Self-consistent light-front quark model analyis of meson structure

Ahmad Jafar Arifi

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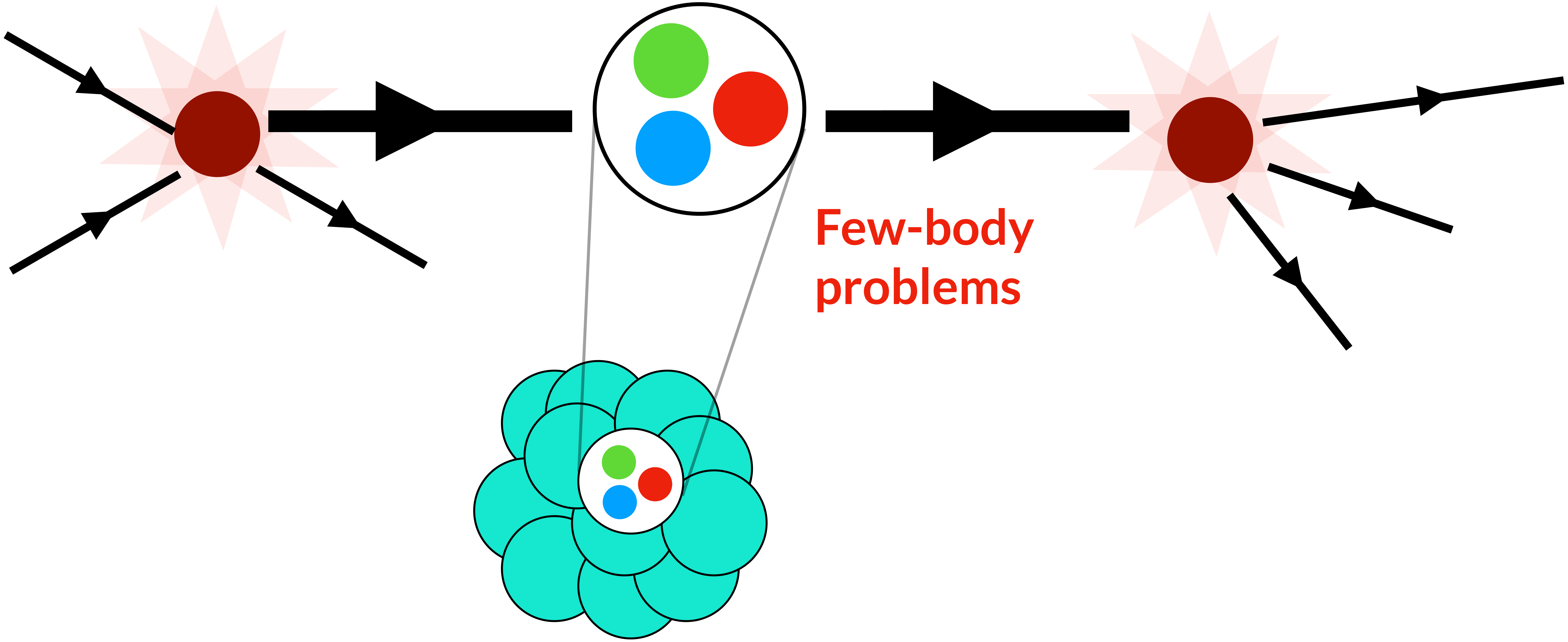
Hadron studies



Accelerator based Experiment

Strong, weak, & EM probe

Few-body problems

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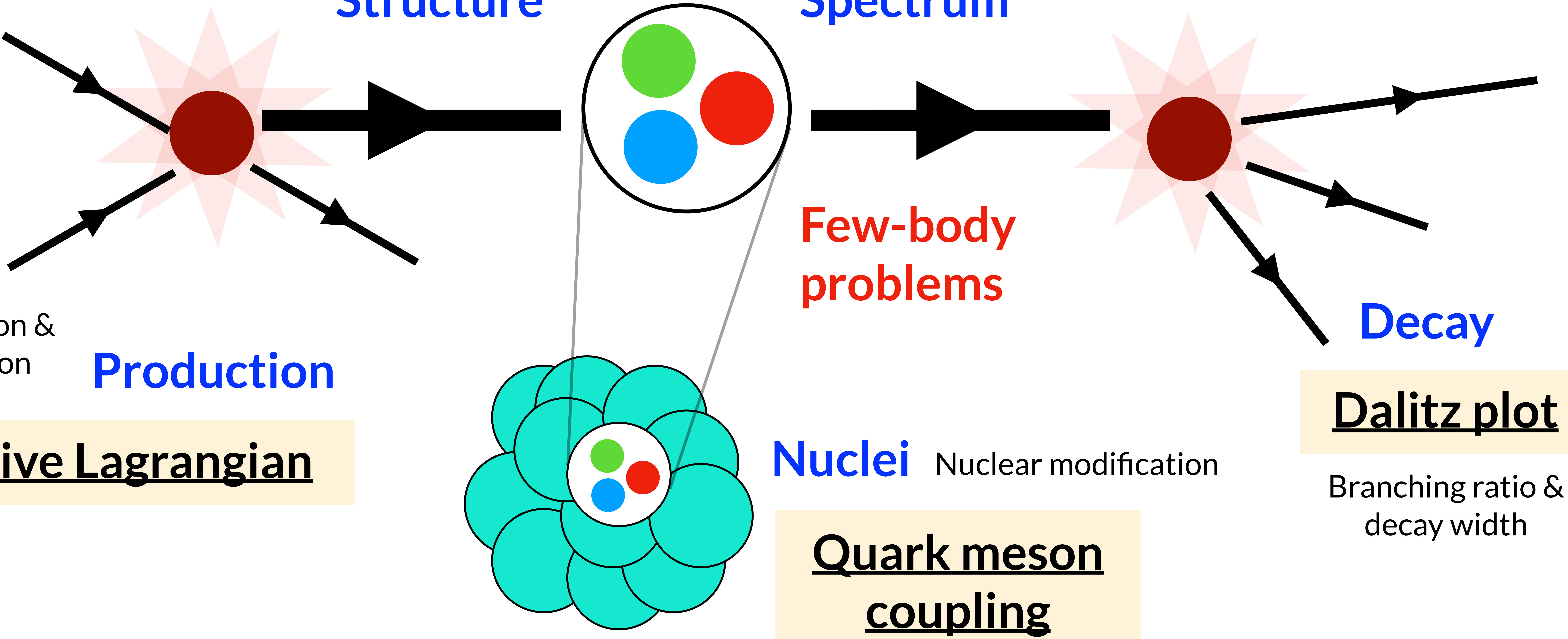

Accelerator based Experiment

Strong, weak, & EM probe

Light-front model

Form factor & quark distribution

Structure

Quark model

Mass, splitting, & resonance

Spectrum

Few-body problems

Decay

Dalitz plot

Branching ratio & decay width

Cross section & polarization

Production

Effective Lagrangian

Nuclei Nuclear modification

Quark meson coupling

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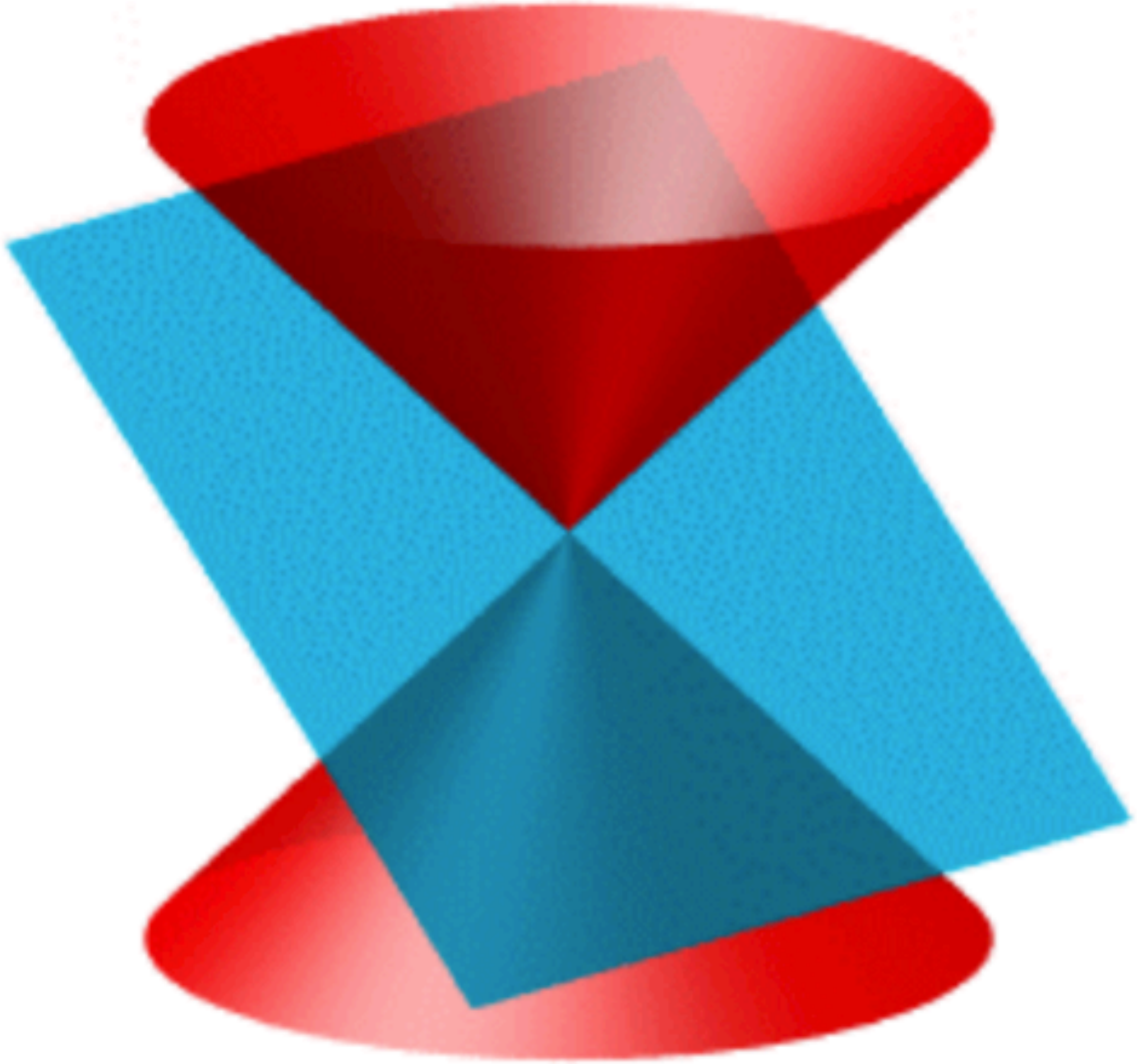
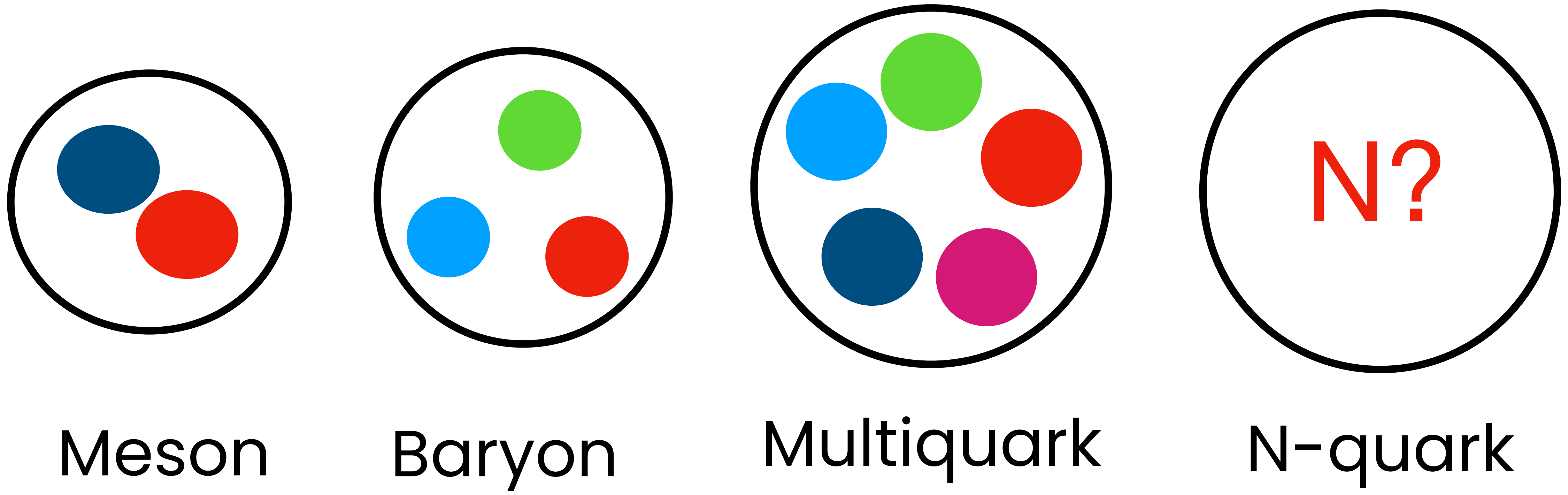
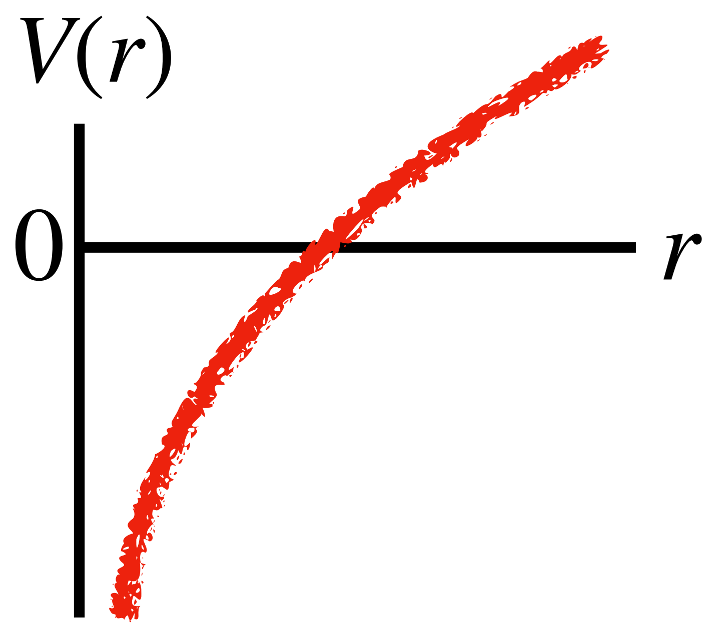


