Some progress on perturbative correction for quarkonium processes

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Outline:

1. NRQCD factorization
2. Charmonium production at B factory
3. Bottomonium decay into double charmonia
4. B_c decay constant at three loop
5. Summary

1. NRQCD factorization

QCD is the foundamental theory to describe strong interaction

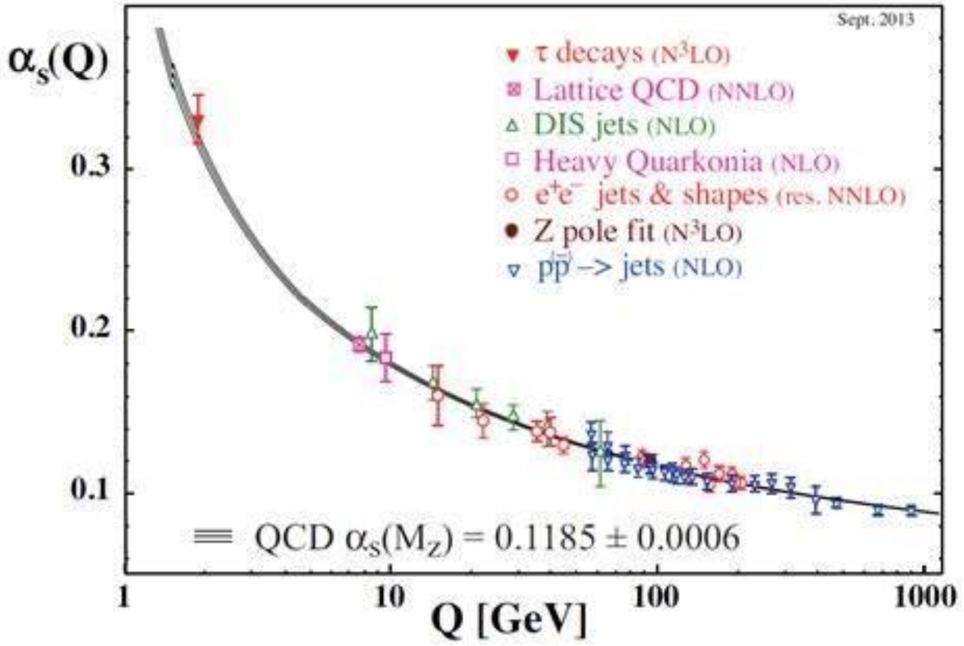
$$\mathcal{L}_{\rm QCD} = \bar{\psi}_i \left(i\gamma^\mu (D_\mu)_{ij} - m\,\delta_{ij}\right)\psi_j - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a$$
$$= \bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - gG^a_\mu\bar{\psi}_i\gamma^\mu T^a_{ij}\psi_j - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a\ ,$$

1. Asymptotic freedom

perturbative at high energy (short distance)

2. Color confinement

nonperturbative at low energy (long distance)



QCD running coupling constant

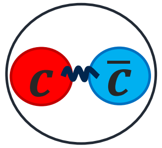
There involve both long-distance and short-distance dynamics for hadron production and decay! How can we separate them?

1. NRQCD factorization

Various QCD Effective Field Theory(EFT)

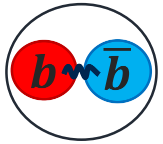
1. NRQCD factorization

Quarkonium is a nonrelativistic system, which is composed of a pair of heavy quark and antiquark.



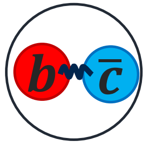
粲偶素

J/ψ, η_c



底偶素

Υ, η_b



B_c, B^*_c

Quarkonium is the simplest hadron, which provides an ideal platform to study the perturbative and nonperturbative property of QCD

