Investigation of a three - body system with Dunkl operator and considering some new transitions

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Introduction to Dunkl operator
Deformed functions
Three body systems

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Tow particles in an infinite well: Preamble

Classical Particles study by **Maxwell-Boltzmann** and Wavefunction of two particles in a one dimensional box is:

$$\Psi_d(x_1, x_2) = \Psi_{n_1}(x_1)\Psi_{n_2}(x_2)$$
$$\Psi_{n_1}(x_1)\Psi_{n_2}(x_2) \neq \Psi_{n_1}(x_2)\Psi_{n_2}(x_1)$$

Bosons (with integer spin) study by **Bos-Einstein** and Wavefunction of two particles in a one dimensional box is:

$$\Psi_b = \frac{1}{\sqrt{2}}\{\Psi_{n_1}(x_1)\Psi_{n_2}(x_2) + \Psi_{n_1}(x_2)\Psi_{n_2}(x_1)\}$$

Fermions (with half-integer spin) study by **Fermi-Dirac** and Wavefunction of two particles in a one dimensional box is:

$$\Psi_f = \frac{1}{\sqrt{2}}\{\Psi_{n_1}(x_1)\Psi_{n_2}(x_2) - \Psi_{n_1}(x_2)\Psi_{n_2}(x_1)\}$$

Tow particles in an infinite well: Preamble

We new consider a quantum particle in the potential well.

$$V(x) = \begin{cases} 0 & if\ |x| < a \\ \infty & if\ |x| \geq a \end{cases}$$

And the **Hamiltonian** $\widehat{H}$:

$$\widehat{H} = \frac{p^2}{2\,m} + V(x)$$

And has explicit symmetry with respect **space reflection**

$$[\widehat{H}, \widehat{R}] = 0$$

The eigenvalue are ± 1

$$\begin{cases} \Psi_n^+(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{n\,\pi}{2\,a} x\right), & E_n = Kn^2 & with\ odd\ ns \\ \Psi_n^-(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{n\,\pi}{2\,a} x\right), & E_n = Kn^2 & with\ even\ ns \end{cases}$$

Eigenvalue and degeneracy

$$\left(K = \frac{\pi^2 \hbar^2}{2mL^2}\right)$$

Classical Particles

$gs : n_1, n_2 = 1 \rightarrow E_{11} = 2k \qquad 1$

$fes : (1{,}2), (2{,}1) \rightarrow E_{12} = 5k \qquad 2$

$ses : (2{,}2) \quad \rightarrow \quad E_{22} = 8k \qquad 1$

Bosons

$n_1, n_2 = 1 \rightarrow E_{11} = 2k \qquad 1$

$(1{,}2), (2{,}1) \rightarrow E_{12} = 5k \qquad 1$

the same function

$(2{,}2) \quad \rightarrow \quad E_{22} = 8k \qquad 1$

Fermions

$n_1, n_2 = 1 \rightarrow E_{11} \cancel{= 2k} \qquad 1$

$(1{,}2), (2{,}1) \rightarrow E_{12} = 5k \qquad 1$

$(2{,}2) \quad \rightarrow \quad \cancel{E_{22} = 8k} \qquad 1$

$\psi_{12} \neq \psi_{21} \rightarrow$ phase different

$\psi_{11} \neq \psi_{22} = 0$ (PEP)

Time reversal

- From the **experimental** view:
  - Time reversal makes the transformation $t \to -t$

  - for the reaction: $a+b \to c+d$ the total cross section is σ_1
  - for the reaction: $c+d \to a+b$ the total cross section is σ_2

- **If the σ_1/σ_2 goes to 1, then this reaction is reversible and the time reversal is valid for it.**
- The operation of time reversal changes the sign of momentum p and of the direction of the total angular momentum J:

$$P \xrightarrow{T} P' = -P \qquad J \xrightarrow{T} J' = -J$$

Space inversion operator:

We suppose that the eigenstates of coordinate and **momentum operators** are: $|\Psi_x\rangle$ and $|\phi_x\rangle$

$$R|\Psi_{\pm x}\rangle = |\Psi_{\mp x}\rangle$$
$$R|\phi_{\pm p}\rangle = |\phi_{\mp p}\rangle$$

Using the **normalization properties** of coordinate and momentum eigenstates:

$$\langle\Psi_{\pm x'}|\Psi_{\pm x}\rangle = \delta(x - x')$$
$$\langle\phi_{\pm p'}|\phi_{\pm p}\rangle = \delta(p - p')$$

and the unitary condition for operators R:

$$R^T R = I$$

The **parity operator** is an Hermition operator and equal to its inverse:

$$R^T = R = R^{-1}$$

Hermitivity of the operator indicates that the eigenvalue of the operator should be real numbers.

$$R|\Psi_{\pm x}\rangle = \pm|\Psi_{\pm x}\rangle$$

States referred to as positive and negative parity states

Ans also:

$$[\hat{x}, \hat{R}] = 2\hat{x}\hat{R}$$
$$[\hat{p}, \hat{R}] = 2\hat{p}\hat{R}$$

Dunkl operator

- From the historical view:

Model	**Name**	**Year**	**Deformed Parameter**	**Refs**
$[\mathrm{x}, p]_v = \mathrm{i}(1 + 2\mathrm{\nu} P_x)$	Wigner	1950	ν	[1]
$P_c = \frac{\hbar}{i}(\partial_x - \frac{c}{2x} R_x)$	Yang	1951	c	[2]
$P_\mu = \frac{\hbar}{i}\left(\partial_x - \frac{\mu}{x}(1 - R_x)\right)$	Dunkl	1989	μ	[3]
$P_{\alpha,\beta,\gamma} = \frac{\hbar}{i}\left(\partial_x + \frac{\alpha}{x} + \frac{\beta}{x} R_x + \gamma \partial_x R_x\right)$	Generalized Dunkl	2021	α, β, γ	[4]

[1] E. P. Wigner, Phys. Rev., 77 (1950) 711. [2] L. M.Yang , Phys. Rev., 84 (1951) 788. [3] C. F. Dunkl , Trans. Amer. Math. Soc., 311 (1989) 167.
[4]W. S. Chung and H. Hassanabadi, Eur. Phys. J. Plus, 136 (2021) 239.

Dunkl operator

- Four **Dunkl-momentum** operator:

$$P_\mu = \frac{\hbar}{i}\partial_\mu = \frac{\hbar}{i}\left(\partial_\mu - \frac{\nu}{x_\mu}(1 - R_\mu)\right)$$

R_0 is **time reversal** operator.

- $R_x, R_y and\ R_z$ are called a reflection operators $R_x\, f(x) = f\ (-x)$
- The **Wigner parameter** $\mathcal{V}$ should be $\nu > -1/2$
- For the **even** function: $R_x\, f_e\,(x) = f_e\,(x)$
- For the **odd** function: $R_x\, f_o\,(x) = -\ f_o\,(x)$

◉Dunkl derivative - one dimension :

- The **one** of the Dunkl derivative:

$$D_x^\nu = \frac{d}{dx} + \frac{\nu}{x_\mu} = \frac{\hbar}{i}(\,1 - R_\mu)$$

- The **square** of the Dunkl derivative:

$$(D_x^\nu)^2 = \frac{d^2}{dx^2} + \frac{2\nu}{x_\mu}\frac{d}{dx} - \frac{\nu}{x_\mu^2}(\,1 - R_\mu)$$

- Then the **Heisenberg relation** is deformed as

$$[\mathrm{x}, p\,]_\nu = \mathrm{i}(1 + 2\nu P_x)$$

Dunkl operator

The Dunkl derivative is linear because

$$D_x\big(af(x)+bg(x)\big)=aD_xf(x)+bD_xg(x)$$

Acting the Dunkl derivative on the **monomial** gives

$$D_xx^n=[n]_\nu\, x^{n-1}$$

where ν-deformed number is defined by

$$[n]_\nu= n+\nu(1-(-1)^n)$$

Generally we have

$$[2k]_\nu= 2k, \qquad [2k+1]_\nu= 2k+1+2\nu \qquad (k=0,1,2,\dots)$$

The Dunkl derivative satisfies the following **Leibniz rule**

$$D_x\big(f(x)g(x)\big)=(D_xf(x))g(x)+f(x)D_xg(x)-\frac{\nu}{x}[(1-P)f(x)][(1-P)g(x)]$$