Nonrelativistic quark model

- **Quark model** is one of the succesful models in describing spectrum of hadrons.
- You need to solve the Schrodinger equation:
  - Gaussian expansion method (GEM), etc





Confinement

$$V(r) \propto -\frac{\alpha_s}{r} + br$$

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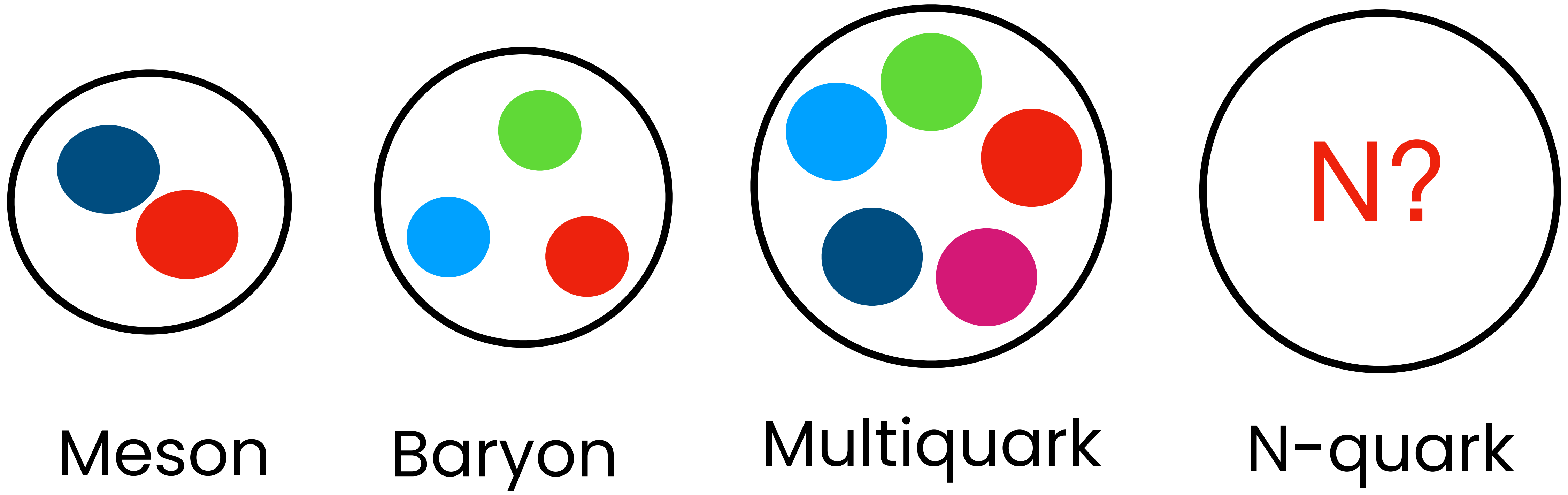

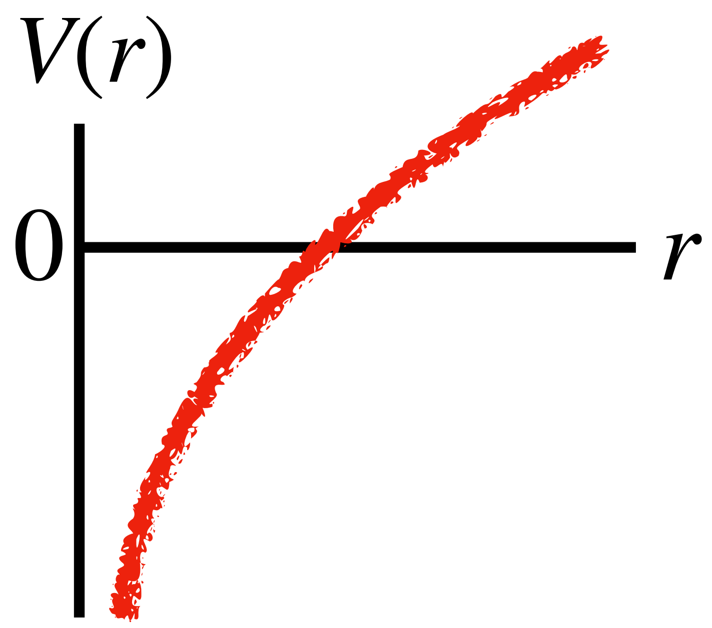

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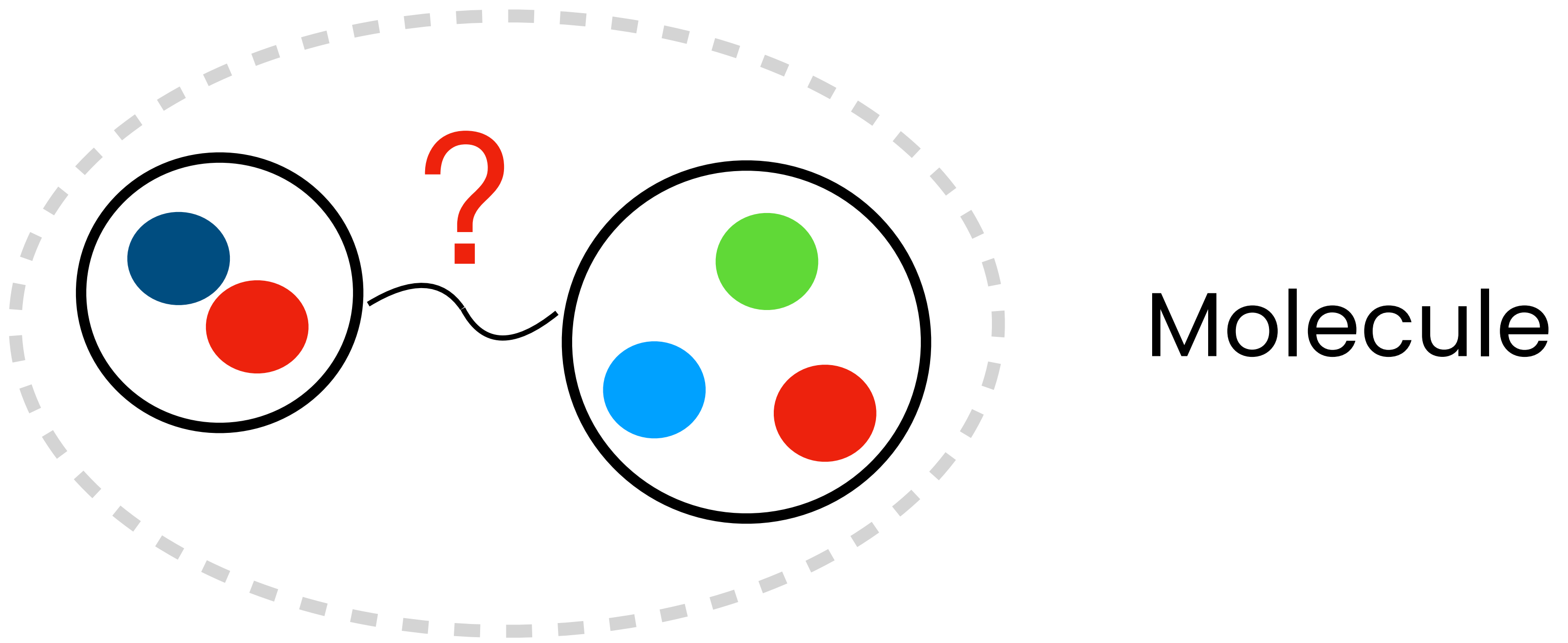
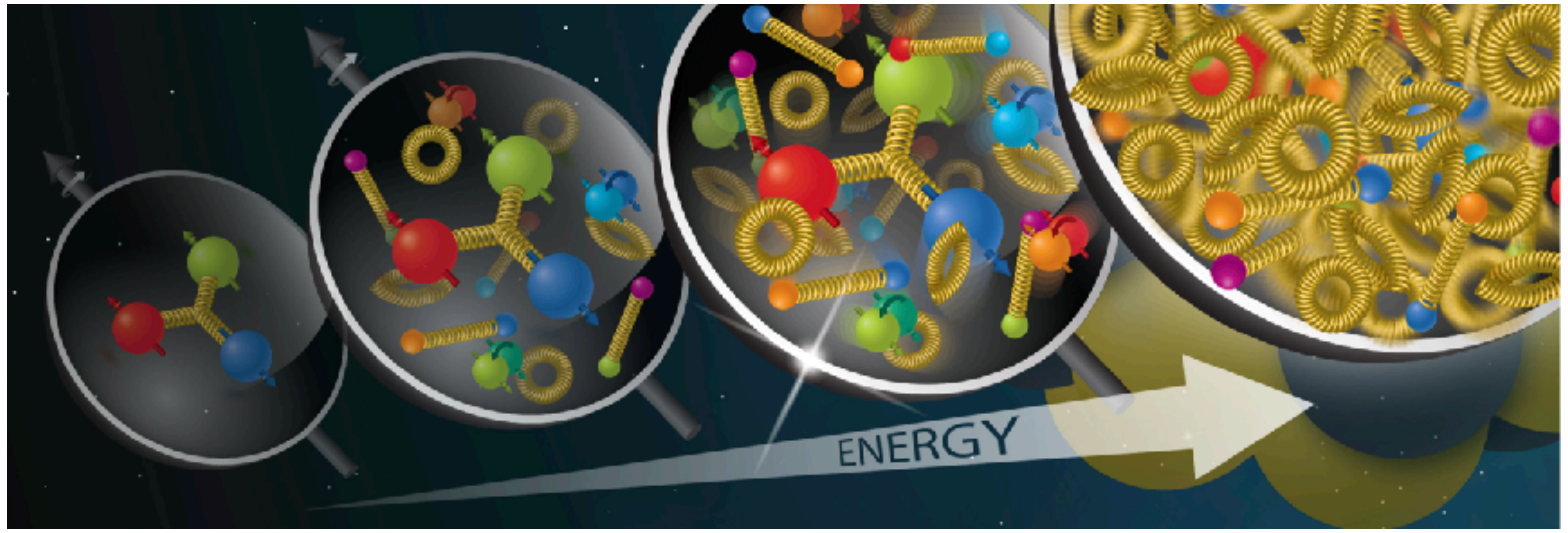
Exotic configuration



Interaction is less known

- Coupled channel effect?
- Compositeness?
- Pole position?

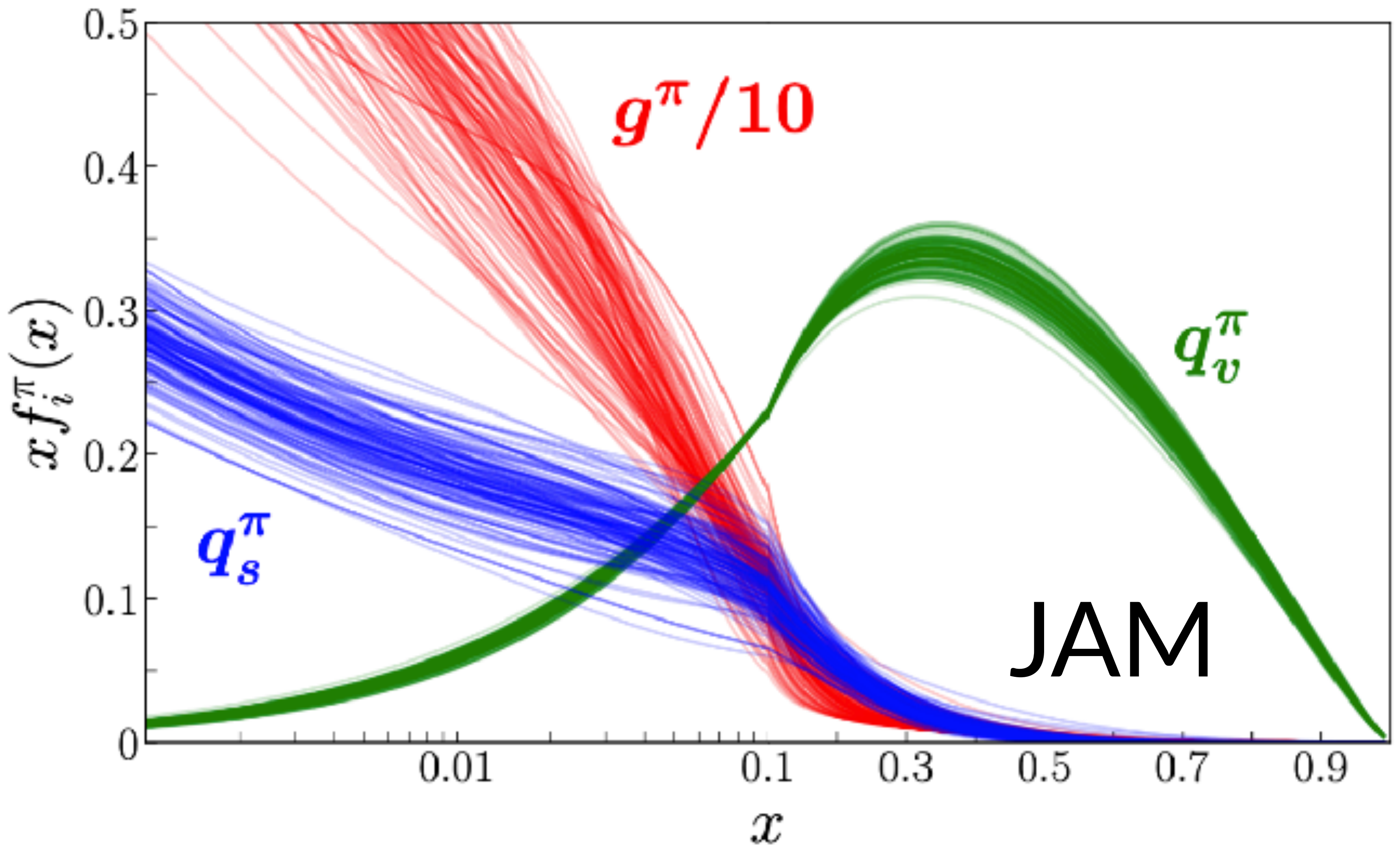
Quark and gluon distribution



- Hadron depends on the scale
  - In low energy → the valence quarks are dominant
  - In high energy → gluon starts to dominate