Hierarchy of the typical energy scales

	$c\bar{c}$	$b\bar{b}$	$t\bar{t}$
M	1.5 GeV	4.7 GeV	180 GeV
Mv	0.9 GeV	1.5 GeV	16 GeV
Mv^2	0.5 GeV	0.5 GeV	1.5 GeV

$$M \gg Mv \gg Mv^2$$

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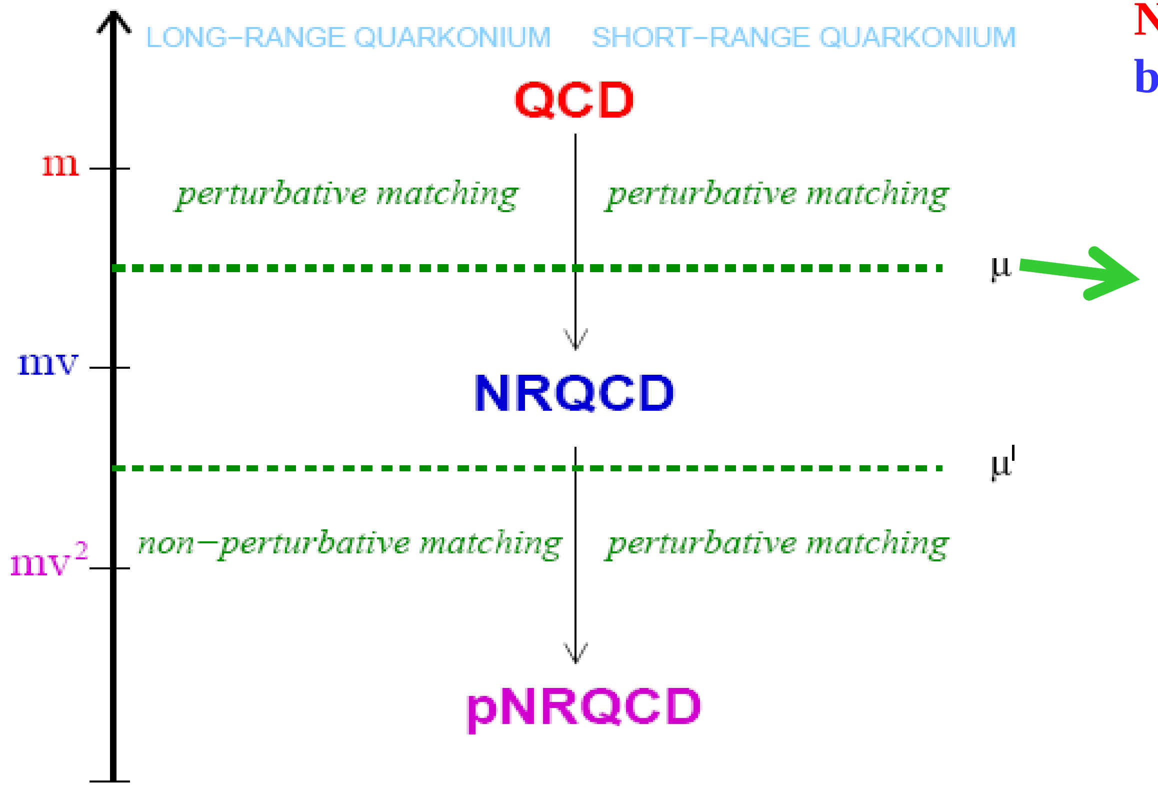
1. NRQCD factorization

Caswell, Lepage (1986); Bodwin, Braaten, Lepage (1995)



