infty$ and $t' \gg t$:

$$R_{\boldsymbol{p},-\boldsymbol{p}}(t', t) = \langle 0 | T \phi_\pi(t', \boldsymbol{p}) \phi_\pi(t, -\boldsymbol{p}) A_\mu(0) | 0 \rangle$$

$$\sim \sum_n e^{-w(\boldsymbol{p})t' - (E_n - w(\boldsymbol{p}))t} \langle 0 | \phi_\pi(0, \boldsymbol{p}) | \boldsymbol{p} \rangle \langle \boldsymbol{p} | \phi_\pi(0, -\boldsymbol{p}) | n \rangle \langle n | A_\mu(0) | 0 \rangle + \cdots$$

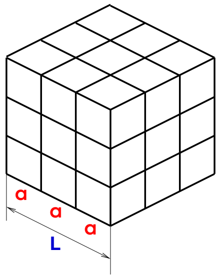
- The energy collapses towards threshold:

  The state with the minimum energy: $E_n \to 2M_\pi < 2w(\boldsymbol{p}) = 2\sqrt{M_\pi^2 + \boldsymbol{p}^2}$

  $\hookrightarrow \langle n | A_\mu(0) | 0 \rangle$ is not related to the form factor

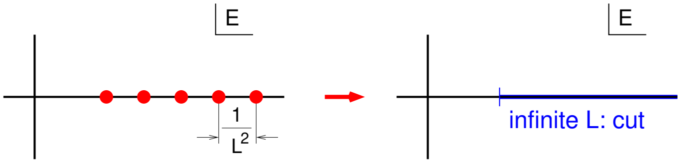
"Scattering" in a finite volume

- Impose (periodic) boundary conditions
- The spatial size of the box, L, is finite
- Assume the temporal size $L_t \gg L$, $L_t \to \infty$
- Three-momenta are quantized $\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{n}\,, \quad \boldsymbol{n} \in \mathbb{Z}^3$
- Discrete energy levels: $E_{n+1} - E_n = O(L^{-2})$
- In a finite volume, the three-momentum is quantized
  $\hookrightarrow$ states lying above threshold can be reached



There is no free lunch…

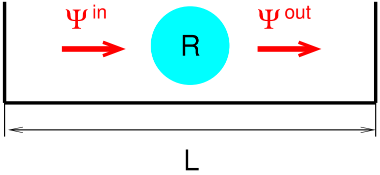
- The structure of spectrum is different in a finite and infinite volume:



- No asymptotic scattering states in the infinite volume
- No regular infinite-volume limit at fixed energy for the calculated matrix elements

How does one extract the scattering observables:
phase shifts, cross sections, resonance poles,…
from the measured quantities on the lattice?

The place where lattice meets NREFT and few-body physics



Scale separation: use EFT to describe the large-distance behavior of hadrons:

- When $R \ll L$, well-separated hadrons can be formed, $\Psi_{in/out}$ are close to asymptotic states
- Justifying the use of the non-relativistic EFT: since $p \sim 1/L$ and $R \sim 1/m$, then $p \ll m$

    Polarization effects, caused by creation/annihilation of the particles, are exponentially small and can be neglected

Non-relativistic EFT: essentials

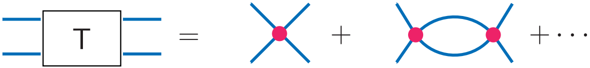
- Propagator:

$$\frac{1}{m^2 - p^2} = \underbrace{\frac{1}{2w(\boldsymbol{p})(w(\boldsymbol{p}) - p^0 - i\varepsilon)}}_{\text{particle}} + \underbrace{\frac{1}{2w(\boldsymbol{p})(w(\boldsymbol{p}) + p^0 - i\varepsilon)}}_{\text{anti-particle}}$$

- The vertices in the Lagrangian conserve particle number:

$$\mathscr{L} = \phi^\dagger(i\partial_t - w)(2w)\phi + \frac{C_0}{4}\,\phi^\dagger\phi^\dagger\phi\phi + \frac{D_0}{36}\,\phi^\dagger\phi^\dagger\phi^\dagger\phi\phi\phi + \cdots$$

- Only bubble diagrams:



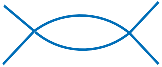
K-matrix

Relation to the potential scattering theory

- Non-relativistic EFT *is* the potential scattering theory for the short-range potential:
  - NREFT couplings $\sim$ potentials expanded in Taylor series.
  - Regularization is used to render all integrals ultraviolet-convergent.
  - In a given order, all couplings are matched to the physical observables (scattering length, effective radius,...).
  - Two-body T matrix obeys Lippmann-Schwinger equation.
  - Three-body T-matrix obeys Faddeev equations and so on...
  - Particle number is conserved. The sectors with different number of particles do not talk to each other.
- Finite volume, $R \ll L$: the energy spectrum can be calculated by using the *same* EFT in a finite volume (decoupling theorem)
  - Couplings remain the same, only three-momenta are discretized.