In general, Dunkl derivative **dose not obey chain rule** because

$$D_xf(u(x))=\frac{df}{du}\frac{du}{dx}+\frac{\nu}{x}[f(u(x))-f(u(-x))]$$

$\boldsymbol{v}$ -deformed function

we want to find the **$\boldsymbol{v}$ -exponential** function obeying

$$D_x e_v(ax) = a\, e_v(ax), \qquad e_v(0)=1$$

We consider the **$\boldsymbol{v}$ -deformed differential** equation

$$D_x y(x) = a\, y(x), \qquad y(0)=1$$

Let us set

Even Function

odd Function

$$y(x) = y_e(x) + y_o(x)$$

Considering these equations for the **even** and **odd** part, we have:

$$\frac{dy_o(x)}{dx} = a y_o(x)$$

$$\frac{dy_o(x)}{dx} + \frac{2\text{v}}{x} y_o(x) = a y_e(x)$$

Let us set

$$y_e(x) = \sum_{n=0}^{\infty} a_n x^{2n}$$

$$y_o(x) = \sum_{n=0}^{\infty} b_n x^{2n+1}$$

By using above equations:

$$2(n+1)a_{n+1} = ab_n$$

$$(2n+1+2\text{v})b_n = aa_n$$

thus, we have

$$a_n = \frac{1}{n!\,(\text{v}+\frac{1}{2})_n}\left(\frac{a}{2}\right)^{2n}$$

$$b_n = \frac{1}{n!\,(\text{v}+\frac{1}{2})_{n+1}}\left(\frac{a}{2}\right)^{2n+1}$$

thus, we have

$$y(x) = e_\text{v}(ax) = \cos h_\text{v}(ax) + \sin h_\text{v}(ax)$$

- ν-deformed **Exponential** function:

$$e_\nu(ax) = \cosh_\nu(ax) + \sinh_\nu(ax) \quad , \quad e_\nu(iax) = \cos_\nu(ax) + i\sin_\nu(ax),$$

$$\cosh_\nu(ax) = \sum_{n=0}^{\infty} \frac{a_0}{n!(\nu+\frac{1}{2})_n}(\frac{ax}{2})^{2n} \quad , \quad \cos_\nu(ax) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\nu+\frac{1}{2})_n}(\frac{ax}{2})^{2n},$$

$$\sinh_\nu(ax) = \sum_{n=0}^{\infty} \frac{a_0}{n!(\nu+\frac{1}{2})_{n+1}}(\frac{ax}{2})^{2n+1} \quad , \quad \sin_\nu(ax) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\nu+\frac{1}{2})_{n+1}}(\frac{ax}{2})^{2n+1}.$$

- ν-deformed **Hermite polynomial**:

$$H^\nu_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k}[2n]_\nu!}{[2k]_\nu!(n-k)!}(2x)^{2k},$$

$$\begin{aligned}
H^\nu_0 &= 1,\\
H^\nu_2 &= 4x^2 - 2(1+2\nu),\\
H^\nu_4 &= 16x^4 - 16(3+2\nu)x^2 + 4(3+2\nu)(1+2\nu),\\
H^\nu_6 &= 64x^6 - 96(5+2\nu)x^4 + 48(5+2\nu)(3+2\nu)x^2 - 8(5+2\nu)(3+2\nu)(1+2\nu),
\end{aligned}$$

$$H^\nu_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k}[2n+1]_\nu!}{[2k+1]_\nu!(n-k)!}(2x)^{2k+1},$$

$$\begin{aligned}
H^\nu_1 &= 2x,\\
H^\nu_3 &= 8x^3 - 4(3+2\nu)x,\\
H^\nu_5 &= 32x^5 - 32(5+2\nu)x^3 + 8(5+2\nu)(3+2\nu)x,\\
H^\nu_7 &= 128x^7 - 192(7+2\nu)x^5 + 96(7+2\nu)(5+2\nu)x^3 - 16(7+2\nu)(5+2\nu)(3+2\nu)x\,.
\end{aligned}$$

Dunkl operator

- ν -deformed **Generating** function:

$$g_\nu(p,t)=\sum_{n=0}^{\infty}\frac{1}{[n]_\nu!}H_n^\nu(p)t^n=e^{-t^2}e_\nu^{2pt},$$

$$g_\nu^e(x,t)=\sum_{n=0}^{\infty}\frac{1}{[2n]_\nu!}H_{2n}^\nu(p)t^{2n}=e^{-t^2}\cosh_\nu(2pt),$$

$$g_o(x,t)=\sum_{n=0}^{\infty}\frac{1}{[2n+1]_\nu!}H_{2n+1}^\nu(p)t^{2n+1}=e^{-t^2}\sinh_\nu(2pt)\,.$$

- ν -deformed **Fourier** transform:

$$u(p)=\int_{-\infty}^{\infty}dx\,|x|^{2\nu}\psi(x)K_\nu(p,x),$$

$$K_\nu(p,x)=\frac{1}{\sqrt{2\pi}}|px|^{-\nu}e^{-ipx},$$

$$\mathscr{F}_\nu(\psi(x))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\,|x|^{2\nu}|px|^{-\nu}e^{-ipx}\psi(x)=u(p),$$

$$\mathscr{F}_\nu^{-1}(u(p))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dp\,|p|^{2\nu}|px|^{-\nu}e^{ipx}u(p)=\psi(x)\,.$$

- ν -deformed **Bessel** functions:

$$J_{\nu-\frac{1}{2}}(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^{2m+\nu-\frac{1}{2}},$$

$$J_{\nu+\frac{1}{2}}(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+\frac{3}{2})}\left(\frac{x}{2}\right)^{2m+\nu+\frac{1}{2}}$$

Particle in a well potential by Dunkl operator

- For a particle in a box:

$$V(x) = \begin{cases} 0 & (-L < x < L) \\ \infty & \text{elsewhere} \end{cases}$$

- The **Dunkl-Schrödinger** equation is:

$$-\frac{1}{2m}D_x^2\psi = E\psi \quad \text{or} \quad -\frac{1}{2m}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x - \frac{\nu}{x^2}(1-R)\right)\psi = E\psi$$

- For the **even parity** solution by ψ_+ we get

$$-\frac{1}{2m}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x\right)\psi_+ = E_+\psi_+$$

$$\psi = \psi_+ + \psi_-$$

- For the **odd parity** solution by ψ_- we get

$$-\frac{1}{2m}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x - \frac{2\nu}{x^2}\right)\psi_- = E_-\psi_-$$

- For the even parity solution, If we set

$$\psi_+^\lambda = \sum_{n=0}^{\infty} a_n^+ x^{2n}|x|^\lambda$$

- with insert it into the Schrödinger equation, we have the recurrence relation:

$$a_{n+1}^+ = -\frac{2mE_+}{(2n+2+\lambda)(2n+1+\lambda+2\nu)}a_n^+$$

Particle in a well by Dunkl operator

- with a characteristic equation:

$$\lambda(\lambda - 1 + 2\nu) = 0$$

- Thus we obtain the wave function and energy for an even parity solution as:

$$\psi_+ = N_+ x^{\frac{1}{2}-2\nu} J_{\nu-\frac{1}{2}}(\sqrt{2mE_+}\,x) \qquad E_n^+ = \frac{1}{2mL^2} a^2_{\nu-\frac{1}{2},n} \qquad n = 1,2,\ldots$$

- For the odd parity solution, If we set

$$\psi_-^\lambda = \sum_{n=0}^{\infty} a_n^- x^{2n+1} |x|^\lambda$$

- with insert it into the Schrödinger equation, we have the recurrence relation:

$$a_{n+1}^- = -\frac{2mE_-}{(2n+2+\lambda)(2n+3+\lambda+2\nu)} a_n^-$$

- with a characteristic equation:

$$\lambda(\lambda + 1 + 2\nu) = 0$$

- Thus we obtain the wave function and energy for an odd parity solution as:

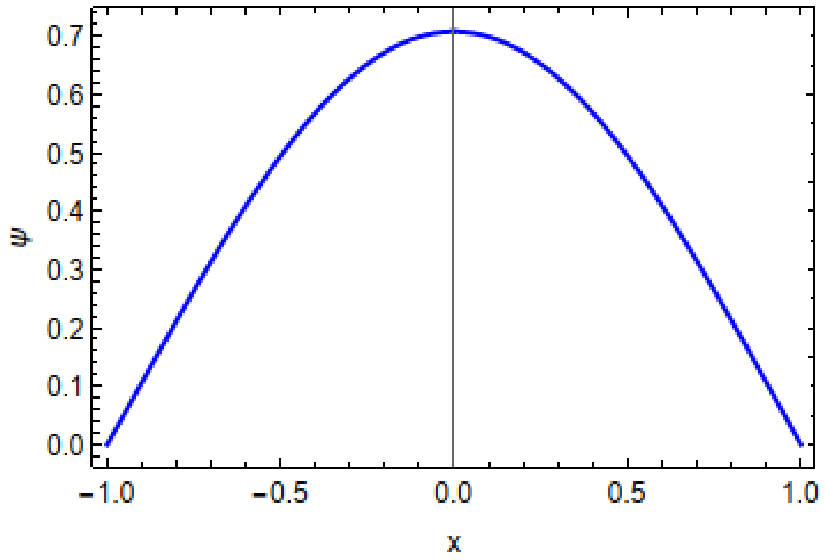
$$\psi_- = N_- x^{\frac{1}{2}-\nu} J_{\nu+\frac{1}{2}}(\sqrt{2mE_-}\,x) \qquad E_n^- = \frac{1}{2mL^2} a^2_{\nu+\frac{1}{2}n} \qquad n = 1,2,\ldots$$

Particle in a well by Dunkl operator

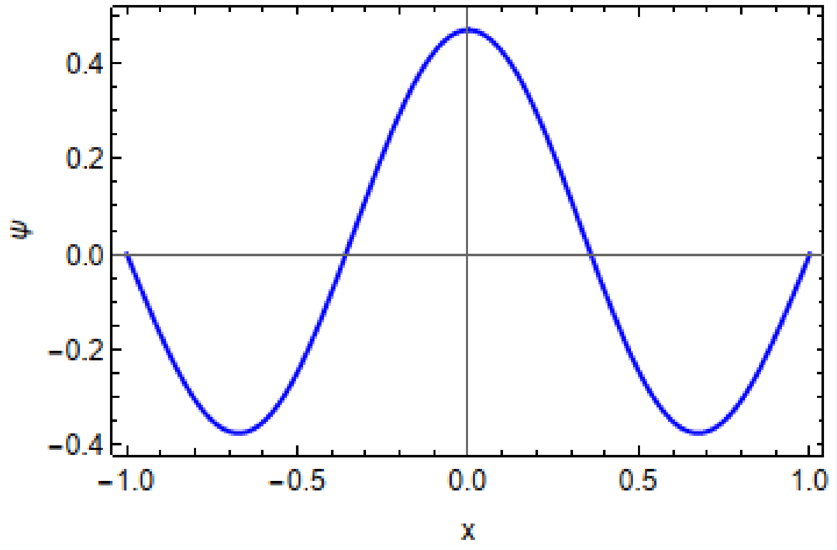
$$\Psi_+ = N_+ x^{\frac{1}{2}-\nu} J_{\nu-\frac{1}{2}}(\sqrt{2mE_+}x)$$

$$E_n^+ = \frac{1}{2mL^2}\alpha^2_{\nu-1/2,n}$$

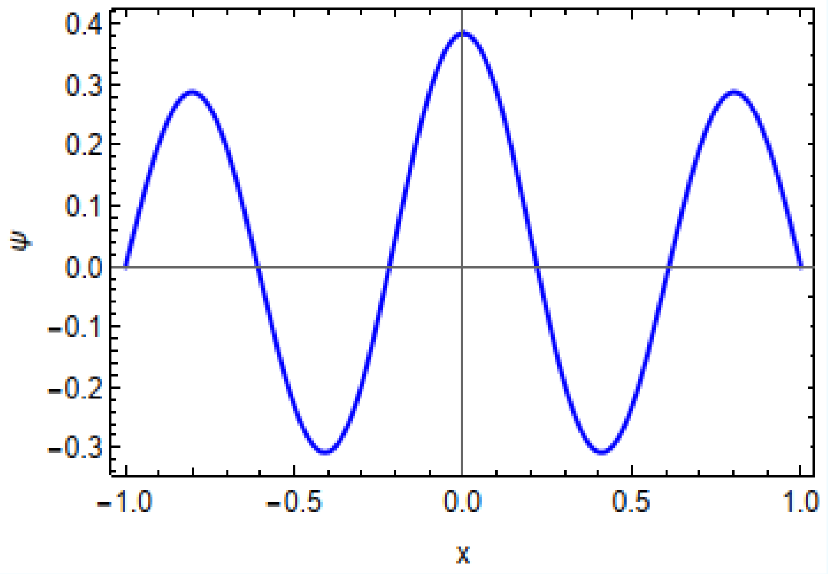
$n = 1$



$n = 2$



$n = 3$

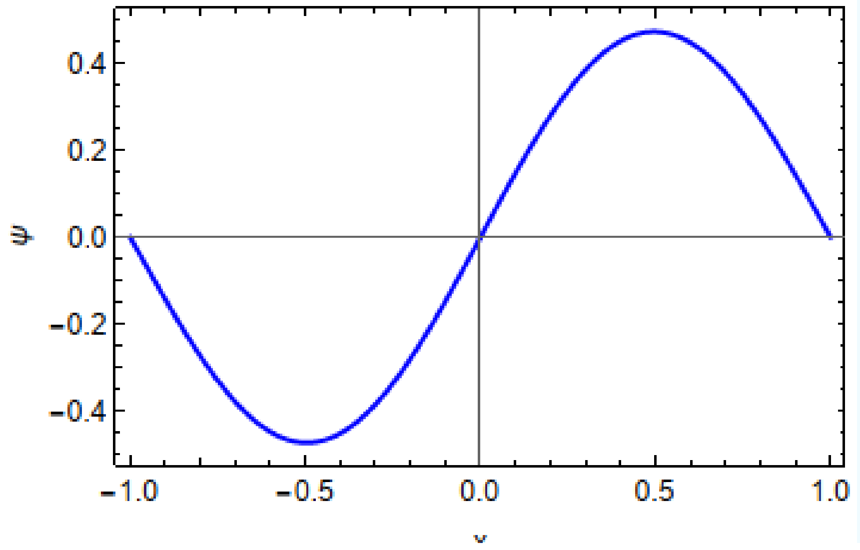


$\nu = 0.1$
$m = 1$
$N = 1$

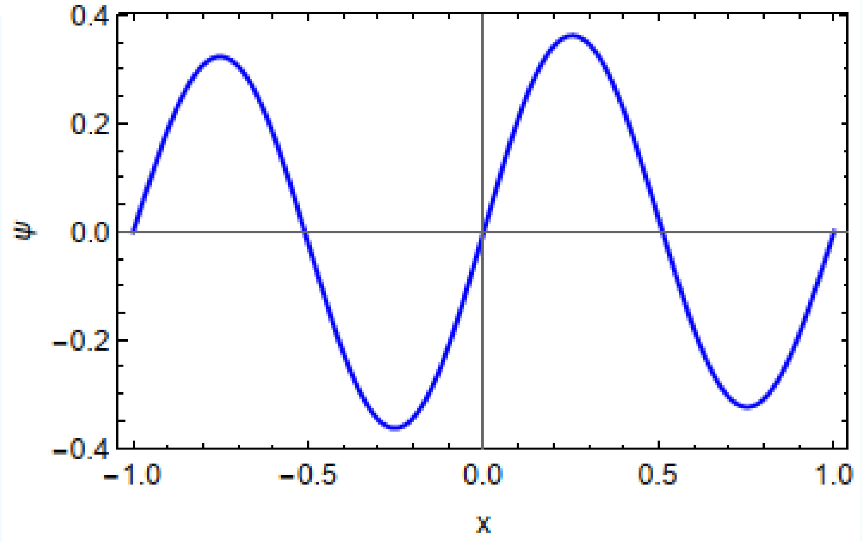
$$\Psi_- = N_- x^{\frac{1}{2}-\nu} J_{\nu+\frac{1}{2}}(\sqrt{2mE_-}x)$$

