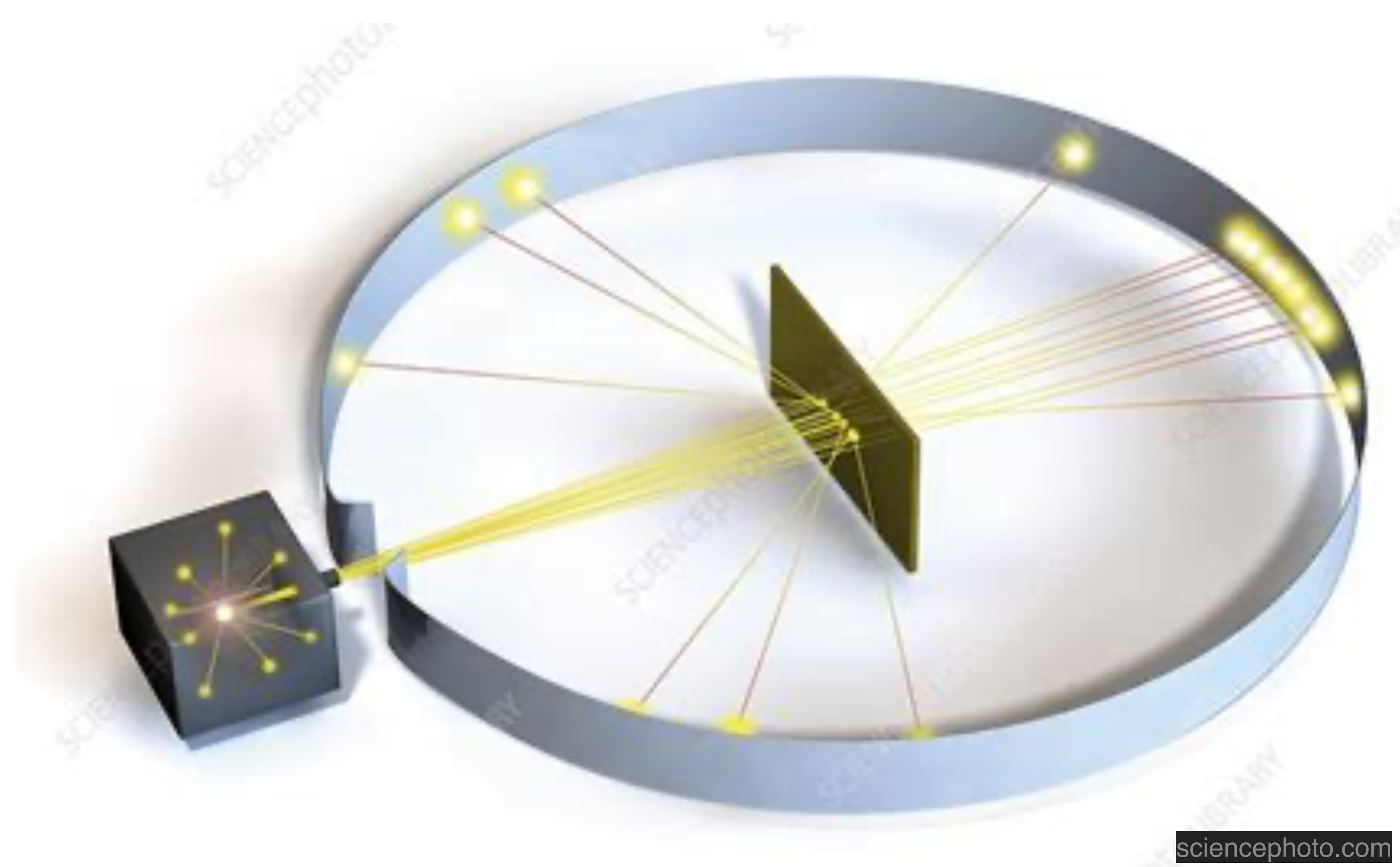
Nucleons in a finite volume

from ground states to the continuum

Hang Yu

Sept. 26

From Scattering Experiments to EFT



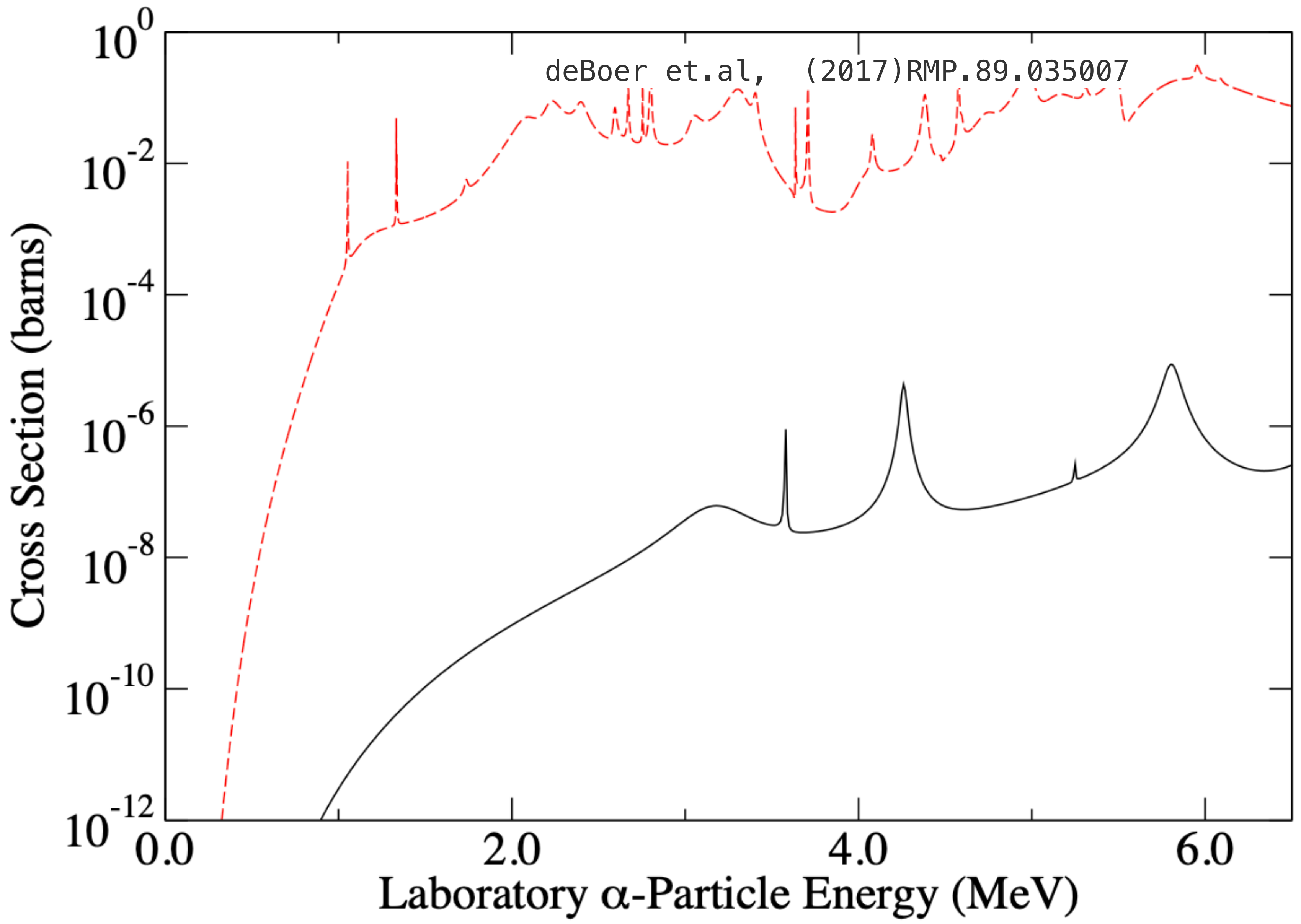
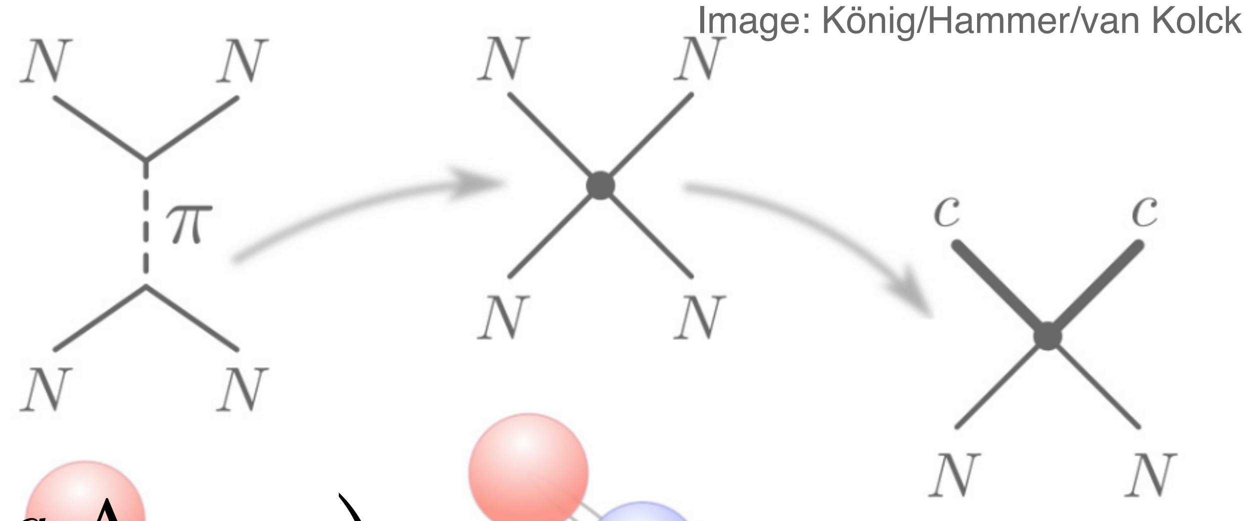
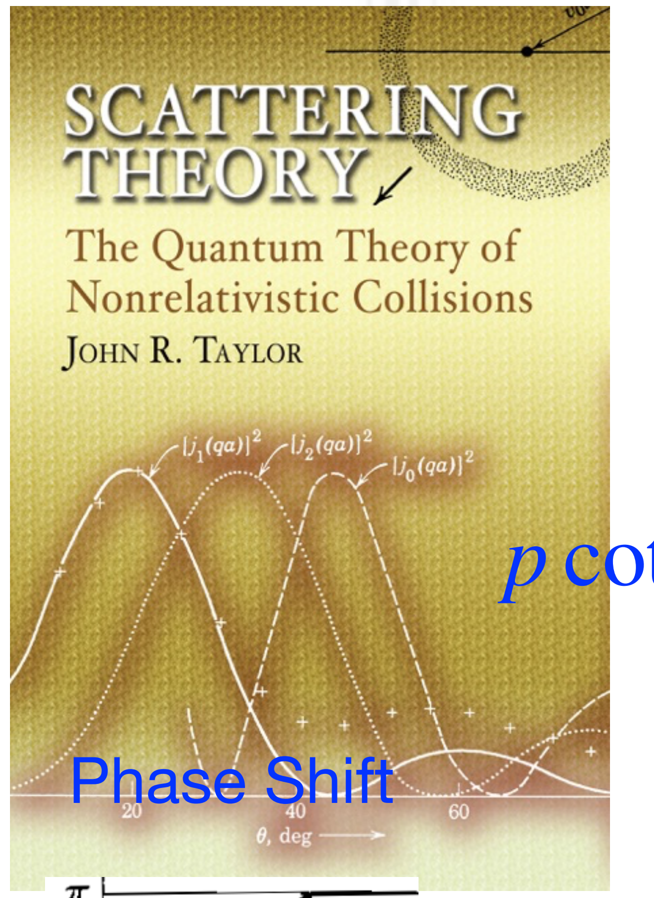


Image: König/Hammer/van Kolck

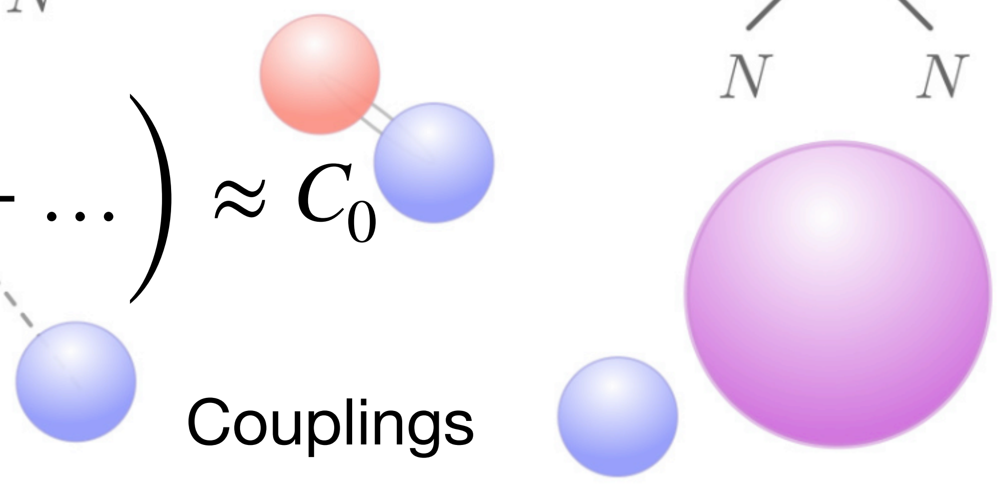




$$\sigma_l(p) = \frac{4\pi(2l+1)}{p^2}\sin^2\delta_l(p)$$

$$p\cot(\delta_0(p)) = -a_0^{-1} + \frac{1}{2}r_0p^2 + \ldots$$

$$\frac{8\pi}{m}\left(1 + \frac{2a_0\Lambda}{\pi} + \ldots\right) \approx C_0$$

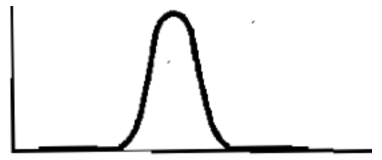


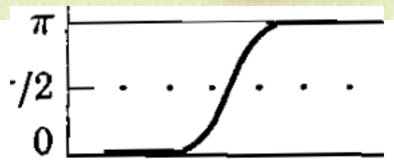
Cross Section

Phase Shift

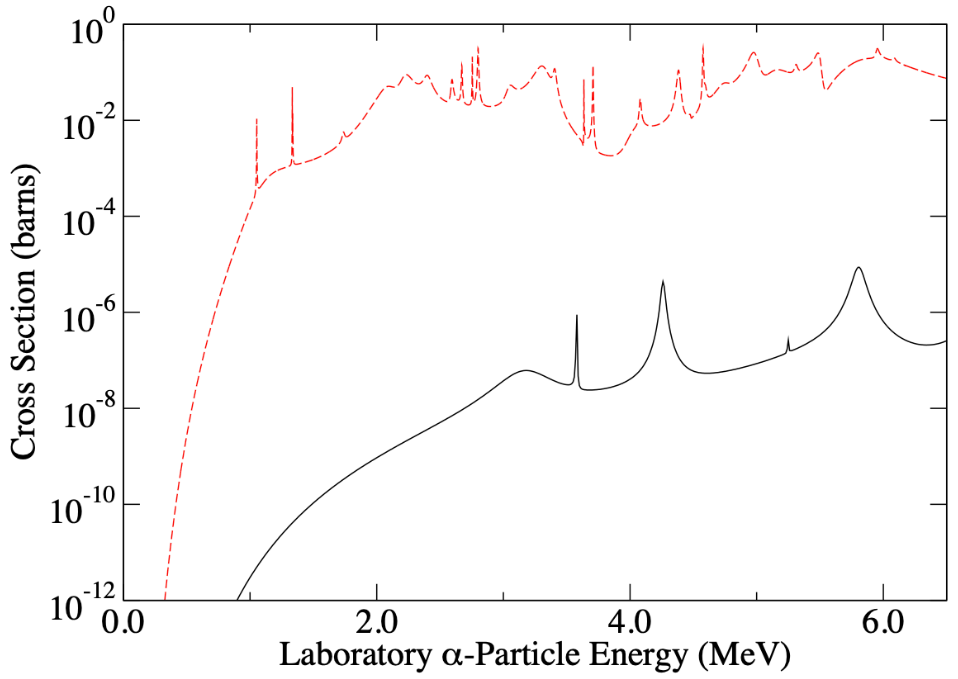
ERE

Couplings

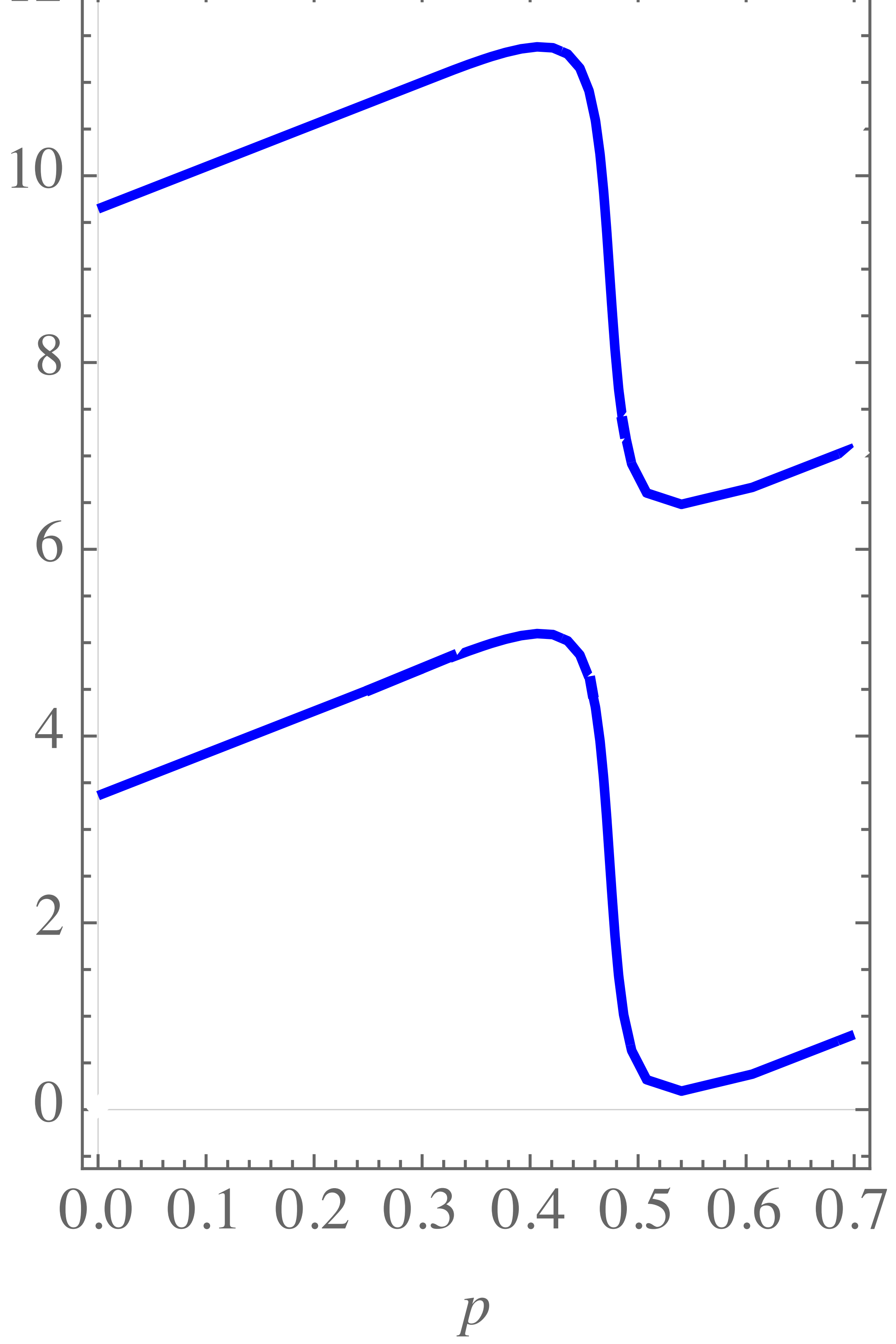




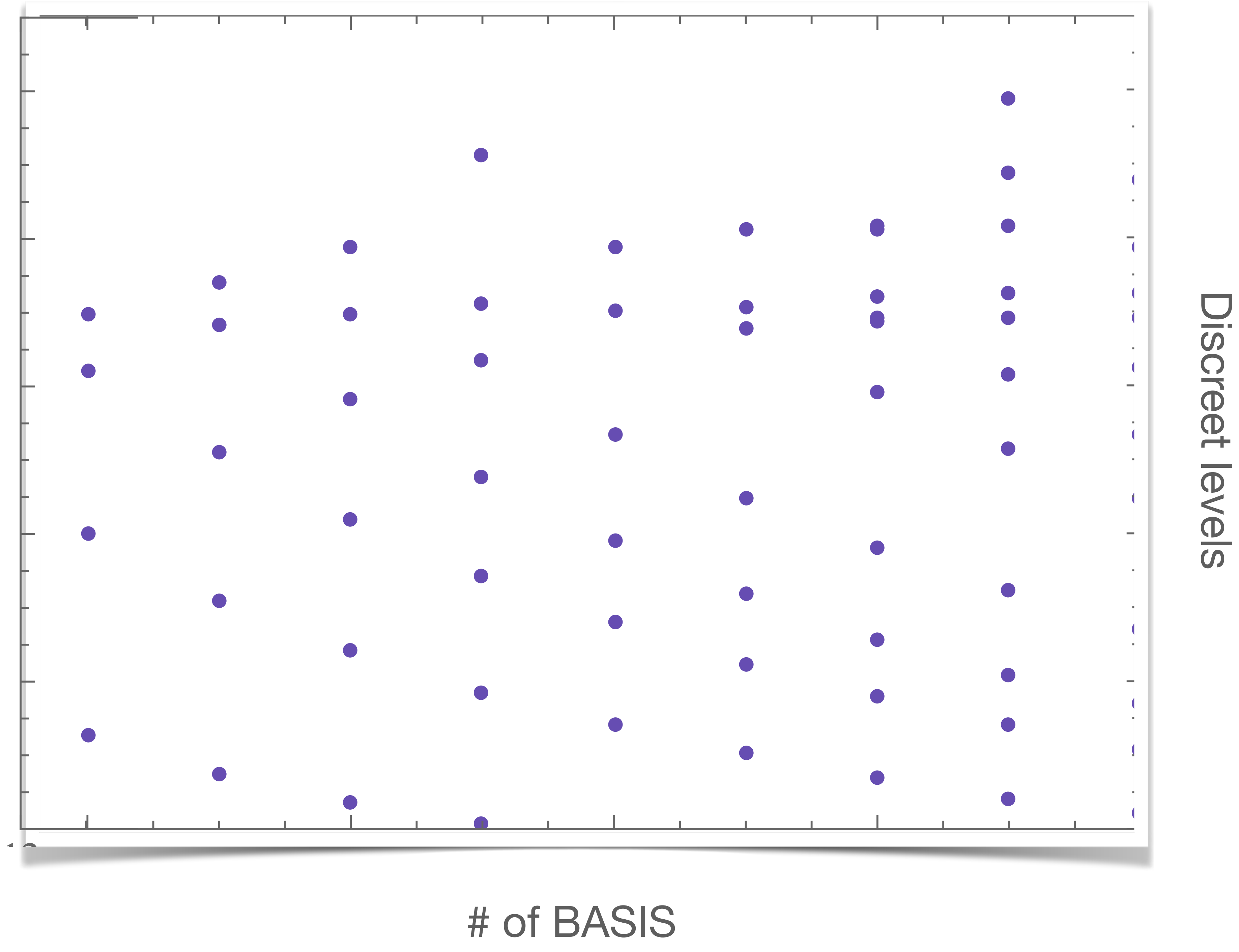
From Theory to Experiments?



$$p \cot(\delta_0(p)) = -a_0^{-1} + \frac{1}{2} r_0 p^2 + \ldots$$

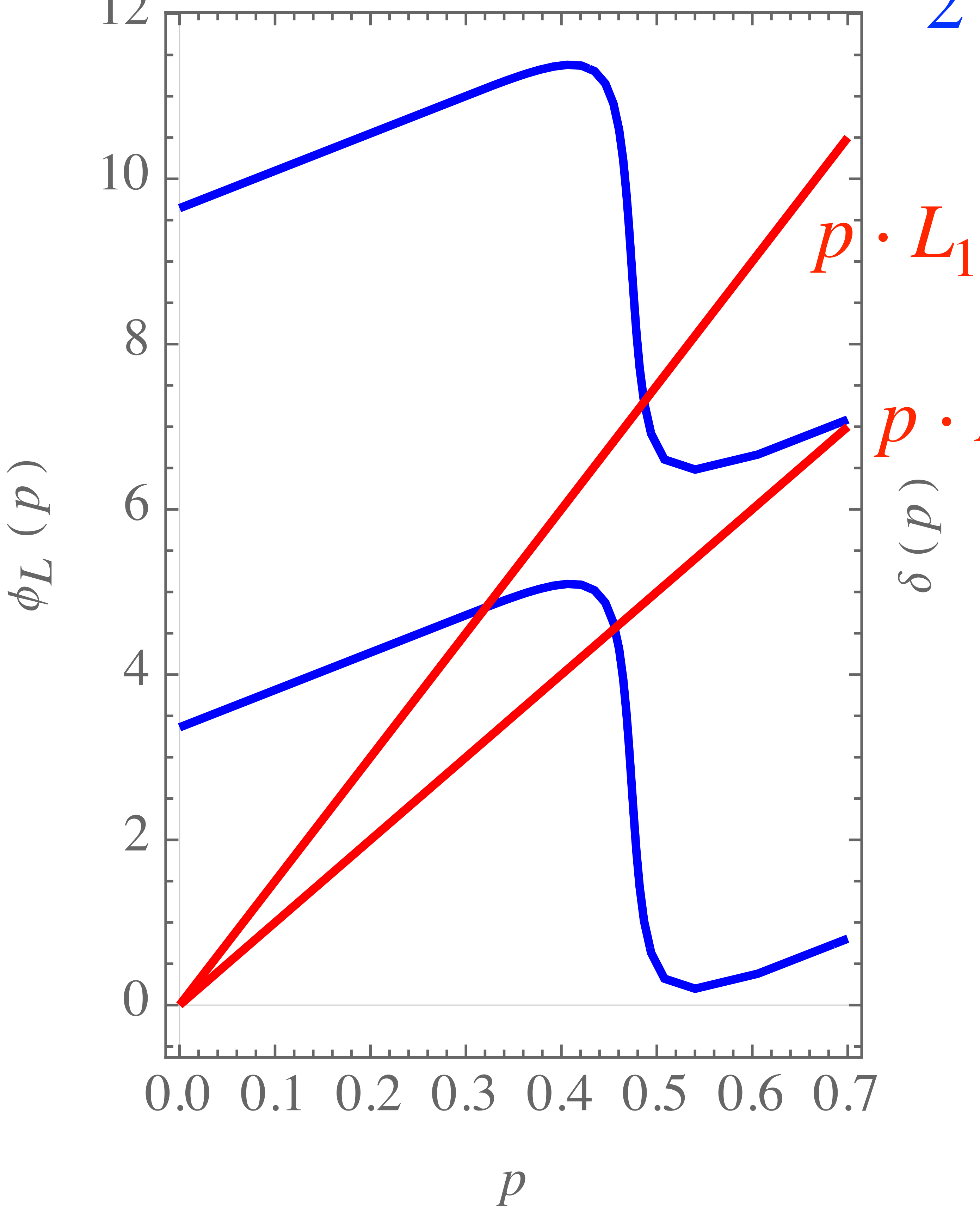


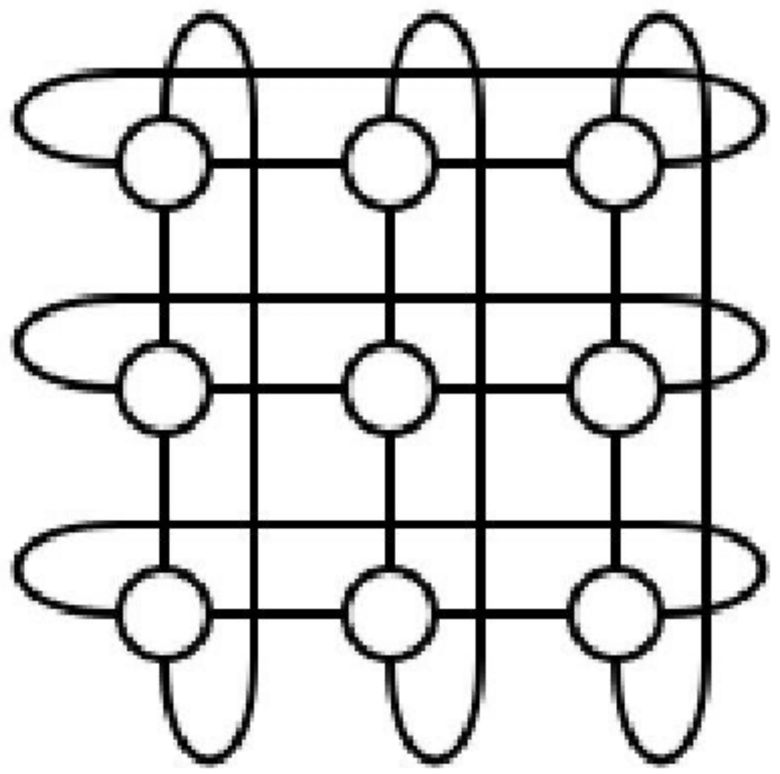
$$H = \sum_{pq} K_{pq} a_p^\dagger a_q + \frac{1}{4} \sum_{pqrs} V_{pqrs} a_p^\dagger a_q^\dagger a_s a_r$$