A loop in a finite volume

- The energy spectrum is given by the poles of the T-matrix in a finite volume
- Loop diagram in a finite volume



$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\boldsymbol{k}}, \qquad \boldsymbol{k}_n = \frac{2\pi}{L}\boldsymbol{n},\ \ \boldsymbol{n}\in\mathbb{Z}^3$$

$$J_L(P) = \frac{1}{L^3}\sum_{\boldsymbol{k}}\int\frac{dk^0}{2\pi i}\frac{1}{2w(\boldsymbol{k})(w(\boldsymbol{k})-k^0-i\varepsilon)2w(\boldsymbol{P}-\boldsymbol{k})(w(\boldsymbol{P}-\boldsymbol{k})-P^0+k^0-i\varepsilon)}$$

$$J_L(P) \propto \frac{2}{\sqrt{\pi}L\gamma} Z_{00}^{\boldsymbol{P}}(1;q_0^2)\,, \qquad q_0 = \frac{pL}{2\pi}\,, \qquad q_0^2 = \frac{P^2}{4}-m^2\,, \quad \gamma = \frac{P^0}{\sqrt{P^2}}$$

(irregular function, poles at free two-particle energies)

The Lüscher equation (Lüscher, 1991)

- The Lüscher equation (in the absence of partial-wave mixing):

$$T \propto \frac{1}{p\cot\delta(p) - ip} \to \frac{1}{p\cot\delta(p) - \frac{2}{\sqrt{\pi}L\gamma} Z_{00}^{\mathbf{P}}(1;q_0^2)}$$

$$\hookrightarrow \underbrace{p\cot\delta(p)}_{\text{short-range}} = \frac{2}{\sqrt{\pi}L\gamma} \underbrace{Z_{00}^{\mathbf{P}}(1;q_0^2)}_{\text{geometry of a box}}$$

$\hookrightarrow$ measuring energy levels, one extracts phase shift at the same energy

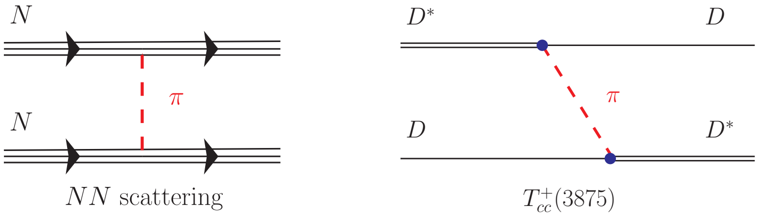
- Relativistic-invariant: can be used in moving frames $\mathbf{P} \neq 0$
- Resonances: analytic continuation into the complex plane

NREFT serves as a bridge between finite and infinite volume

Further milestones

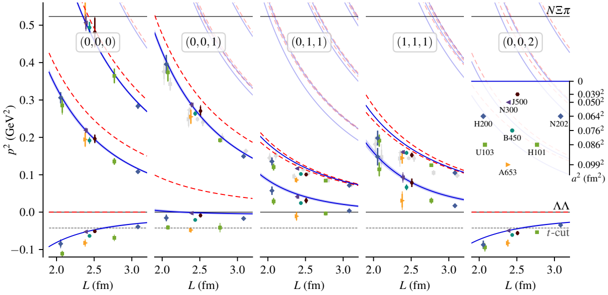
- Three-particle quantization condition
  - Polejaeva, Hammer, Pang & AR (2012-2017)
  - Hansen & Sharpe (2014)
  - Mai & Döring (2017)
- Two-particle decays:
  - Lellouch & Lüscher (2001)
- Three-particle decays:
  - Müller & AR (2020)
  - Hansen, Romero-Lopez & Sharpe (2021)
- Explicitly Lorentz-invariant formulation of the three-particle problem in a finite volume
  - Bubna, Hammer, Müller, Pang, AR & Wu (2021-2023)
- and many more. . .

What it the force is long-ranged?



- Left hand cut close to threshold: the energy levels below the left-hand branch point cannot be used
- Slowly converging partial-wave expansion: expecting strong admixture of higher partial waves in the quantization condition (Meng & Epelbaum, 2021)
- Exponentially suppressed corrections still sizable

Left-hand cut: case of NN scattering



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$$V = \frac{1}{2}\int_{-1}^{1} d\cos\theta \frac{g^2}{M_\pi^2 + (\boldsymbol{p} - \boldsymbol{q})^2}$$

- Left-hand cut: $-\infty < s \leq \underbrace{(2m_N)^2 - M_\pi^2}_{=(1875\,\text{MeV})^2}$; right-hand cut: $\underbrace{(2m_N)^2}_{=(1880\,\text{MeV})^2} \leq s < +\infty$
- Phase shift *real* below the left-hand branch point?