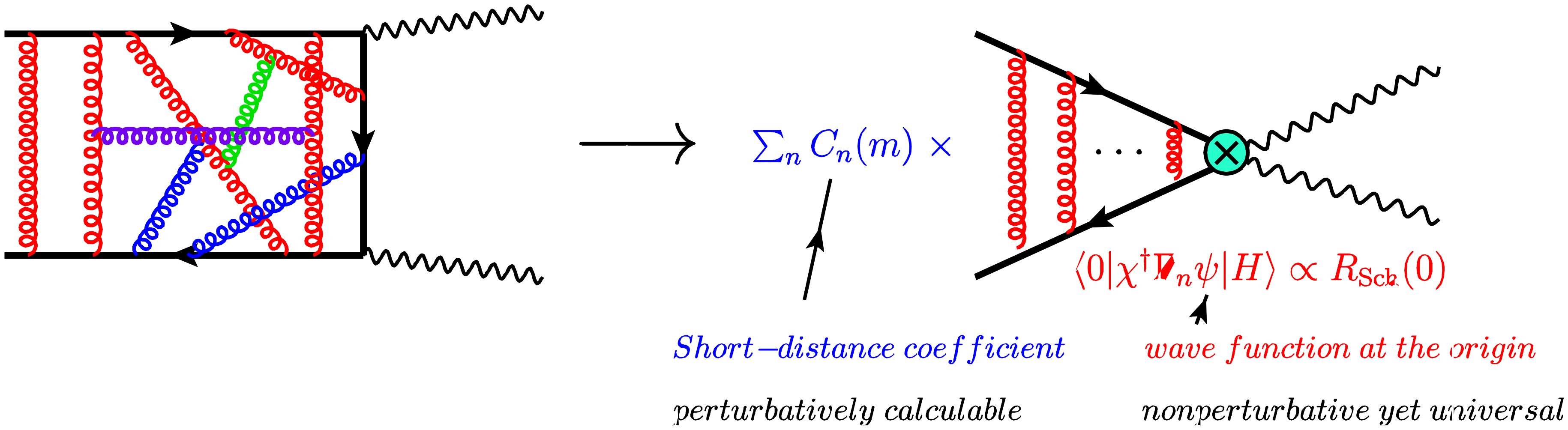
NRQCD factorization is viewed as being first principle of QCD

This scale separation is usually referred to as **NRQCD factorization.**

The NRQCD short-dist. coefficients can be computed in perturbation theory, order by order

1. NRQCD factorization

Quarkonium is a QCD bound state involving several distinct scales

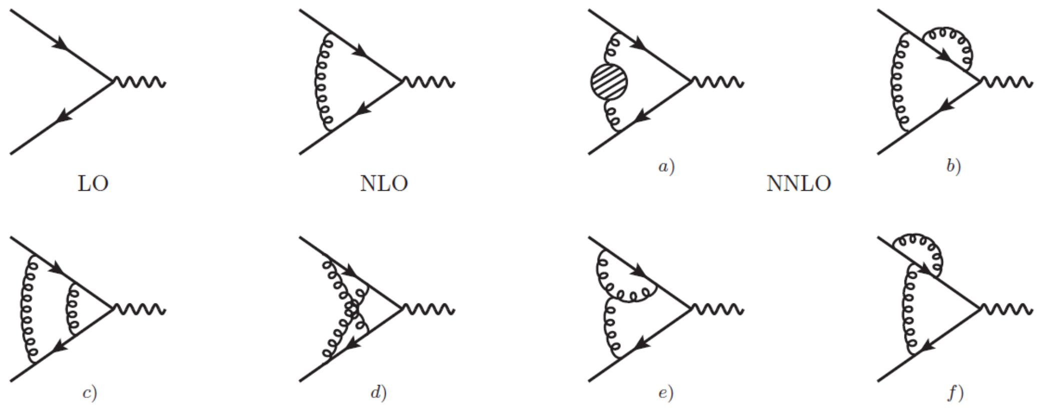


Separate the short-distance effect and long-distance dynamics

Asymptotic freedom: $\alpha_s(m) << 1$, one can invoke perturbation theory

1. NRQCD factorization

$$J/\psi \to \gamma^* \to e^+e^-$$



$$\Gamma(^3S_1 \to e^+e^-) = \frac{F_{ee}(^3S_1)}{m^2}\left|\langle 0|\chi^\dagger \boldsymbol{\sigma}\psi|^3S_1\rangle\right|^2 \quad \mathcal{O}(v^0)$$

$$+ \frac{G_{ee}(^3S_1)}{m^4}\,\mathrm{Re}\left[\langle ^3S_1|\psi^\dagger\boldsymbol{\sigma}\chi|0\rangle \cdot \langle 0|\chi^\dagger\boldsymbol{\sigma}(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}})^2\psi|^3S_1\rangle\right] \quad \mathcal{O}(v^2)$$

$$+ \frac{H^1_{ee}(^3S_1)}{m^6}\langle ^3S_1|\psi^\dagger\boldsymbol{\sigma}(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}})^2\chi|0\rangle \cdot \langle 0|\chi^\dagger\boldsymbol{\sigma}(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}})^2\psi|^3S_1\rangle \quad \mathcal{O}(v^4)$$

$$+ \frac{H^2_{ee}(^3S_1)}{m^6}\,\mathrm{Re}\left[\langle ^3S_1|\psi^\dagger\boldsymbol{\sigma}\chi|0\rangle \cdot \langle 0|\chi^\dagger\boldsymbol{\sigma}(-\tfrac{i}{2}\overleftrightarrow{\mathbf{D}})^4\psi|^3S_1\rangle\right] \quad \mathcal{O}(v^4)$$