- QCD evolution (DGLAB)
  - Scale evolution of parton distribution function

- Momentum distribution of quark & gluon
  - Parton distribution function (PDF)



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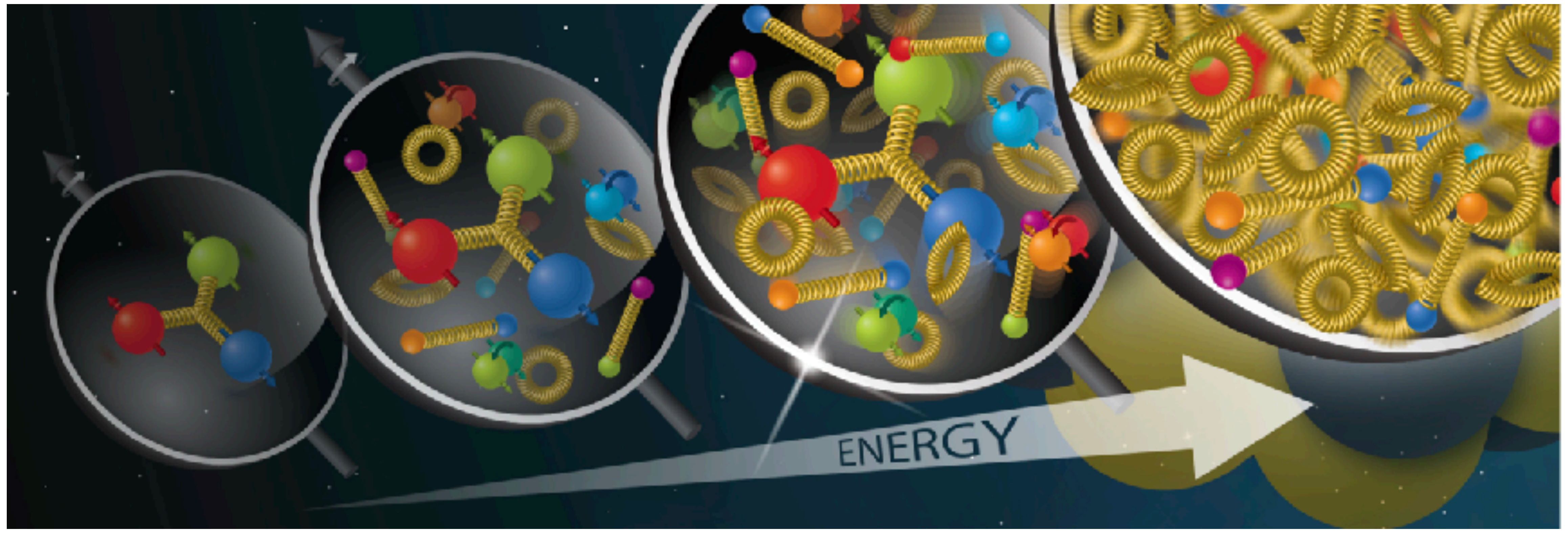

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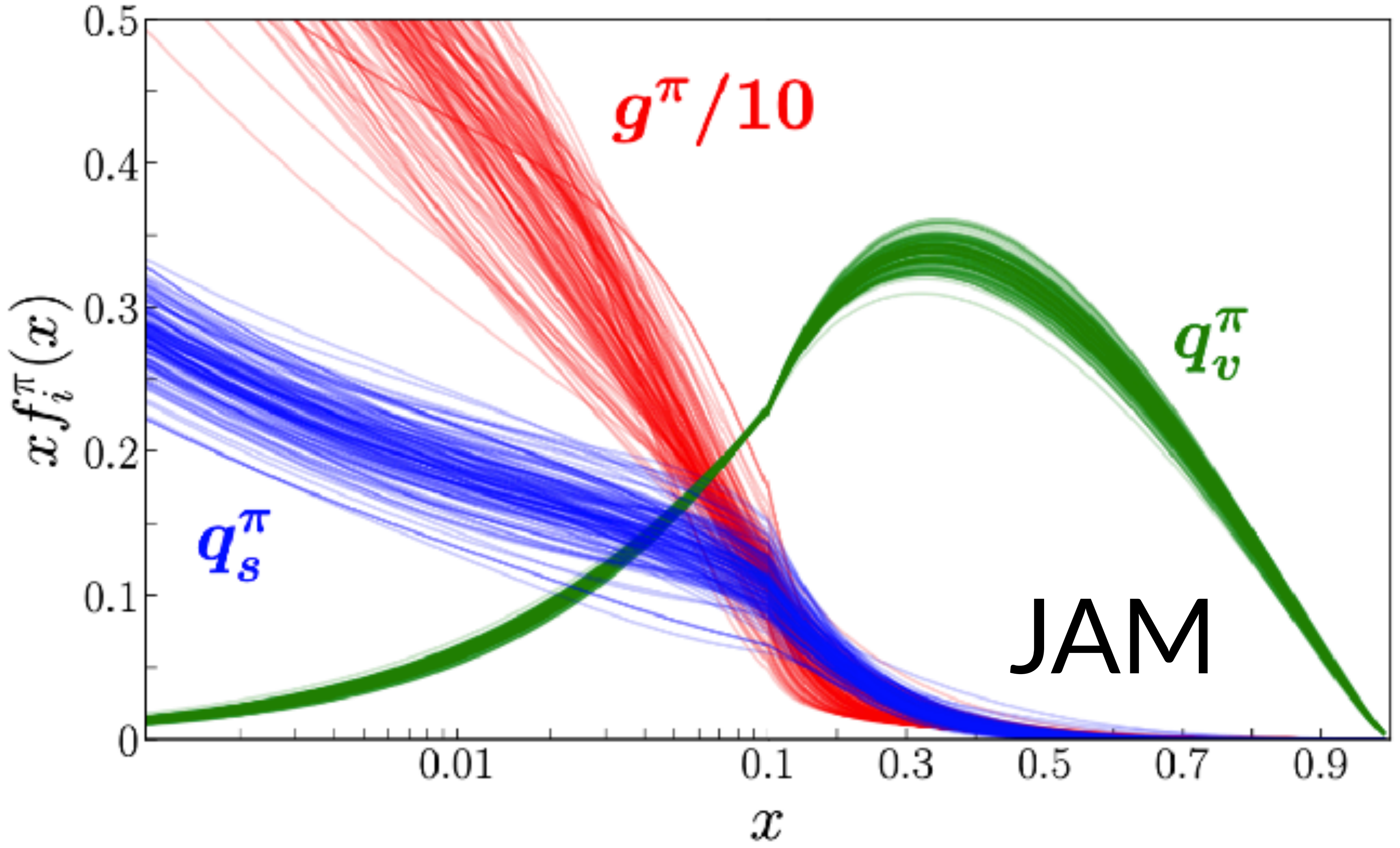
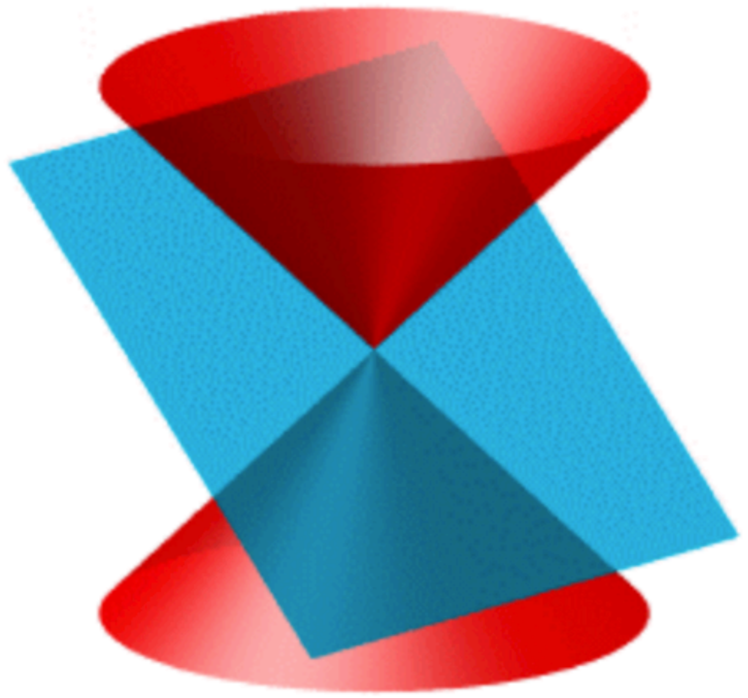


How to connect mass spectra & partonic observables?

Light-front dynamics

Formalism

- Proposed by Dirac (1949)
- Equivalent to Infinite momentum frame (IMF)



Why LFD?

- Handle relativistic effect properly
- Relevant for high-energy process
- Vacuum becomes simpler, and so on

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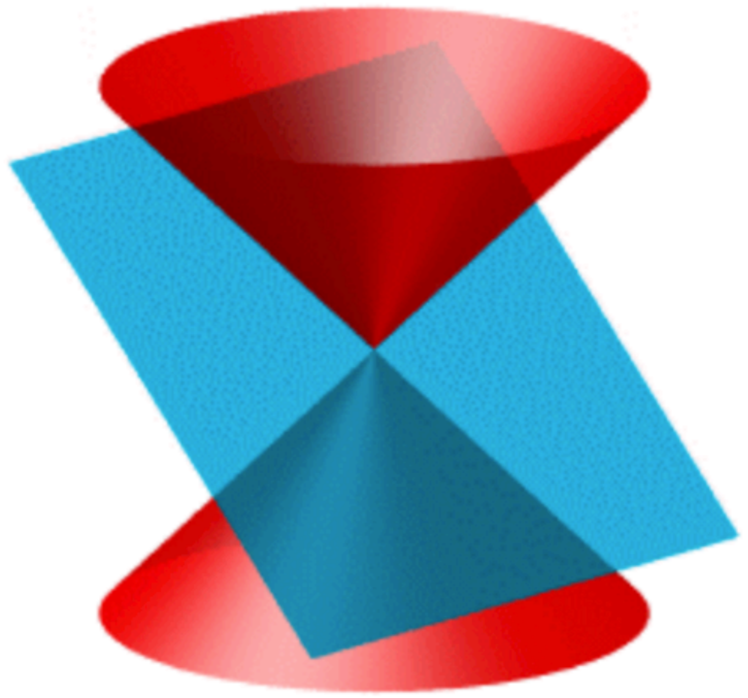

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	Instant form	Light-Front form
Time	x^0	$x^+ = x^0 + x^3$
Space	x^1, x^2, x^3	$x^- = x^0 - x^3, \ \ \mathbf{x}_\perp = (x^1, x^2)$
Hamiltonian	p^0	$p^- = p^0 - p^3$
Momentum	p^1, p^2, p^3	$p^+ = p^0 + p^3, \ \ \mathbf{p}_\perp = (p^1, p^2)$
Product	$x \cdot p = x^0 p^0 - \mathbf{x} \cdot \mathbf{p}$	$x \cdot p = (x^+ p^- + x^- p^+)/2 - \mathbf{x}_\perp \cdot \mathbf{p}_\perp$
Vacuum	very complex	can only contain zero-mode excitations

Light-front wave function approach

Goal: compute *hadron structure and properties* from LFWFs.

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LFWFs

- Frame independent & Boost invariant
- Current matrix element → LFWF overlaps

$$\Psi(x,k_\perp)=\underset{\substack{\downarrow\\ \text{Radial part}}}{\Phi(x,k_\perp)}\underset{\substack{\downarrow\\ \text{Spin-orbit part}}}{\mathcal{R}(x,k_\perp)}$$

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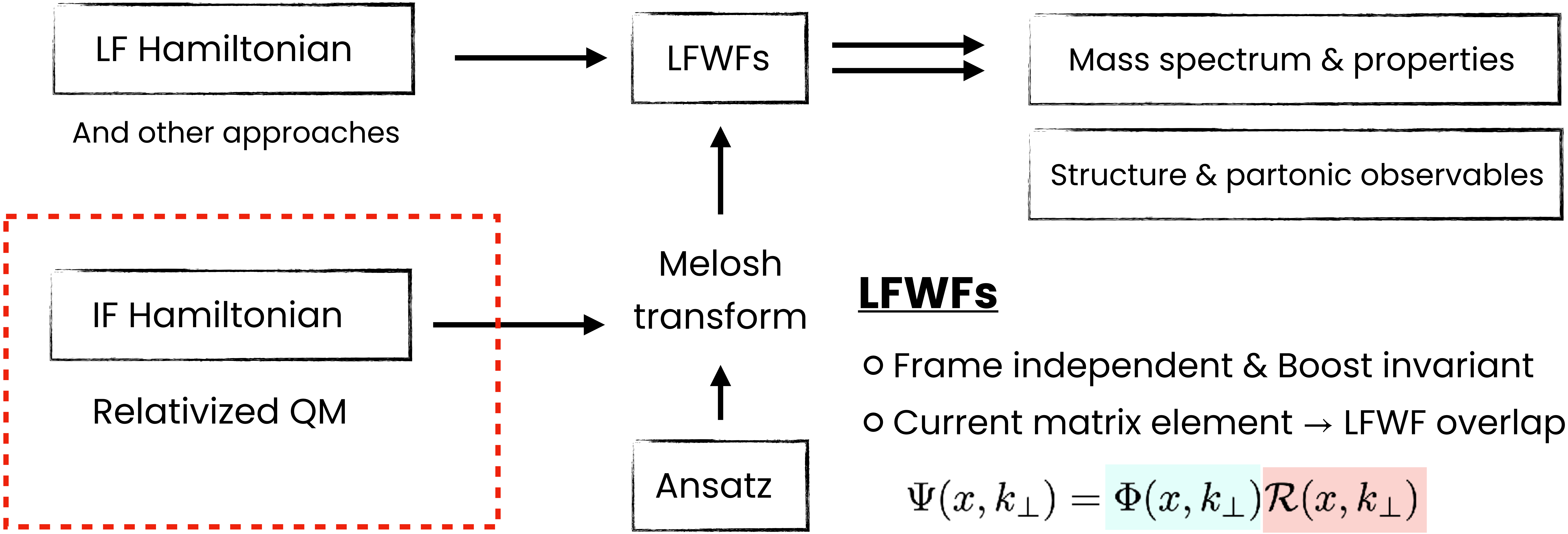