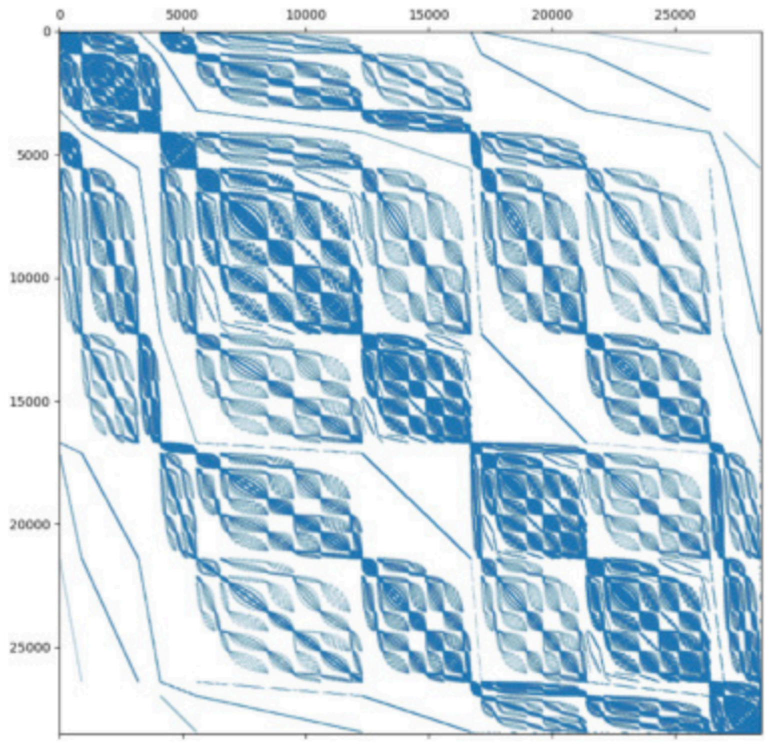
Analytical tools: Finite Volume effects

$$p\cot(\delta_0(p)) = -a_0^{-1} + \frac{1}{2}r_0p^2 + \dots$$





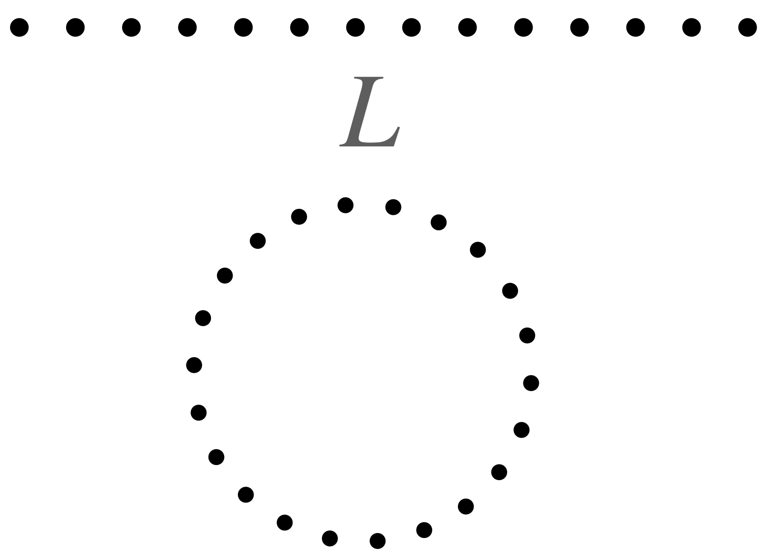
$$H =$$

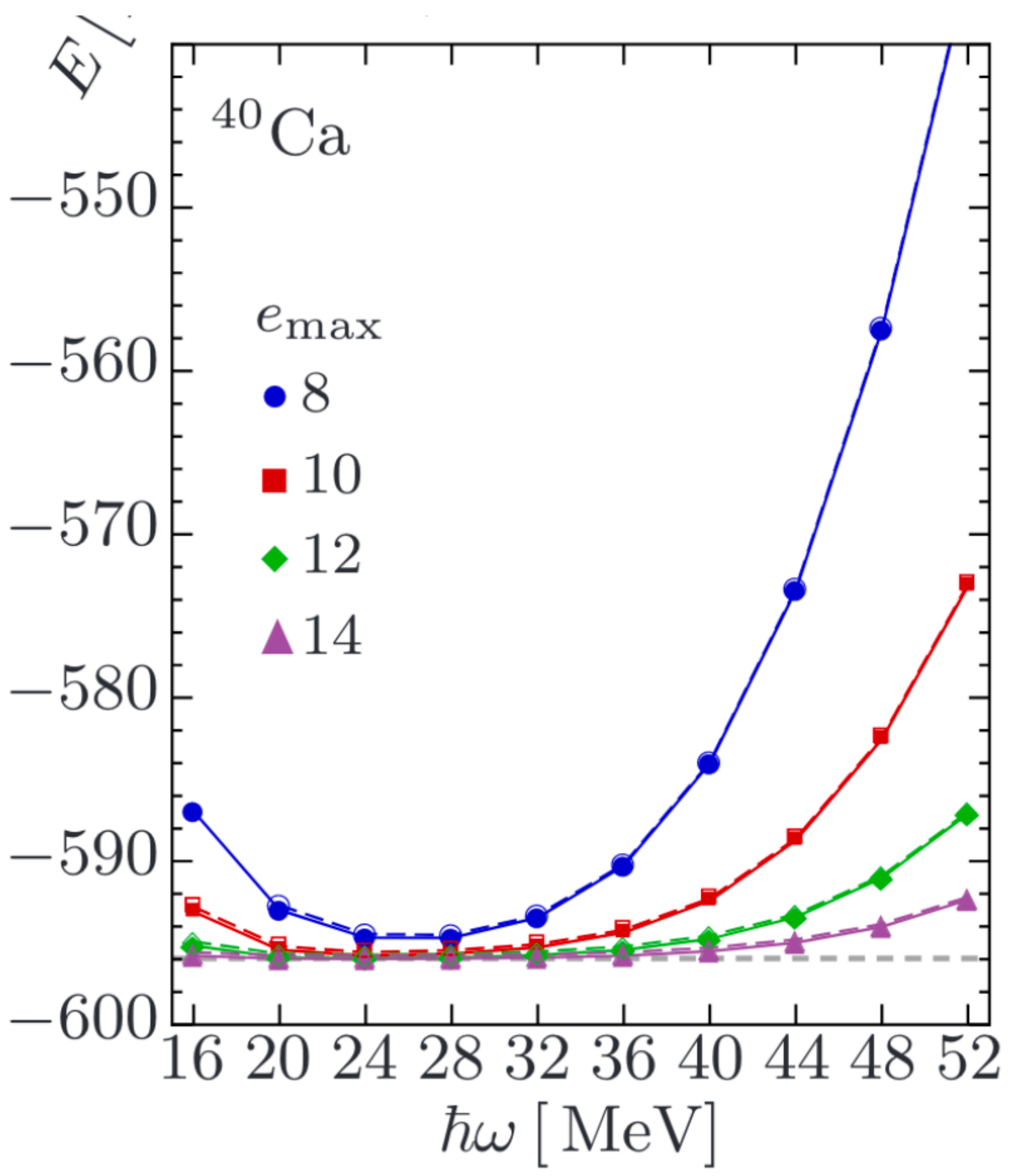


N. Shimizu Comp. Phys. Comm. 244, 372 (2019)

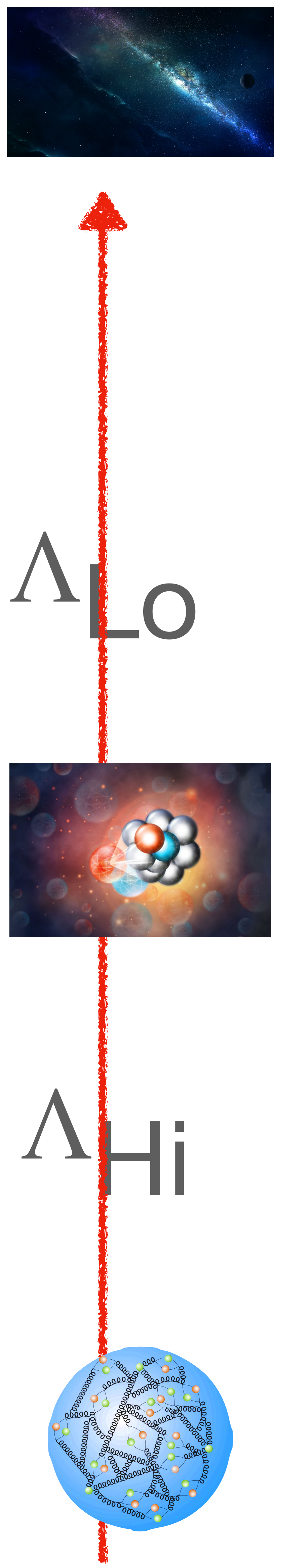
Boundary Condition!

$$\alpha\cos(p \cdot L/2) + \beta\sin(p \cdot L/2) = 0$$





H. Hergert j.physrep.2015.12.007



$$e^{-i\delta_0(p)} = S_0(p) \propto \frac{C_0^2}{ip - \kappa} - \frac{1}{a_0^*} + \frac{1}{2} r_0^* (ip - \kappa)^2 + \ldots$$

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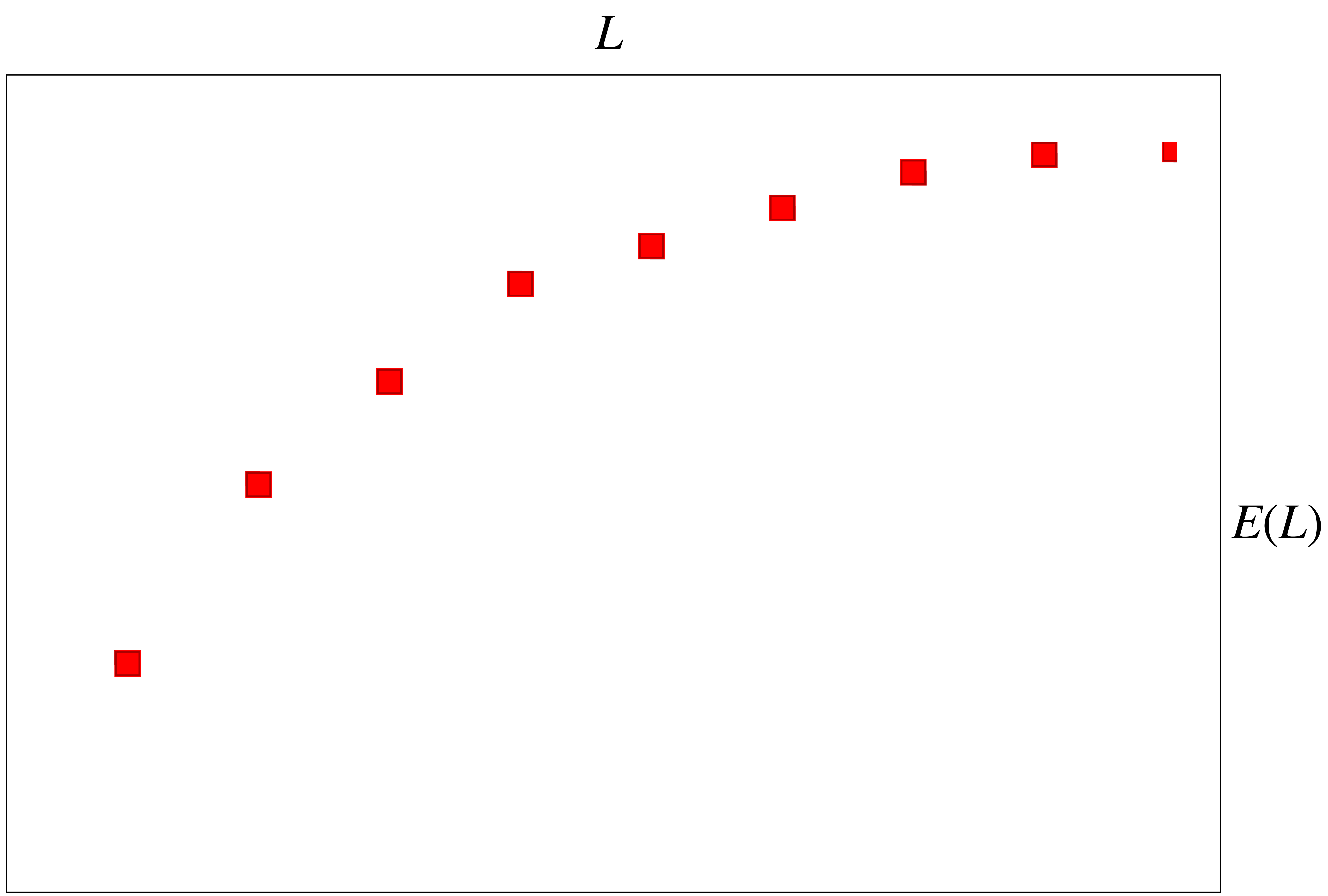

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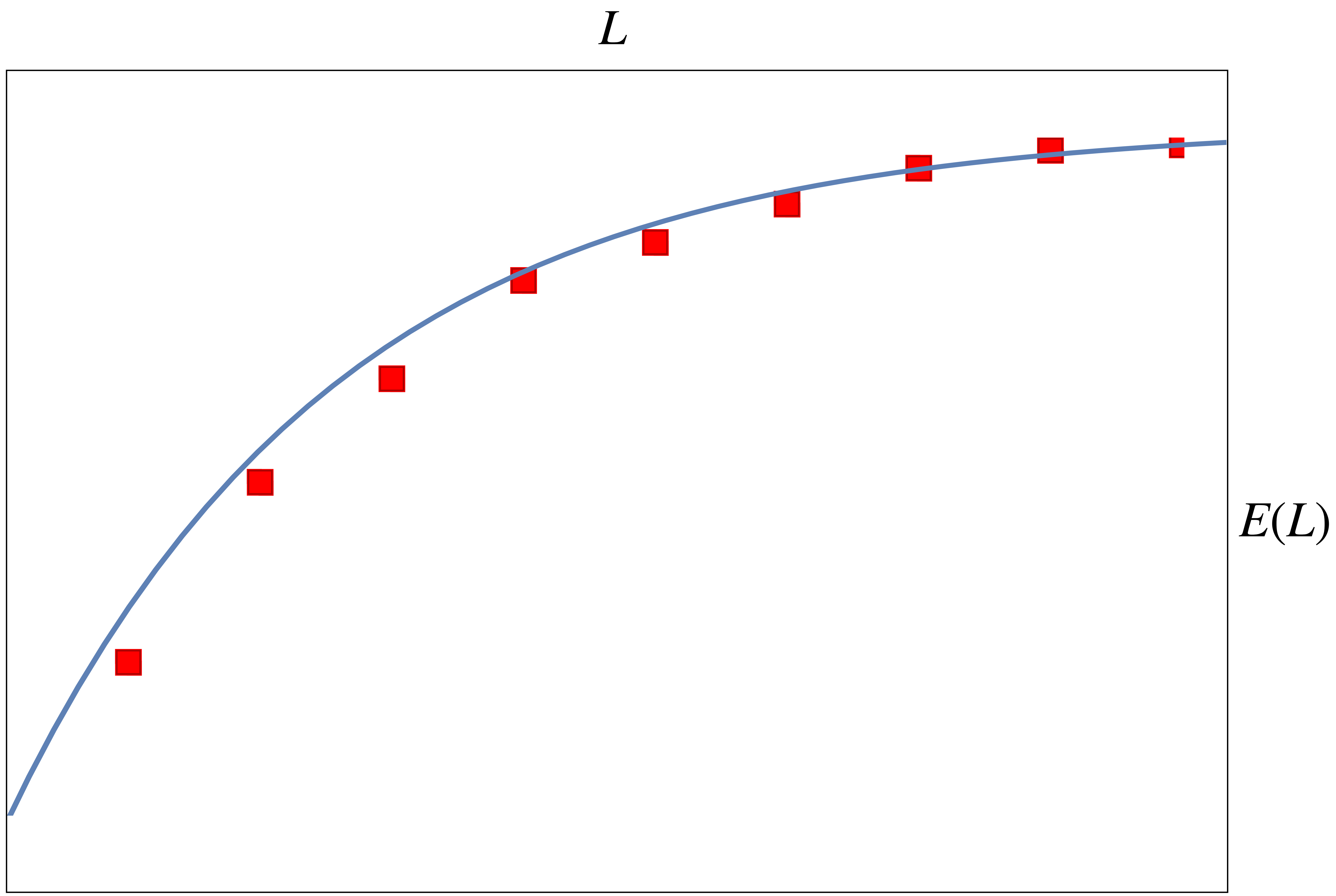

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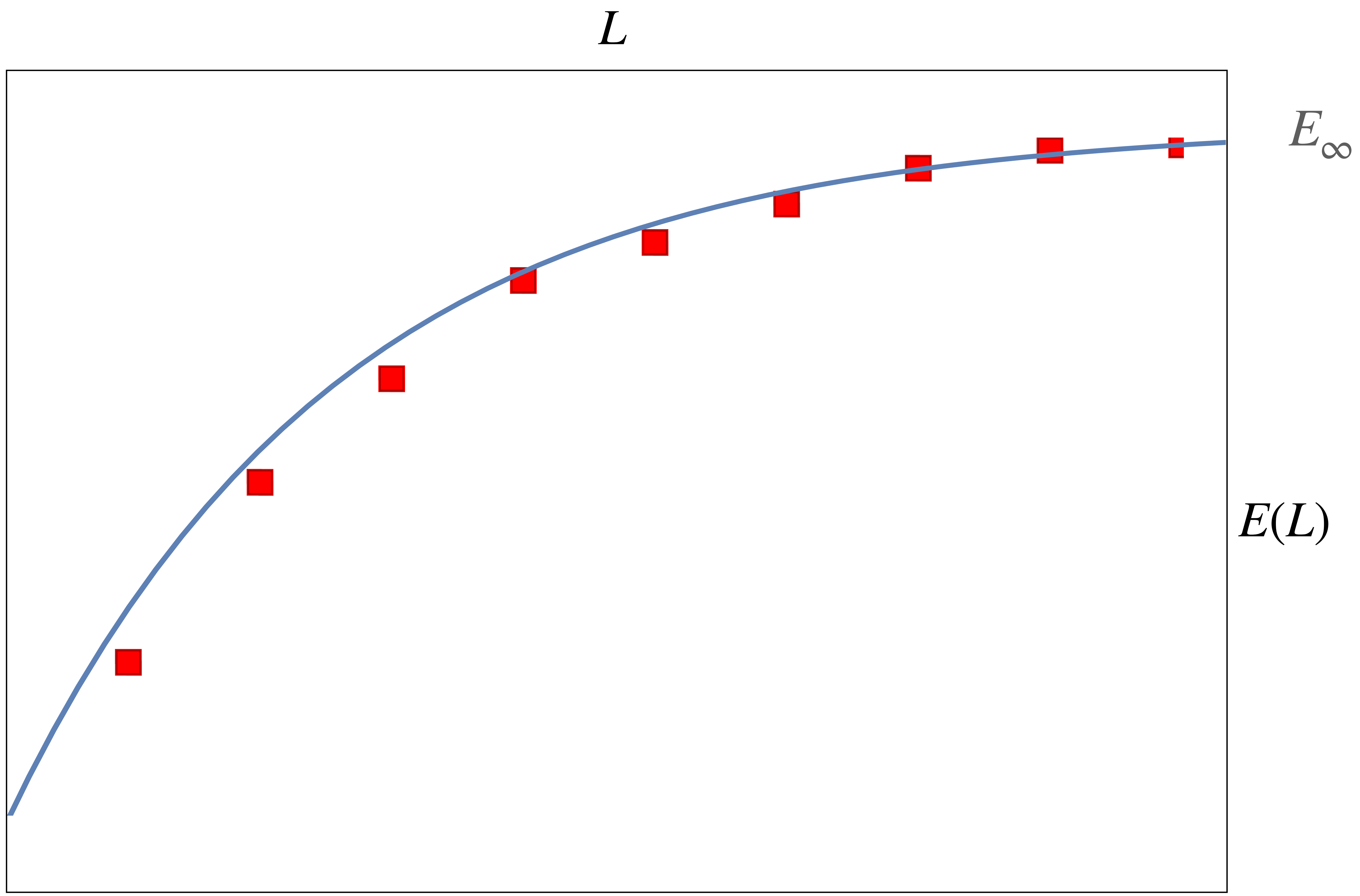

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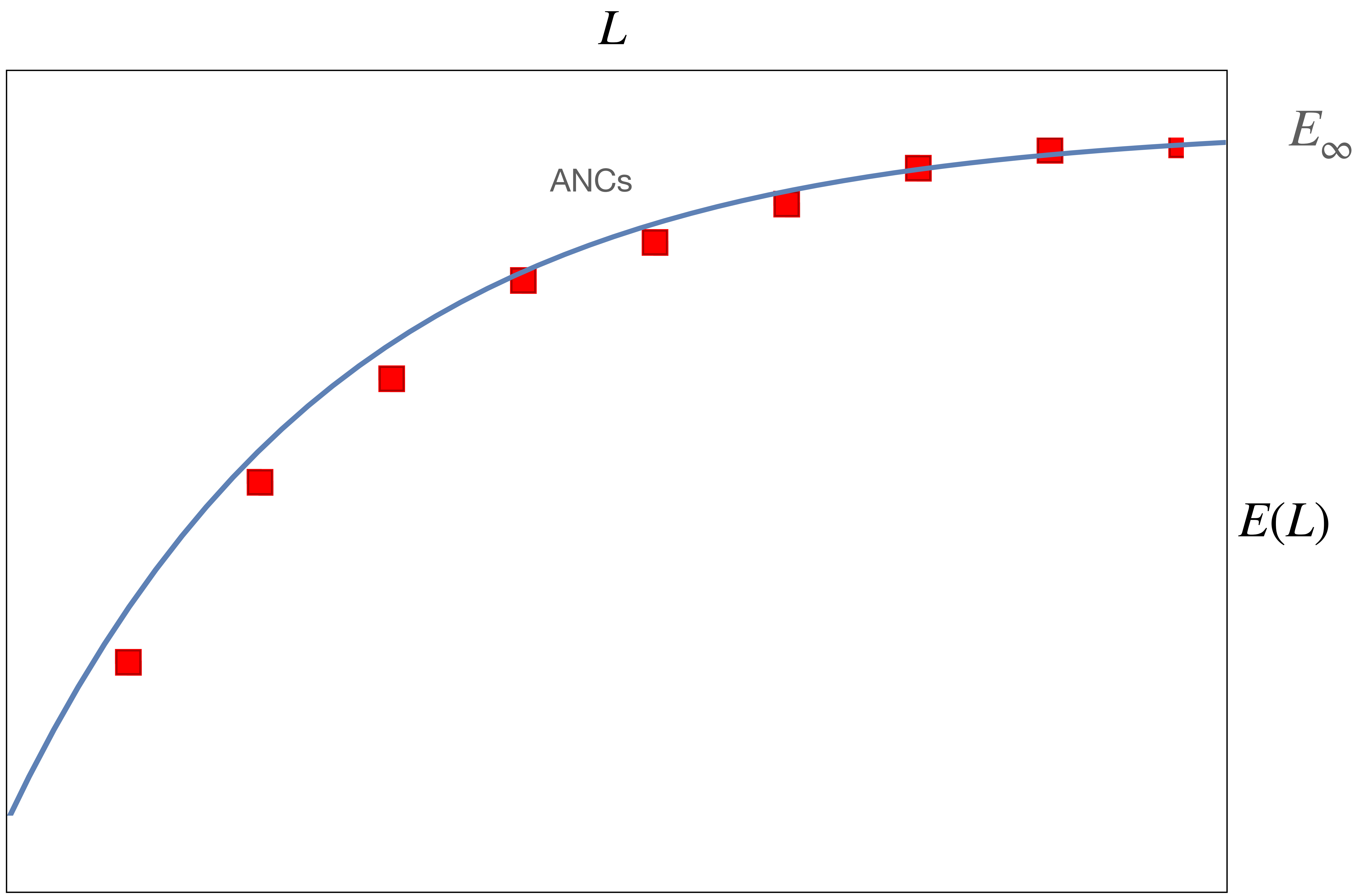