$$E_n^- = \frac{1}{2mL^2}\alpha^2_{\nu+1/2,n}$$

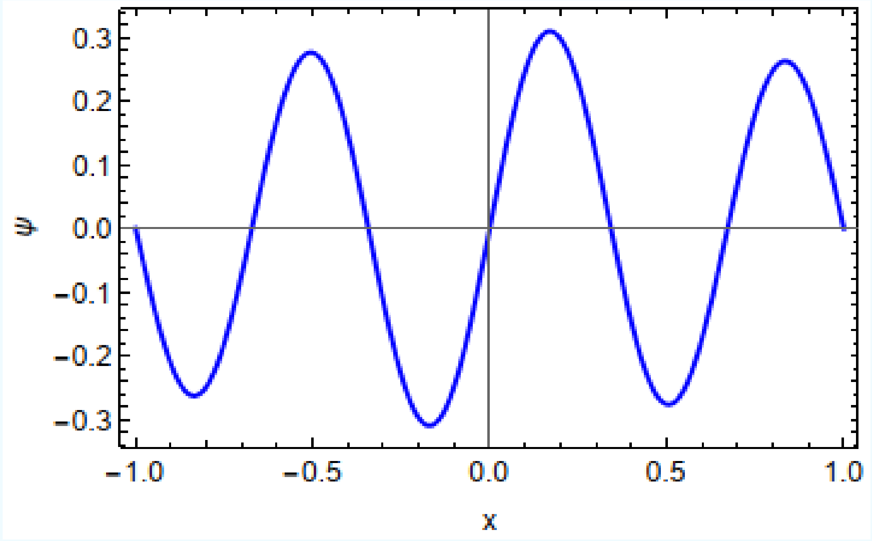
$n = 1$



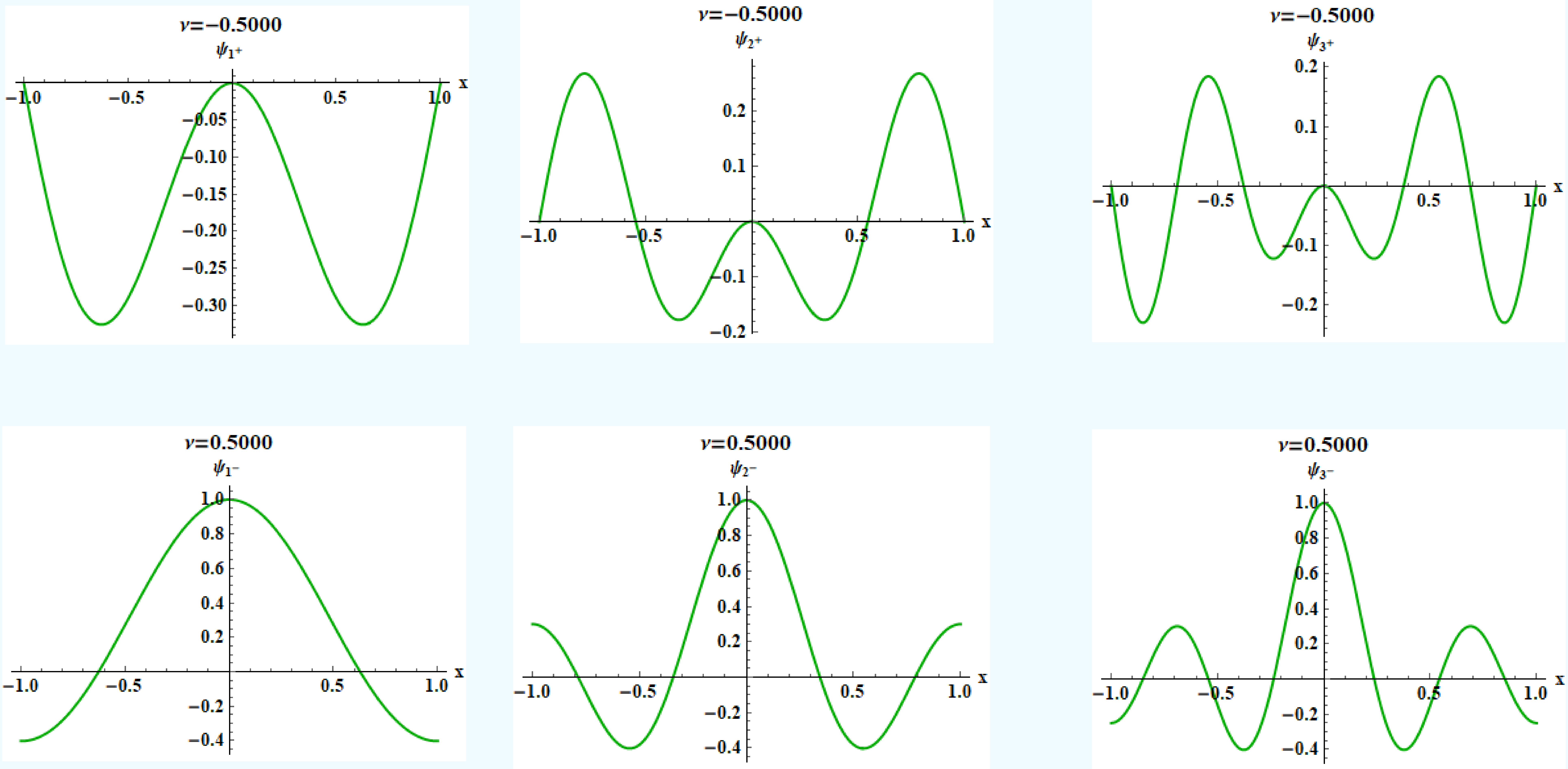
$n = 2$



$n = 3$



States of three particles



Particle in a well by Dunkl operator

Properties of the energy

Here we compare the transition and the parity and the energy of the emitted particle:

$$\begin{cases} E_n^+ = k\left(\alpha_{v-\frac{1}{2},n}\right)^2 \\ E_n^- = k\left(\alpha_{v+\frac{1}{2},n}\right)^2 \end{cases}$$