Quantization condition in the presence of a long-range force

- Describe the system in terms of the parameters of the effective Lagrangian which, by definition, encode only faraway singularities (Meng & Epelbaum, 2021)

$$
\begin{aligned}
V(\boldsymbol{p},\boldsymbol{q}) &= -\left(\frac{g_A}{2F_\pi}\right)^2 \frac{(\boldsymbol{\sigma}_1\cdot\boldsymbol{k})(\boldsymbol{\sigma}_2\cdot\boldsymbol{k})}{M_\pi^2+\boldsymbol{k}^2} + C_S + \frac{C_1}{4}(\boldsymbol{p}+\boldsymbol{q})^2 + C_2\boldsymbol{k}^2 + \cdots \\
\boldsymbol{k} &= \boldsymbol{p}-\boldsymbol{q}
\end{aligned}
$$

- Work in the plane wave basis; do not resort to the partial-wave expansion
- Alternative approaches
  - Splitting long- and short-range forces: Hansen & Raposo (2023)
  - Embedding two-body problem in the three-body framework in case of DD^* scattering: Hansen, Romero-Lopez & Sharpe (2024)
  - HAL QCD approach: Lyu *et al.* (2023)
  - Modified effective range expansion: Bubna, Hammer, Müller, Pang, AR & Wu (2024)

Modified effective range expansion (van Haeringen & Kok, 1982)

- Lüscher equation is based on the assumption $R \sim M^{-1} \ll L$
  ... violated by a long-range force with a small M!
- Splitting of the potential

$$V(r) = \underbrace{V_L(r)}_{\text{known, local}} + \underbrace{V_S(r)}_{\text{unknown}}$$

- Effective-range expansion: very small radius of convergence

$$q^{2\ell+1} \cot \delta_\ell(q) = -\frac{1}{a_\ell} + \frac{1}{2} r_\ell q^2 + O(q^4)$$

- Define modified effective-range function:

$$K_\ell^M(q^2) = M_\ell(q) + \frac{q^{2\ell+1}}{|f_\ell(q)|^2} \left(\cot(\delta_\ell(q) - \sigma_\ell(q)) - i\right)$$

Jost functions and all that

- Jost function for the long-range interaction:

$$f_\ell(q) = \frac{q^\ell e^{-i\ell\pi/2}(2\ell+1)}{(2\ell+1)!!} \lim_{r\to 0} r^\ell f_\ell(q,r)$$

- The function $M_\ell(q)$:

$$M_\ell(q) = \frac{1}{\ell!}\left(-\frac{iq}{2}\right)^\ell \lim_{r\to 0} \frac{d^{2\ell+1}}{dr^{2\ell+1}} \frac{f_\ell(q,r)}{f_\ell(q)}$$

- Larger radius of convergence for the modified effective-range function:

$$K_\ell^M(q^2) = -\frac{1}{\tilde{a}_\ell} + \frac{1}{2}\tilde{r}_\ell q^2 + O(q^4)$$

- Relation between $K_\ell^M(q^2)$ and the full phase $\delta_\ell(q)$ is *algebraic*

Requirements on the potential

- The long-range potential $V_L(r)$ is *local*
- The long-range potential must be *superregular*

$$\left| \lim_{r \to 0} r^{-2\ell} V_L(r) \right| < \infty$$

For example, sharp cutoff for the Yukawa potential:

$$V_L(r) = \theta(r - r_0) \frac{g e^{-M_\pi r}}{r}$$

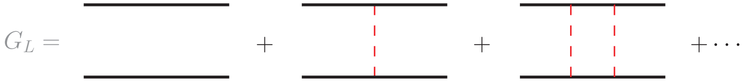
- The short-range potential is a low-energy polynomial:

$$\langle \boldsymbol{p} | V_S | \boldsymbol{q} \rangle = C_0^{00} + 3 C_1^{00} \boldsymbol{p}\boldsymbol{q} + C_0^{10} (\boldsymbol{p}^2 + \boldsymbol{q}^2) + \cdots$$

Scattering on two potentials: the EFT framework

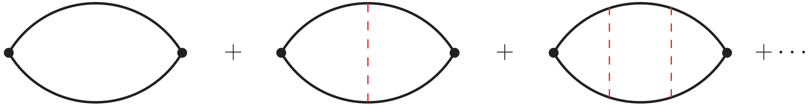
$$T = T_L + (1 + T_L G_0) T_S (1 + G_0 T_L)$$
$$T_S = V_S + V_S G_L T_S$$