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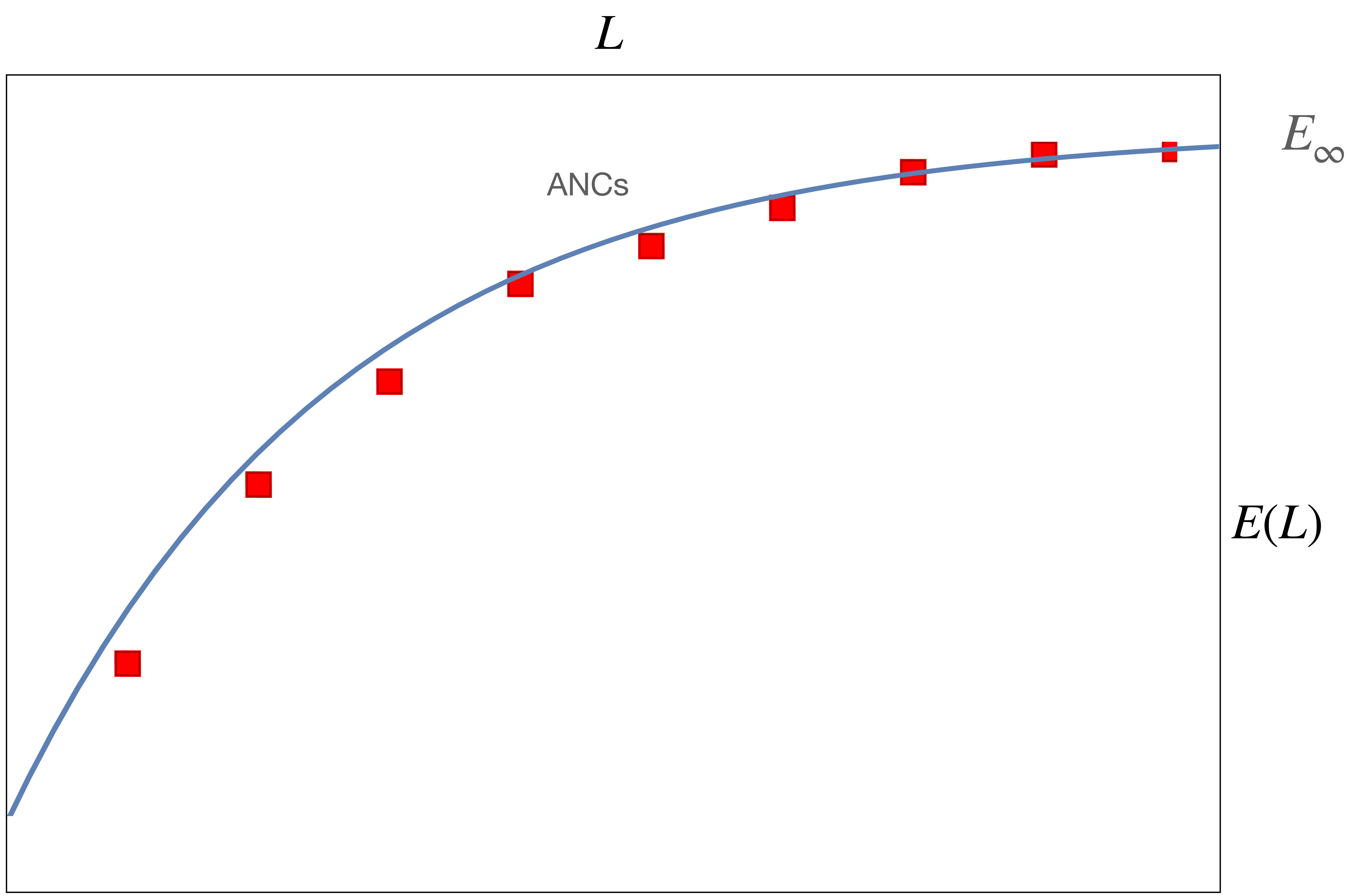

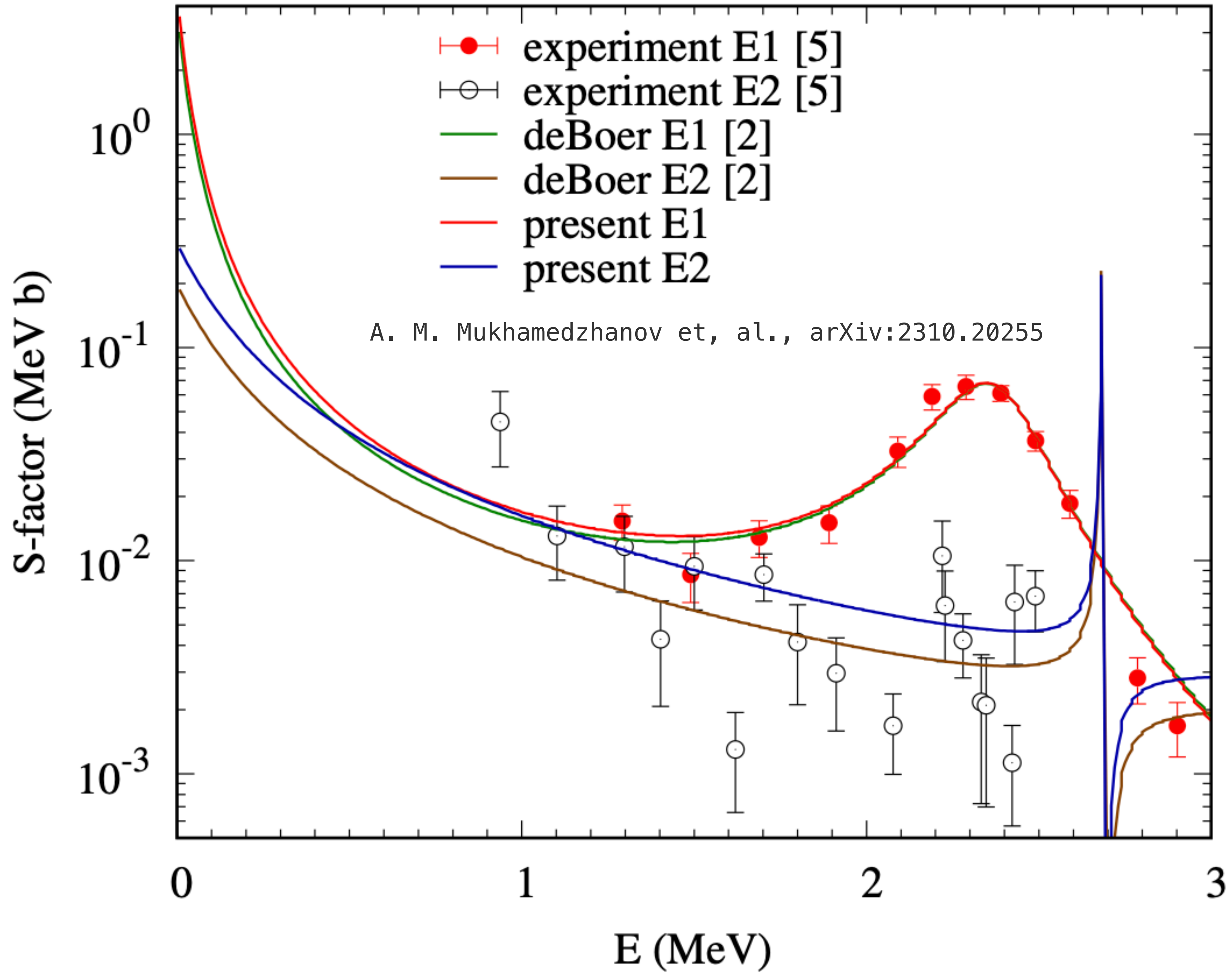

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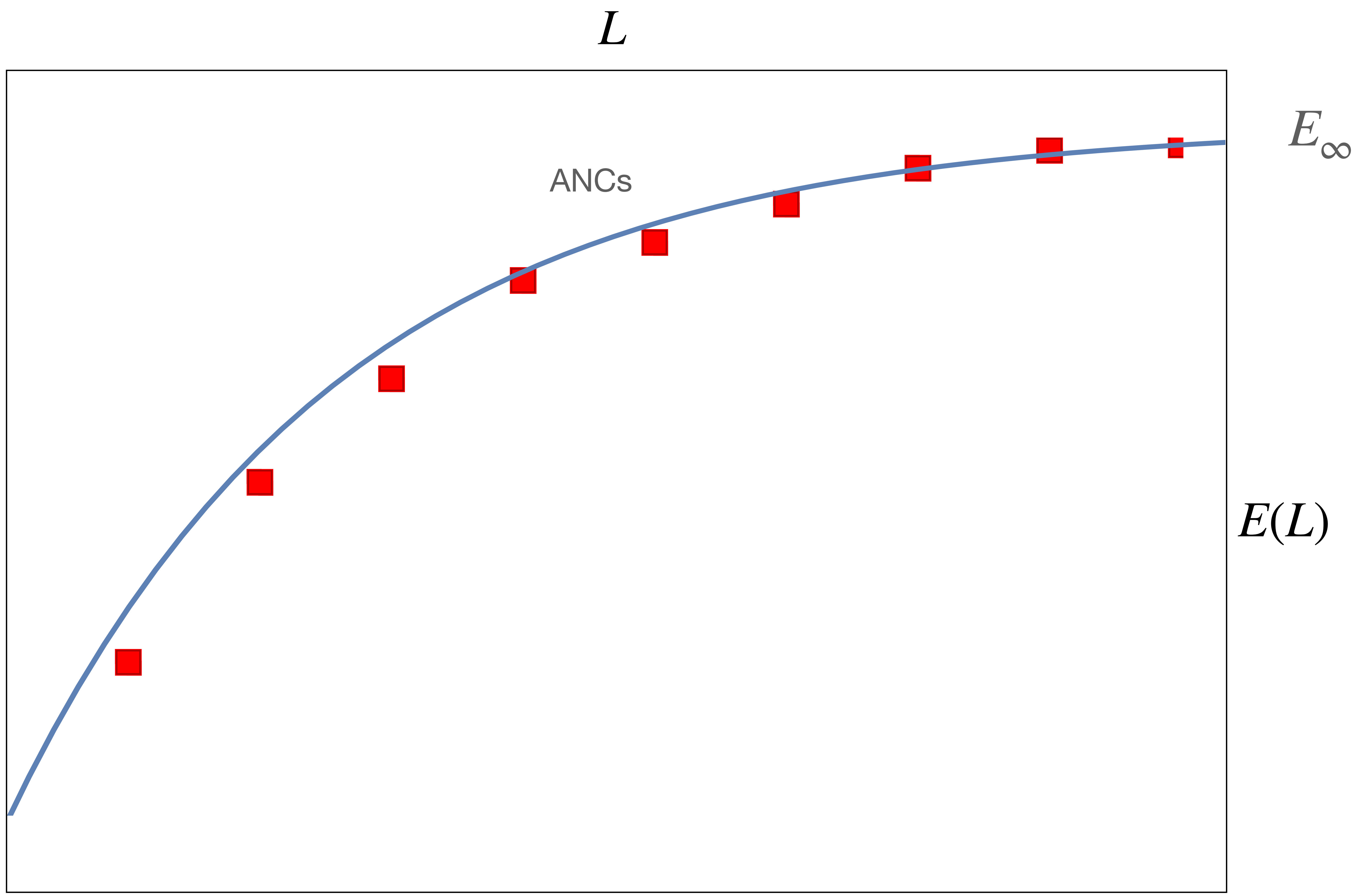
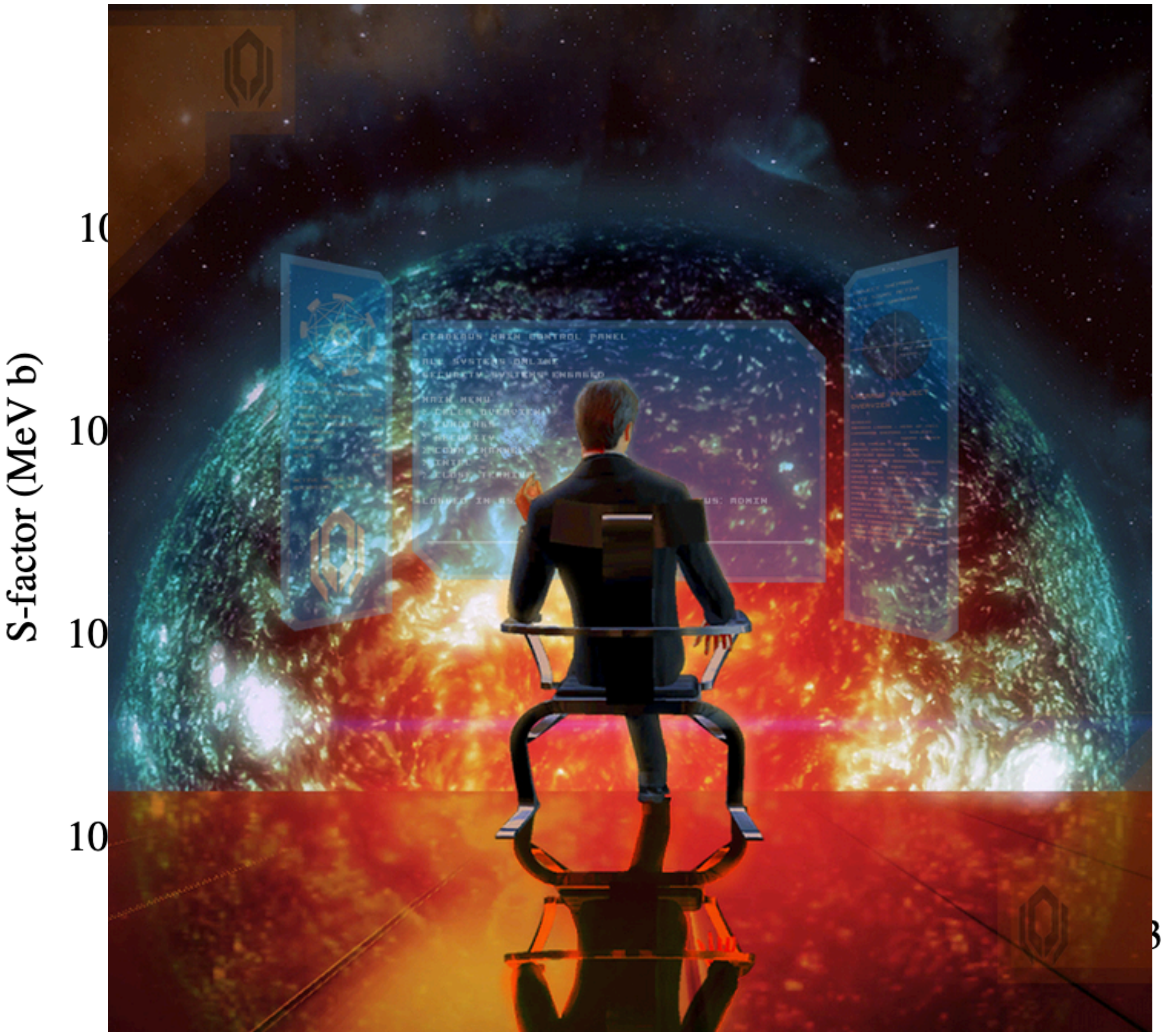




Summary: Why Finite Volume?

- Consequence of lattice regularization

  Discreet levels of spectra do not reproduce scattering info

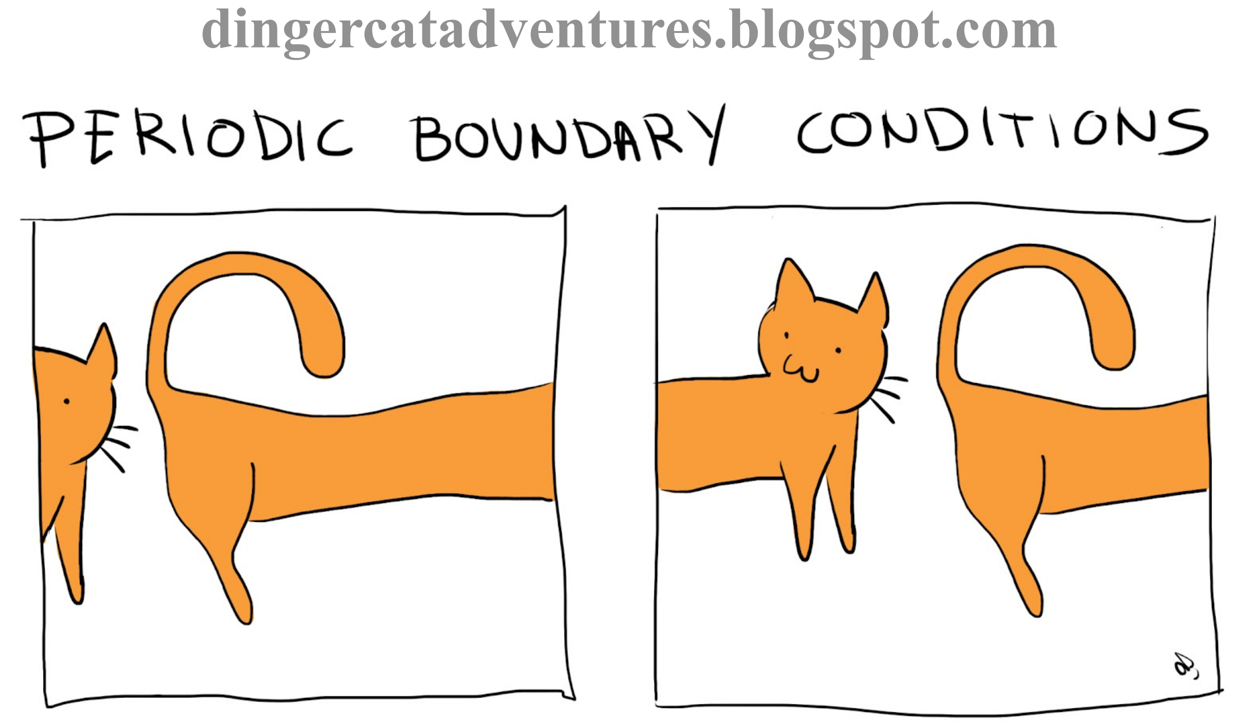
- Scattering matrices

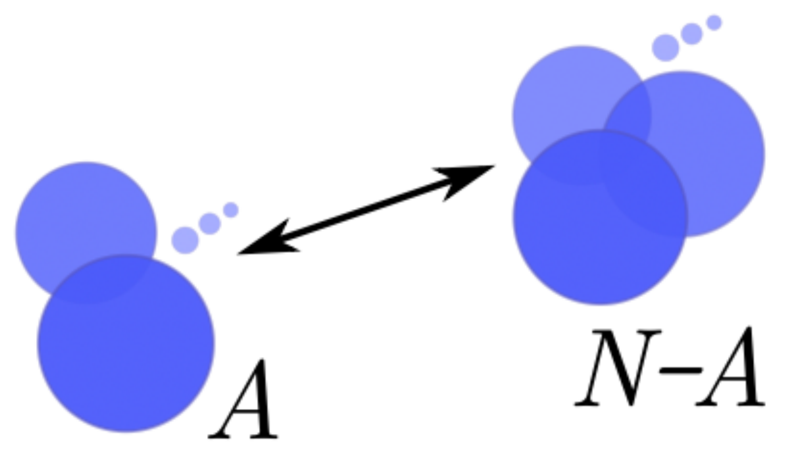
  Lüscher, Commun.Math.Phys.**104** 153(1986)

  Lüscher, Commun.Math.Phys.**104** 177(1986)

- Nuclear EFT and reaction theory

  S.R. Beane et al., PLB 585 106 (2004)
  S. Koenig et al.,PRL, 107 112001(2011),
  Meißner et.al.,PRL, 114 091602(2015).
  S. Koenig , D. Lee, PLB 779 9 (2018),



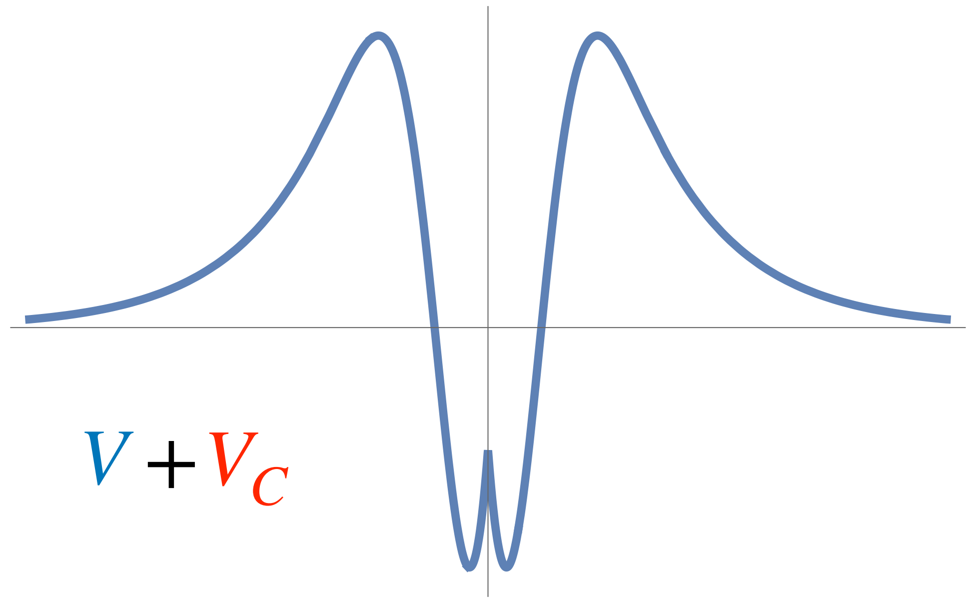


Assumption: Interactions are Short-Ranged

Our setup

HY, S. König, D. Lee, PRL, **131,** 212 502 (2023)

We are considering a non-relativistic system short-range interactions + repulsive Coulomb interactions

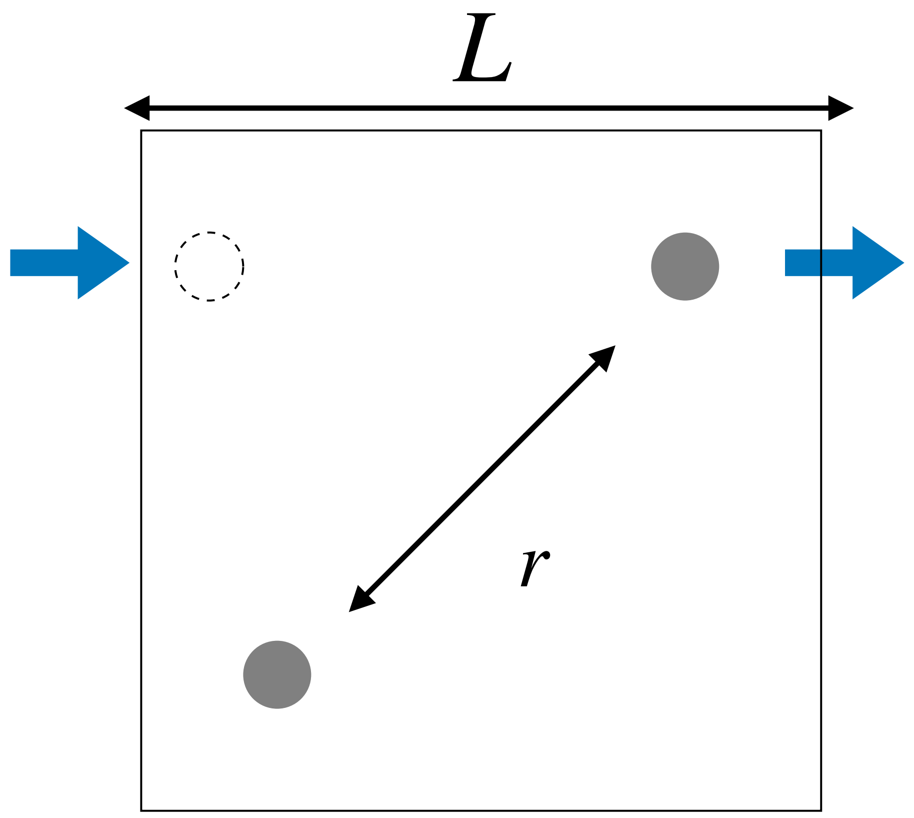


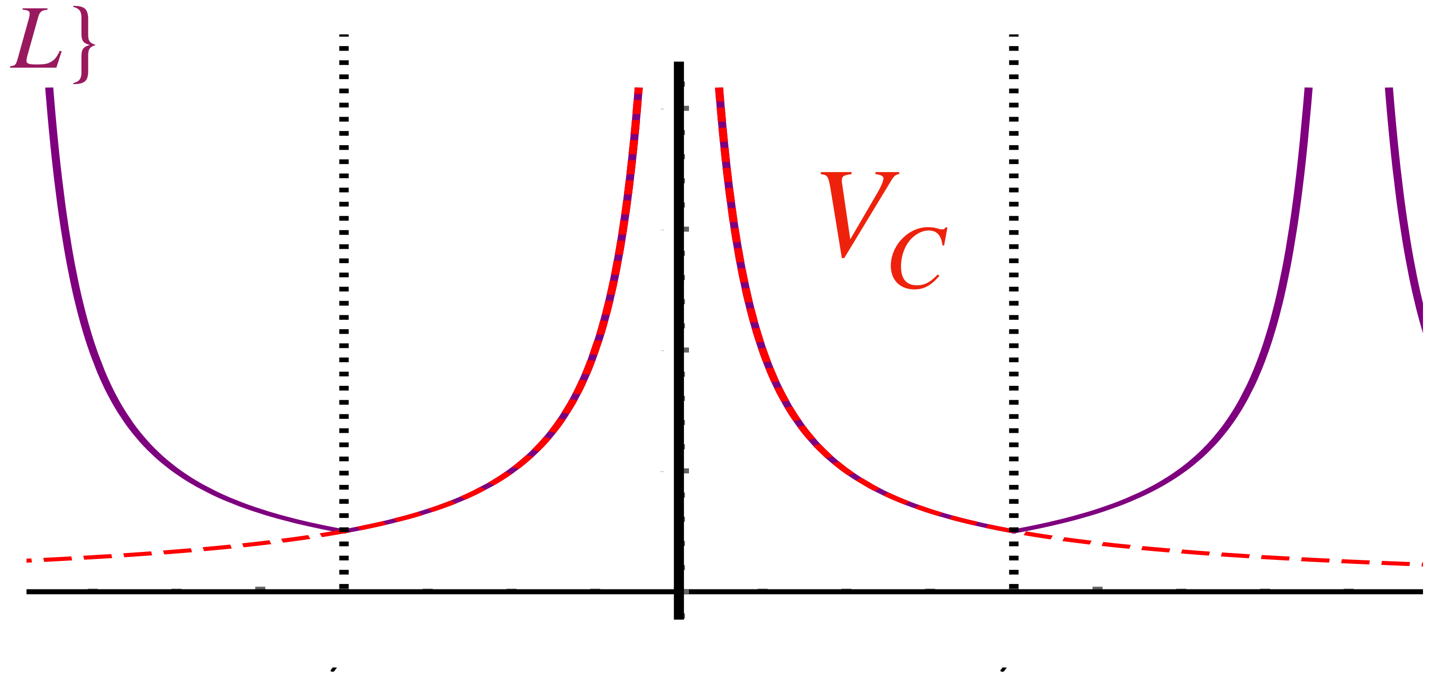
$$H = -\frac{1}{2\mu}\nabla^2 + V + V_C$$

$$V_L(x) = \sum_{\mathbf{n}} V(x + \mathbf{n}L)$$

$$H_L = -\frac{1}{2\mu}\nabla^2 + V_L + V_{C,\{L\}}$$

In finite volume





P. Guo, PRC ***103*** *064611 (2021),*
P. Guo et. al., PRD ***103*** *094520 (2021).*

With COM frame coordinates:

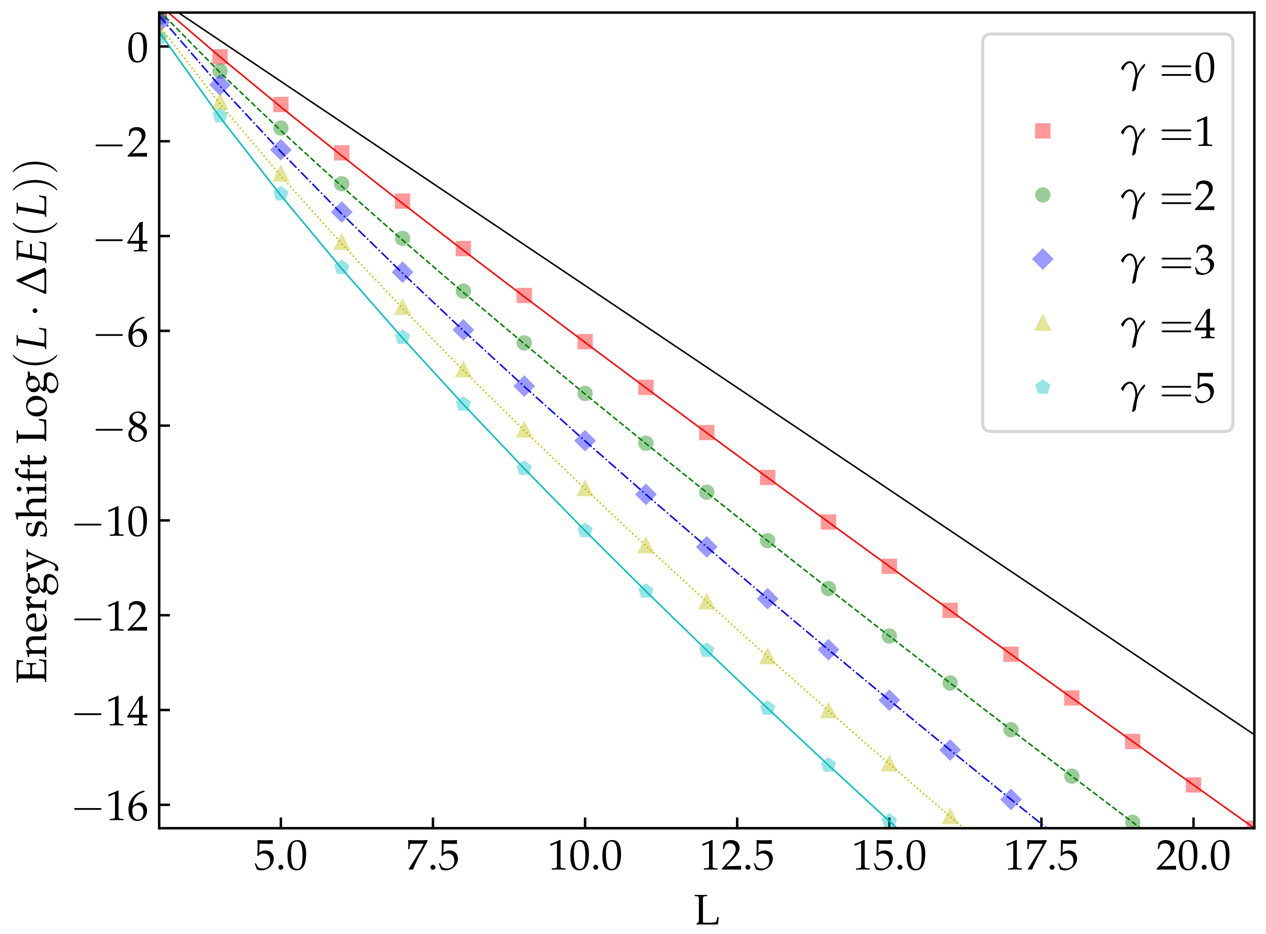
$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$

$$V_C(r) = \frac{\gamma}{2\mu r}, \quad \gamma = 2\mu\alpha Z_1 Z_2 > 0$$

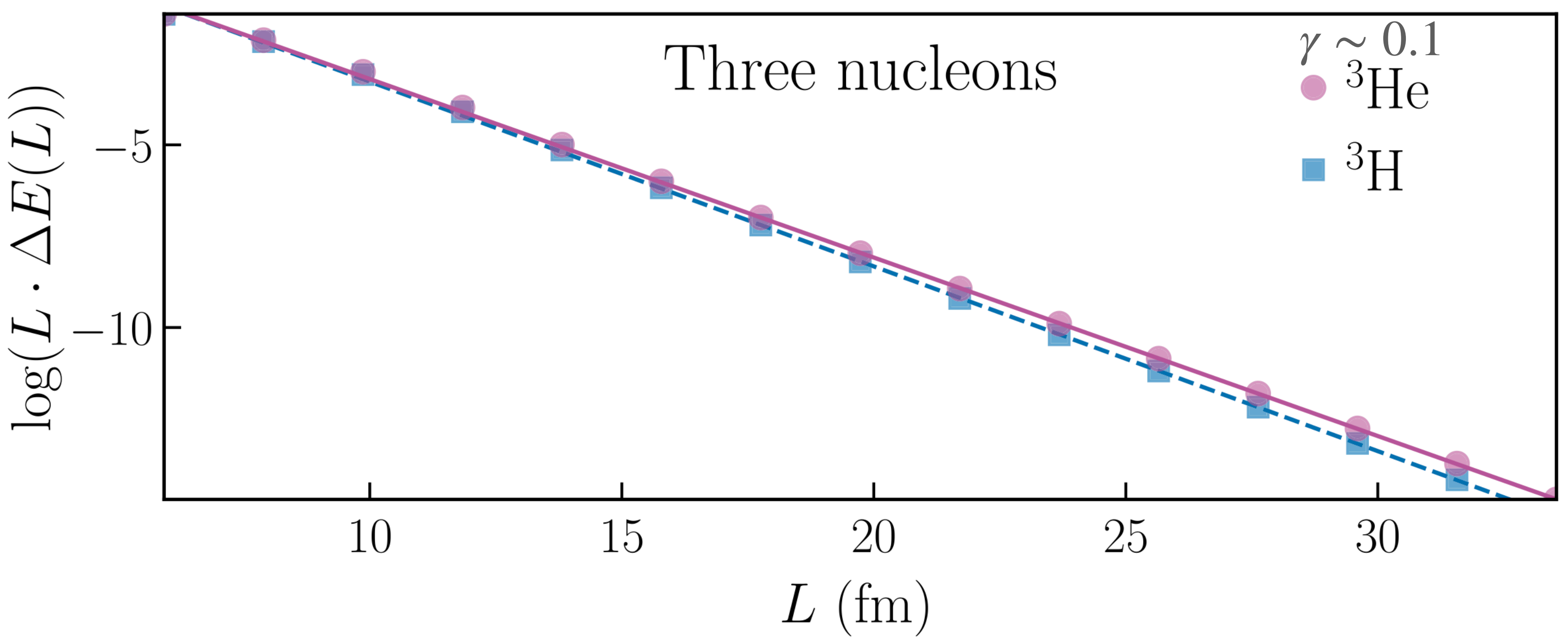
$$\psi_\infty(r) \to \frac{C_0}{\sqrt{4\pi r}} W_{-\bar{\eta},1/2}(2\kappa r)$$

$$\bar{\eta} = \frac{\gamma}{2\kappa}$$

Main result



	Finite-volume fit			Continuum result	
γ	κ_∞	C_0	L range	κ_∞	C_0
1.0	0.8610(3)	5.039(2)	17–28	0.861	5.049
2.0	0.8607(3)	11.71(4)	15–26	0.860	11.79
3.0	0.8605(7)	29.95(20)	14–24	0.859	30.31
4.0	0.8604(1)	83.14(10)	14–22	0.858	84.76
5.0	0.8604(2)	247.9(5)	14–18	0.857	255.4



$$\Delta E(L) = -\frac{3C_0^2}{\mu L}\left[W_{-\bar{\eta},\frac{1}{2}}(\kappa L)\right]^2 + \Delta\tilde{E}(L) + \Delta\tilde{E}'(L) + \mathcal{O}(e^{-\sqrt{2}\kappa L})$$

HY, S. König, D. Lee, PRL, **131,** 212 502 (2023)

Milestones of NLEFT

- First ab initio Calculation of 12C Holye state

  E. Epelbaum, et. al. PRL ***106*** *192501 (2011)*

- First ab initio alpha - alpha scattering

  S. Elhatisari, et. al. Nature **528** 111 (2015)

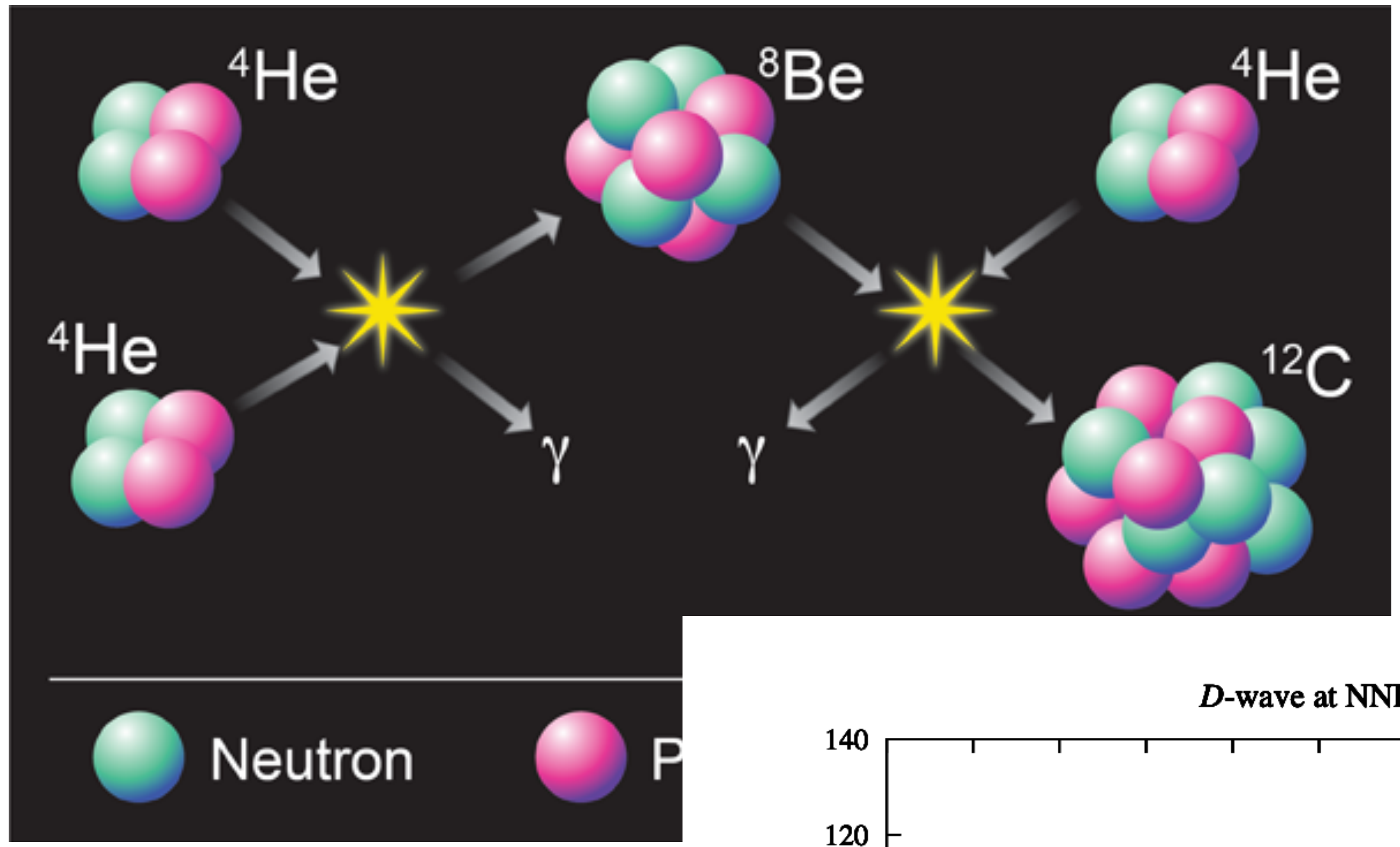
- Accessing correlated densities

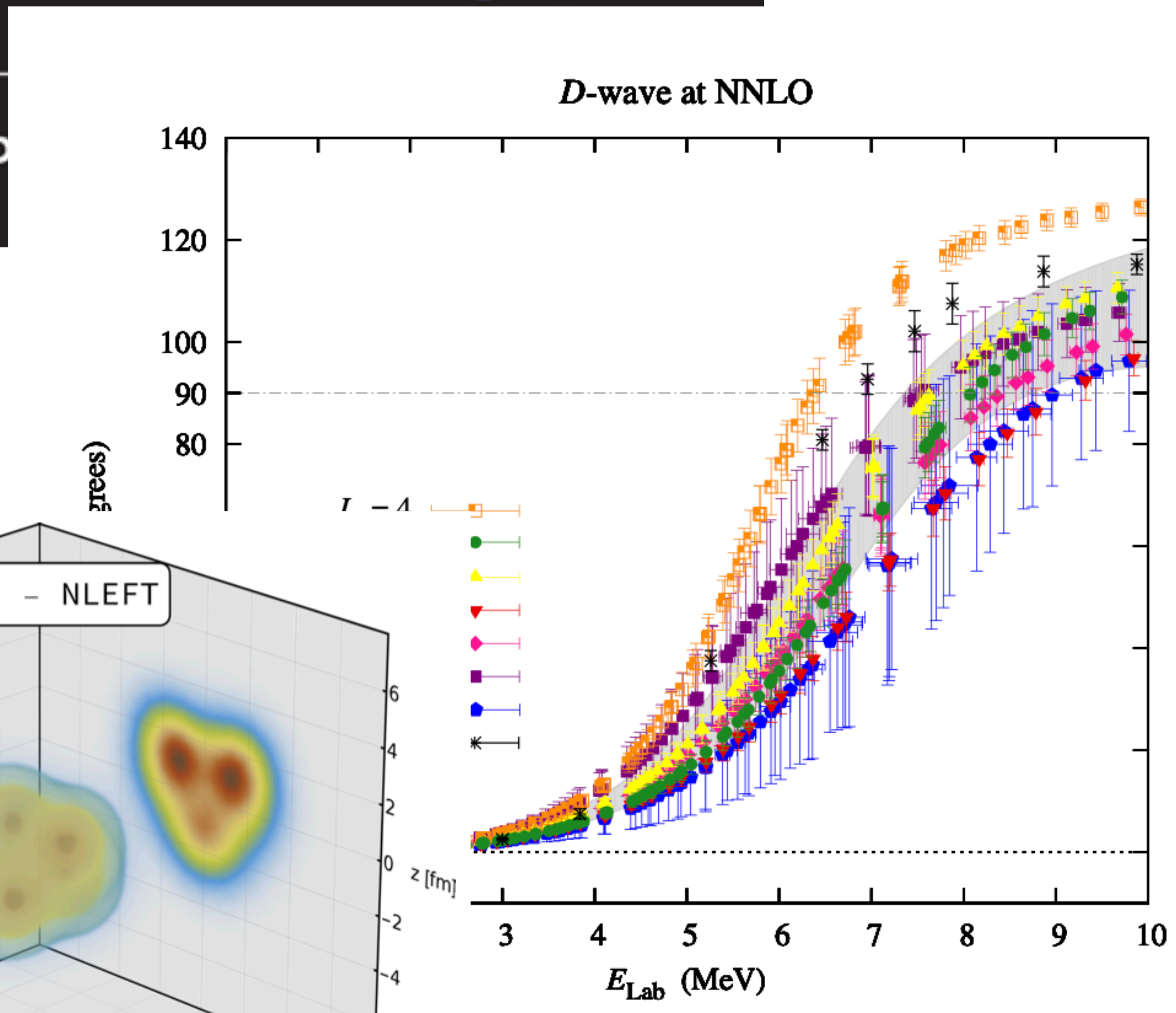
  *S. Elhatisari, et. al. PRL***119**, 222505 (2017)

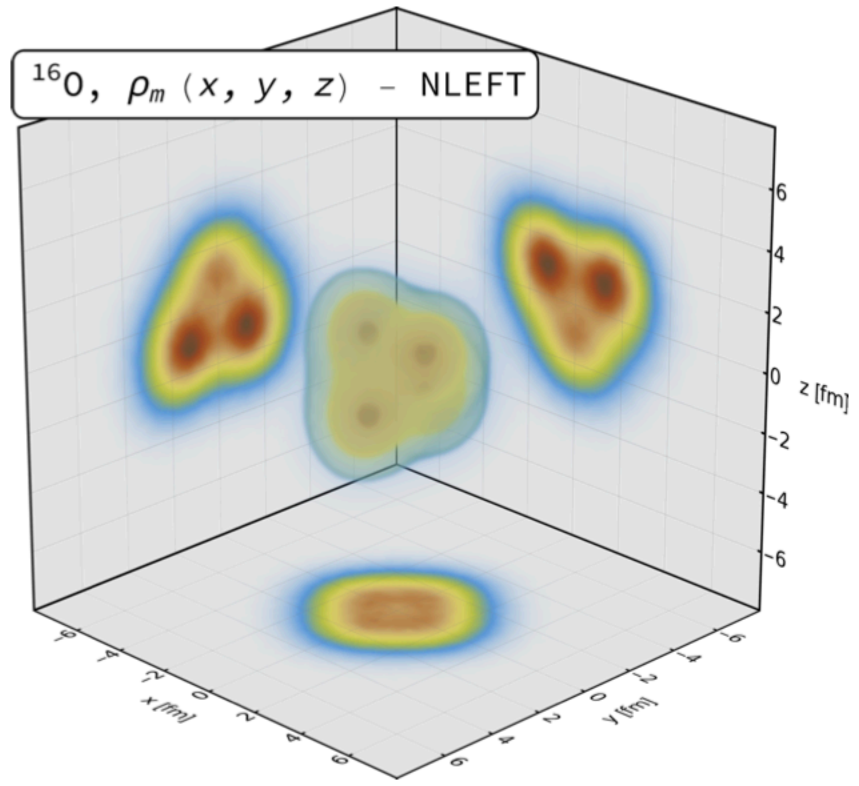
- Now: Free from sign problem
  (Wave function matching + SU(4) invariant interactic

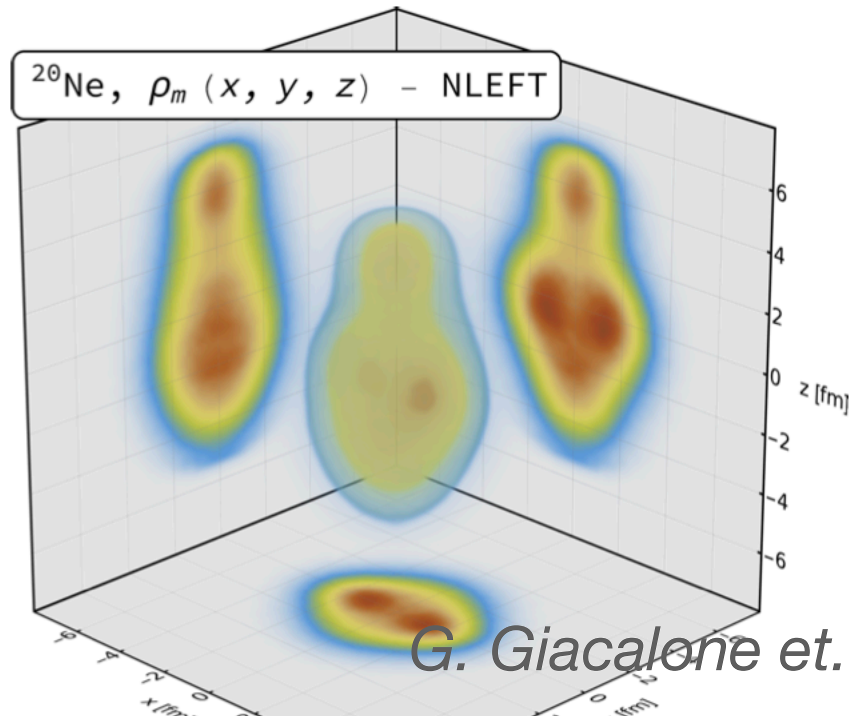
S. Elhatisari, et. al. Nature **630** 8015 (2024)

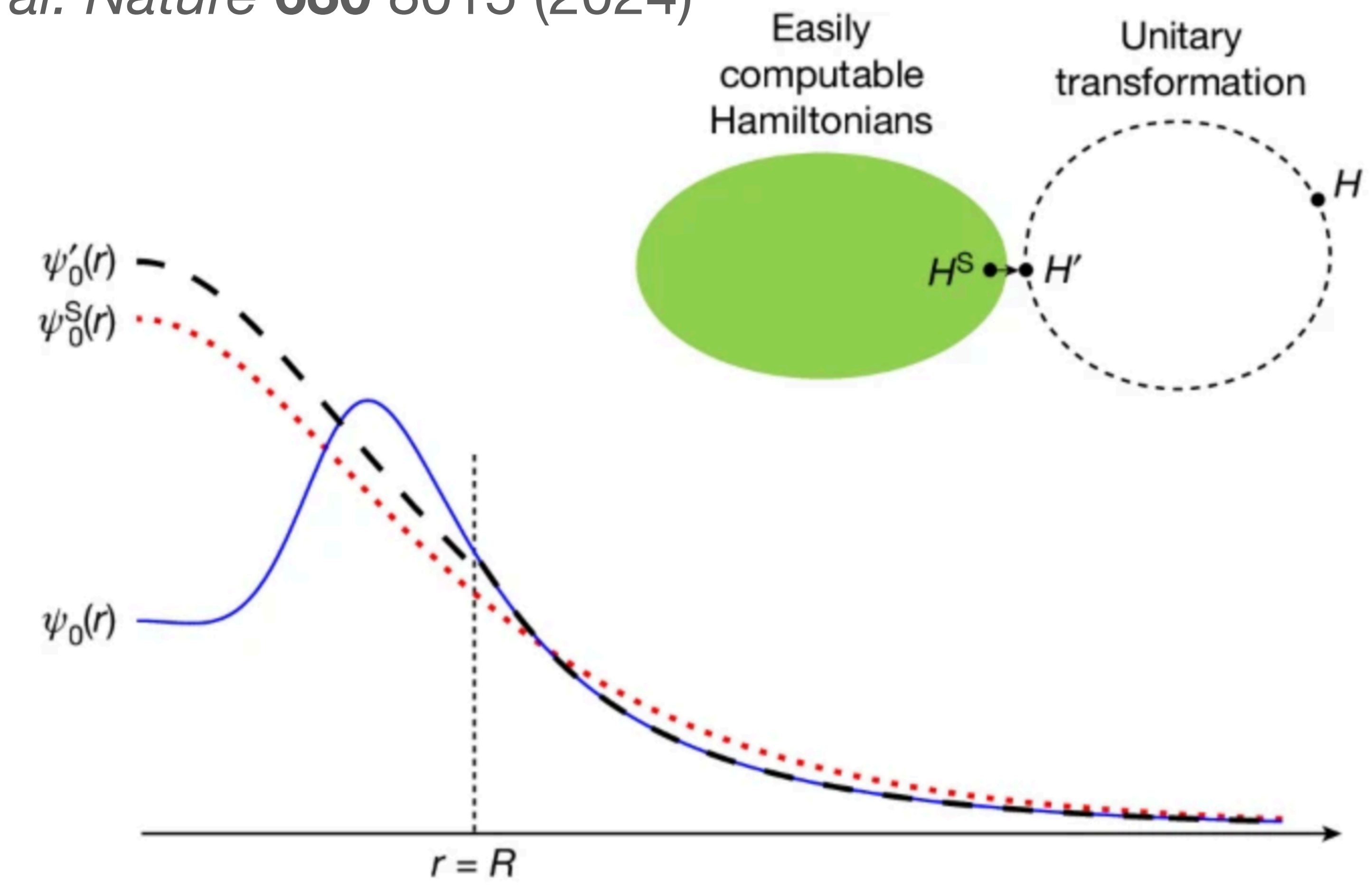
B. Lu, et. al. PLB, **797** 134863(2019)