LF Hamiltonian

And other approaches

LFWFs

Mass spectrum & properties

Structure & partonic observables

IF Hamiltonian

Relativized QM

Melosh transform

Ansatz

LFWFs

- Frame independent & Boost invariant
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$$\Psi(x,k_\perp) = \underbrace{\Phi(x,k_\perp)}_{\text{Radial part}}\underbrace{\mathcal{R}(x,k_\perp)}_{\text{Spin-orbit part}}$$

- Large Momentum EFT (LAMET) → Lattice QCD

Spin-orbit wave function (Meson)

Pseudoscalar $$\mathcal{R}^{00}_{\lambda_q\lambda_{\bar{q}}} = \frac{1}{\sqrt{2}\tilde{M}_0}\bar{u}_{\lambda_q}(p_q)\gamma_5 v_{\lambda_{\bar{q}}}(p_{\bar{q}})$$

Vector $$\mathcal{R}^{1h}_{\lambda_q\lambda_{\bar{q}}} = -\frac{1}{\sqrt{2}\tilde{M}_0}\bar{u}_{\lambda_q}(p_q)\epsilon_\mu(h)\left[\gamma^\mu - \frac{(p_q-p_{\bar{q}})^\mu}{D}\right]v_{\lambda_{\bar{q}}}(p_{\bar{q}})$$

Orthonormality

$$\sum_{\lambda_q,\lambda_{\bar{q}}}\left\langle \mathcal{R}^{Jh}_{\lambda_q\lambda_{\bar{q}}} \middle| \mathcal{R}^{J'h'}_{\lambda_q\lambda_{\bar{q}}}\right\rangle = \delta_{JJ'}\delta_{hh'}$$

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Pseudoscalar (spin 0)

$$\mathcal{R}^{00}_{\lambda_1\lambda_2}(x,\boldsymbol{k}_\perp) = \frac{\mathcal{R}_0}{\sqrt{2}}\begin{pmatrix} -k^L & \mathcal{A} \\ -\mathcal{A} & -k^R \end{pmatrix},$$

$$\frac{1}{\sqrt{2}}(\psi_{\uparrow\downarrow} - \psi_{\downarrow\uparrow})$$

Vector (spin 1)

$$\mathcal{R}^{1+1}_{\lambda_1\lambda_2}(x,\boldsymbol{k}_\perp) = \mathcal{R}_0\begin{pmatrix} \mathcal{A} + \frac{k_\perp^2}{D} & k^R\frac{\mathcal{M}_1}{D} \\ -k^R\frac{\mathcal{M}_2}{D} & -\frac{(k^R)^2}{D} \end{pmatrix}, \qquad \psi_{\uparrow\uparrow}$$

$$\mathcal{R}^{10}_{\lambda_1\lambda_2}(x,\boldsymbol{k}_\perp) = \frac{\mathcal{R}_0}{\sqrt{2}}\begin{pmatrix} k^L\frac{\mathcal{M}}{D} & \mathcal{A} + \frac{2k_\perp^2}{D} \\ \mathcal{A} + \frac{2k_\perp^2}{D} & -k^R\frac{\mathcal{M}}{D} \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}(\psi_{\uparrow\downarrow} + \psi_{\downarrow\uparrow})$$

$$\mathcal{R}^{1-1}_{\lambda_1\lambda_2}(x,\boldsymbol{k}_\perp) = \mathcal{R}_0\begin{pmatrix} -\frac{(k^L)^2}{D} & k^L\frac{\mathcal{M}_2}{D} \\ -k^L\frac{\mathcal{M}_1}{D} & \mathcal{A} + \frac{k_\perp^2}{D} \end{pmatrix}, \qquad \psi_{\downarrow\downarrow}$$

$$\mathscr{R}_0 = \frac{1}{\sqrt{A^2 + k_\perp^2}} \qquad k^{R(L)} = k_x \pm ik_y \qquad A = xm_2 + (1-x)m_1 \qquad D = M_0 + m_1 + m_2$$

Gaussian expansion method

A numerical method to solve Schrödinger equation.

PPNP51, 223 (2003)

- Solving Schrödinger equation

$$H\left|\psi\right\rangle = E\left|\psi\right\rangle$$

- Gaussian basis functions

$$\psi = \sum_{n=1}^{\max} c_n \phi_n^G$$

$$\phi_n^G(r) = \frac{(2\nu_n)^{3/4}}{\pi^{3/4}} \mathrm{e}^{-\nu_n r^2}$$

- Generalized Eigenvalue equation

$$\boldsymbol{H}_h \boldsymbol{c} = M_h \boldsymbol{S} \boldsymbol{c}$$

$$\boldsymbol{H} = \left\langle \phi_n^G \middle| \hat{H} \middle| \phi_m^G \right\rangle$$

$$\boldsymbol{S} = \left\langle \phi_n^G \middle| \phi_m^G \right\rangle$$

- Geometric progression $[r_1, r_{\max}]$

$$\nu_n = \frac{1}{r_n^2} \quad r_n = r_1 a^{n-1} \quad a = \left(\frac{r_{\max}}{r_1}\right)^{\frac{1}{n_{\max}-1}}$$

- Normalization

$$\left\langle \psi \middle| \psi \right\rangle = \sum_{m,n} c_n^* S_{nm} c_m = 1$$

Simple variational analysis

A simple approach

$$\begin{pmatrix}\Phi_{1S}\\ \Phi_{2S}\\ \Phi_{3S}\end{pmatrix}=\begin{pmatrix}c_1^{1S} & c_2^{1S} & c_3^{1S}\\ c_1^{2S} & c_2^{2S} & c_3^{2S}\\ c_1^{3S} & c_2^{3S} & c_3^{3S}\end{pmatrix}\begin{pmatrix}\phi_{1S}^{\mathrm{HO}}\\ \phi_{2S}^{\mathrm{HO}}\\ \phi_{3S}^{\mathrm{HO}}\end{pmatrix}$$

$$\boldsymbol{R}=\begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & 0 & s_{13}\\ 0 & 1 & 0\\ -s_{13} & 0 & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix}$$

Harmonic oscillator wave function

$$\phi_{1S}^{\mathrm{HO}}(\boldsymbol{k})=\frac{1}{\pi^{3/4}\beta^{3/2}}e^{-k^2/2\beta^2},$$

$$\phi_{2S}^{\mathrm{HO}}(\boldsymbol{k})=\frac{(2k^2-3\beta^2)}{\sqrt{6}\pi^{3/4}\beta^{7/2}}e^{-k^2/2\beta^2},$$

$$\phi_{3S}^{\mathrm{HO}}(\boldsymbol{k})=\frac{(15\beta^4-20\beta^2k^2+4k^4)}{2\sqrt{30}\pi^{3/4}\beta^{11/2}}e^{-k^2/2\beta^2},$$

- More accurate method:
  - Using Gaussian-expansion method (GEM)

Orthonormality

$$\langle\Phi_{nS}|\Phi_{n'S}\rangle=\delta_{nn'}$$

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$$\boldsymbol{R} = \begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix} \begin{pmatrix}c_{13} & 0 & s_{13}\\ 0 & 1 & 0\\ -s_{13} & 0 & c_{13}\end{pmatrix} \begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix}$$

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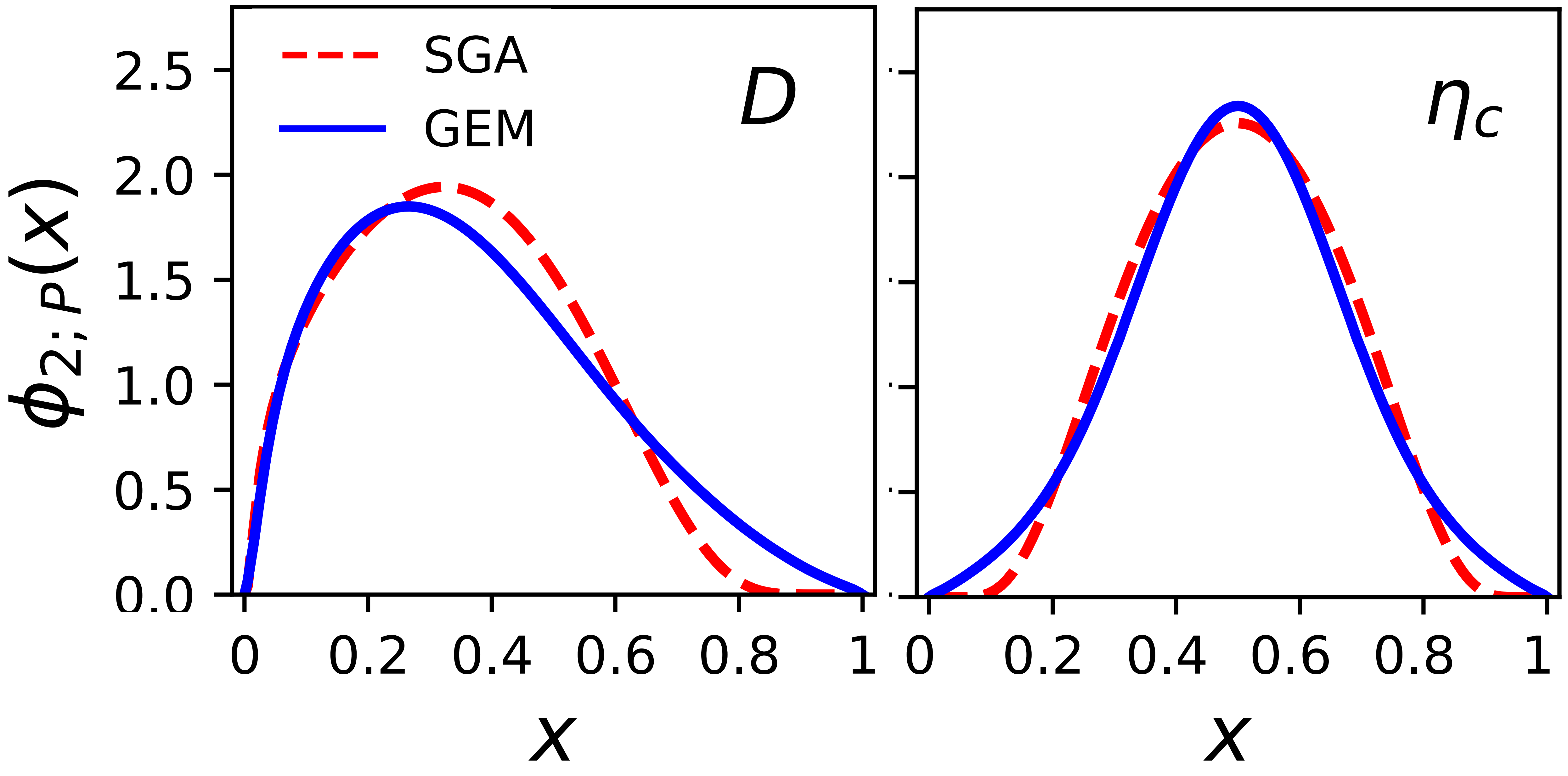
- Variable transformation $(k_z, k_\perp) \rightarrow (x, k_\perp)$

$$\Phi_{nS}(x, \boldsymbol{k}_\perp) = \sqrt{2(2\pi)^3}\sqrt{\frac{\partial k_z}{\partial x}}\Phi_{nS}(\boldsymbol{k})$$

$$k_z = \left(x - \frac{1}{2}\right) M_0 + \frac{(m_{\bar{q}}^2 - m_q^2)}{2M_0} \qquad \frac{\partial k_z}{\partial x} = \frac{M_0}{4x(1-x)}\left[1 - \frac{(m_q^2 - m_{\bar{q}}^2)^2}{M_0^4}\right]$$

Distribution amplitude: GEM vs SGA

$$\langle 0|\bar{q}(z)\gamma^+\gamma_5 q(-z)|\mathcal{P}(P)\rangle = i f_{\mathcal{P}} P^+ \int_0^1 \mathrm{d}x\ e^{i\xi P\cdot z}\phi_{2,\mathcal{P}}(x),$$



Arifi, Happ, Ohno, Oka. PRD110, 014020 (2024)