Relativistic corrections

$$F_{ee}(^3S_1) = d^{(0)} + \frac{\alpha_s}{\pi}d^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 d^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 d^{(3)} + \mathcal{O}(\alpha_s^4)$$

Radiative corrections

1. NRQCD factorization

Matching the short-distance coeffient (SDC)

The SDC is insensitive to nonperturbative (long-distance) physics. Thus one can use free quark pair instead of hadron state to compute the SDC.

$$\Gamma(J/\psi \to e^+e^-) = \frac{F(^3S_1)}{m^2}\left|\langle 0|\chi^\dagger \boldsymbol{\sigma}\psi|J/\psi\rangle\right|^2 + \mathcal{O}(v^2\Gamma)$$

⇓

$$\Gamma(c\bar{c}(^3S_1) \to e^+e^-) = \frac{F(^3S_1)}{m^2}\left|\langle 0|\chi^\dagger \boldsymbol{\sigma}\psi|c\bar{c}(^3S_1)\rangle\right|^2$$

perturbative → **solve the SDC**

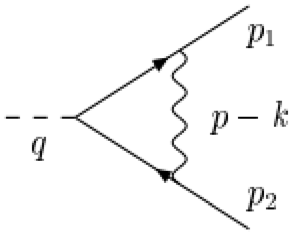
1. NRQCD factorization

It is very challenging to match the SDC at high loop, following the procedure of matching.

One usually use the "method of region"

Beneke & Smirnov: hep-ph/9711391

$$\Gamma(c\bar{c}(^3S_1) \to e^+e^-) = \frac{F(^3S_1)}{m^2} \left| \langle 0|\chi^\dagger \boldsymbol{\sigma} \psi | c\bar{c}(^3S_1)\rangle \right|^2$$



$$y \equiv m^2 - \frac{q^2}{4} = p^2 \ll q^2.$$

$$I_1 \equiv \int \frac{[dk]}{(k^2 + q\cdot k - y)(k^2 - q\cdot k - y)(k-p)^2},$$

hard (h): $k_0 \sim q,\ \vec{k} \sim q,$

$$I_1^h = \int \frac{[dk]}{k^2(k^2 + q\cdot k)(k^2 - q\cdot k)} = e^{\epsilon\gamma_E}\left(\frac{4}{q^2}\right)^{1+\epsilon}\left(-\frac{1}{2}\right)\frac{\Gamma(\epsilon)}{1+2\epsilon}.$$

potential (p): $k_0 \sim y/q,\ \vec{k} \sim \sqrt{y},$

soft (s): $k_0 \sim \sqrt{y},\ \vec{k} \sim \sqrt{y},$

or *ultrasoft* (us): $k_0 \sim y/q,\ \vec{k} \sim y/q.$

1. NRQCD factorization

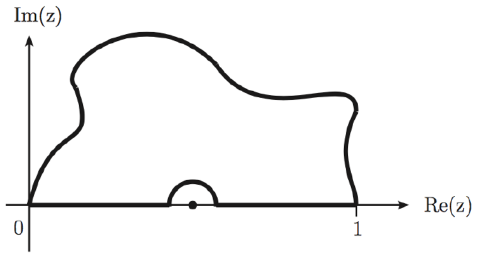
The techniques used in computing master integrals

- Some subtlety in Sector Decomposition

$$\mathcal{F}(a_1,\cdots,a_n)=\int\cdots\int\frac{d^dk_1\cdots d^dk_l}{E_1^{a_1}\cdots E_n^{a_n}}$$

where k_i are the loop momenta and the denominators E_i are either quadratic or linear with respect to the loop momenta k_i of the graph.