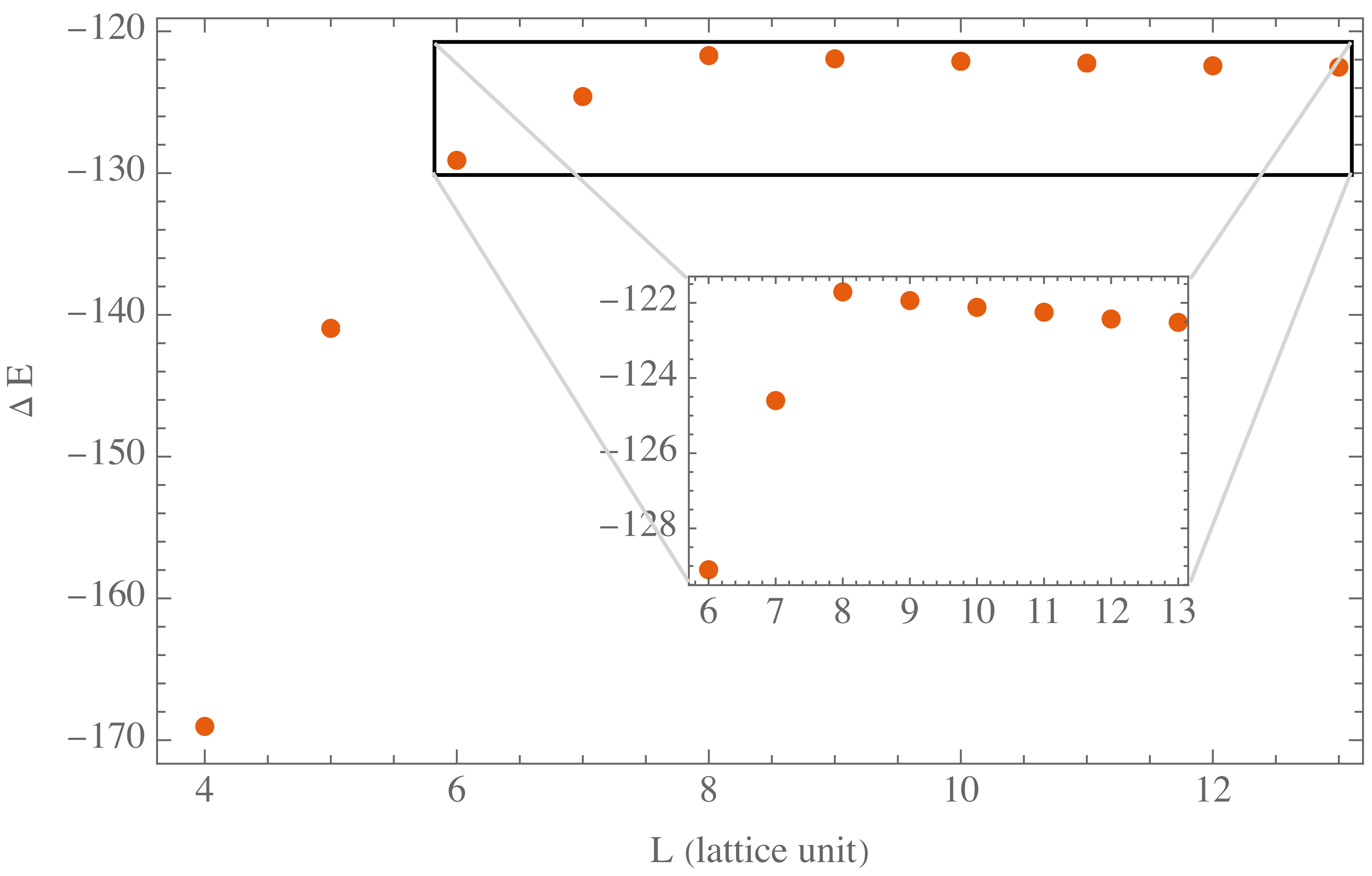
G. Giacalone et. al. arXiv:2402.05995

16O: a Multi-Channel Case

Prelim results & PhD Thesis

16O: a Multi-Channel Case

Prelim results & PhD Thesis



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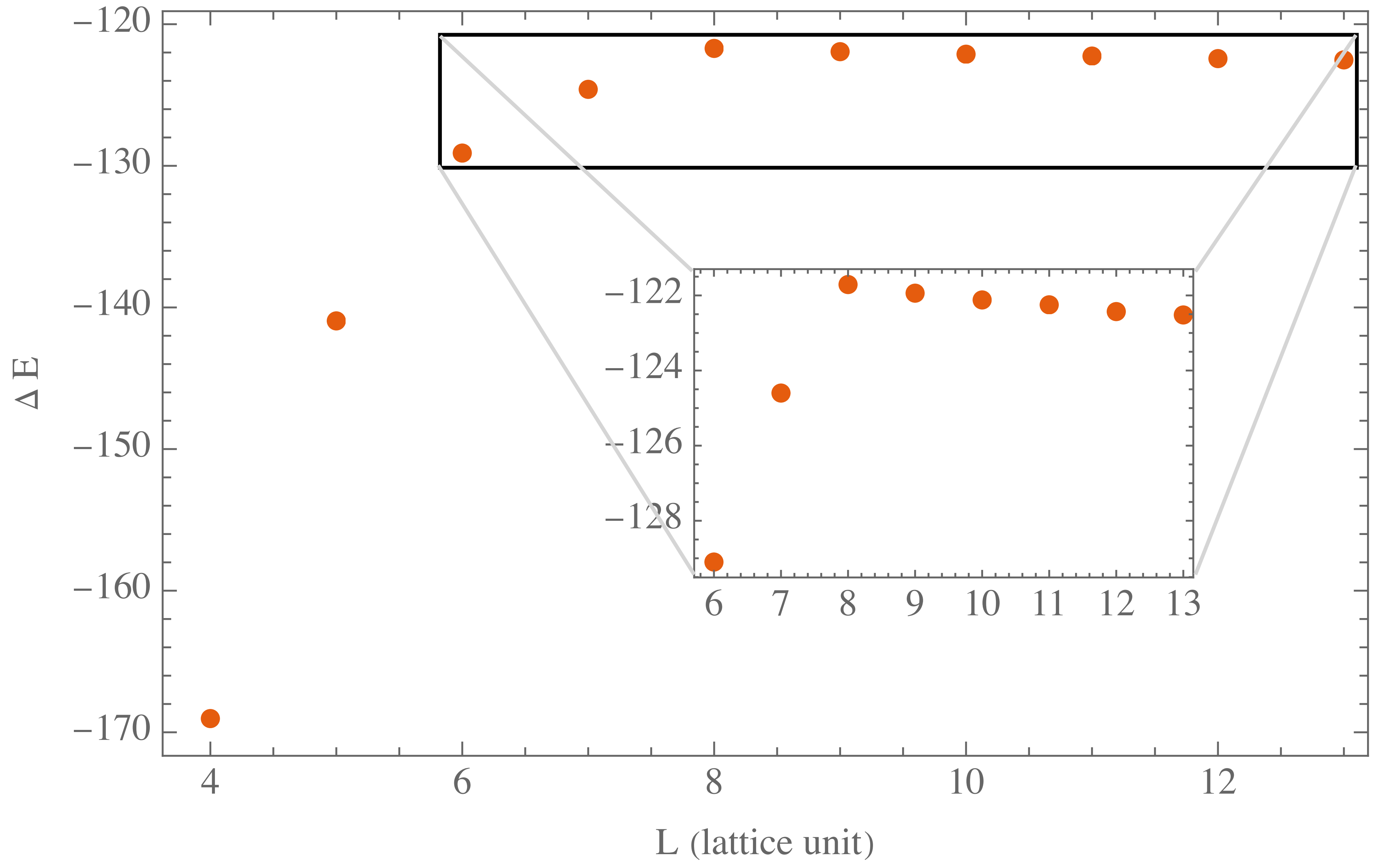
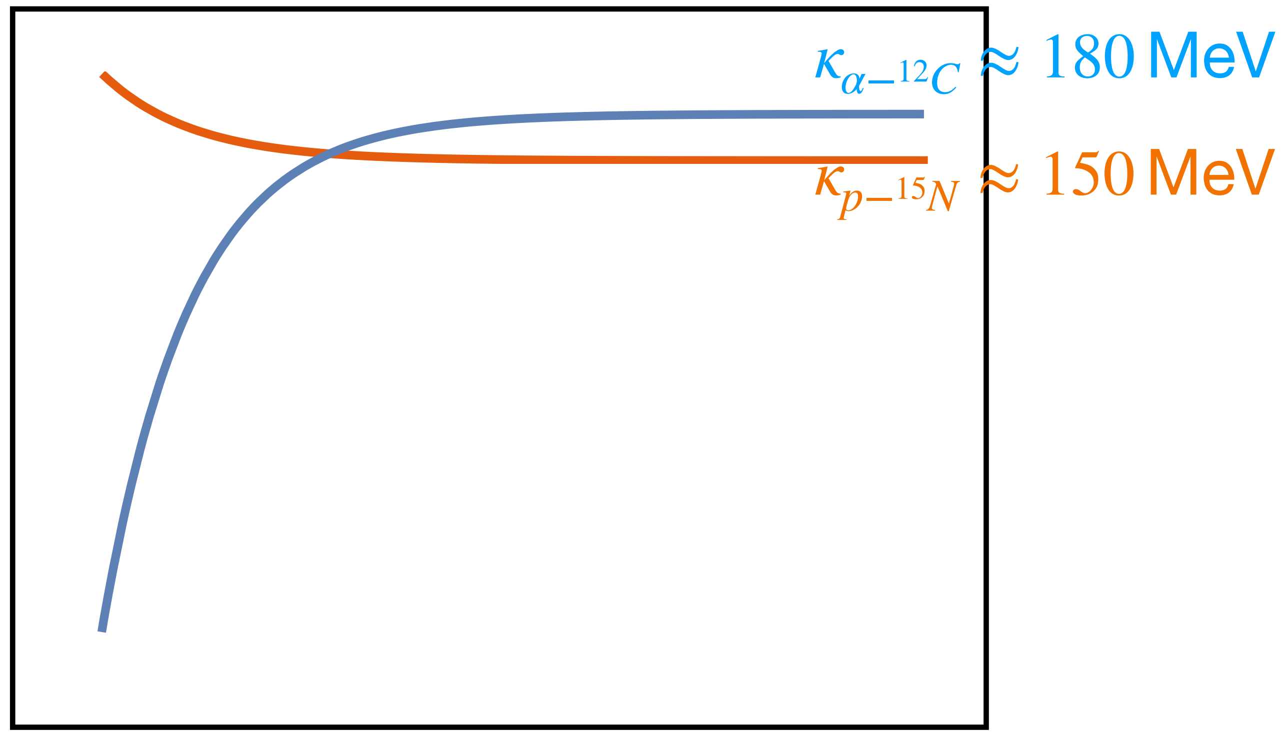
Prelim results & PhD Thesis





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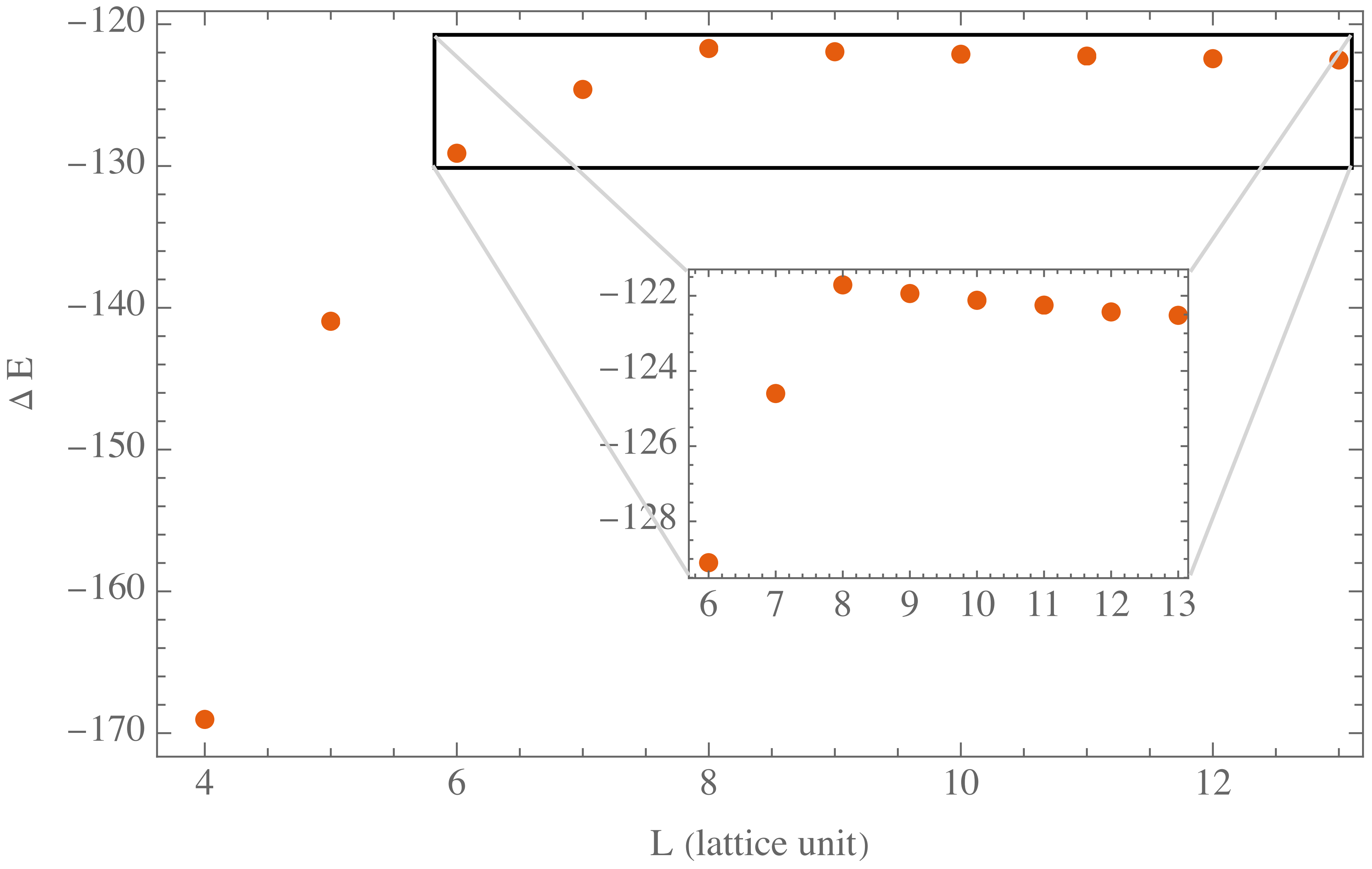

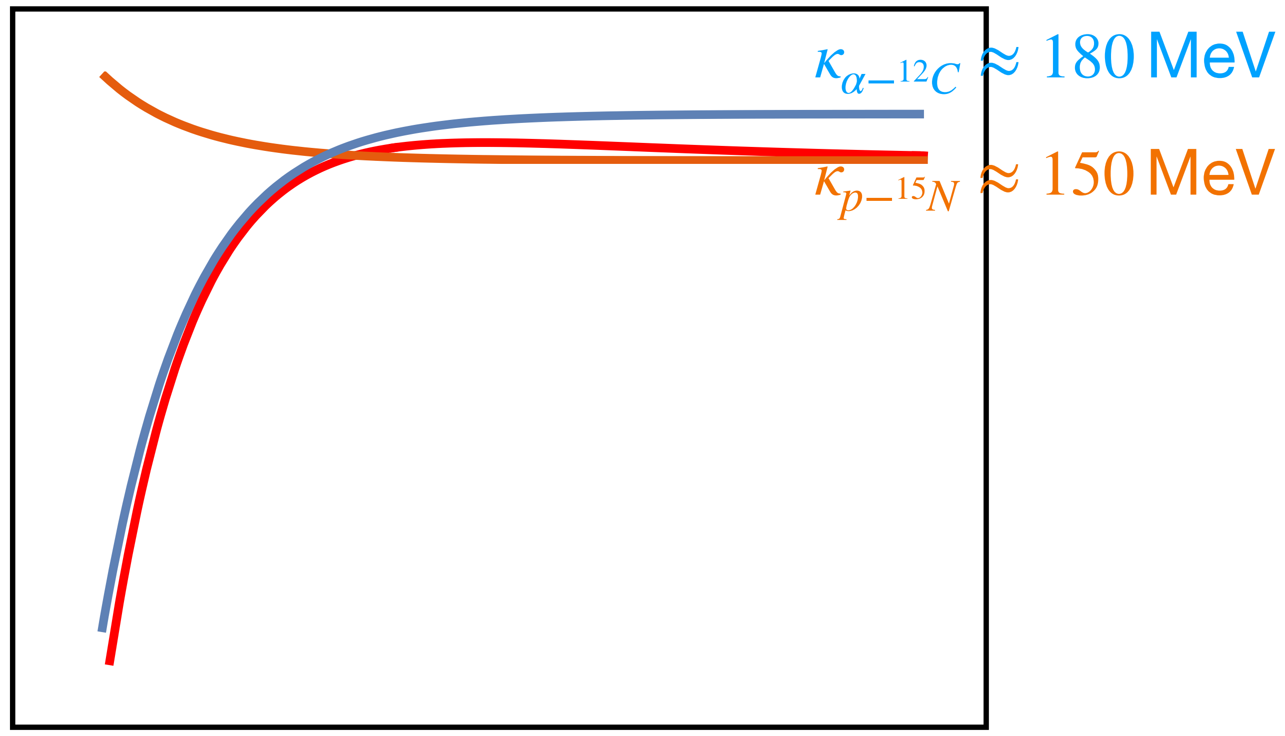

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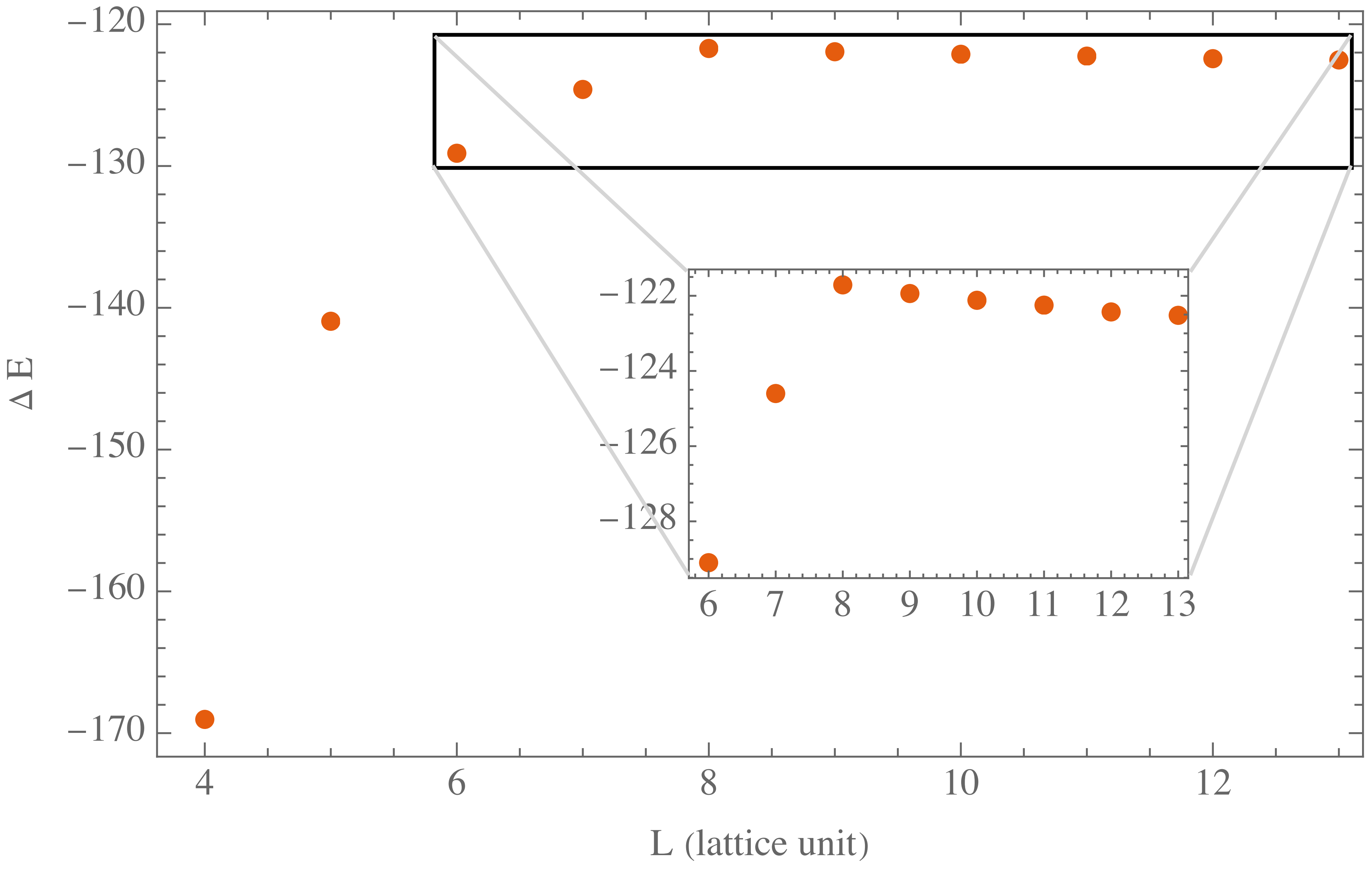

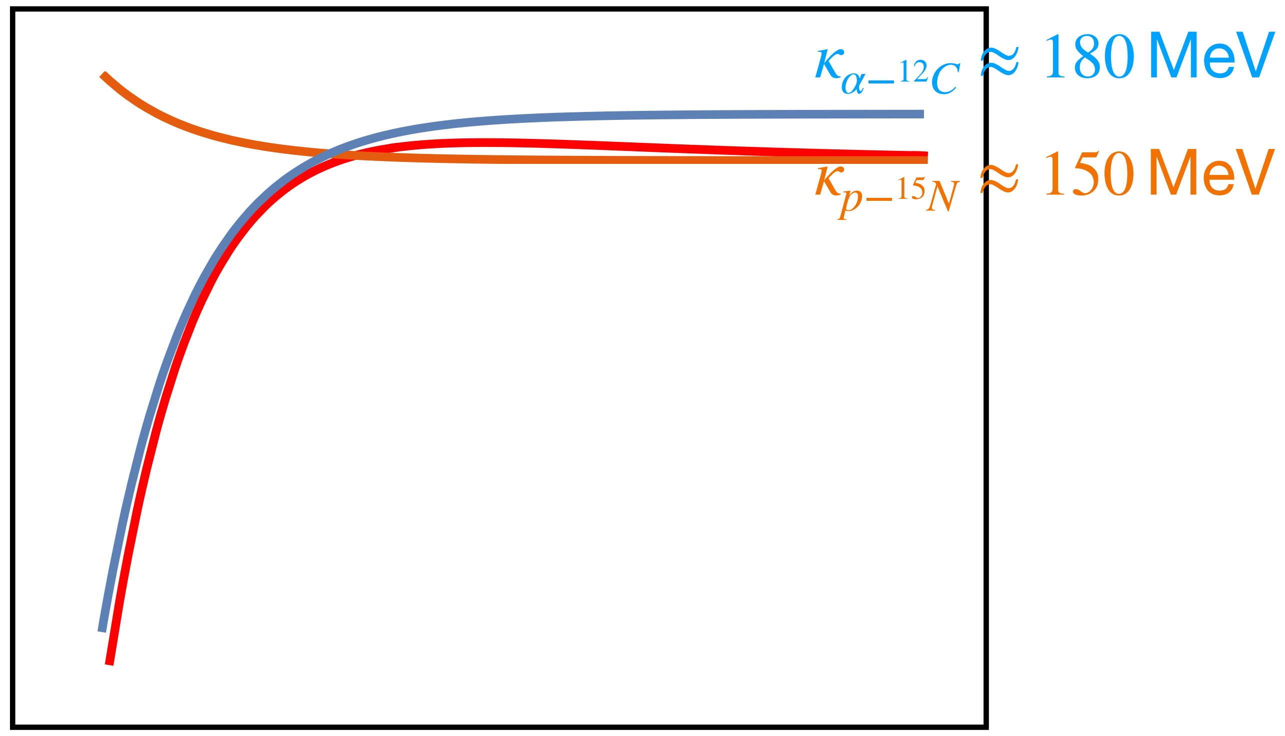

10% difference in Binding Energy!

16O: a Multi-Channel Case

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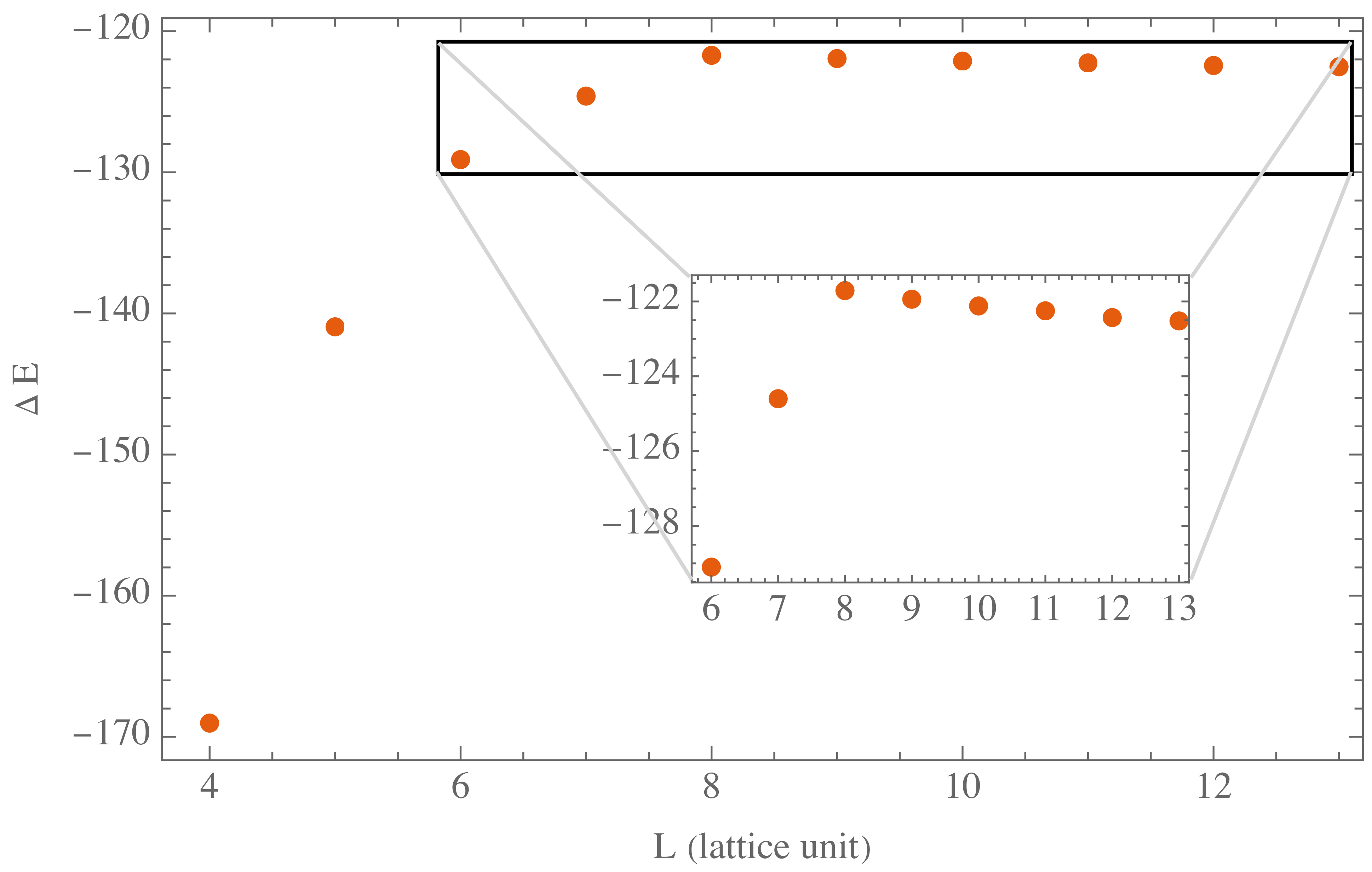