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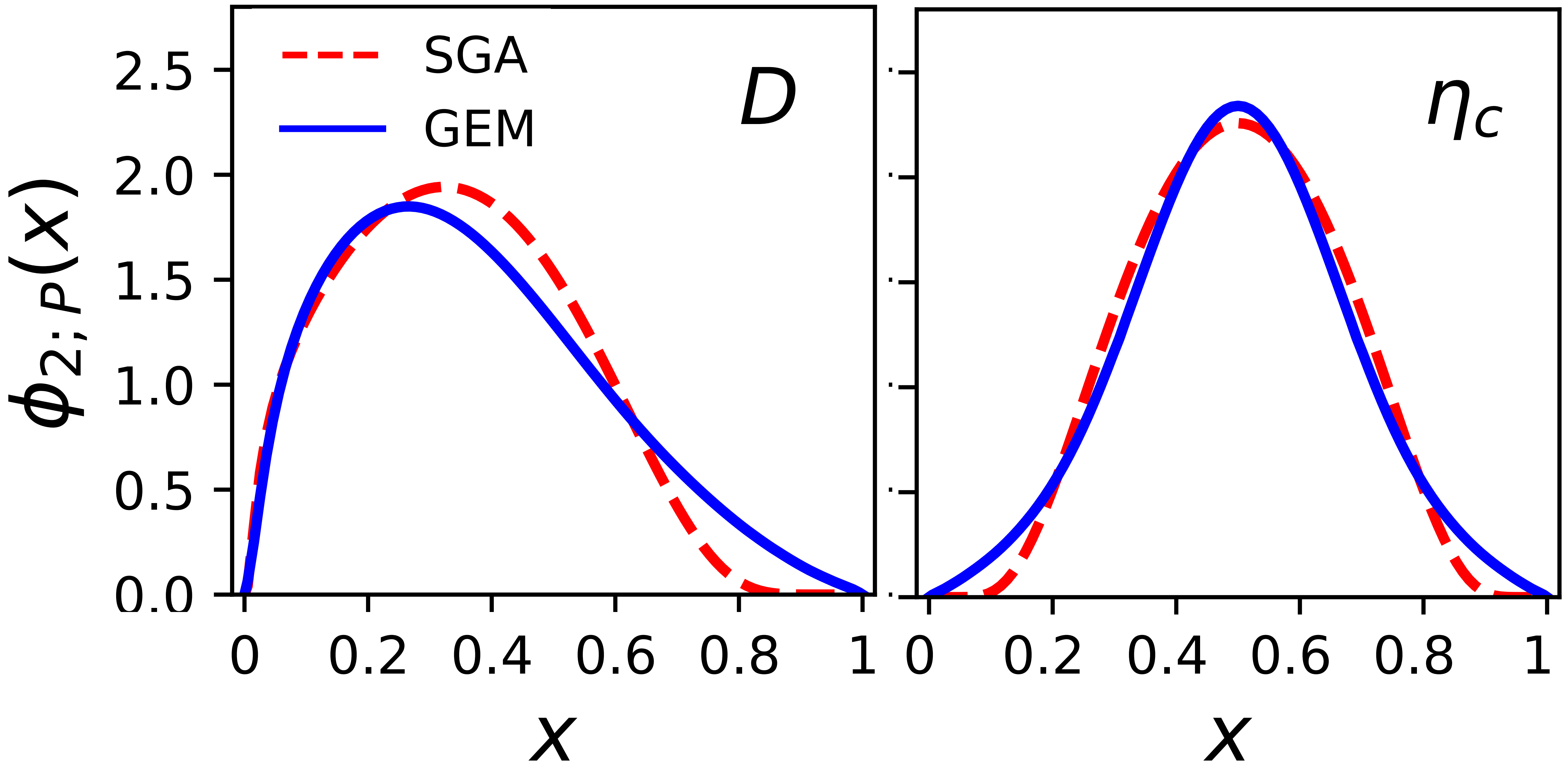
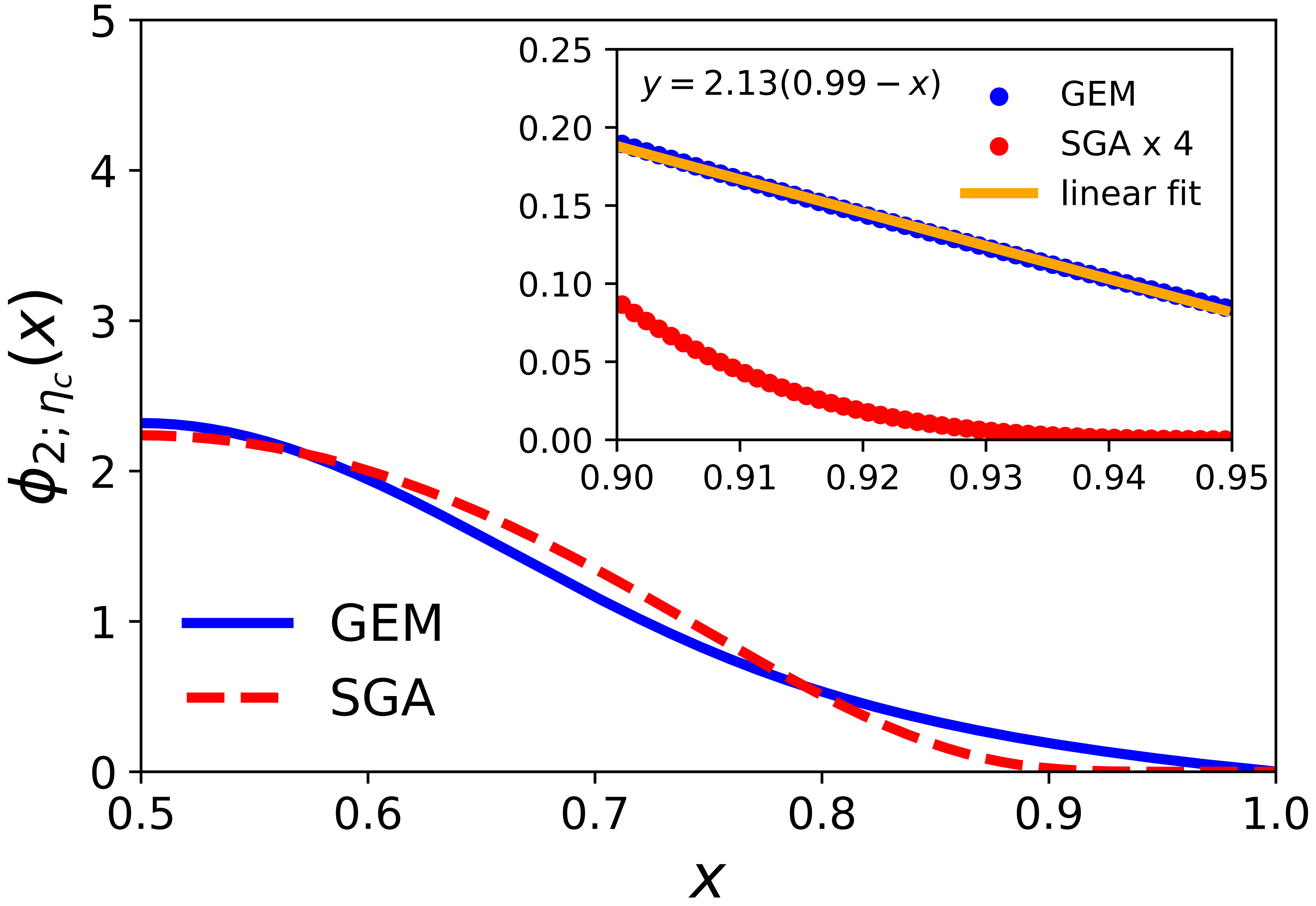


Arifi, Happ, Ohno, Oka. PRD110, 014020 (2024)

$$\phi(x) = x^\alpha(1-x)^\beta$$

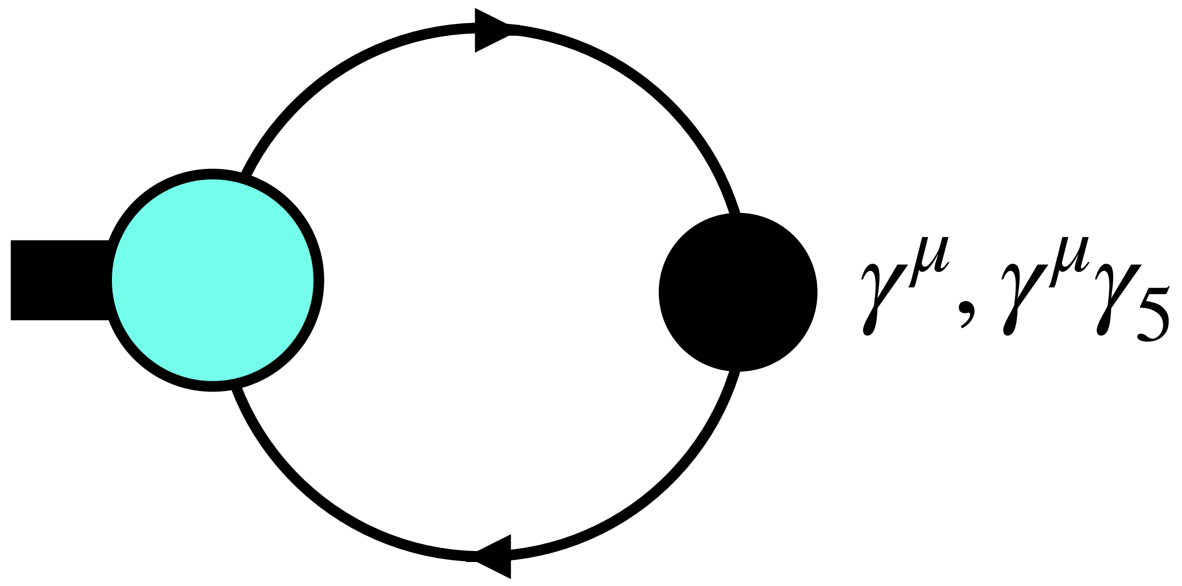


Consistent with the perturbative QCD.

$$\phi_{2,M}(x \to 1) \propto (1-x)^1$$

Decay constants

- Let us consider pseudoscalar and vector meson decay constants



$$\langle 0|\bar{q}(0)\gamma^\mu\gamma_5 q(0)|P(P)\rangle = i f_P P^\mu$$

$$\langle 0|\bar{q}(0)\gamma^\mu q(0)|V(P,h)\rangle = f_V M \epsilon^\mu$$

l.h.s: $\mathcal{J}^\mu = \langle 0|\bar{q}(0)\Gamma q(0)|\mathcal{M}\rangle$

r.h.s: $\mathcal{G}_P^\mu = iP^\mu \quad \mathcal{G}_V^\mu = M\epsilon^\mu$

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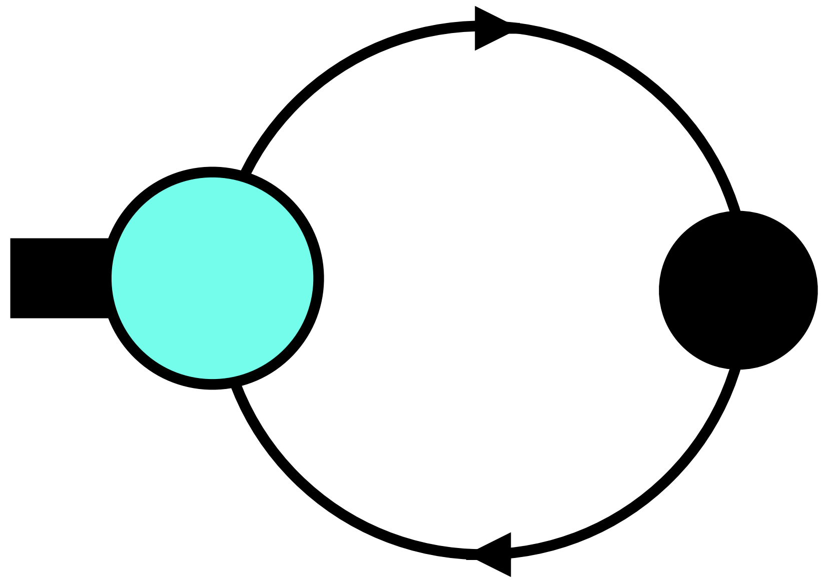

$\gamma^\mu, \gamma^\mu\gamma_5$

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- Decay constant calculation

  In the nonrelativistic limit ~ $|\psi(r=0)|$

$$f_{P(V)} = \frac{\mathcal{J}^\mu}{\mathcal{G}^\mu_{P(V)}}$$

$$\mu = \pm, \perp$$

$$h = 0, \pm 1$$

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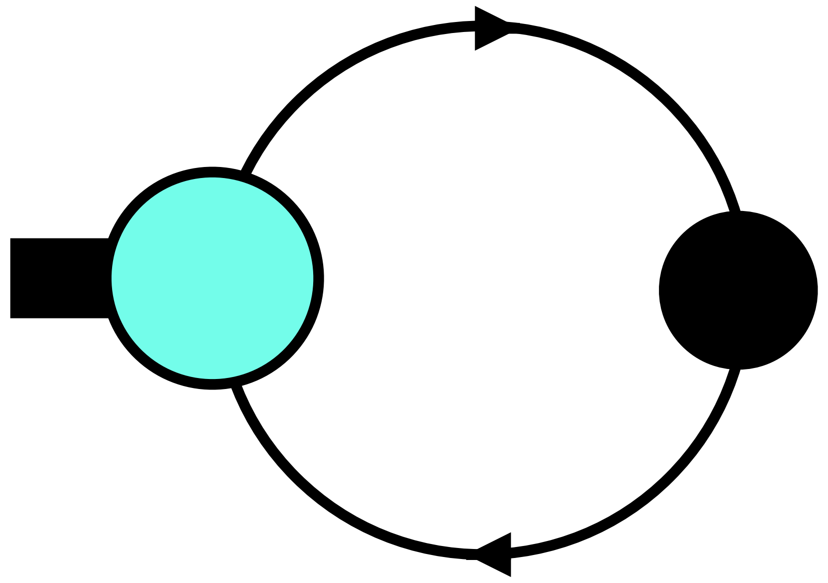

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$\mu = \pm, \perp$

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$$f_{P(V)} = \sqrt{3}\int \frac{dxd^2k_\perp}{2(2\pi)^3}\frac{\Phi(x,k_\perp)}{\mathcal{G}_{P(V)}}\sum_{\lambda_1,\lambda_2}\mathcal{R}^{Jh}_{\lambda_1\lambda_2}(x,k_\perp)\left[\frac{\bar{\nu}_{\lambda_2}(p_2)}{\sqrt{x_2}}\Gamma\frac{u_{\lambda_1}(p_1)}{\sqrt{x_1}}\right]$$

Noninteracting quark basis

$$\langle 0|\bar{q}(0)\gamma^\mu q(0)|V(P,h)\rangle = f_V M \epsilon^\mu$$

External variable

$$\mathcal{G}_V^\mu = M\epsilon^\mu$$

$$f_V = \frac{\mathcal{J}^\mu}{\mathcal{G}_V^\mu}$$

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$\downarrow$ External variable

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- To obtain the self-consistency

Physical mass

Invariant mass

$$M^2 \longrightarrow M_0^2 = \frac{m_1^2 + k_{\perp}^2}{x} + \frac{m_2^2 + k_{\perp}^2}{1-x}$$

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↓

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- In Light-front Bethe-Salpheter approach

$$S = S_{on} + S_{inst} + S_{z.m.}$$

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- In Light-front Bethe-Salpheter approach

$$S = S_{on} + S_{inst} + S_{z.m.}$$

- Physical mass should be replaced by the invariant mass

$$f_{P(V)} \propto \frac{1}{M}\int \frac{dx d^2k_\perp}{2(2\pi)^3} \longrightarrow f_{P(V)} \propto \int \frac{dx d^2k_\perp}{2(2\pi)^3}\frac{1}{M_0(x,k_\perp)}$$