Deformation of the integration contour



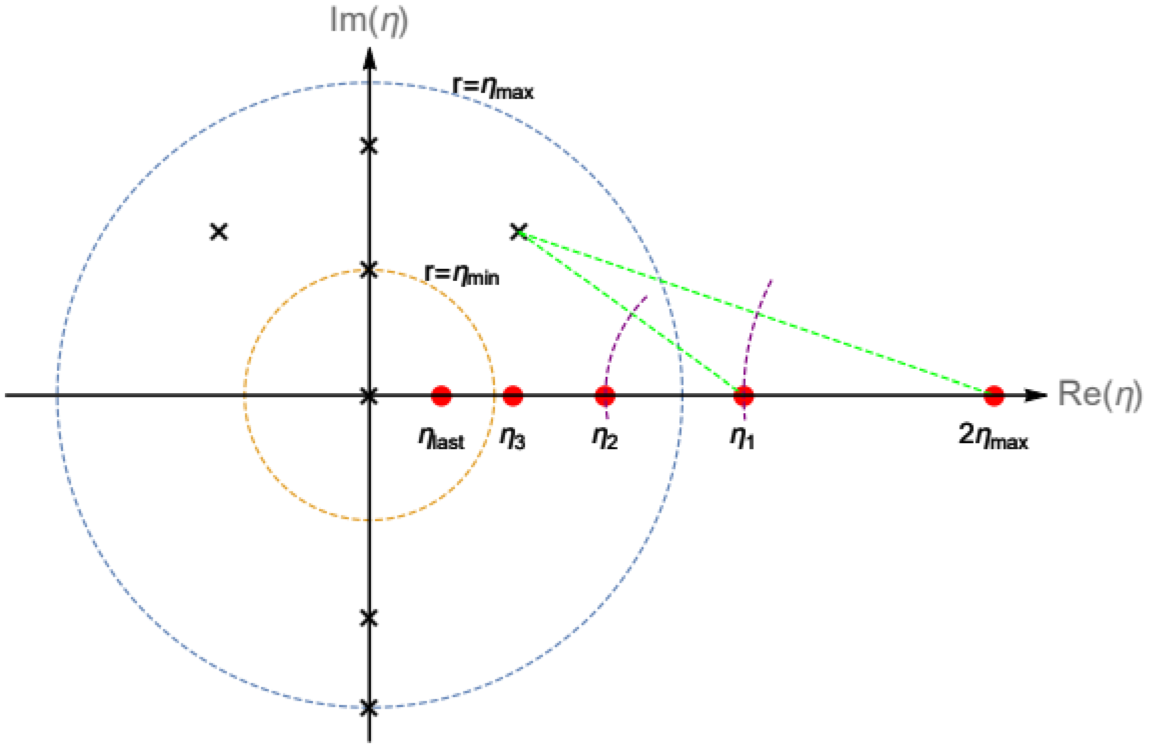
Perform Feynman parametrization and integrate over the loop momenta

$$\mathcal{F}=(i\pi^{d/2})^l\frac{\Gamma(A-ld/2)}{\Pi_{j=1}^n\Gamma(a_j)}\int_{x_j\geq 0}dx_1\cdots dx_n\delta(1-\sum x_i)(\Pi x_j^{a_j-1})\frac{U^{A-(l+1)d/2}}{(F-i0+)^{A-ld/2}}$$

The singularity in the endpoints can be treated by sector decomposition. However *F* **may dispear at some intermediate *x* points** !