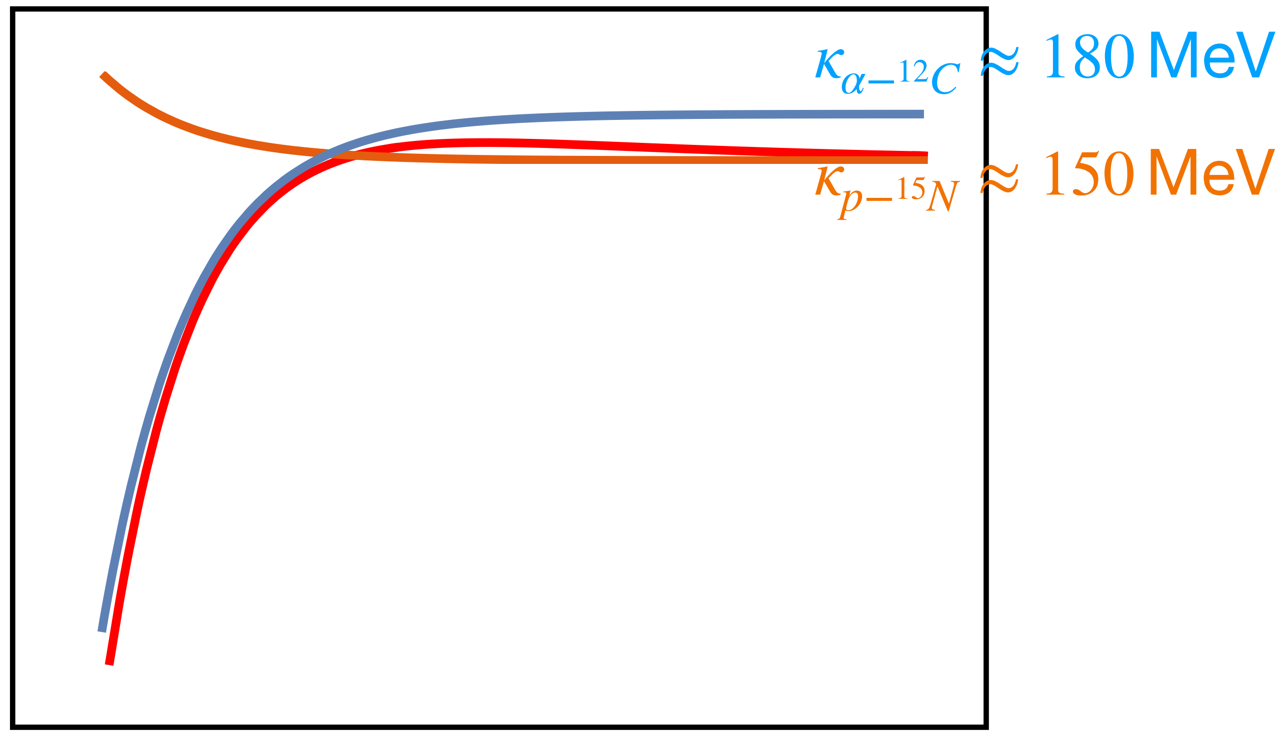

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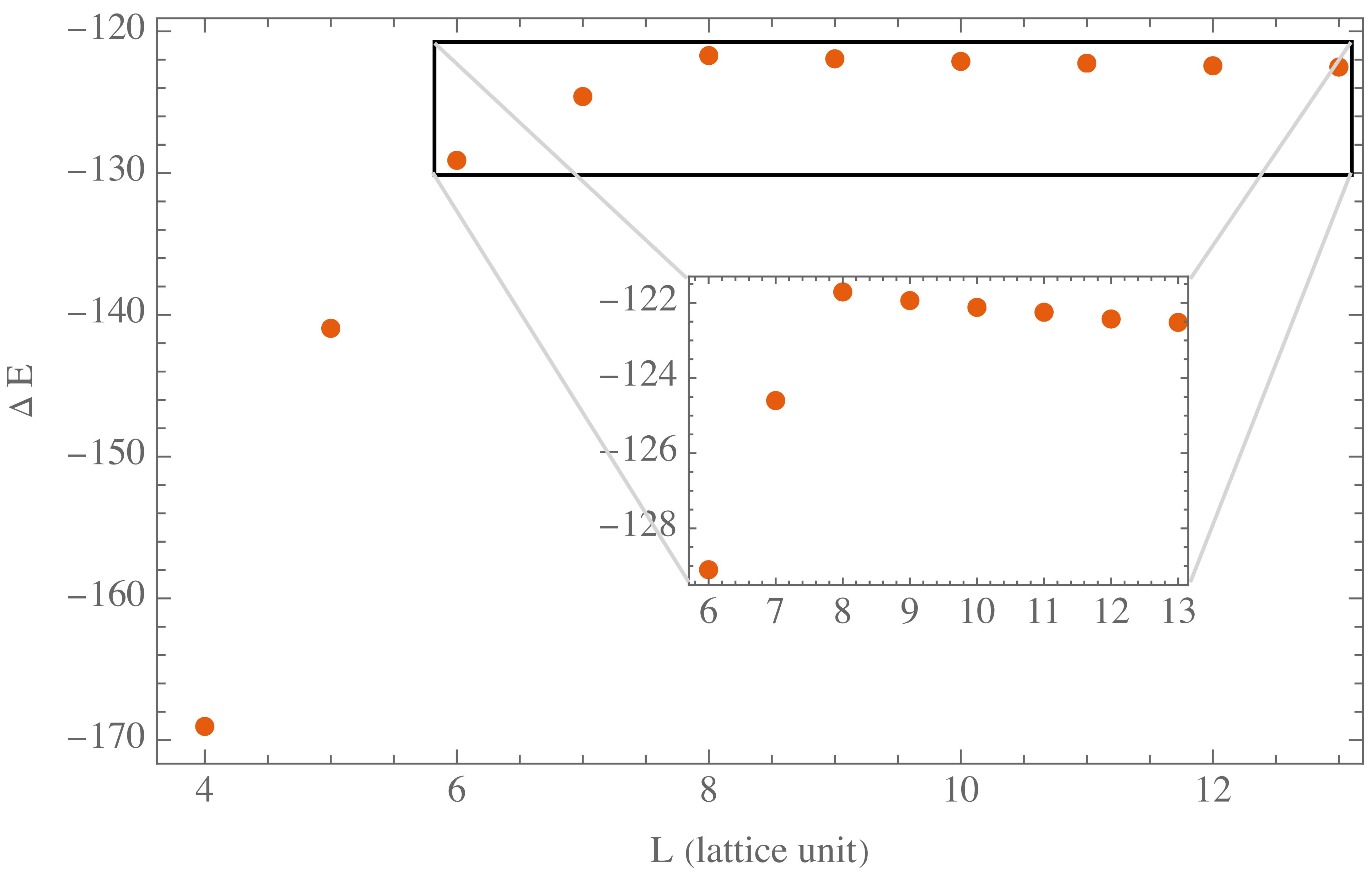
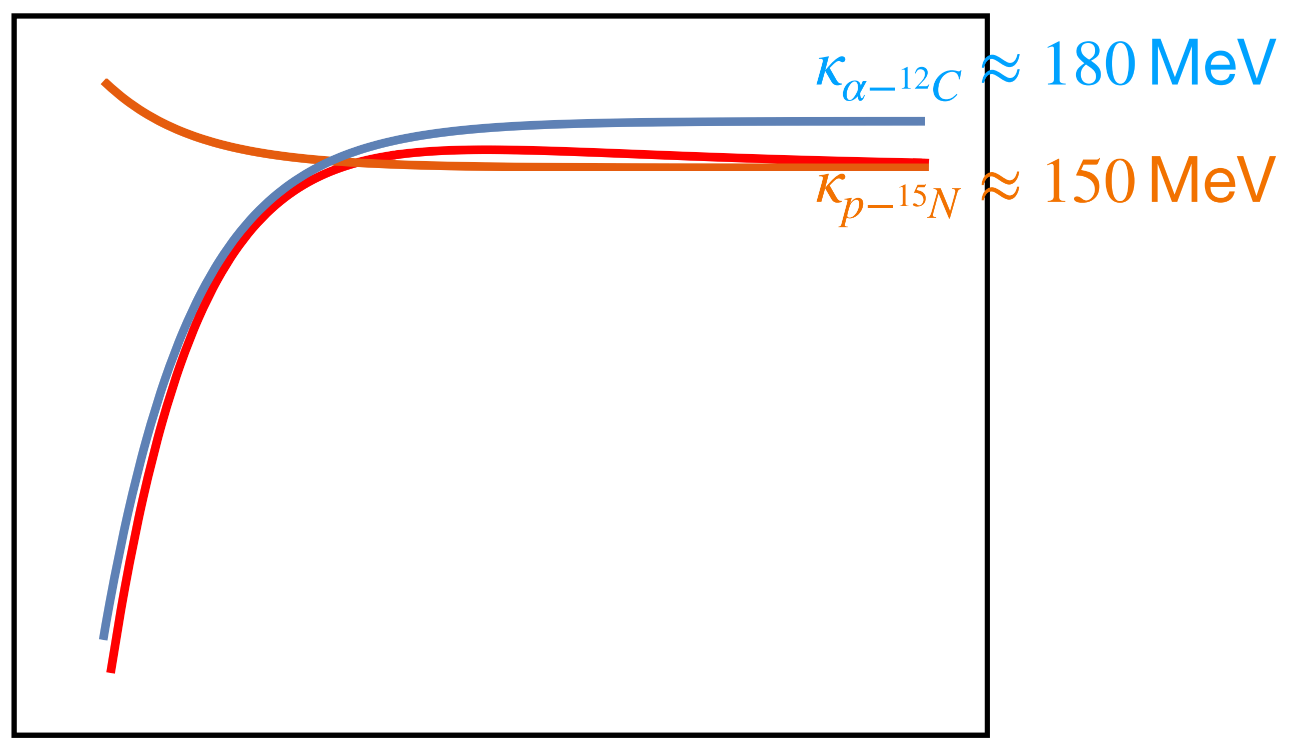
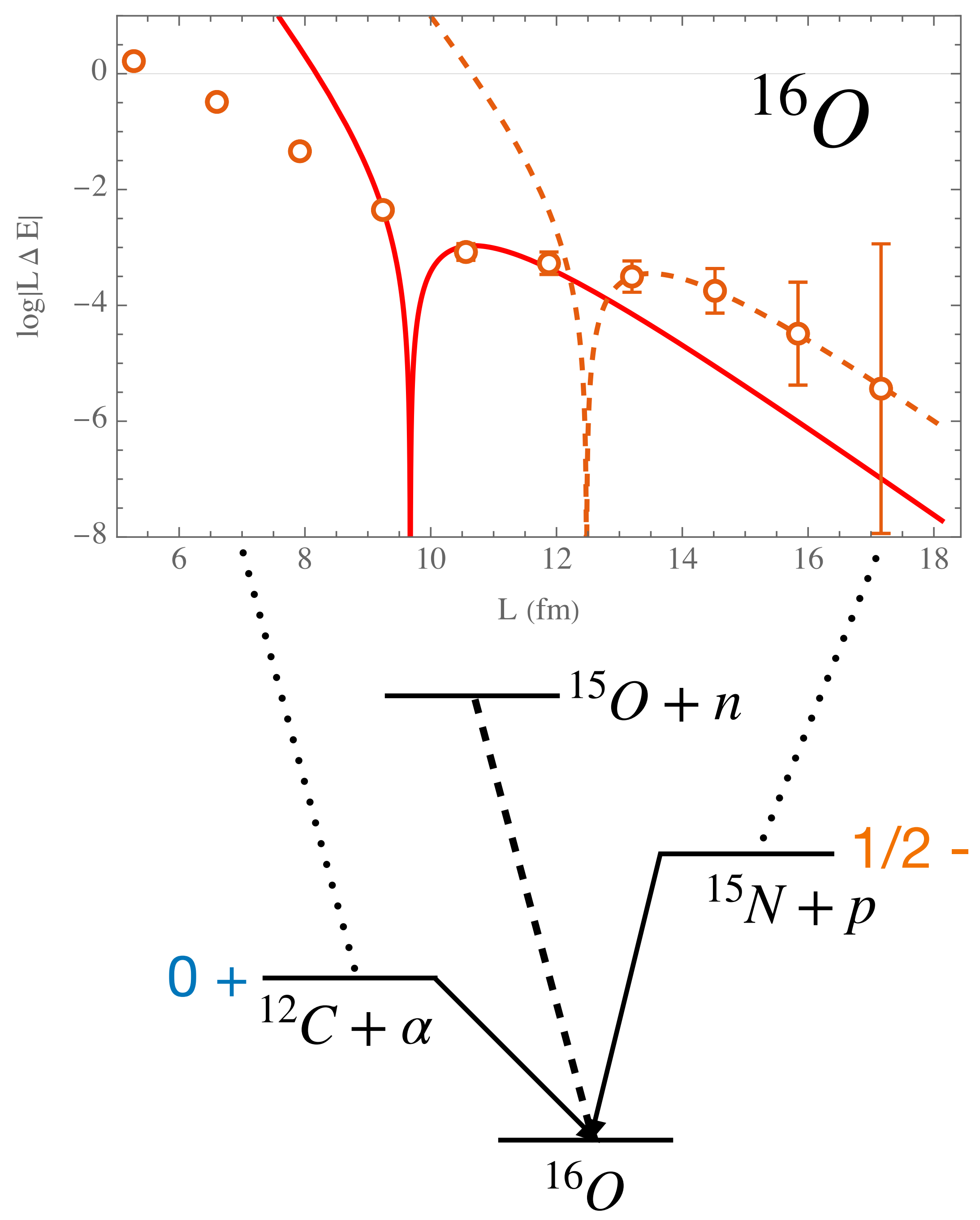
Problem: overfitting

16O: a Multi-Channel Case

Prelim results & PhD Thesis







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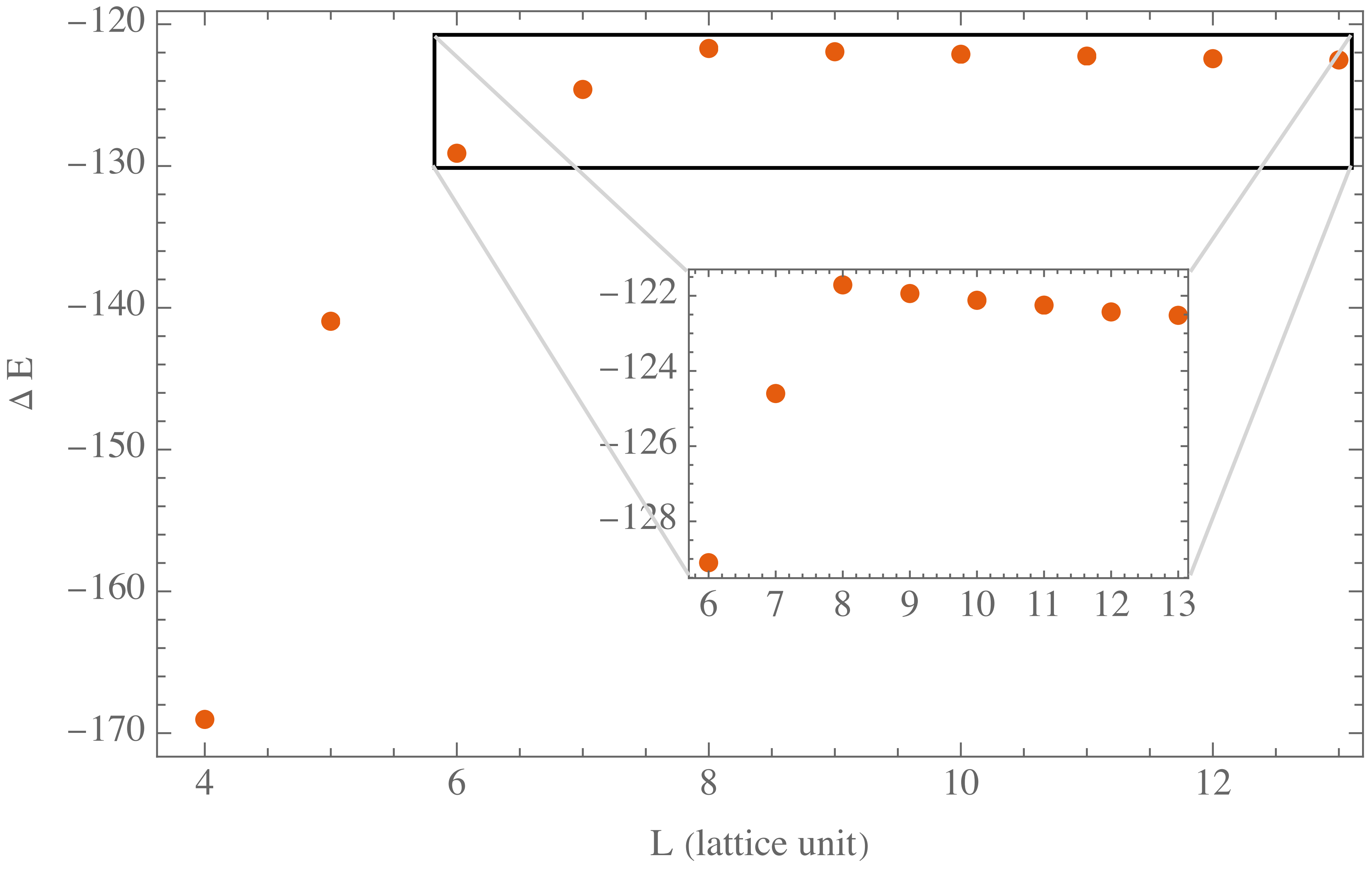

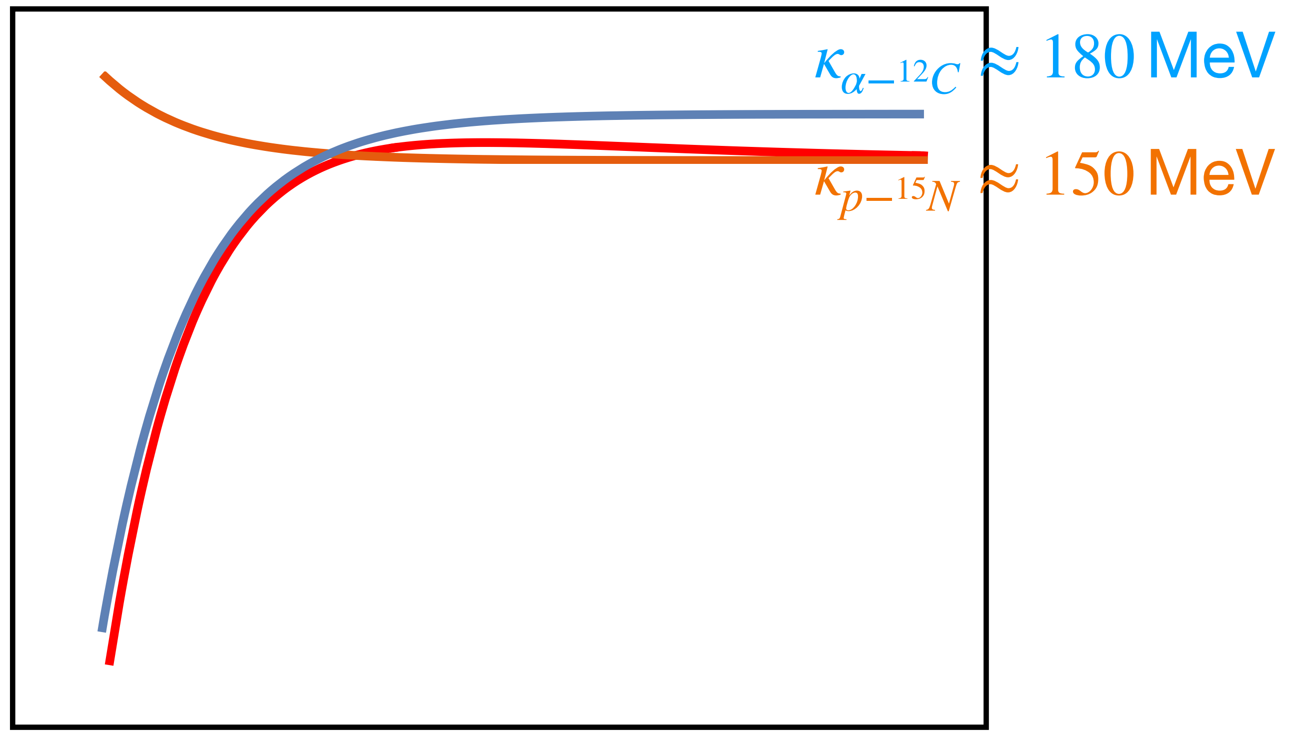

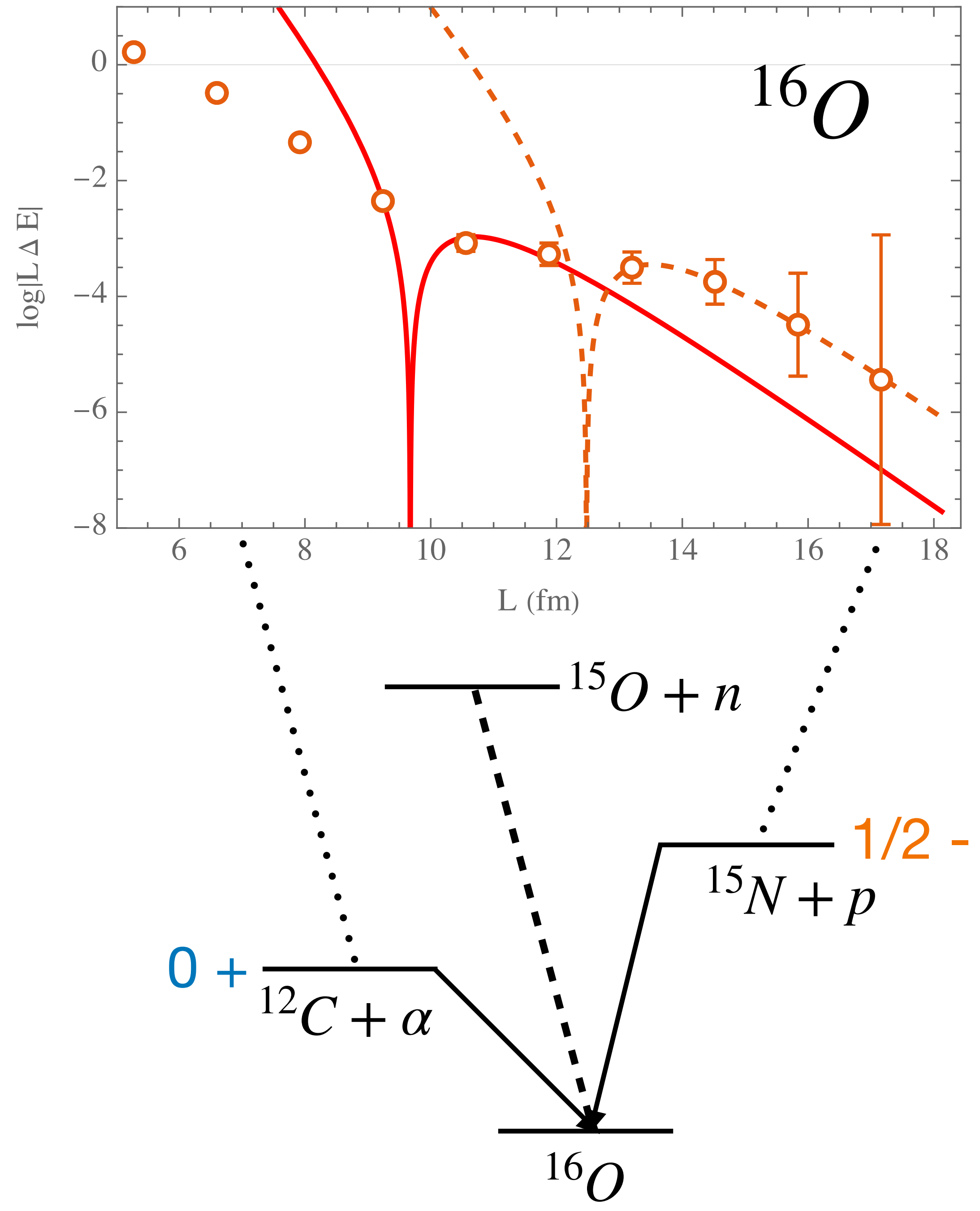

10% difference in Binding Energy!