Operators

The explicit expression is given by

$$f_{P(V)} = \sqrt{6}\int \frac{dx\,d^2k_\perp}{2(2\pi)^3}\frac{\Phi(x,k_\perp)}{\sqrt{A^2+k_\perp^2}}\mathcal{O}^{\mu}_{P(V)}(h)$$

Arifi, Choi, Ji, Oh. PRD107, 053003 (2023)

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Pseudoscalar meson

$$\mathcal{O}_P^+ = 2A \qquad \mathcal{O}_P^- = 2A$$

Vector meson

$$\mathcal{O}_V^+(0) = 2\left[A + \frac{2k_\perp^2}{D}\right]$$

$$\mathcal{O}_V^\perp(\pm 1) = \left[\frac{A^2 + k_\perp^2}{x(1-x)M_0} - \frac{2k_\perp^2}{D}\right]$$

M1 Radiative decays

- Let us consider $V \to P\gamma$ decay:

$$\langle P(P')|J_{em}(0)|V(P,h)\rangle = ie\epsilon^{\mu\nu\rho\sigma}\epsilon_\nu q_\rho P_\sigma F_{VP\gamma}(Q^2) \longrightarrow \boxed{F_{VP\gamma}(Q^2) = \frac{\mathcal{J}^\mu}{\mathcal{G}^\mu}}$$

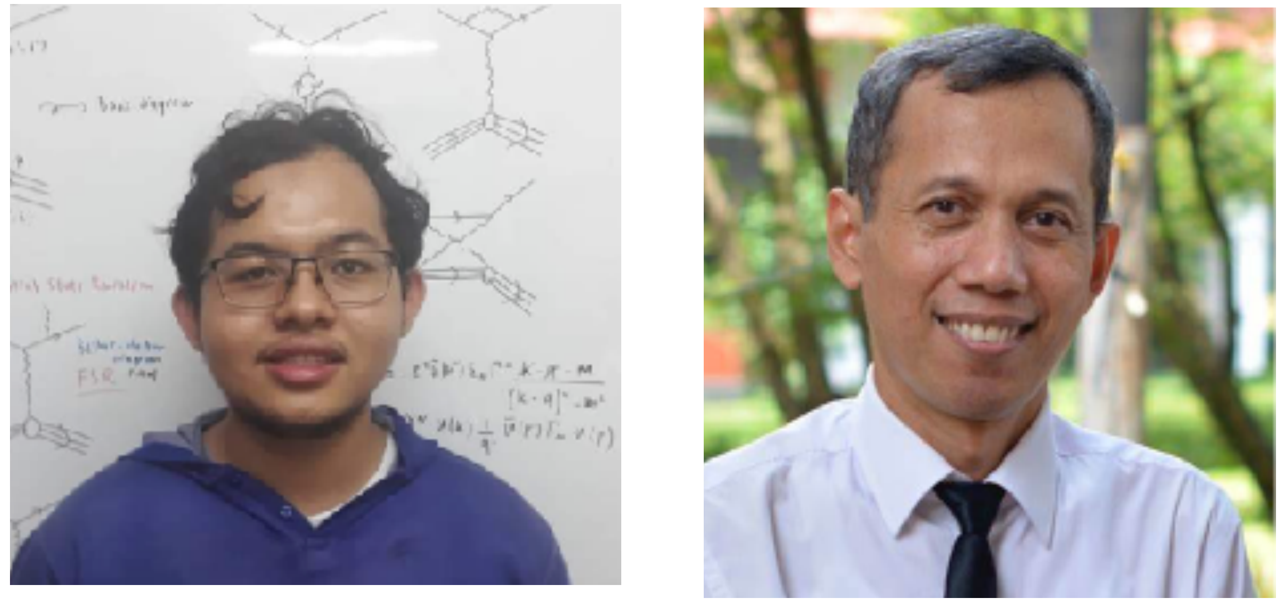
EM vector current

Lorentz structure

Form factor

l.h.s: $\mathcal{J}^\mu = \langle P(P')|J_{em}(0)|V(P,h)\rangle$

r.h.s: $\mathcal{G}^\mu = ie\epsilon^{\mu\nu\rho\sigma}\epsilon_\nu q_\rho P_\sigma$



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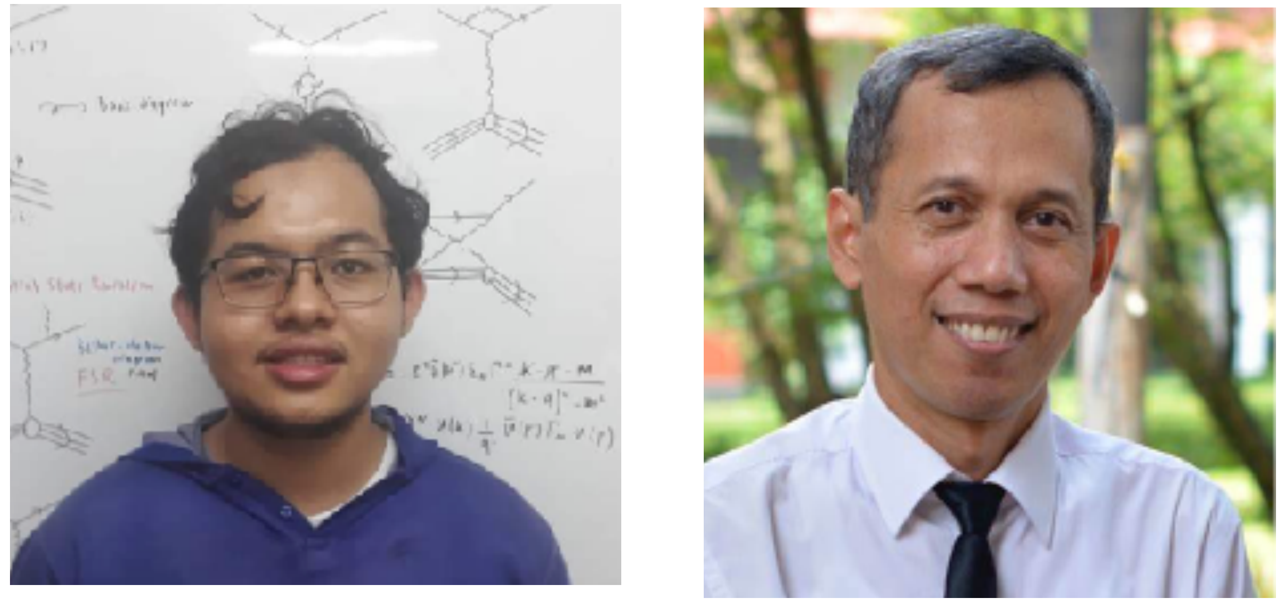

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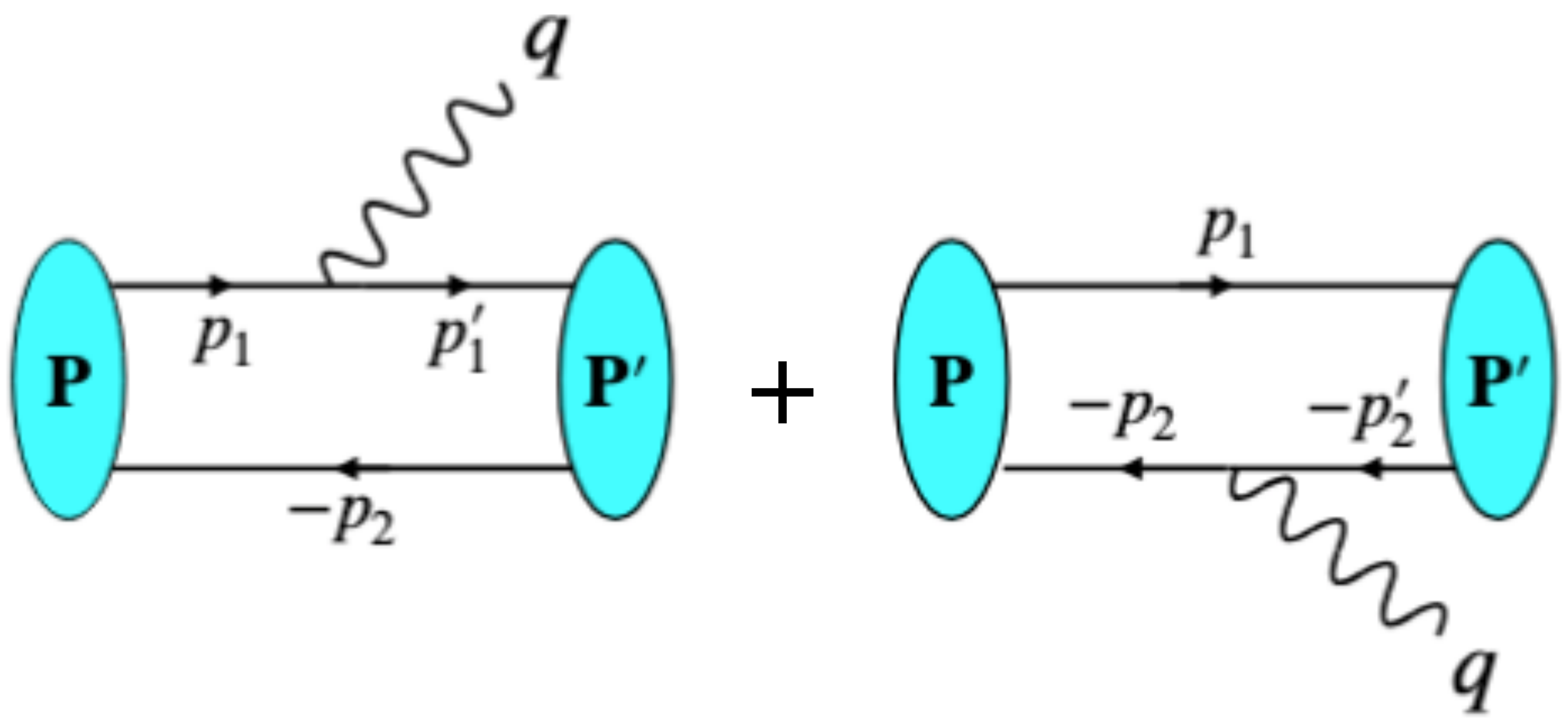
- Feynman Diagrams



Operators

$$F_{VP\gamma}(Q^2) = \frac{\mathcal{J}^\mu}{\mathcal{G}^\mu}$$

$$F_{VP\gamma}(Q^2) = e_q I_q^\mu + e_{\bar{q}} I_{\bar{q}}^\mu$$

$$I_j = \int \frac{dx d^2 k_\perp}{2(2\pi)^3} \frac{\Phi(x, k'_\perp)}{\sqrt{A^2 + k_\perp^2}} \frac{\Phi(x, k_\perp)}{\sqrt{A^2 + k_\perp^2}} \mathcal{O}^\mu_{VP\gamma}(h)$$

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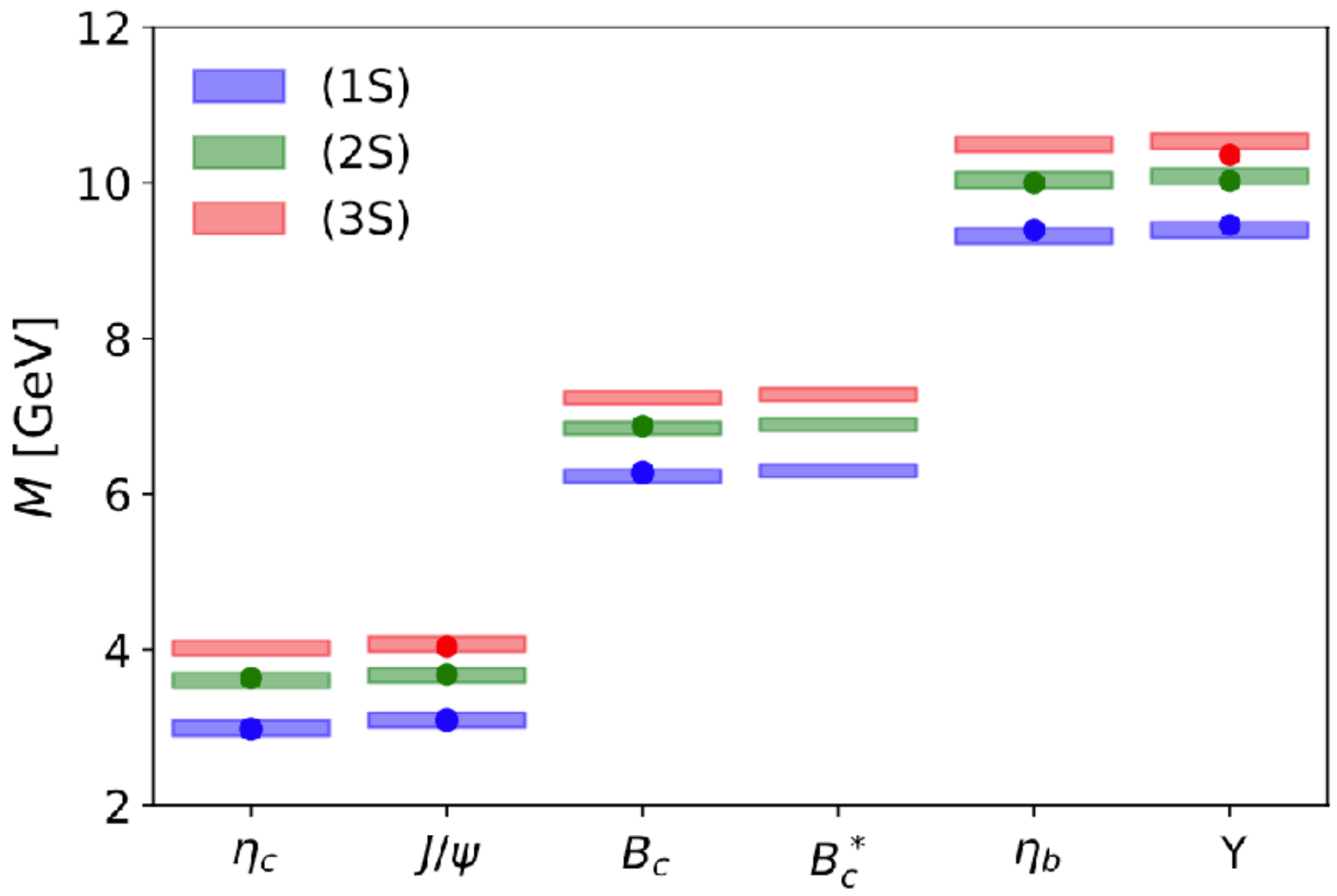
μ	$\epsilon(h)$	$\mathcal{O}$
$+$	$\epsilon(0)$	$\ldots$
$+$	$\epsilon(\pm 1)$	$2(1-x)\left[\mathcal{A} + \frac{2}{\mathcal{D}_0}\left(\boldsymbol{k}_\perp^2 - \frac{(\boldsymbol{k}_\perp \cdot \boldsymbol{q}_\perp)^2}{\boldsymbol{q}_\perp^2}\right)\right]$
$R(L)$	$\epsilon(0)$	$\frac{1}{xM_0}\left\{\mathcal{A}\left(\mathcal{A} + \frac{2\boldsymbol{k}_\perp^2}{\mathcal{D}_0}\right) + \frac{\mathcal{M}}{\mathcal{D}_0}\left[(1-2x)\boldsymbol{k}_\perp^2 + (1-x)\left((\boldsymbol{k}_\perp \cdot \boldsymbol{q}_\perp) - \frac{2(\boldsymbol{k}_\perp \cdot \boldsymbol{q}_\perp)^2}{\boldsymbol{q}_\perp^2}\right)\right]\right\}$
$R(L)$	$\epsilon(-1)[\epsilon(+1)]$	$\frac{2}{x(M_0^2 - M_0'^2 - \boldsymbol{q}_\perp^2)}\left[(\boldsymbol{k}_\perp \cdot \boldsymbol{q}_\perp)\left(\mathcal{A} + \frac{x\boldsymbol{k}_\perp^2}{\mathcal{D}_0} - \frac{\mathcal{A}\mathcal{M}_1}{\mathcal{D}_0}\right) + (1-x)(\boldsymbol{k}_\perp \cdot \boldsymbol{q}_\perp - \boldsymbol{q}_\perp^2)\left(\mathcal{A} + \frac{\boldsymbol{k}_\perp^2}{\mathcal{D}_0}\right)\right]$
$R(L)$	$\epsilon(+1)[\epsilon(-1)]$	$2(1-x)\left[\mathcal{A} + \frac{2}{\mathcal{D}_0}\left(\boldsymbol{k}_\perp^2 - \frac{(\boldsymbol{k}_\perp \cdot \boldsymbol{q}_\perp)^2}{\boldsymbol{q}_\perp^2}\right)\right]$