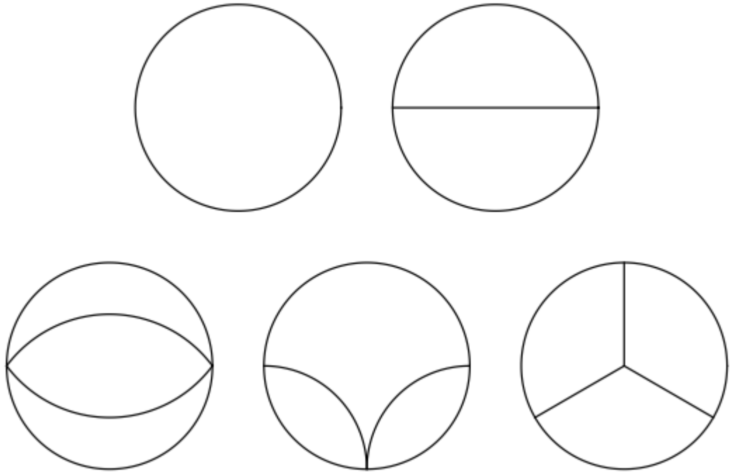
1. NRQCD factorization

➢AMFlow [X. Liu, Y. Q. Ma, C. Y. Wang, arXiv:1711.09572](#)
[X. Liu, Y. Q. Ma, arXiv:2107.01864](#)
[Z. F. Liu, Y. Q. Ma, arXiv:2201.11637](#)(PRL)
[X. Liu, Y. Q. Ma, arXiv:2201.11669](#)(Package)



$$I(D;\{\nu_\alpha\};\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}}$$

$$\frac{\partial}{\partial \eta}\vec{I}(\eta) = A(\eta)\vec{I}(\eta)$$



1. NRQCD factorization

To make phenomenology prediction, we should further determine the long-distance matrix elements (LDMEs).

We approximate LDMEs at $\mu_\Lambda \approx m_c v_c \approx m_b v_b \approx 1\ \mathrm{GeV}$ by the Schrödinger radial wave function at the

$$\langle J/\psi|\psi^\dagger \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_{J/\psi}\chi(\mu_\Lambda)|0\rangle \approx \sqrt{\frac{N_c}{2\pi}}R_{1S}(0),$$

$$\langle \eta_c|\psi^\dagger \chi(\mu_\Lambda)|0\rangle \approx \sqrt{\frac{N_c}{2\pi}}R_{1S}(0),$$

$$\langle \chi_{cJ}|\psi^\dagger \mathcal{K}_{^3P_J}\chi(\mu_\Lambda)|0\rangle \approx \sqrt{\frac{3N_c}{2\pi}}R'_{1P}(0),$$

In the phenomenological analysis, we adopt Buchmuller-Tye (BT) **potential**

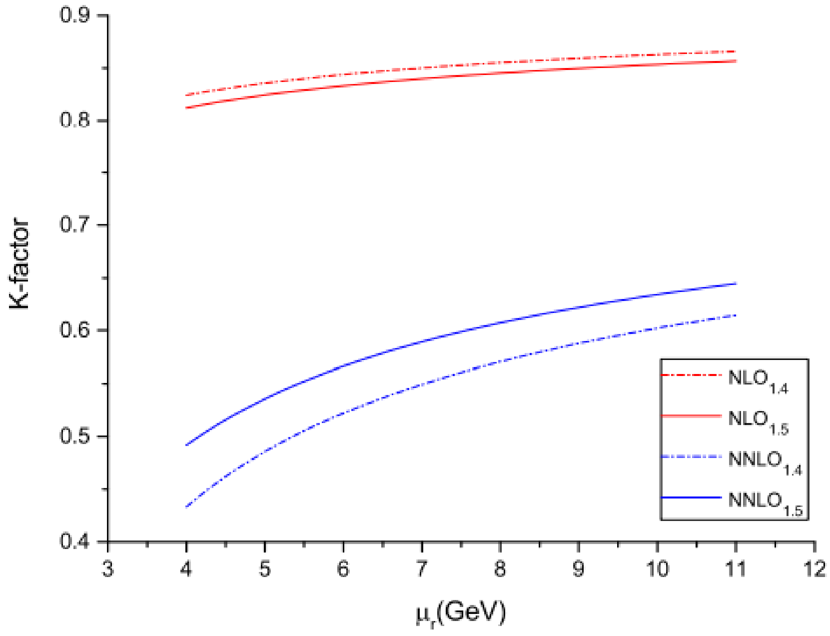
2. Charmonium production at B factory

$$e^+e^- \to \eta_c + \gamma$$

Chen, Liang, Qiao, JHEP(2018)

NLO=LO+$\mathcal{O}(\alpha_s)$
NNLO=LO+$\mathcal{O}(\alpha_s)$+$\mathcal{O}(\alpha_s^2)$

σ(fb)	LO	NLO	NNLO
$\eta_c(1.4)$	89.7	75.2	44.6
$\eta_c(1.5)$	82.8	68.5	45.2
$\eta_b(4.7)$	2.50	1.77	1.75
$\eta_b(4.8)$	2.07	1.47	1.46



The NLO & NNLO corrections are considerable, however not so huge! So the convergence may be not so worse.