Problem: overfitting

Solution: Applying for grants

ANCs of 16O and 20Ne

Isotope States	ANCs ($fm^{-1/2}$)	Reference
$^{16}O(0^+; G.S.)$	11.01	Orlov [2021]
	58	deBoer et al. [2017]
	709	Sayre et al. [2012]
	13.9(24)	Adhikari and Basu [2009]
	750-4000	Morais and Lichtenthäler [2011]
	380 ± 80	Pinhole algorithm
	210 ± 20	This work (w/o fitting separation energy)
	980 ± 100	This work (w/ fitting separation energy)
$^{20}Ne(0^+; G.S.)$	$3.3(1) \times 10^3$	Orlov [2021]
	2500 (1150)	Costantini et al. [2010]
	3400(700)	Costantini et al. [2010]; Mao et al. [1996]
	$3.80(95) \times 10^3$	Pinhole algorithm
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HY, et. al. 2024 (WIP) SOON™

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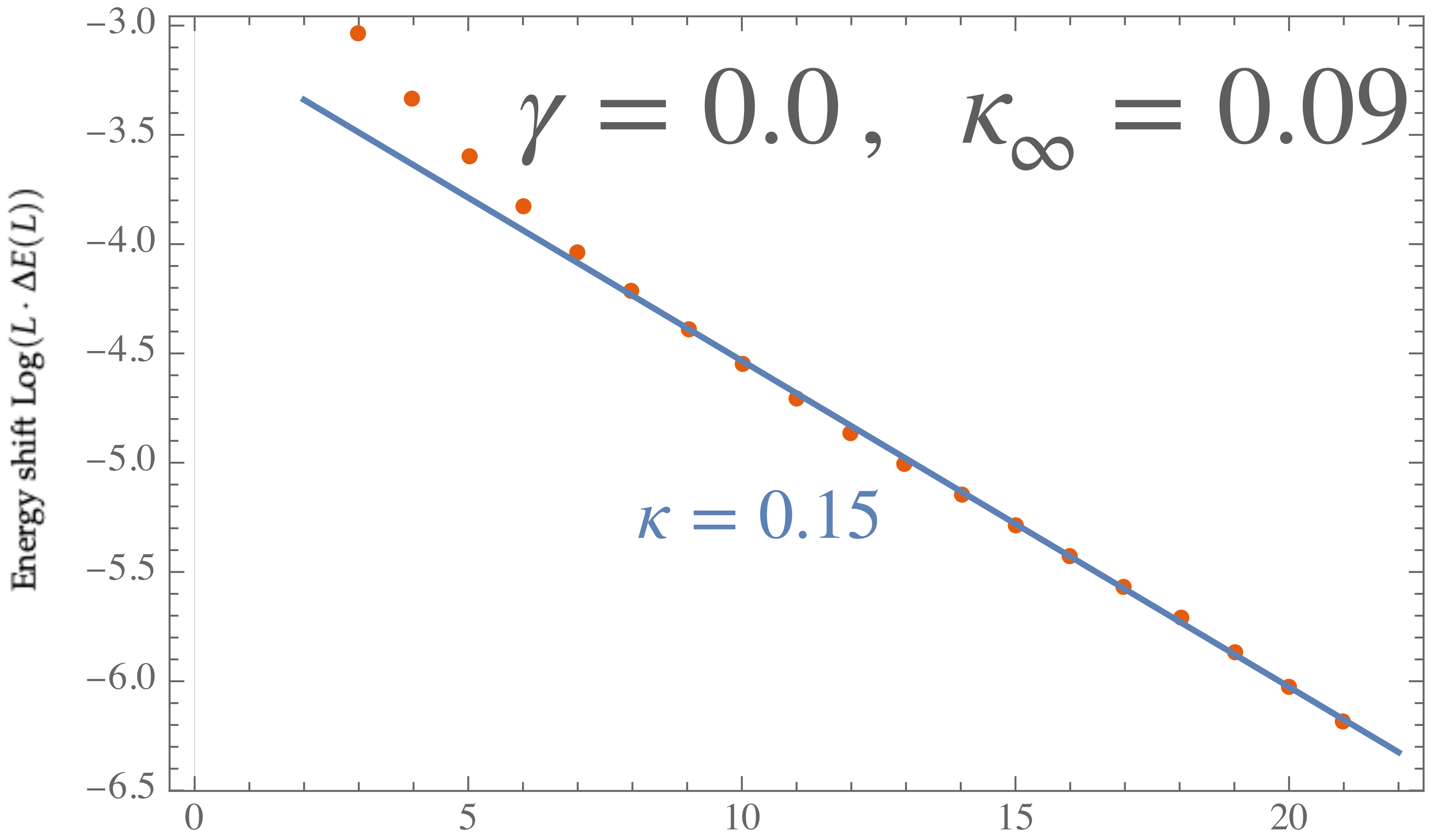
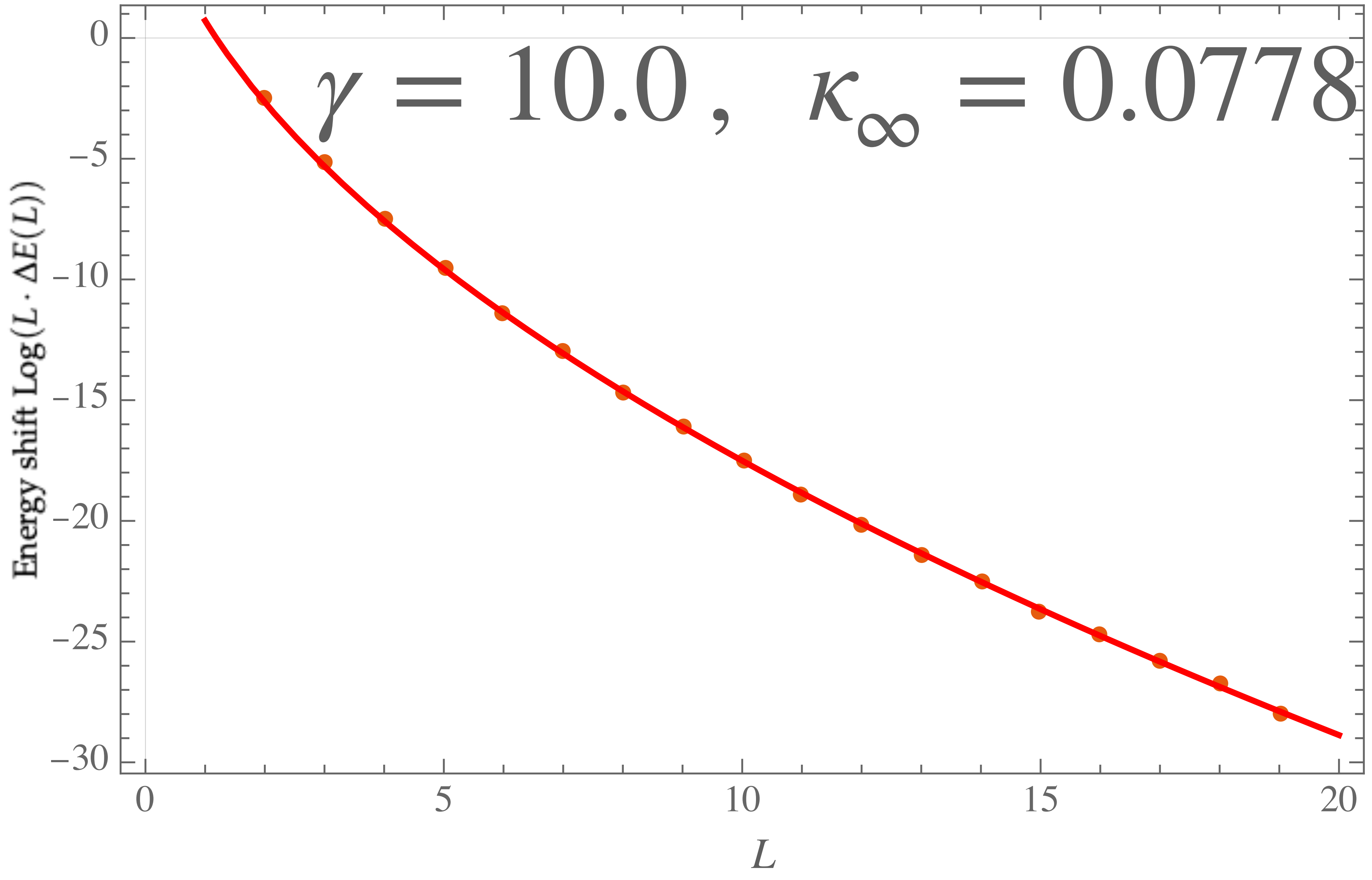
+ Geometric Factors

HY, et. al. 2024 (WIP) SOON™

So, it is working. And…

Exotic bound-states with coulomb:





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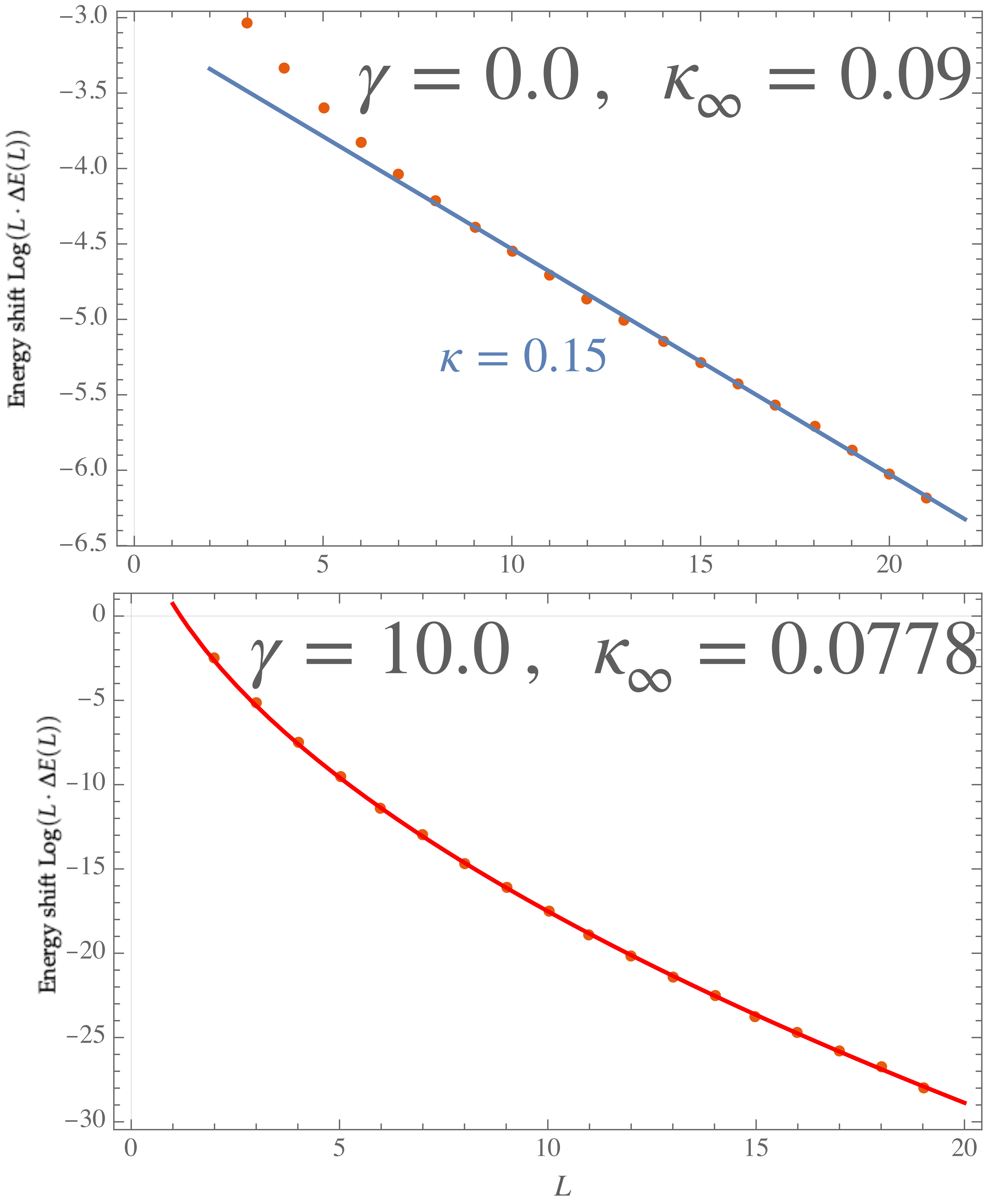

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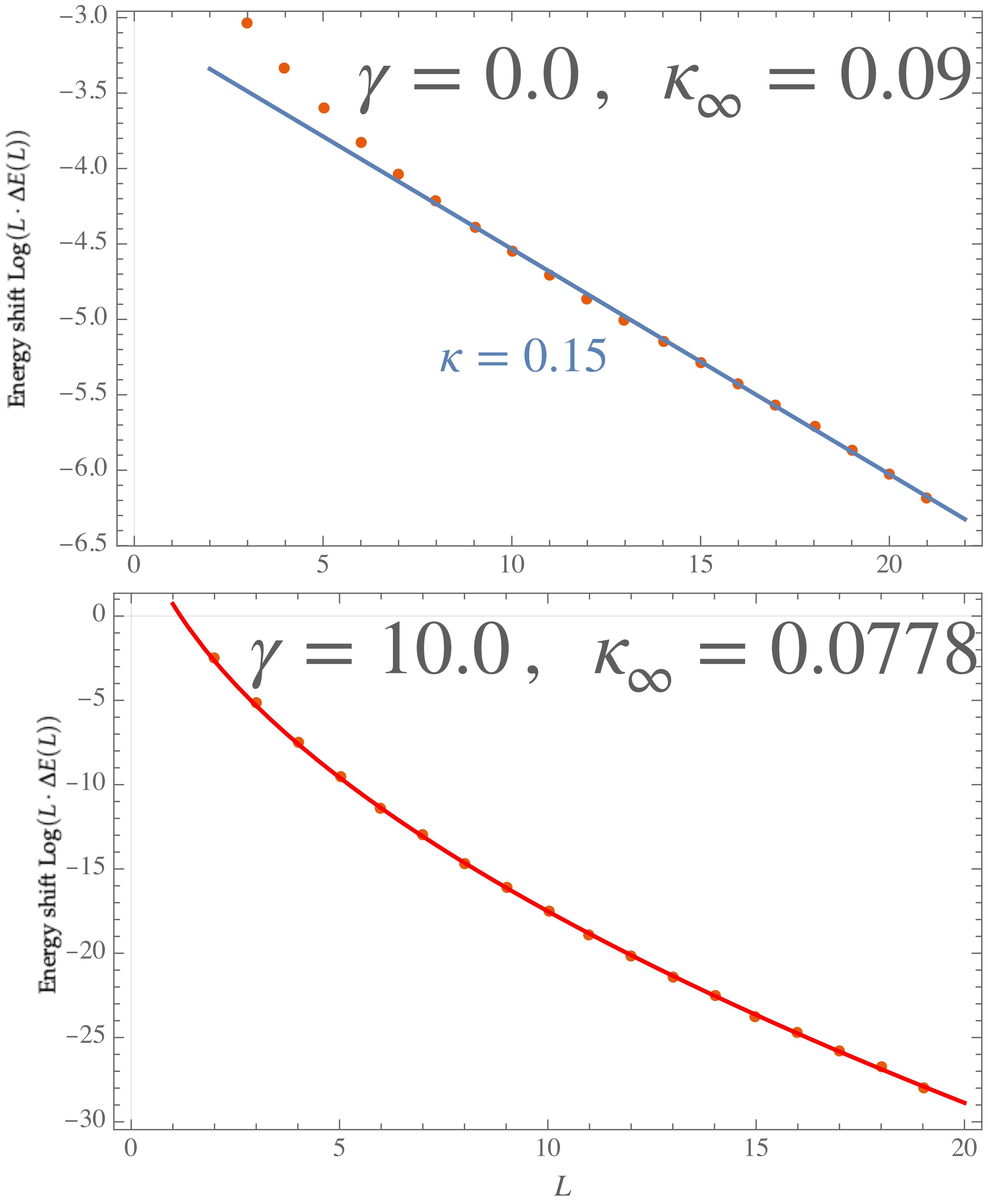

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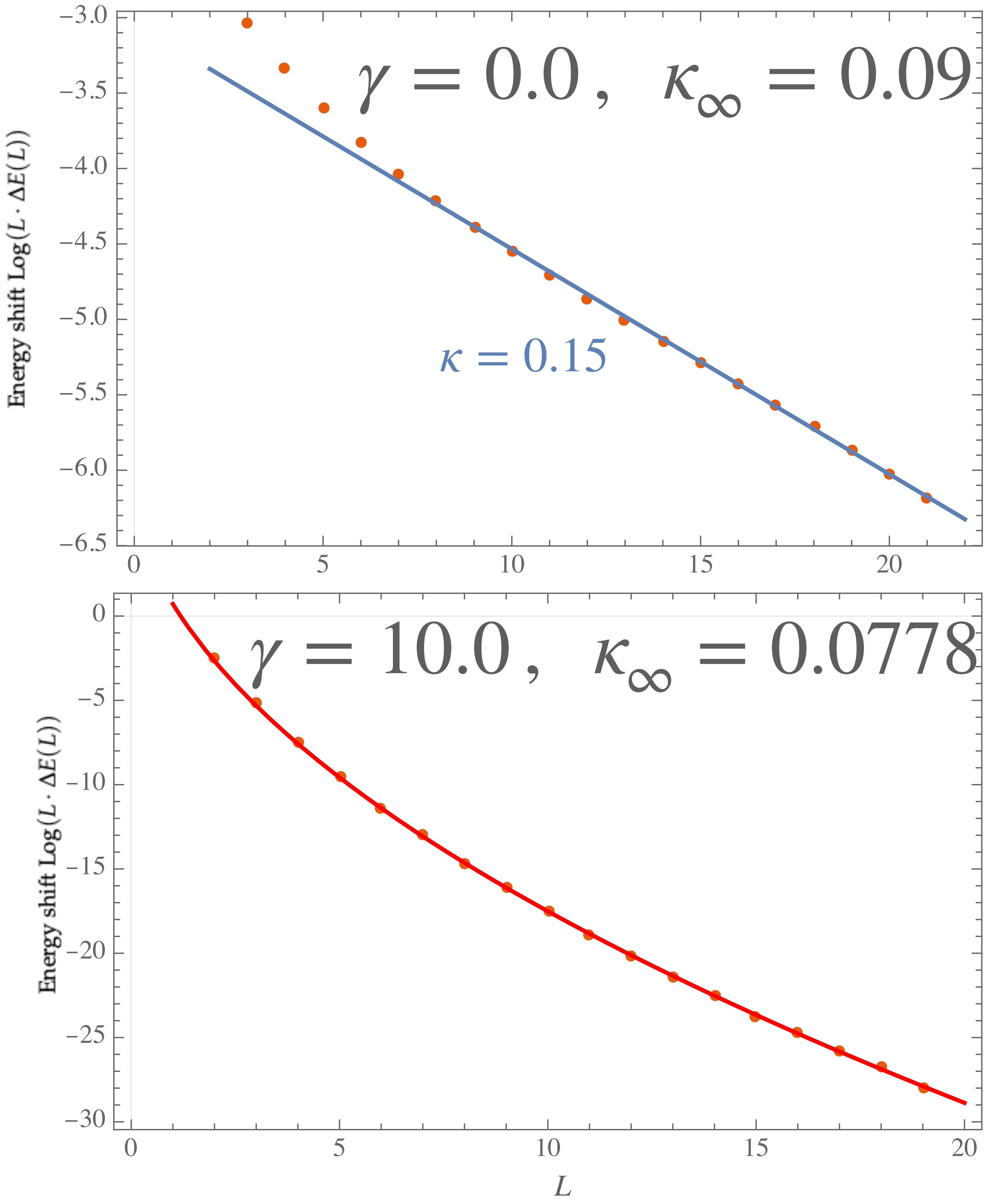

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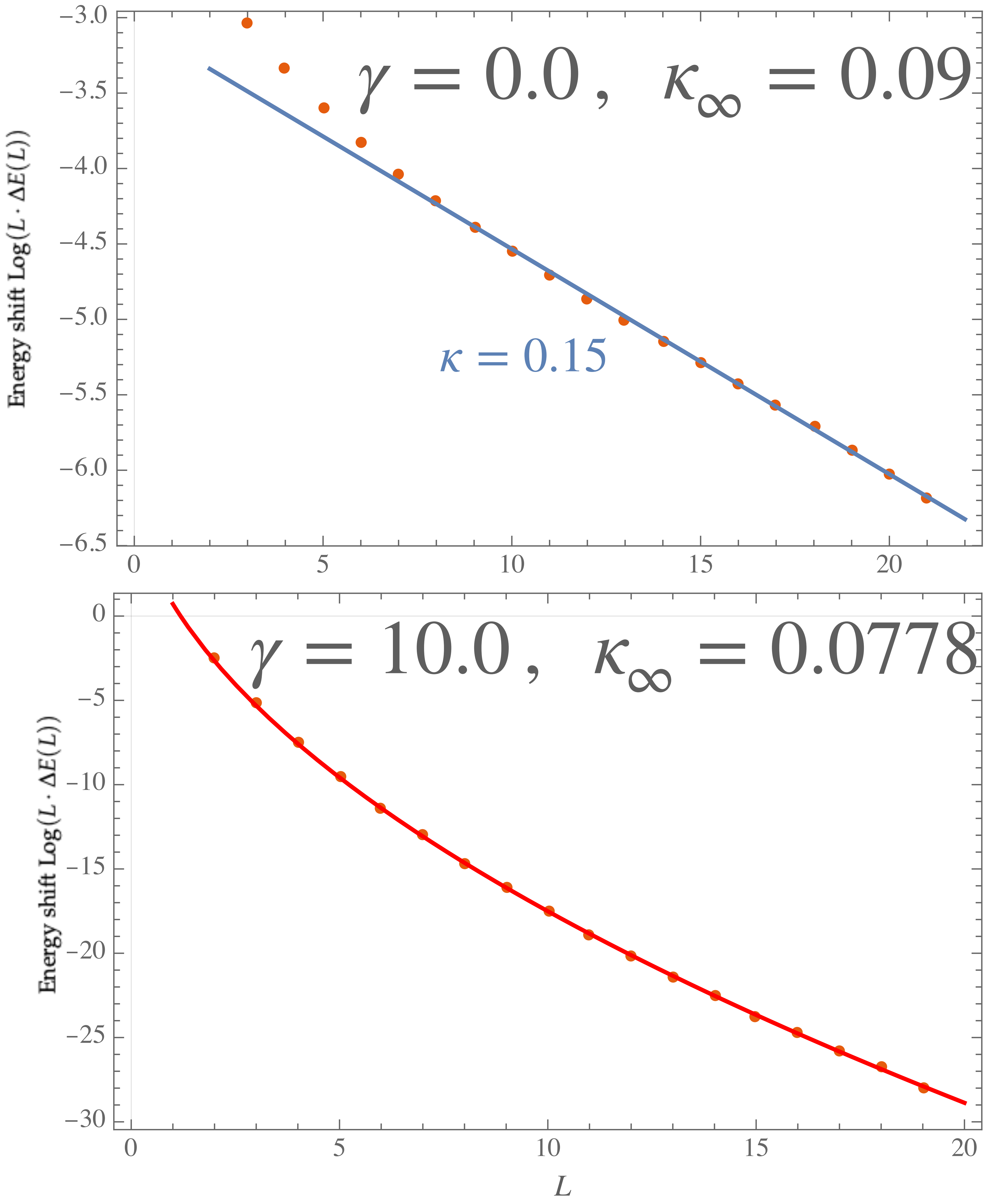

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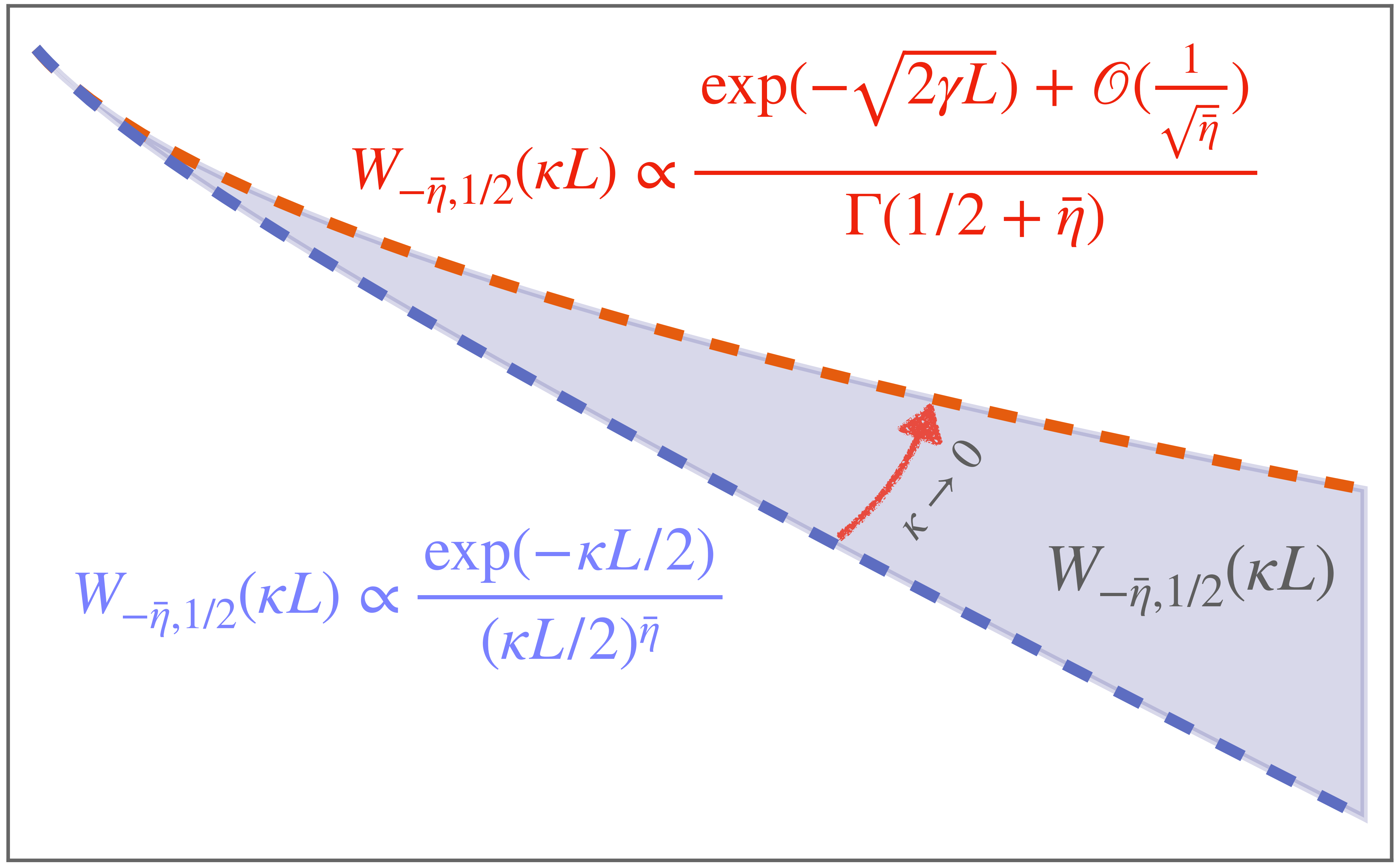

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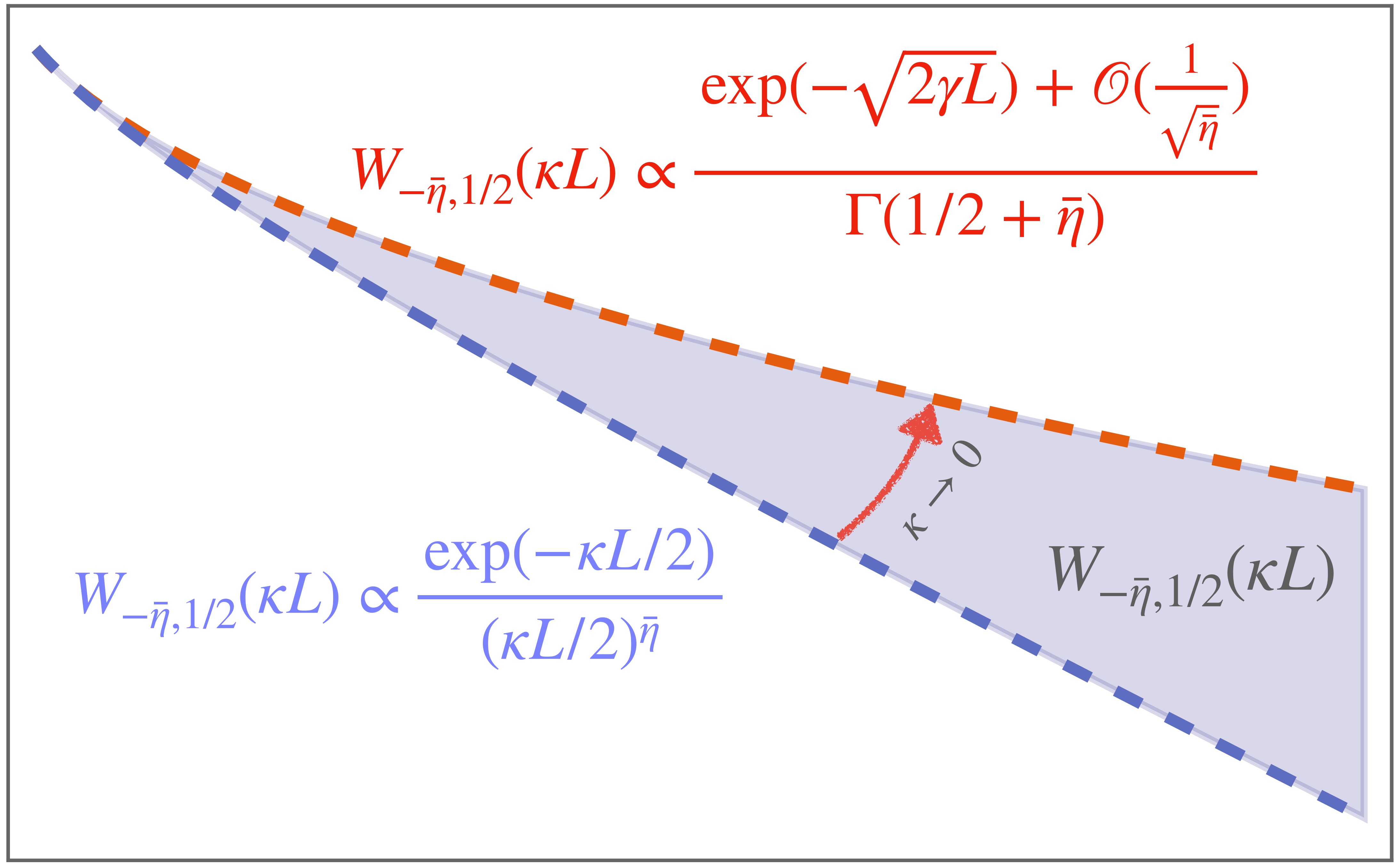