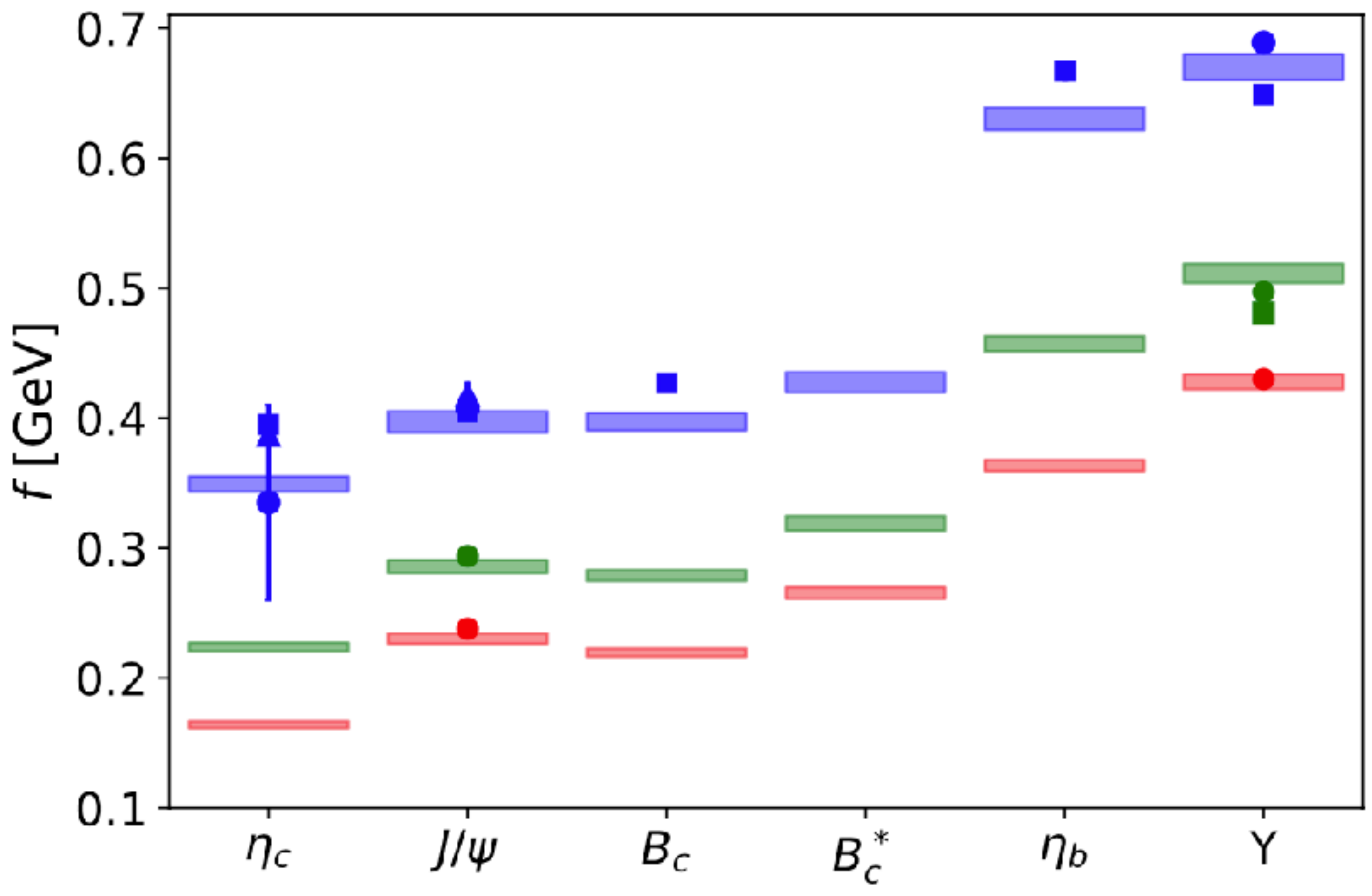
Heavy quarkonia & B_c mesons

Mass spectra



$$\Delta M_{err} = \left| \frac{M_{theo} - M_{exp}}{M_{exp}} \right| \times 100\,\% \approx 0.6\,\%$$

Decay constants



$$\Delta f_{err} = \left| \frac{f_{theo} - f_{exp}}{f_{exp}} \right| \times 100\,\% \approx 2.7\,\%$$

Self-consistency (different polarizations)

Decay constant

$$\Delta\mathcal{O}_1 = \mathcal{O}_V^{\perp}(\pm 1) - \mathcal{O}_V^{+}(0)$$

$$\Delta\mathcal{O}_1 = -\frac{2}{D}(k_{\perp}^2 - 2k_z^2)$$

Self-consistency (different polarizations)

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M1 radiative decay

$$\Delta\mathcal{O}_2 = \mathcal{O}_{PV}^{+}(\pm 1) - \mathcal{O}_{PV}^{\perp}(0)$$

$$\Delta\mathcal{O}_2 = -\frac{2}{D}\left[\frac{2k_z^2 k_\perp^2}{m^2 + k_\perp^2} + k_\perp^2 - 2k_z^2\right]$$

Self-consistency (different polarizations)

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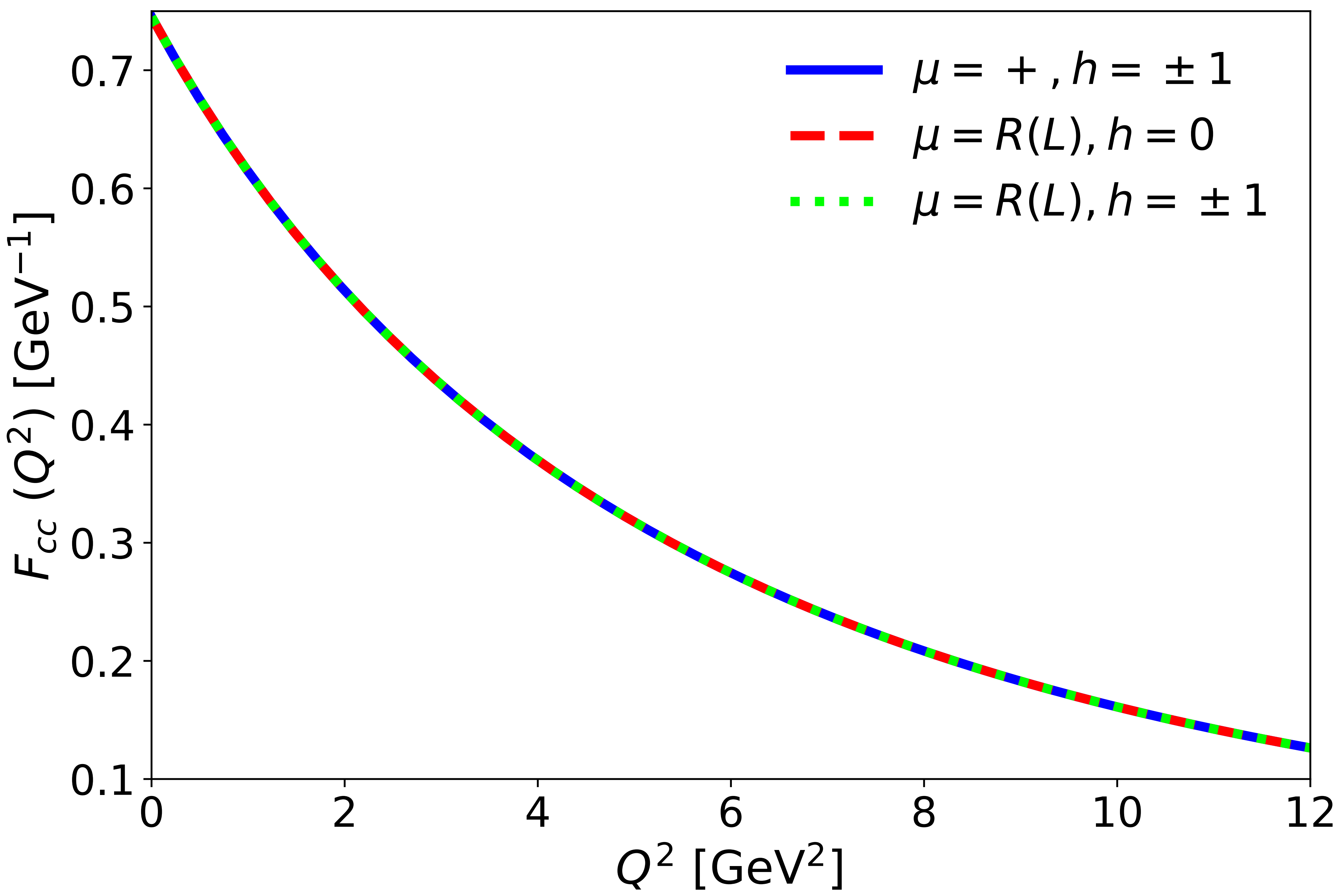

Probing the rotational symmetry breaking.