Where α indicate the **zeros of the Bessel function**. also

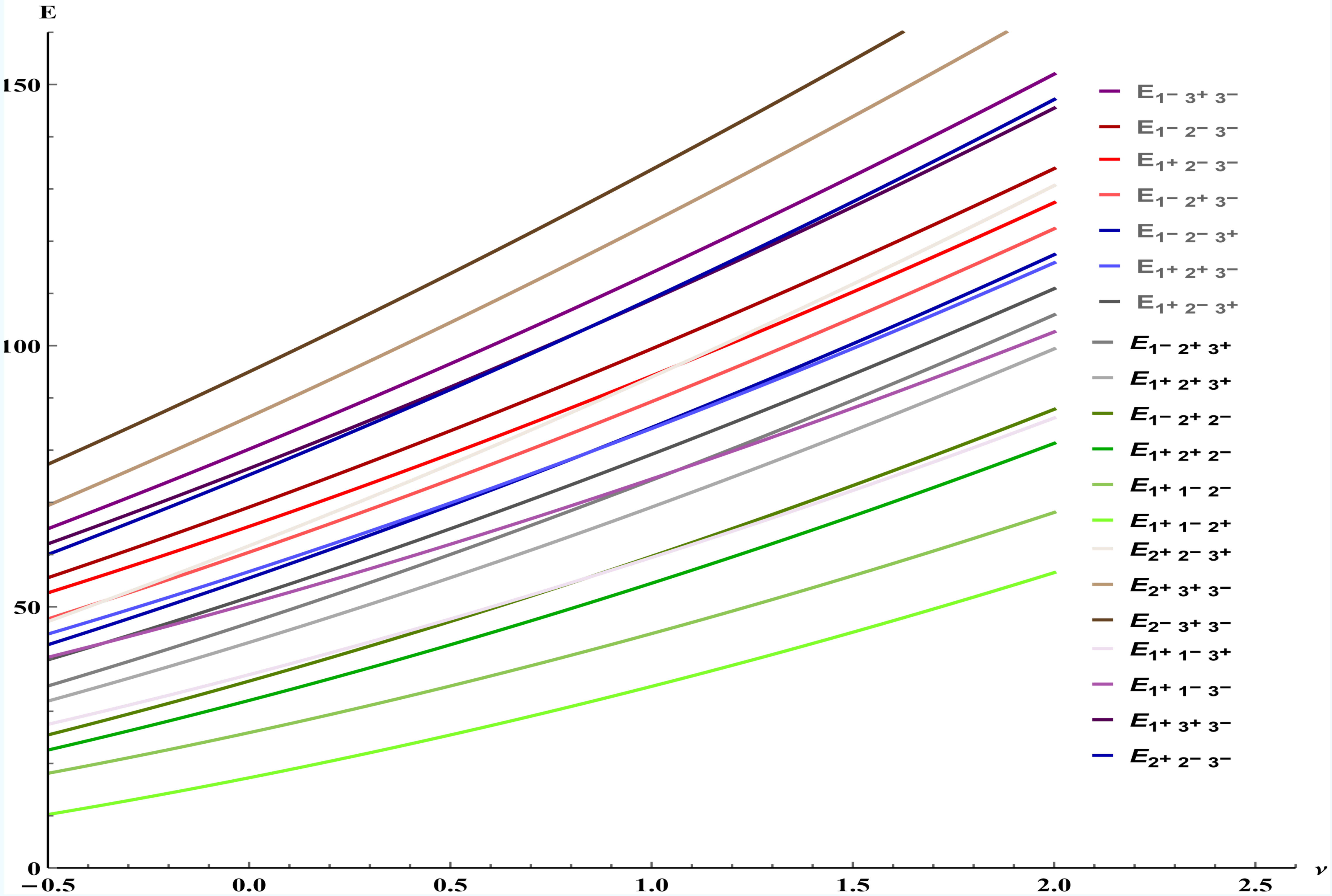
$$\alpha_{v-\frac{1}{2},n} < \alpha_{v+\frac{1}{2},n} \quad \Rightarrow \quad E_n^+ < E_n^-$$

$$\alpha_{v+\frac{1}{2},n} < \alpha_{v-\frac{1}{2},n+1} \quad \Rightarrow \quad E_n^- < E_{n+1}^+$$

for $v=\frac{1}{2}$

$\alpha_{0,1} = 2.40$
$\alpha_{0,2} = 5.52$
$\alpha_{0,3} = 8.65$
$\alpha_{1,1} = 3.83$
$\alpha_{1,2} = 7.01$
$\alpha_{1,3} = 10.17$

Energy of three particles states



Particle in a well by Dunkl operator

$E_{n_1^{\pm}n_2^{\pm}n_3^{\pm}}$	Just considering n=1 and 2
E^{+-+}_{112}	$\frac{1}{10}((2.40)^2 + (3.83)^2 + (5.52)^2) = 5.09$
E^{+--}_{112}	$\frac{1}{10}((2.40)^2 + (3.83)^2 + (7.01)^2) = 6.95$
E^{++-}_{122}	$\frac{1}{10}((2.40)^2 + (5.52)^2 + (7.01)^2) = 8.53$
E^{-+-}_{122}	$\frac{1}{10}((3.83)^2 + (5.52)^2 + (7.01)^2) = 9.42$

$E_{n_1^{\pm}n_2^{\pm}n_3^{\pm}}$	Just considering n=1,2 and 3
E^{+-+}_{113}	$\frac{1}{10}((2.40)^2 + (3.83)^2 + (8.65)^2) = 9.52$
E^{+++}_{123}	$\frac{1}{10}((2.40)^2 + (5.52)^2 + (8.65)^2)=11.10$
E^{-++}_{123}	$\frac{1}{10}((3.83)^2 + (5.52)^2 + (8.65)^2)=11.99$
E^{+--}_{113}	$\frac{1}{10}((2.40)^2 + (3.83)^2 + (10.17)^2) = 12.38$
E^{+-+}_{123}	$\frac{1}{10}((2.40)^2 + (7.01)^2 + (8.65)^2)=12.97$
E^{--+}_{123}	$\frac{1}{10}((3.83)^2 + (7.01)^2 + (8.65)^2) = 13.86$
E^{++-}_{123}	$\frac{1}{10}((2.40)^2 + (5.52)^2 + (10.17)^2)=13.96$
E^{-+-}_{123}	$\frac{1}{10}((3.83)^2 + (5.52)^2 + (10.17)^2)=14.85$
E^{+-+}_{223}	$\frac{1}{10}((5.52)^2 + (7.01)^2 + (8.65)^2)=15.52$
E^{+--}_{123}	$\frac{1}{10}((2.40)^2 + (7.01)^2 + (10.17)^2)=15.83$
E^{---}_{123}	$\frac{1}{10}((3.83)^2 + (7.01)^2 + (10.17)^2)=16.72$
E^{+--}_{223}	$\frac{1}{10}((5.52)^2 + (7.01)^2 + (10.17)^2) = 18.30$
E^{++-}_{133}	$\frac{1}{10}((2.40)^2 + (8.65)^2 + (10.17)^2) = 18.40$
E^{-+-}_{133}	$\frac{1}{10}((3.83)^2 + (8.65)^2 + (10.17)^2)=19.29$
E^{++-}_{233}	$\frac{1}{10}((5.52)^2 + (8.65)^2 + (10.17)^2)=20.87$
E^{-+-}_{233}	$\frac{1}{10}((7.01)^2 + (8.65)^2 + (10.17)^2)=22.73$

Particle in a well by Dunkl operator

Three particles with two levels n=1 and 2 all the antisymmetric states: **Nothing**

Transition: **Nothing**

Three particles with three levels n=1,2 and 3 all the antisymmetric states: **One State**

$$\psi_{n_1 n_2 n_3}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{n_1}(1)\left[\psi_{n_2}(2)\psi_{n_3}(3)\right]-\psi_{n_2}(1)\left[\psi_{n_1}(2)\psi_{n_3}(3)-\psi_{n_3}(2)\psi_{n_1}(3)\right]+\psi_{n_3}(1)\left[\psi_{n_1}(2)\psi_{n_2}(3)-\psi_{n_2}(2)\psi_{n_1}(3)\right]\}$$

Transition: **Nothing**

But in the Dunkl formalism for the same quantum numbers (1, 2 and 3), we have the following antisymmetric states:

States of three particles

All the antisymmetric cases for three particle for n=1,2 and 3:

$$\psi_{1^+1^-2^+}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^+}(1)[\psi_{1^-}(2)\psi_{2^+}(3)]-\psi_{1^-}(1)[\psi_{1^+}(2)\psi_{2^+}(3)-\psi_{2^+}(2)\psi_{1^+}(3)]+\psi_{2^+}(1)[\psi_{1^+}(2)\psi_{1^-}(3)-\psi_{1^-}(2)\psi_{1^+}(3)]\}$$

$$\psi_{1^+1^-2^-}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^+}(1)[\psi_{1^-}(2)\psi_{2^-}(3)]-\psi_{1^-}(1)[\psi_{1^+}(2)\psi_{2^-}(3)-\psi_{2^-}(2)\psi_{1^+}(3)]+\psi_{2^-}(1)[\psi_{1^+}(2)\psi_{1^-}(3)-\psi_{1^-}(2)\psi_{1^+}(3)]\}$$

$$\psi_{1^+2^+2^-}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^+}(1)[\psi_{2^+}(2)\psi_{2^-}(3)]-\psi_{2^+}(1)[\psi_{1^+}(2)\psi_{2^-}(3)-\psi_{2^-}(2)\psi_{1^+}(3)]+\psi_{2^-}(1)[\psi_{1^+}(2)\psi_{2^+}(3)-\psi_{2^+}(2)\psi_{1^+}(3)]\}$$