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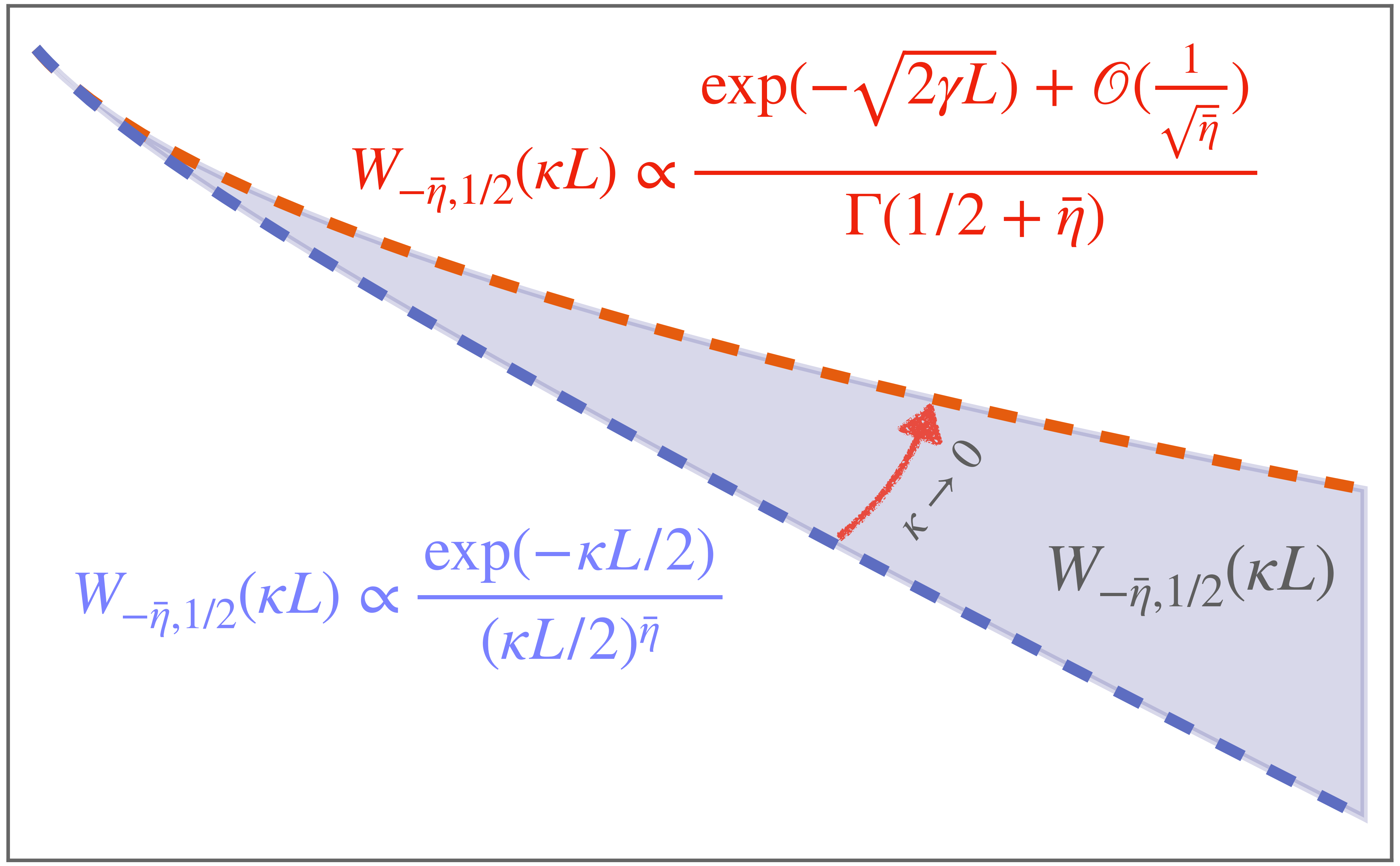

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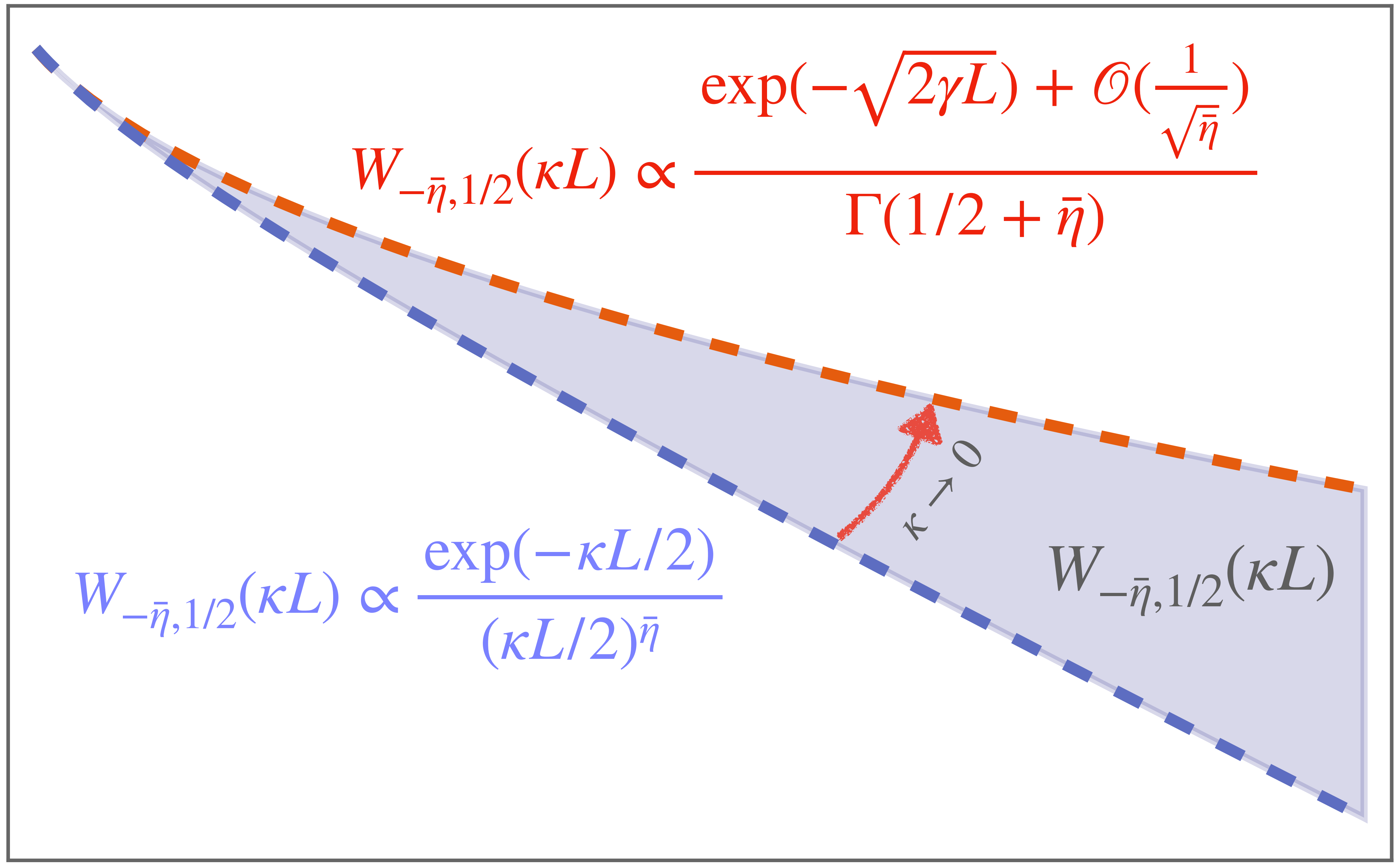

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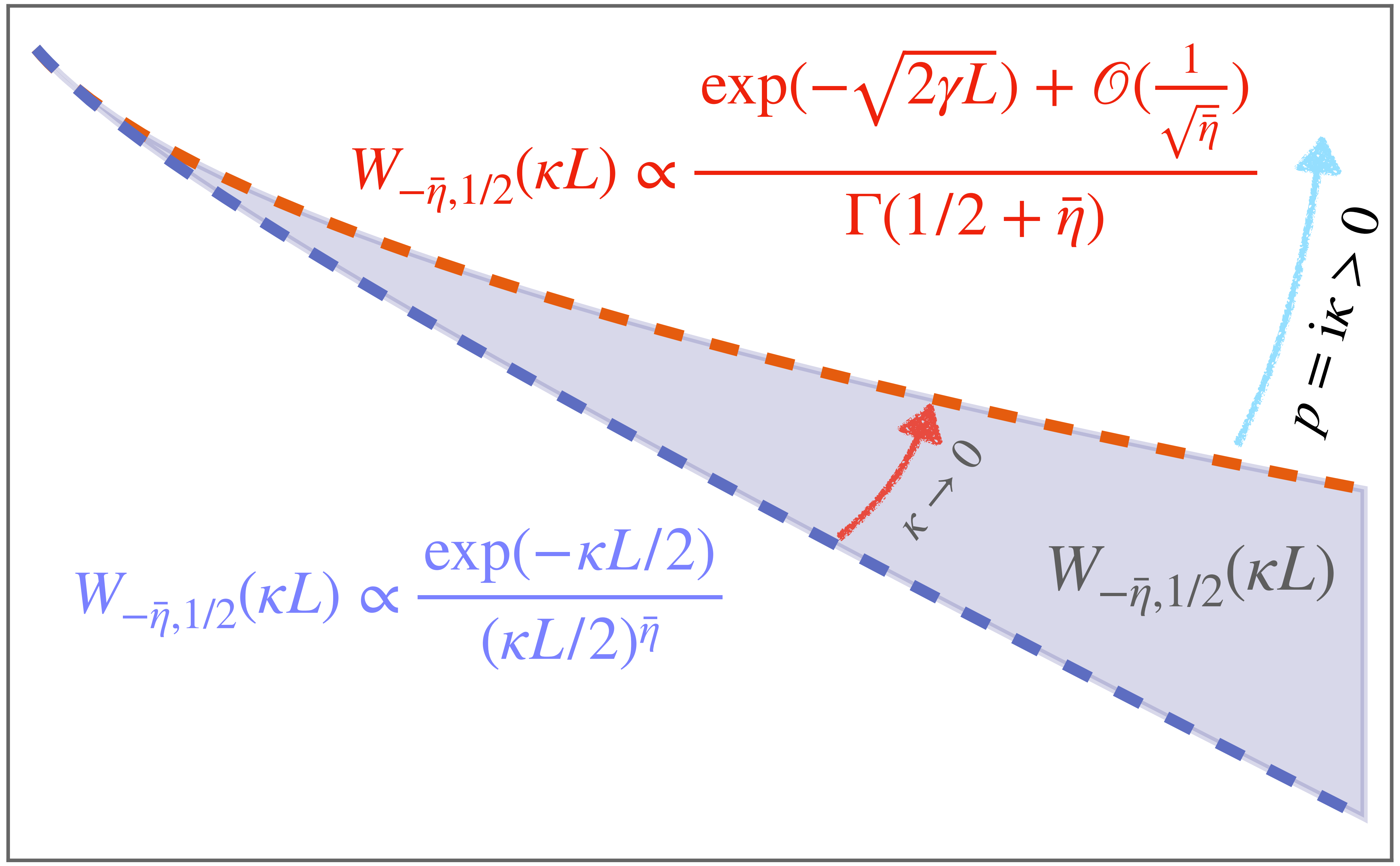

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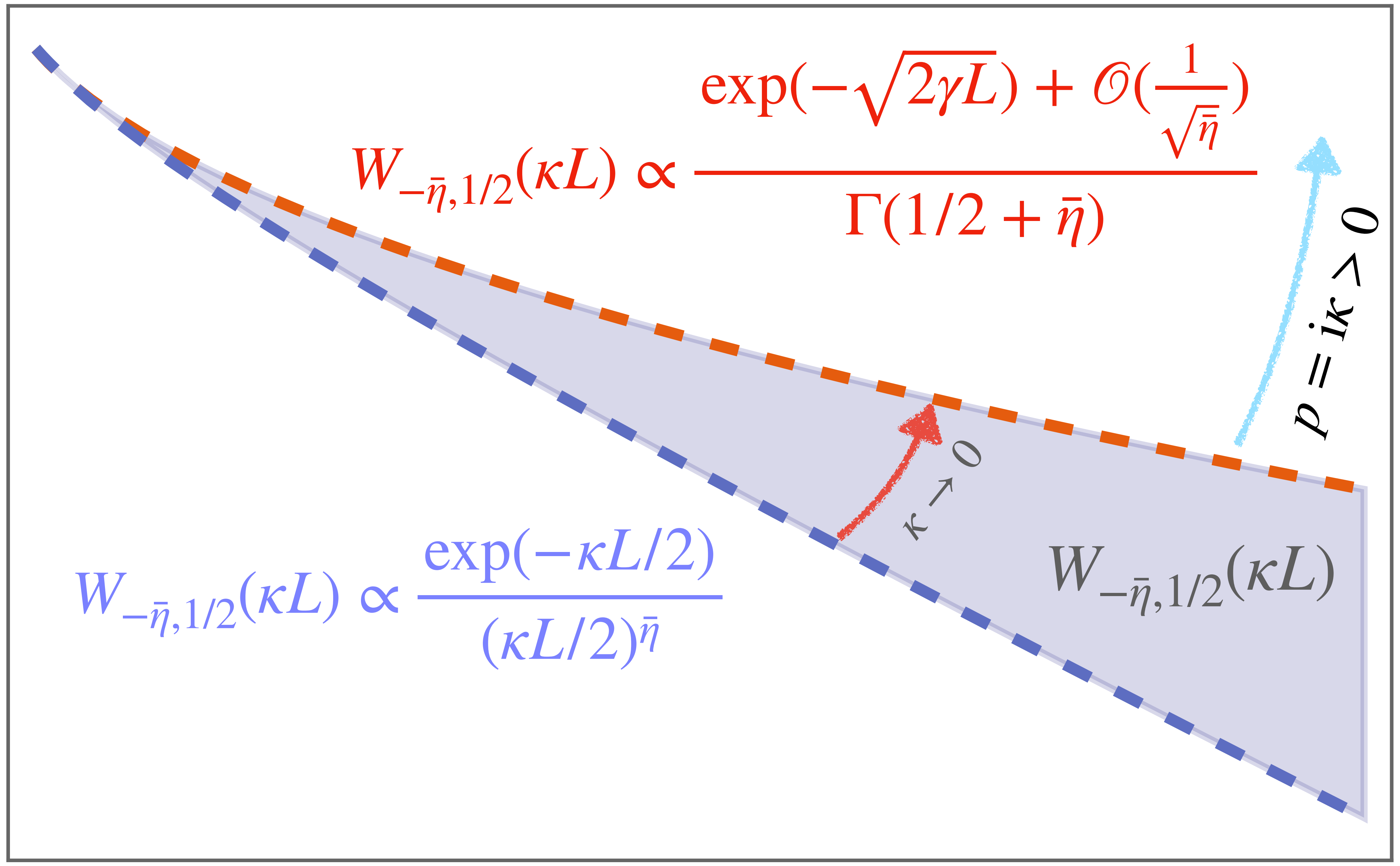

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To the Continuum!

Wrapping up…

Volume dependences:
from ground states to the continuum

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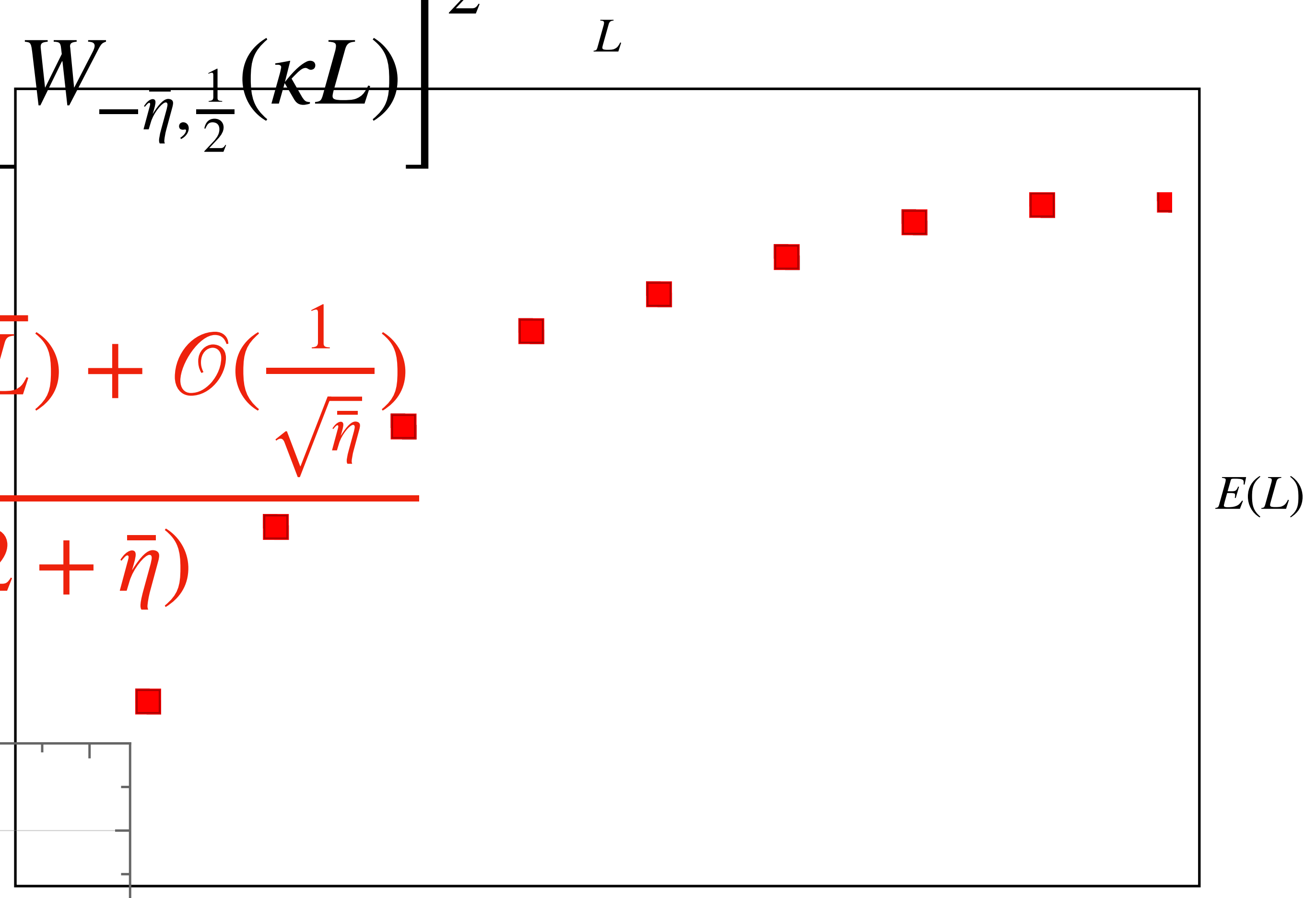

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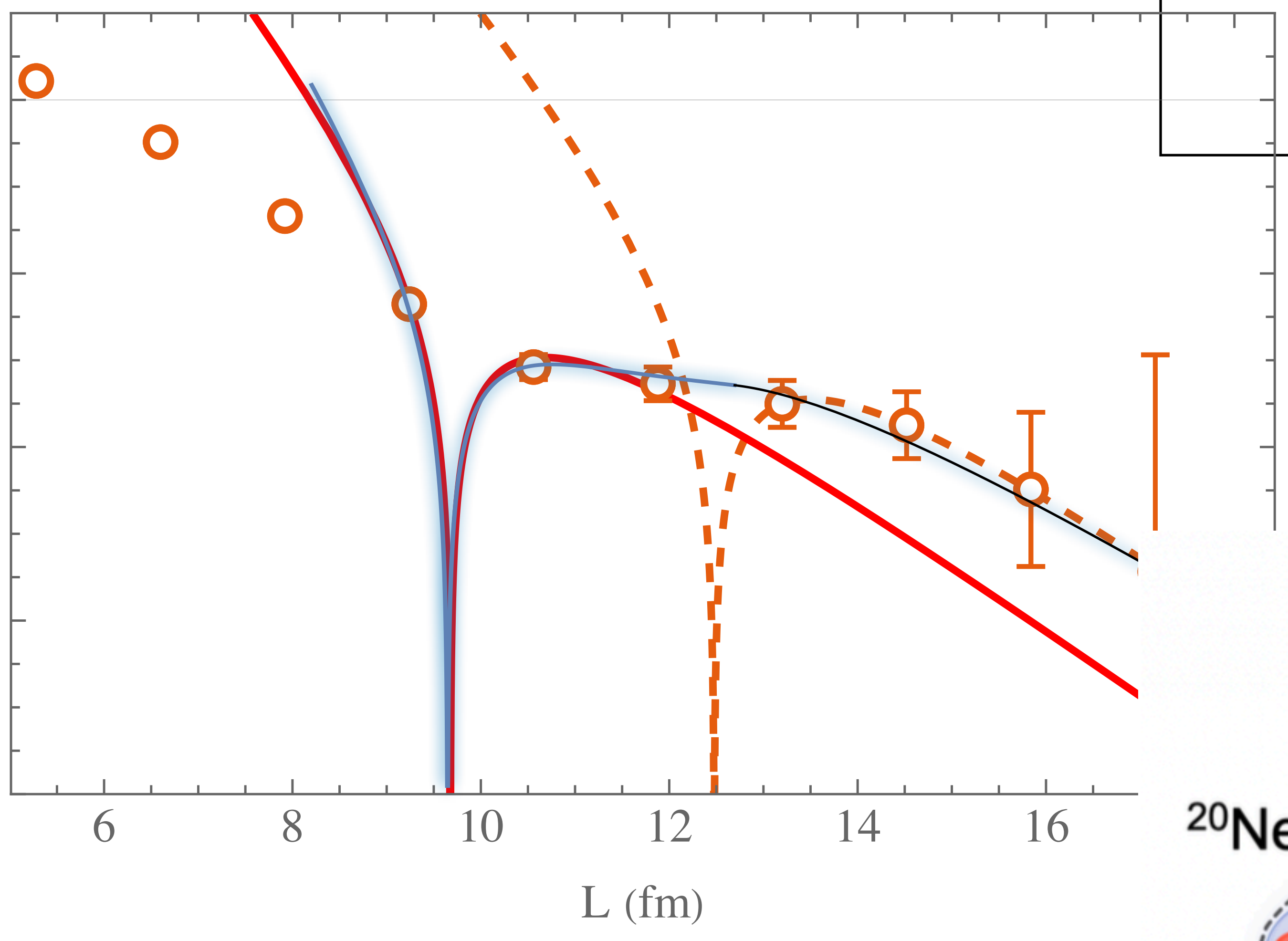
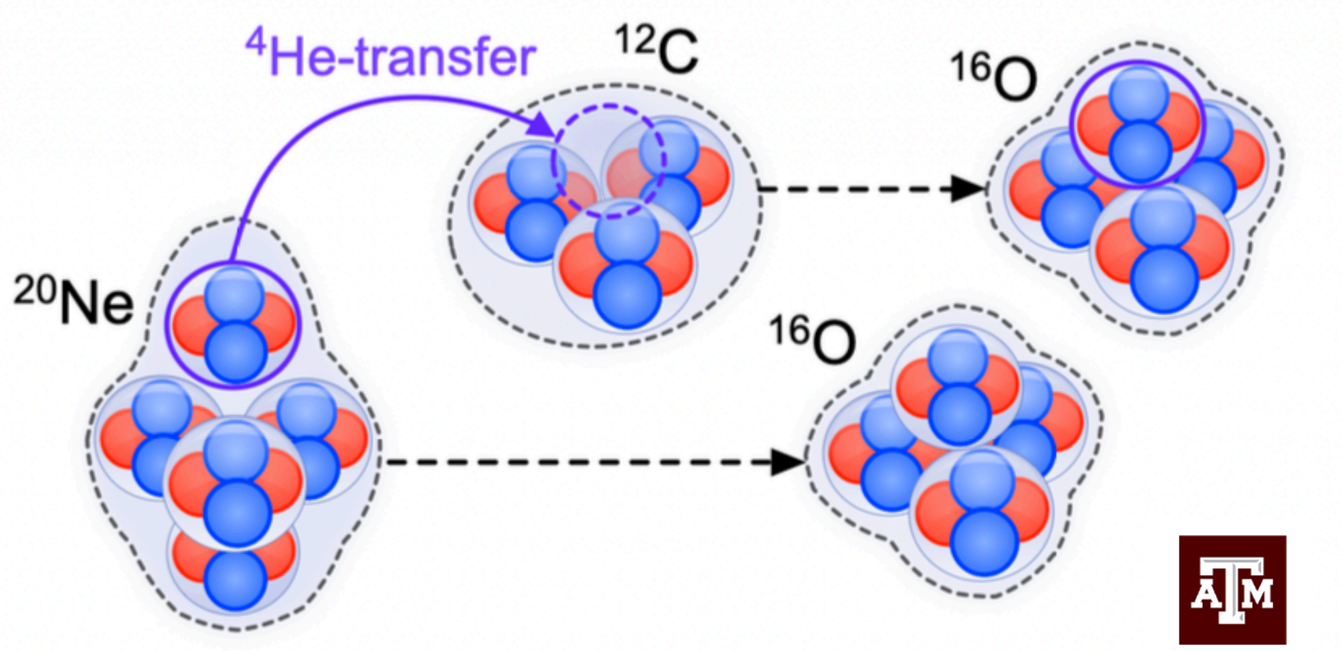
- We include the difficult coulomb interactions (no-PT) for **bound states** in the volume dependencies
- This difficult interaction opens doors to the **continuum**
- Application: *ab initio* calculations (work in progress)
- …





Thank you!