$$\langle k_\perp^2\rangle = \langle 2k_z^2\rangle$$

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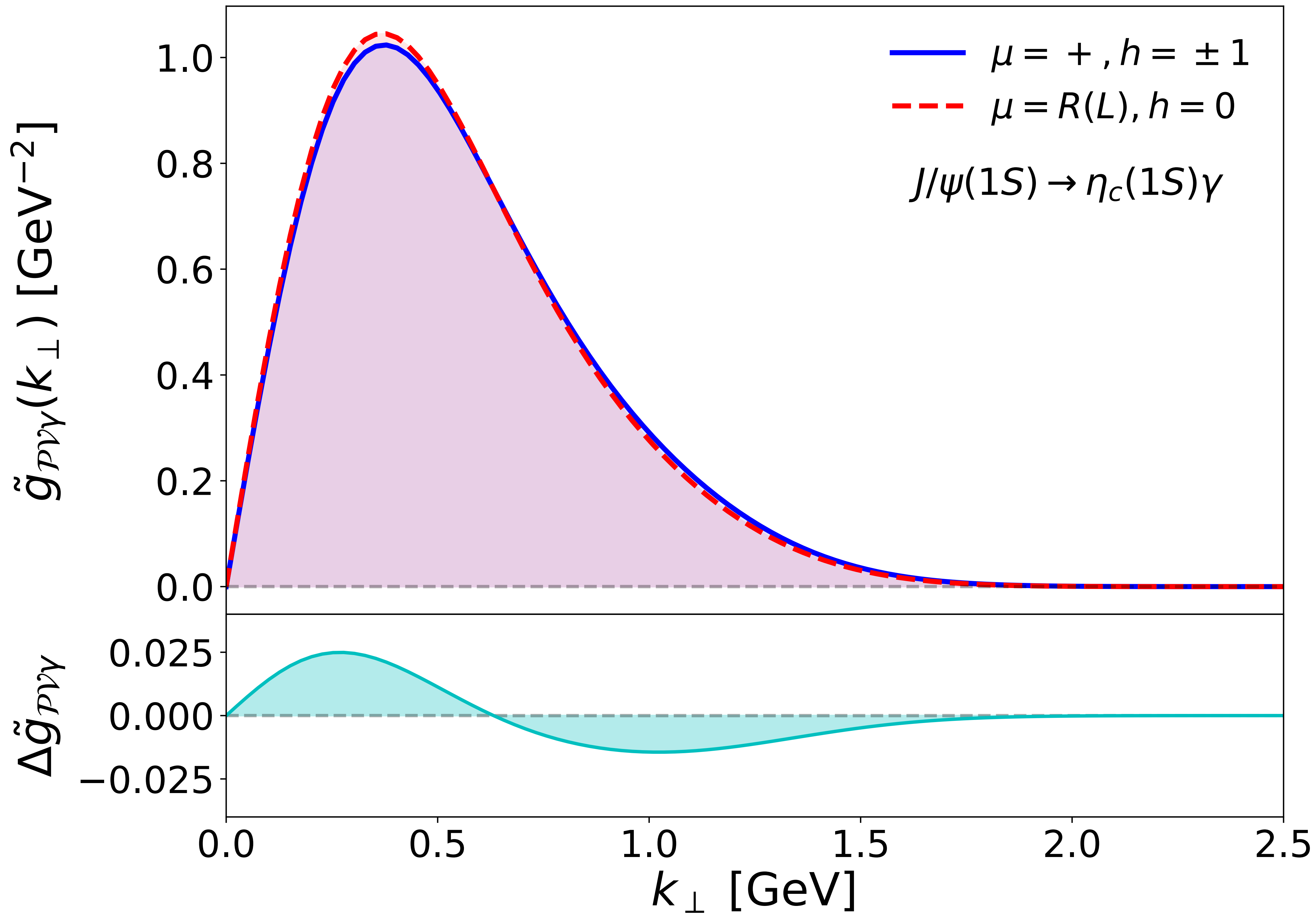
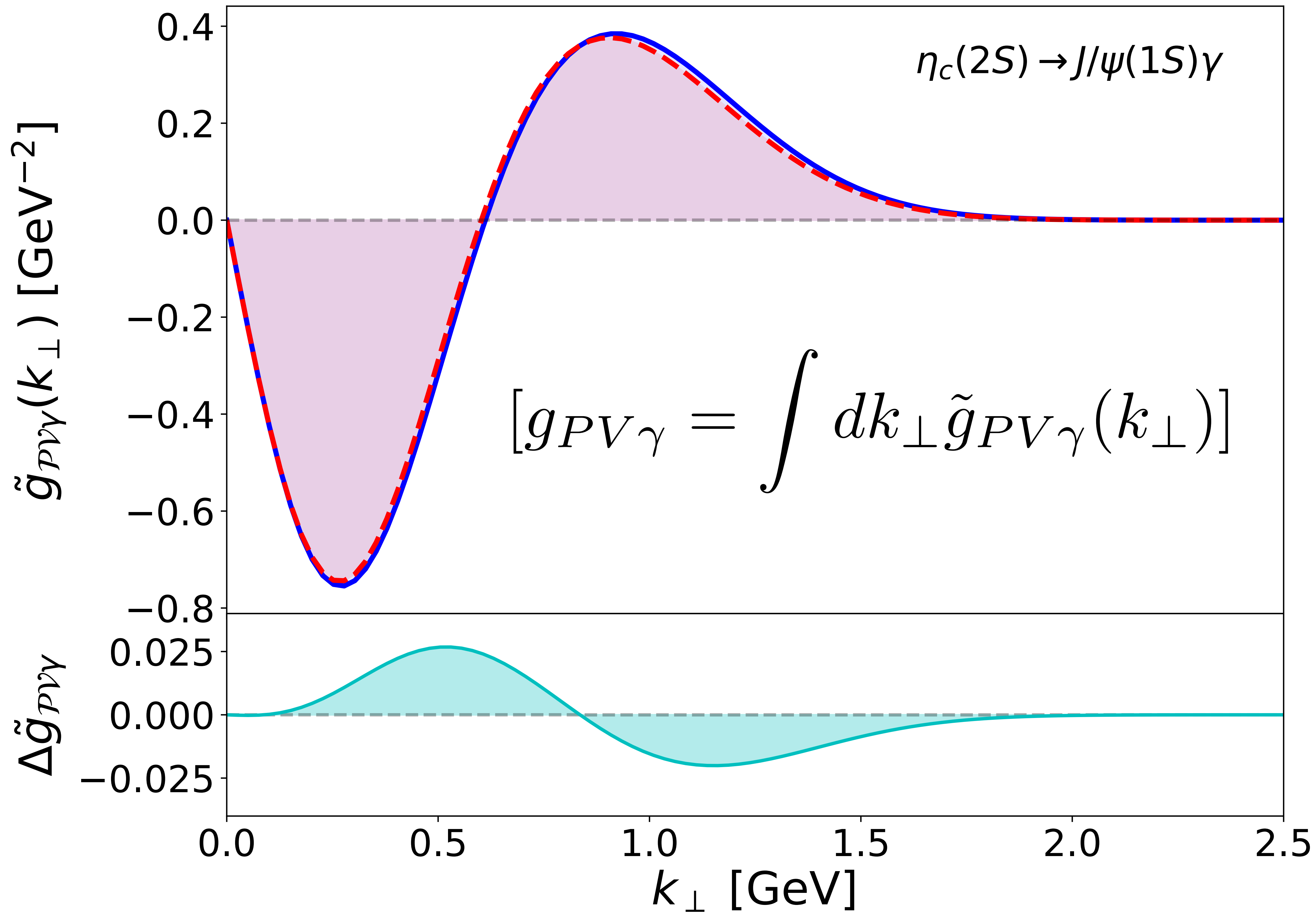
Integrands

Allowed case $(n' = n)$



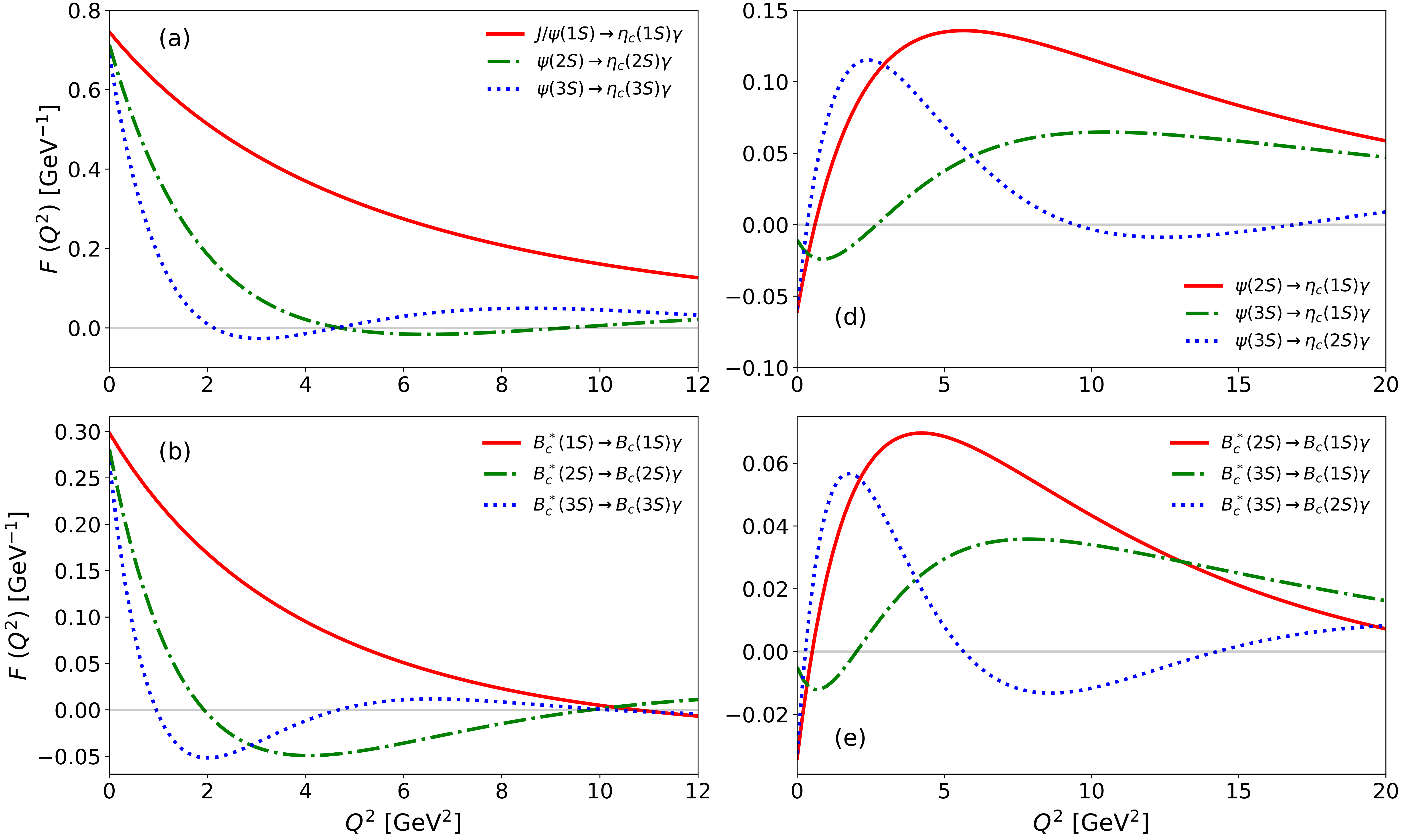
Overlap of wave function

Hindered case $(n' \neq n)$



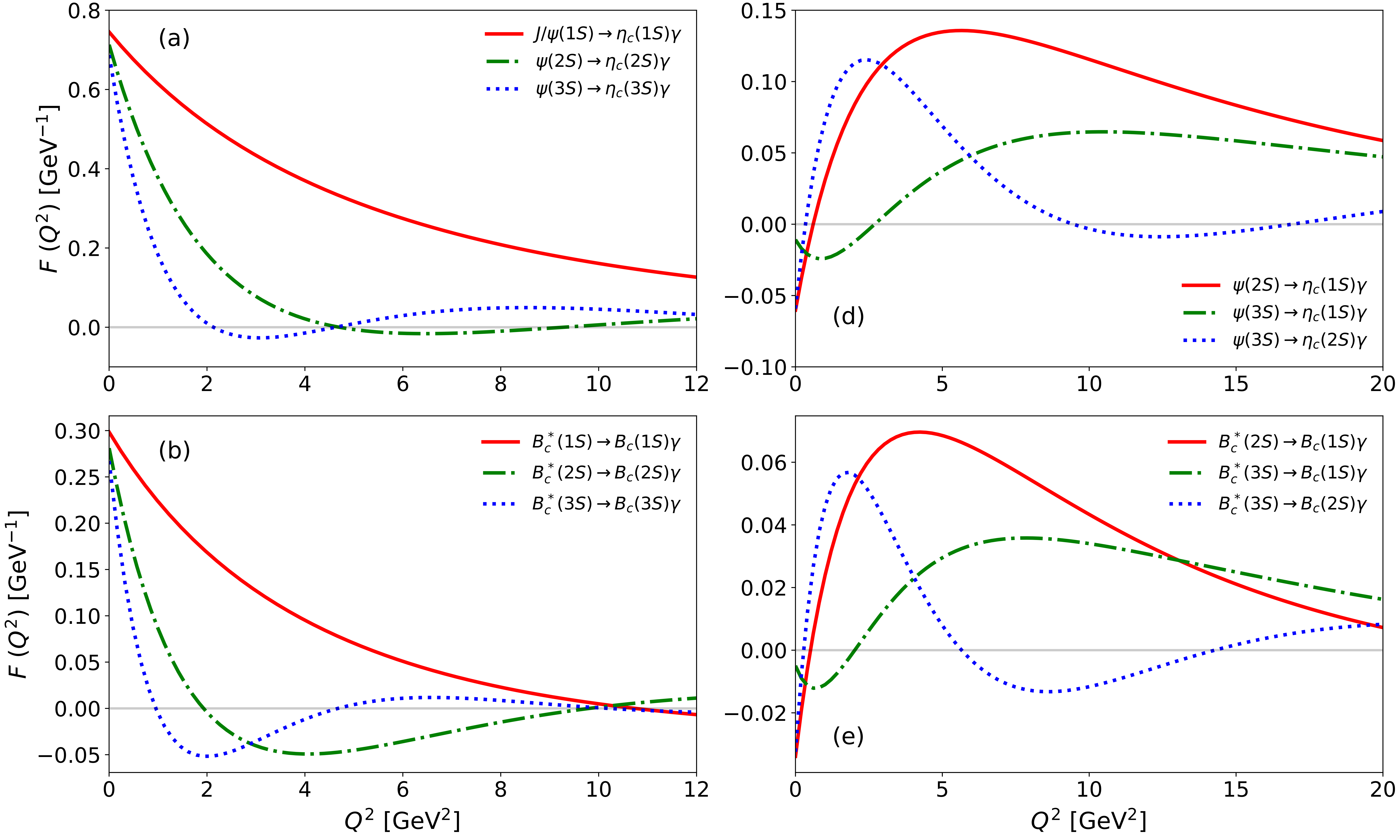
Orthogonality of wave function

Transition form factor



Selected transition form factors

Transition form factor



Selected transition form factors

Coupling constant

$$g_{PV\gamma} = F_{PV\gamma}(Q^2 \to 0)$$

(Real photon)

Transition	Our	Expt.	[38]	[17]
$J/\psi \to \eta_c(1S)\gamma$	0.745(15)	0.670	0.873	0.690
$\psi(2S) \to \eta_c(2S)\gamma$	0.713(14)	0.884	0.739	0.680
$\psi(3S) \to \eta_c(3S)\gamma$	0.688(12)	…	…	…
$\psi(2S) \to \eta_c(1S)\gamma$	−0.0605(37)	−0.040	−0.144	−0.056
$\psi(3S) \to \eta_c(2S)\gamma$	−0.0595(33)	…	…	…
$\psi(3S) \to \eta_c(1S)\gamma$	−0.0108(4)	…	…	…
$\eta_c(2S) \to J/\psi\gamma$	−0.0605(37)	…	−0.022	…
$\eta_c(3S) \to \psi(2S)\gamma$	−0.0595(33)	…	…	…
$\eta_c(3S) \to J/\psi\gamma$	−0.0108(4)	…	…	…

(allowed)>>(hindered)

Summary

- Hadron spectra & structure
  - Quark model & light-front dynamics

- Discussions
  - Realistic LFWF with GEM: consistent with perturbative QCD
  - Self-consistency with different polarizations
  - Prediction has reasonable agreement with the data

- Outlook
  - Further works on the obervables and self-consistency
  - Extension to baryon and beyond

Thank you for your attention!

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