$$\psi_{1^-2^+2^-}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^-}(1)[\psi_{2^+}(2)\psi_{2^-}(3)]-\psi_{2^+}(1)[\psi_{1^-}(2)\psi_{2^-}(3)-\psi_{2^-}(2)\psi_{1^-}(3)]+\psi_{2^-}(1)[\psi_{1^-}(2)\psi_{2^+}(3)-\psi_{2^+}(2)\psi_{1^-}(3)]\}$$

$$\psi_{1^+2^+3^+}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^+}(1)[\psi_{2^+}(2)\psi_{3^+}(3)]-\psi_{2^+}(1)[\psi_{1^+}(2)\psi_{3^+}(3)-\psi_{3^+}(2)\psi_{1^+}(3)]+\psi_{3^+}(1)[\psi_{1^+}(2)\psi_{2^+}(3)-\psi_{2^+}(2)\psi_{1^+}(3)]\}$$

$$\psi_{1^-2^+3^+}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^-}(1)[\psi_{2^+}(2)\psi_{3^+}(3)]-\psi_{2^+}(1)[\psi_{1^-}(2)\psi_{3^+}(3)-\psi_{3^+}(2)\psi_{1^-}(3)]+\psi_{3^+}(1)[\psi_{1^-}(2)\psi_{2^+}(3)-\psi_{2^+}(2)\psi_{1^-}(3)]\}$$

$$\psi_{1^+2^-3^+}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^+}(1)[\psi_{2^-}(2)\psi_{3^+}(3)]-\psi_{2^-}(1)[\psi_{1^+}(2)\psi_{3^+}(3)-\psi_{3^+}(2)\psi_{1^+}(3)]+\psi_{3^+}(1)[\psi_{1^+}(2)\psi_{2^-}(3)-\psi_{2^-}(2)\psi_{1^+}(3)]\}$$

$$\psi_{1^+2^+3^-}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^+}(1)[\psi_{2^+}(2)\psi_{3^-}(3)]-\psi_{2^+}(1)[\psi_{1^+}(2)\psi_{3^-}(3)-\psi_{3^-}(2)\psi_{1^+}(3)]+\psi_{3^-}(1)[\psi_{1^+}(2)\psi_{2^+}(3)-\psi_{2^+}(2)\psi_{1^+}(3)]\}$$

$$\psi_{1^-2^-3^+}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^-}(1)[\psi_{2^-}(2)\psi_{3^+}(3)]-\psi_{2^-}(1)[\psi_{1^-}(2)\psi_{3^+}(3)-\psi_{3^+}(2)\psi_{1^-}(3)]+\psi_{3^+}(1)[\psi_{1^-}(2)\psi_{2^-}(3)-\psi_{2^-}(2)\psi_{1^-}(3)]\}$$

$$\psi_{1^-2^+3^-}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^-}(1)[\psi_{2^+}(2)\psi_{3^-}(3)]-\psi_{2^+}(1)[\psi_{1^-}(2)\psi_{3^-}(3)-\psi_{3^-}(2)\psi_{1^-}(3)]+\psi_{3^-}(1)[\psi_{1^-}(2)\psi_{2^+}(3)-\psi_{2^+}(2)\psi_{1^-}(3)]\}$$

$$\psi_{1^+2^-3^-}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^+}(1)[\psi_{2^-}(2)\psi_{3^-}(3)]-\psi_{2^-}(1)[\psi_{1^+}(2)\psi_{3^-}(3)-\psi_{3^-}(2)\psi_{1^+}(3)]+\psi_{3^-}(1)[\psi_{1^+}(2)\psi_{2^-}(3)-\psi_{2^-}(2)\psi_{1^+}(3)]\}$$

$$\psi_{1^-2^-3^-}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^-}(1)[\psi_{2^-}(2)\psi_{3^-}(3)]-\psi_{2^-}(1)[\psi_{1^-}(2)\psi_{3^-}(3)-\psi_{3^-}(2)\psi_{1^-}(3)]+\psi_{3^-}(1)[\psi_{1^-}(2)\psi_{2^-}(3)-\psi_{2^-}(2)\psi_{1^-}(3)]\}$$

$$\psi_{1^-3^+3^-}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^-}(1)[\psi_{3^+}(2)\psi_{3^-}(3)]-\psi_{3^+}(1)[\psi_{1^-}(2)\psi_{3^-}(3)-\psi_{3^-}(2)\psi_{1^-}(3)]+\psi_{3^-}(1)[\psi_{1^-}(2)\psi_{3^+}(3)-\psi_{3^+}(2)\psi_{1^-}(3)]\}$$

States of three particles

All the antisymmetric cases for three particle for n=1,2 and 3:

$$\psi_{1^{+}1^{-}3^{+}}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^{+}}(1)[\psi_{1^{-}}(2)\psi_{3^{+}}(3)]-\psi_{1^{-}}(1)[\psi_{1^{+}}(2)\psi_{3^{+}}(3)-\psi_{3^{+}}(2)\psi_{1^{+}}(3)]+\psi_{3^{+}}(1)[\psi_{1^{+}}(2)\psi_{1^{-}}(3)-\psi_{1^{-}}(2)\psi_{1^{+}}(3)]\}$$

$$\psi_{1^{+}1^{-}3^{-}}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^{+}}(1)[\psi_{1^{-}}(2)\psi_{3^{-}}(3)]-\psi_{1^{-}}(1)[\psi_{1^{+}}(2)\psi_{3^{-}}(3)-\psi_{3^{-}}(2)\psi_{1^{+}}(3)]+\psi_{3^{-}}(1)[\psi_{1^{+}}(2)\psi_{1^{-}}(3)-\psi_{1^{-}}(2)\psi_{1^{+}}(3)]\}$$

$$\psi_{1^{+}3^{+}3^{-}}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{1^{+}}(1)[\psi_{3^{+}}(2)\psi_{3^{-}}(3)]-\psi_{3^{+}}(1)[\psi_{1^{+}}(2)\psi_{3^{-}}(3)-\psi_{3^{-}}(2)\psi_{1^{+}}(3)]+\psi_{3^{-}}(1)[\psi_{1^{+}}(2)\psi_{3^{+}}(3)-\psi_{3^{+}}(2)\psi_{1^{+}}(3)]\}$$

$$\psi_{2^{+}2^{-}3^{+}}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{2^{+}}(1)[\psi_{2^{-}}(2)\psi_{3^{-}}(3)]-\psi_{2^{-}}[\psi_{2^{+}}(2)\psi_{3^{-}}(3)-\psi_{3^{-}}(2)\psi_{2^{+}}(3)]+\psi_{3^{-}}(1)[\psi_{2^{+}}(2)\psi_{2^{-}}(3)-\psi_{2^{-}}(2)\psi_{2^{+}}(3)]\}$$

$$\psi_{2^{+}2^{-}3^{+}}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{2^{+}}(1)[\psi_{2^{-}}(2)\psi_{3^{+}}(3)]-\psi_{2^{-}}[\psi_{2^{+}}(2)\psi_{3^{+}}(3)-\psi_{3^{+}}(2)\psi_{2^{+}}(3)]+\psi_{3^{+}}(1)[\psi_{2^{+}}(2)\psi_{2^{-}}(3)-\psi_{2^{-}}(2)\psi_{2^{+}}(3)]\}$$

$$\psi_{2^{+}3^{+}3^{-}}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{2^{+}}(1)[\psi_{3^{+}}(2)\psi_{3^{-}}(3)]-\psi_{3^{+}}[\psi_{2^{+}}(2)\psi_{3^{-}}(3)-\psi_{3^{-}}(2)\psi_{2^{+}}(3)]+\psi_{3^{-}}(1)[\psi_{2^{+}}(2)\psi_{3^{+}}(3)-\psi_{3^{+}}(2)\psi_{2^{+}}(3)]\}$$

$$\psi_{2^{-}3^{+}3^{-}}(1,2,3)=\frac{1}{\sqrt{3!}}\{\psi_{2^{-}}(1)[\psi_{3^{+}}(2)\psi_{3^{-}}(3)]-\psi_{3^{+}}[\psi_{2^{-}}(2)\psi_{3^{-}}(3)-\psi_{3^{-}}(2)\psi_{2^{-}}(3)]+\psi_{3^{-}}(1)[\psi_{2^{-}}(2)\psi_{3^{+}}(3)-\psi_{3^{+}}(2)\psi_{2^{-}}(3)]\}$$