Renormalization scale dependence.

2. Charmonium production at B factory

$e^+e^- \to \chi_{cJ} + \gamma$

$$\sigma(e^+e^- \to \chi_{c1} + \gamma) \quad = \quad (17.3^{+4.2}_{-3.9}(\text{stat.}) \pm 1.7(\text{syst.}))fb \ \ @\sqrt{s} = 10.58\text{GeV}$$

Experimental data by BELLE collaboration

PRD98, 092015 (2018)

However, **no** significant excesses for χ_{c0} and χ_{c2}.

OBSERVATION OF $e^+e^- \to \gamma\chi_{c1}$ ...

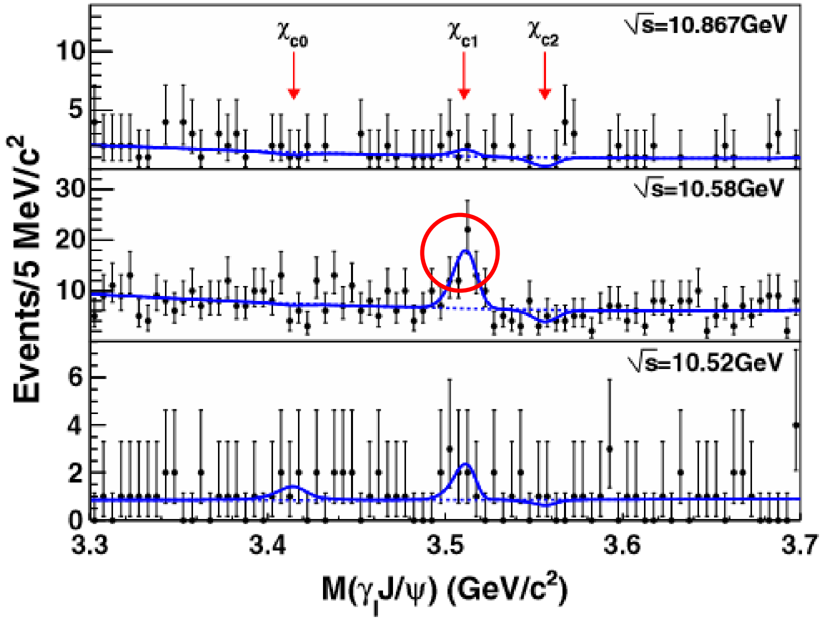
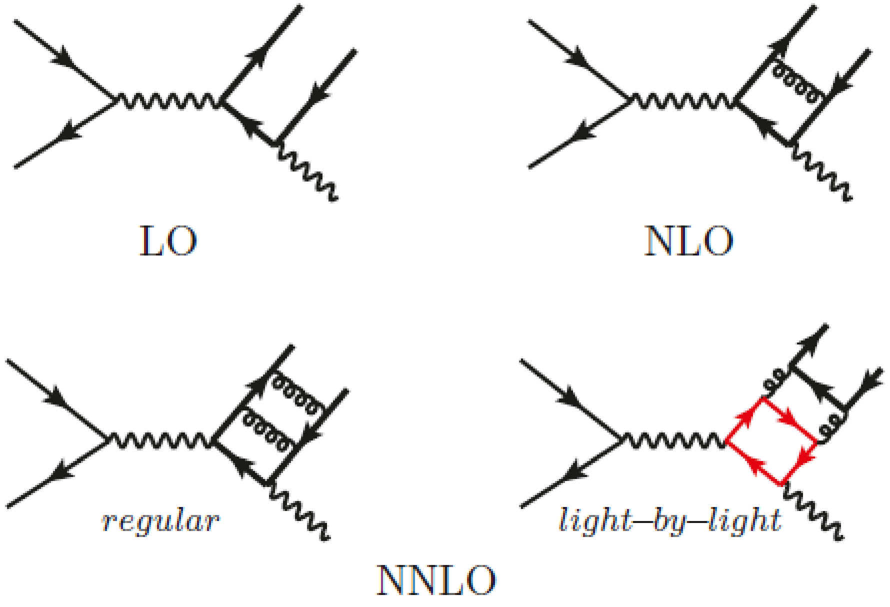


FIG. 2. The $\gamma_l J/\psi$ invariant mass spectra at $\sqrt{s} = 10.52$ (bottom), 10.58 (middle), and 10.867 GeV (top) together with fit results. The points with error bars show the data and the solid curves are the fit functions; the dashed curves show the fitted backgrounds contributions. The arrows show the expected peak positions for the χ_{c0}, χ_{c1}, and χ_{c2} states.