- The Green function with the long-range potential only: $G_L = G_0 + G_0 V_L G_L$



The loop with infinite number of long-range insertions

$\langle G_L \rangle =$



$$\langle \mathbf{r} | G_L(q_0^2) | \mathbf{r}' \rangle = 4\pi \sum_{\ell m} \mathscr{Y}_{\ell m}(\mathbf{r}) \tilde{G}_L^{\ell}(r, r'; q_0^2) \mathscr{Y}^*_{\ell m}(\mathbf{r}') \,, \qquad \langle G_L^{\ell}(q_0^2) \rangle = \lim_{r, r' \to 0} G_L^{\ell}(r, r'; q_0^2)$$

- Relation to the Jost functions:

$$\langle G_L^{\ell}(q_0^2) \rangle = \frac{1}{4\pi((2\ell+1)!!)^2} M_\ell(q_0) + \underbrace{\text{real low-energy polynomial in } q_0^2}_{\text{renormalization prescription}}$$

Modified effective range expansion: EFT framework

- Lowest order: $\langle \boldsymbol{p} | V_S | \boldsymbol{q} \rangle = C_0^{00}$

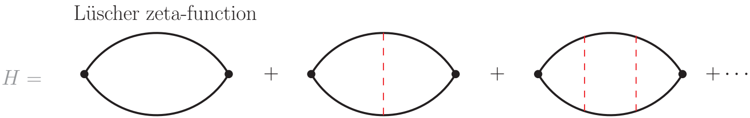
$$\underbrace{4\pi / C_0^{00}}_{= K_0^M(q_0^2) \text{ at lowest order}} = M_0(q_0) + \frac{q_0}{|f_0(q_0)|^2} \left(\cot(\delta_0(q_0) - \sigma_0(q_0)) - i \right)$$

- Higher orders:
  - The quantity $K_0^M(q_0^2)$ is a low-energy polynomial in q_0^2, expressed in terms of couplings $C_0^{00}, C_1^{00}, C_0^{10}, \ldots$
  - In the proof, the locality of $V_L(r)$ plays crucial role. The proof is not valid for a general, non-local potential

Modified Lüscher equation

$$\det \mathscr{A}_{\ell m,\ell' m'} = 0\,, \qquad \mathscr{A}_{\ell m,\ell' m'} = \delta_{\ell\ell'}\delta_{mm'} K_\ell^M(q_0^2) - H_{\ell m,\ell' m'}(q_0)$$

- Modified Lüscher zeta-function, finite volume:

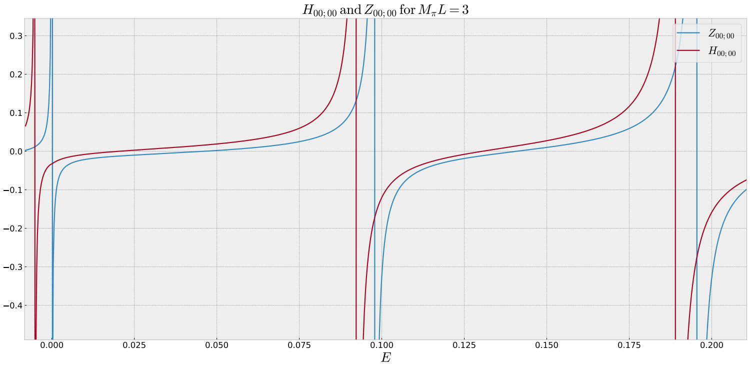


$$H_{\ell m,\ell' m'}(q_0) = \frac{4\pi}{L^6} \sum_{\mathbf{p},\mathbf{q}} \mathscr{Y}^*_{\ell m}(\mathbf{p}) \langle \mathbf{p} | G_L(q_0^2) | \mathbf{q} \rangle \mathscr{Y}_{\ell' m'}(\mathbf{q})$$

- Taking into account the renormalization prescription:

$$H_{\ell m,\ell' m'}(q_0) = (H_{\ell m,\ell' m'}(q_0) - H^\infty_{\ell m,\ell' m'}(q_0)) + \frac{1}{4\pi}\delta_{\ell\ell'}\delta_{mm'} M_\ell(q_0)$$

Modified Lüscher function (S-waves only, preliminary)



Conclusions, outlook

- A novel quantization condition in the presence of the long-range forces has been proposed
  - Solves the left-hand cut problem
  - Reduces partial-wave mixing
  - Relates the energy level to the scattering phase(s) *at the same energy*
- Three-body $\to$ two-body description for stable dimers
- Long-range force in the three-body quantization condition: e.g., NN in NNN
- Long-range three-body force?
- Electromagnetic interactions: is the non-perturbative resummation of the Coulomb photon exchanges needed/possible?