Particle in a well by Dunkl operator

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+1^-2^-\right)=17.64$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+1^-2^+\right)=15.87$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+2^+2^-\right)=14.20$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^-2^+2^-\right)=13.31$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+1^-3^+\right)=13.14$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+2^+3^+\right)=11.63$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^-2^+3^+\right)=10.74$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+1^-3^-\right)=10.35$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+2^-3^+\right)=9.76$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+2^+3^-\right)=8.77$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^-2^-3^+\right)=8.87$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^-2^+3^-\right)=7.88$

$\Delta E\left(2^-3^+3^+ \rightarrow 2^+2^-3^+\right)=7.20$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+2^-3^-\right)=6.90$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^-2^-3^-\right)=6.01$

$\Delta E\left(2^-3^+3^+ \rightarrow 2^+2^-3^-\right)=4.43$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^+3^+3^-\right)=4.33$

$\Delta E\left(2^-3^+3^+ \rightarrow 1^-3^+3^-\right)=3.44$

$\Delta E\left(2^-3^+3^+ \rightarrow 2^+3^+3^-\right)=1.86$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+1^-2^+\right)=15.78$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+1^-2^-\right)=13.91$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+2^+2^-\right)=12.33$

$\Delta E\left(2^+3^+3^- \rightarrow 1^-2^+2^-\right)=11.44$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+1^-3^+\right)=11.34$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+2^+3^+\right)=9.76$

$\Delta E\left(2^+3^+3^- \rightarrow 1^-2^+3^+\right)=8.87$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+1^-3^-\right)=8.49$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+2^-3^+\right)=7.9$

$\Delta E\left(2^+3^+3^- \rightarrow 1^-2^-3^+\right)=7.00$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+2^+3^-\right)=6.90$

$\Delta E\left(2^+3^+3^- \rightarrow 1^-2^+3^-\right)=6.01$

$\Delta E\left(2^+3^+3^- \rightarrow 2^+2^-3^+\right)=5.33$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+2^-3^-\right)=5.03$

$\Delta E\left(2^+3^+3^- \rightarrow 1^-2^-3^-\right)=4.14$

$\Delta E\left(2^+3^+3^- \rightarrow 2^+2^-3^-\right)=2.56$

$\Delta E\left(2^+3^+3^- \rightarrow 1^+3^+3^-\right)=2.47$

$\Delta E\left(2^+3^+3^- \rightarrow 1^-3^+3^-\right)=1.57$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+1^-2^+\right)=14.20$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+1^-2^-\right)=12.33$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+2^+2^-\right)=10.75$

$\Delta E\left(1^-3^+3^- \rightarrow 1^-2^+2^-\right)=9.86$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+1^-3^+\right)=9.76$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+2^+3^+\right)=8.18$

$\Delta E\left(1^-3^+3^- \rightarrow 1^-2^+3^+\right)=7.29$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+1^-3^-\right)=6.91$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+2^-3^+\right)=6.32$

$\Delta E\left(1^-3^+3^- \rightarrow 1^-2^-3^+\right)=5.43$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+2^+3^-\right)=5.32$

$\Delta E\left(1^-3^+3^- \rightarrow 1^-2^+3^-\right)=4.43$

$\Delta E\left(1^-3^+3^- \rightarrow 2^+2^-3^+\right)=3.75$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+2^-3^-\right)=3.46$

$\Delta E\left(1^-3^+3^- \rightarrow 1^-2^-3^-\right)=2.56$

$\Delta E\left(1^-3^+3^- \rightarrow 2^+2^-3^-\right)=0.99$

$\Delta E\left(1^-3^+3^- \rightarrow 1^+3^+3^-\right)=0.89$

$\Delta E\left(1^+3^+3^- \rightarrow 1^+1^-2^+\right)=13.31$

$\Delta E\left(1^+3^+3^- \rightarrow 1^+1^-2^-\right)=11.44$

$\Delta E\left(1^+3^+3^- \rightarrow 1^+2^+2^-\right)=9.86$

$\Delta E\left(1^+3^+3^- \rightarrow 1^-2^+2^-\right)=8.97$

$\Delta E\left(1^+3^+3^- \rightarrow 1^+1^-3^+\right)=8.87$

$\Delta E\left(1^+3^+3^- \rightarrow 1^+2^+3^+\right)=7.29$

$\Delta E\left(1^+3^+3^- \rightarrow 1^-2^+3^+\right)=6.40$

$\Delta E\left(1^+3^+3^- \rightarrow 1^+1^-3^-\right)=6.02$

$\Delta E\left(1^+3^+3^- \rightarrow 1^+2^-3^+\right)=5.42$

$\Delta E\left(1^+3^+3^- \rightarrow 1^-2^-3^+\right)=4.53$

$\Delta E\left(1^+3^+3^- \rightarrow 1^+2^+3^-\right)=4.43$

$\Delta E\left(1^+3^+3^- \rightarrow 1^-2^+3^-\right)=3.54$

$\Delta E\left(1^+3^+3^- \rightarrow 2^+2^-3^+\right)=2.86$

$\Delta E\left(1^+3^+3^- \rightarrow 1^+2^-3^-\right)=2.56$

$\Delta E\left(1^+3^+3^- \rightarrow 1^-2^-3^-\right)=1.67$

$\Delta E\left(1^+3^+3^- \rightarrow 2^+2^-3^-\right)=0.98$

Particle in a well by Dunkl operator

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{+}1^{-}2^{+})}=13.21$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{+}1^{-}2^{-})}=11.34$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{+}2^{+}2^{-})}=9.76$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{-}2^{+}2^{-})}=8.87$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{+}1^{-}3^{+})}=8.77$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{+}2^{+}3^{+})}=7.19$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{-}2^{+}3^{+})}=6.30$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{+}1^{-}3^{-})}=5.92$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{+}2^{-}3^{+})}=5.33$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{-}2^{-}3^{+})}=4.44$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{+}2^{+}3^{-})}=4.33$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{-}2^{+}3^{-})}=3.44$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 2^{+}2^{-}3^{+})}=2.76$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{+}2^{-}3^{-})}=2.47$

$\Delta E_{(2^{+}2^{-}3^{-} \rightarrow 1^{-}2^{-}3^{-})}=1.57$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{+}1^{-}2^{+})}=11.63$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{+}1^{-}2^{-})}=9.76$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{+}2^{+}2^{-})}=8.18$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{-}2^{+}2^{-})}=7.29$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{+}1^{-}3^{+})}=7.19$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{+}2^{+}3^{+})}=5.61$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{-}2^{+}3^{+})}=4.72$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{+}1^{-}3^{-})}=4.34$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{+}2^{-}3^{+})}=3.75$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{-}2^{-}3^{+})}=2.86$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{+}2^{+}3^{-})}=2.75$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{-}2^{+}3^{-})}=1.85$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 2^{+}2^{-}3^{+})}=1.18$

$\Delta E_{(1^{-}2^{-}3^{-} \rightarrow 1^{+}2^{-}3^{-})}=0.89$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{+}1^{-}2^{+})}=10.74$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{+}1^{-}2^{-})}=8.87$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{+}2^{+}2^{-})}=7.39$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{-}2^{+}2^{-})}=6.42$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{+}1^{-}3^{+})}=6.30$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{+}2^{+}3^{+})}=4.72$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{-}2^{+}3^{+})}=3.83$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{-}2^{+}3^{+})}=3.83$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{+}1^{-}3^{-})}=3.45$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{-}2^{-}3^{+})}=1.97$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{+}2^{+}3^{-})}=1.86$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 1^{-}2^{+}3^{-})}=0.97$

$\Delta E_{(1^{+}2^{-}3^{-} \rightarrow 2^{-}2^{+}3^{+})}=0.29$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{+}1^{-}2^{+})}=10.64$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{+}1^{-}2^{-})}=8.58$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{+}2^{+}2^{-})}=7.20$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{-}2^{+}2^{-})}=6.30$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{+}1^{-}3^{+})}=6.01$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{+}2^{+}3^{+})}=4.63$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{-}2^{+}3^{+})}=3.74$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{+}21^{-}3^{-})}=3.15$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{+}2^{-}3^{+})}=2.76$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{-}2^{-}3^{+})}=1.87$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{+}2^{+}3^{-})}=1.77$

$\Delta E_{(2^{+}2^{-}3^{+} \rightarrow 1^{-}2^{+}3^{-})}=0.88$

Particle in a well by Dunkl operator

$\Delta E(1^-2^+3^- \rightarrow 1^+1^-2^+)=9.76$

$\Delta E(1^-2^+3^- \rightarrow 1^+1^-2^-)=7.90$

$\Delta E(1^-2^+3^- \rightarrow 1^+2^+2^-)=6.32$

$\Delta E(1^-2^+3^- \rightarrow 1^-2^+2^-)=5.42$

$\Delta E(1^-2^+3^- \rightarrow 1^+1^-3^+)=5.33$

$\Delta E(1^-2^+3^- \rightarrow 1^+2^+3^+)=3.75$

$\Delta E(1^-2^+3^- \rightarrow 1^-2^+3^+)=2.86$

$\Delta E(1^-2^+3^- \rightarrow 1^+1^-3^-)=2.47$

$\Delta E(1^-2^+3^- \rightarrow 1^+2^-3^+)=1.88$

$\Delta E(1^-2^+3^- \rightarrow 1^-2^-3^+)=0.99$

$\Delta E(1^-2^+3^- \rightarrow 1^+2^+3^-)=0.89$

$\Delta E(1^+2^+3^- \rightarrow 1^+1^-2^+)=8.87$

$\Delta E(1^+2^+3^- \rightarrow 1^+1^-2^-)=7.01$

$\Delta E(1^+2^+3^- \rightarrow 1^+2^+2^-)=5.42$

$\Delta E(1^+2^+3^- \rightarrow 1^-2^+2^-)=4.53$

$\Delta E(1^+2^+3^- \rightarrow 1^+1^-3^+)=4.44$

$\Delta E(1^+2^+3^- \rightarrow 1^+2^+3^+)=2.86$

$\Delta E(1^+2^+3^- \rightarrow 1^-2^+3^+)=1.97$

$\Delta E(1^+2^+3^- \rightarrow 1^+1^-3^-)=1.58$

$\Delta E(1^+2^+3^- \rightarrow 1^+2^-3^+)=0.99$

$\boxed{\Delta E(1^+2^+3^- \rightarrow 1^-2^-3^+)=0.10}$

$\Delta E(1^-2^-3^+ \rightarrow 1^+1^-2^+)=8.77$

$\Delta E(1^-2^-3^+ \rightarrow 1^+1^-2^-)=6.90$

$\Delta E(1^-2^-3^+ \rightarrow 1^+2^+2^-)=5.32$

$\Delta E(1^-2^-3^+ \rightarrow 1^-2^+2^-)=4.43$

$\Delta E(1^-2^-3^+ \rightarrow 1^+1^-3^+)=4.33$

$\Delta E(1^-2^-3^+ \rightarrow 1^+2^+3^+)=2.75$

$\Delta E(1^-2^-3^+ \rightarrow 1^-2^+3^+)=1.86$

$\Delta E(1^-2^-3^+ \rightarrow 1^+1^-3^-)=1.48$

$\Delta E(1^-2^-3^+ \rightarrow 1^+2^-3^+)=0.89$

$\Delta E(1^+2^-3^+ \rightarrow 1^+1^-2^+)=7.88$

$\Delta E(1^+2^-3^+ \rightarrow 1^+1^-2^-)=5.99$

$\Delta E(1^+2^-3^+ \rightarrow 1^+2^+2^-)=4.43$

$\Delta E(1^+2^-3^+ \rightarrow 1^-2^+2^-)=3.54$

$\Delta E(1^+2^-3^+ \rightarrow 1^+1^-3^+)=3.44$

$\Delta E(1^+2^-3^+ \rightarrow 1^+2^+3^+)=1.86$

$\Delta E(1^+2^-3^+ \rightarrow 1^-2^+3^+)=0.97$

$\Delta E(1^+2^-3^+ \rightarrow 1^+1^-3^-)=0.59$

Particle in a well by Dunkl operator

$\Delta E_{(1^+1^-3^- \rightarrow 1^+1^-2^+)}=7.29$

$\Delta E_{(1^+1^-3^- \rightarrow 1^+1^-2^-)}=5.42$

$\Delta E_{(1^+1^-3^- \rightarrow 1^+2^+2^-)}=3.84$

$\Delta E_{(1^+1^-3^- \rightarrow 1^-2^+2^-)}=2.95$

$\Delta E_{(1^+1^-3^- \rightarrow 1^+1^-3^+)}=2.85$

$\Delta E_{(1^+1^-3^- \rightarrow 1^+2^+3^+)}=1.27$

$\Delta E_{(1^+1^-3^- \rightarrow 1^-2^+3^+)}=0.38$

$\Delta E_{(1^-2^+3^+ \rightarrow 1^+1^-2^+)}=6.90$

$\Delta E_{(1^-2^+3^+ \rightarrow 1^+1^-2^-)}=5.03$

$\Delta E_{(1^-2^+3^+ \rightarrow 1^+2^+2^-)}=3.45$

$\Delta E_{(1^-2^+3^+ \rightarrow 1^-2^+2^-)}=2.56$

$\Delta E_{(1^-2^+3^+ \rightarrow 1^+1^-3^+)}=2.47$

$\Delta E_{(1^-2^+3^+ \rightarrow 1^+2^+3^+)}=0.89$

$\Delta E_{(1^+2^+3^+ \rightarrow 1^+1^-2^+)}=6.01$

$\Delta E_{(1^+2^+3^+ \rightarrow 1^+1^-2^-)}=4.14$

$\Delta E_{(1^+2^+3^+ \rightarrow 1^+2^+2^-)}=2.56$

$\Delta E_{(1^+2^+3^+ \rightarrow 1^-2^+2^-)}=1.67$

$\Delta E_{(1^+2^+3^+ \rightarrow 1^+1^-3^+)}=1.58$

$\Delta E_{(1^+1^-3^+ \rightarrow 1^+1^-2^+)}=4.43$

$\Delta E_{(1^+1^-3^+ \rightarrow 1^+1^-2^-)}=2.56$

$\Delta E_{(1^+1^-3^+ \rightarrow 1^+2^+2^-)}=0.98$

$\Delta E_{(1^+1^-3^+ \rightarrow 1^-2^+2^-)}=0.97$

$\Delta E_{(1^-2^+2^- \rightarrow 1^+1^-2^+)}=4.33$

$\Delta E_{(1^-2^+2^- \rightarrow 1^+1^-2^-)}=2.47$

$\Delta E_{(1^-2^+2^- \rightarrow 1^+2^+2^-)}=0.89$

$\Delta E_{(1^+2^+2^- \rightarrow 1^+1^-2^+)}=3.44$

$\Delta E_{(1^+2^+2^- \rightarrow 1^+1^-2^-)}=1.58$

$\Delta E_{(1^+2^+2^- \rightarrow 1^+1^-2^+)}=1.90$

Conclusion

- For ν = 0.86 the states$E_{1^-2^-3^+}$and $E_{1^+2^+3^-}$ are degenerate.
- In the interval -0.5< ν <0.86, $E_{1^-2^-3^+} < E_{1^+2^+3^-}$.
- In the interval ν >0.86, $E_{1^-2^-3^+} > E_{1^+2^+3^-}$.
- The lowest transition energy of the three body system in the three level is 0.10.
- The highest transition energy of the three body system in the three level is 17.64.
- In our calculation, we considered ν =0.5, but ν is a fittable parameter, and by some numerical method like the scanning method one can obtain the best amount of it.
- By considering the Dunkl formalism, we see that for each quantum number n there are two positive and negative different states while in the ordinary cases even and odd ns have different parties.
- By considering the Dunkl formalism we see that there are many different new transitions fittable by the experimental data while in the ordinary cases we had only some limit transitions and also for some cases there were not any transitions.
- The above results are expandable to consider interaction between the particles.
- It is possible for us to apply the results to all branches of physics.
- We say welcome to all physicians who want to have cooperation in this area.

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