2. Charmonium production at B factory $e^+e^- \to \chi_{cJ} + \gamma$

SWL, Feng, Jia, JHEP(2020)



$$\sigma(e^+e^- \to \chi_{c1} + \gamma) = (17.3^{+4.2}_{-3.9} \pm 1.7)\text{fb}$$

@BELLE

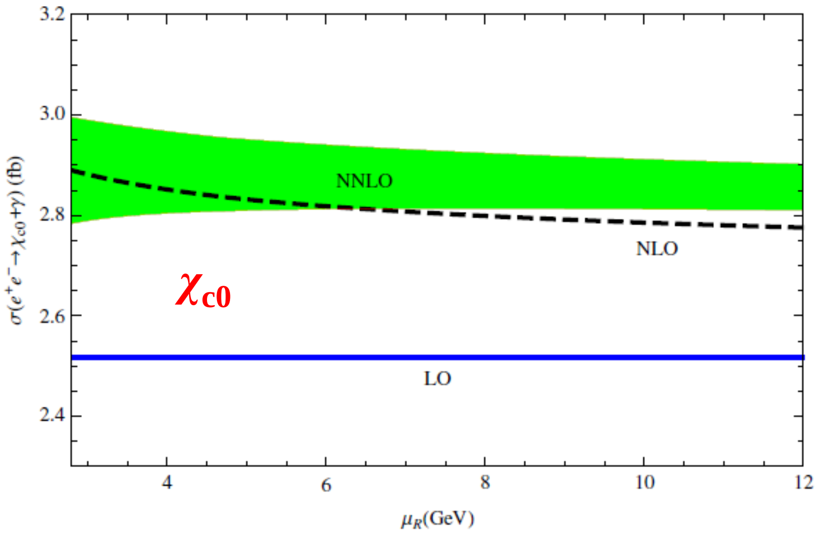
Table 1: NRQCD predictions to $\sigma(\chi_{cJ}+\gamma)$ at various levels of accuracy in α_s at B factory. The LDME $\langle \mathcal{O}(^3P_J)\rangle = 0.107\ \text{GeV}^5$ is taken from Buchmüller-Tye (BT) potential model. The errors are estimated by sliding the renormalization scale μ_R from $2m$ to $\sqrt{s}$.

σ (fb) χ_{cJ} / Order	LO	NLO	NNLO ($\mu_\Lambda = 1$ GeV)	NNLO ($\mu_\Lambda = m$)
$m = 1.40$ GeV				
$\chi_{c0}+\gamma$	2.52	$2.83^{+0.06}_{-0.04}$	$2.96^{+0.05}_{-0.04}$	$2.82^{+0.01}_{-0.03}$
$\chi_{c1}+\gamma$	25.96	$20.72^{+0.75}_{-1.05}$	$17.91^{+0.89}_{-1.21}$	$16.83^{+1.20}_{-1.79}$
$\chi_{c2}+\gamma$	10.02	$4.24^{+0.83}_{-1.16}$	$1.34^{+0.92}_{-1.23}$	$1.03^{+1.01}_{-1.40}$
$m = 1.68$ GeV				
$\chi_{c0}+\gamma$	1.18	$1.39^{+0.03}_{-0.03}$	$1.48^{+0.03}_{-0.03}$	$1.38^{+0.01}_{-0.01}$
$\chi_{c1}+\gamma$	15.98	$12.25^{+0.54}_{-0.50}$	$10.87^{+0.45}_{-0.37}$	$9.84^{+0.75}_{-0.72}$
$\chi_{c2}+\gamma$	6.60	$2.84^{+0.54}_{-0.50}$	$1.03^{+0.58}_{-0.52}$	$0.71^{+0.67}_{-0.63}$

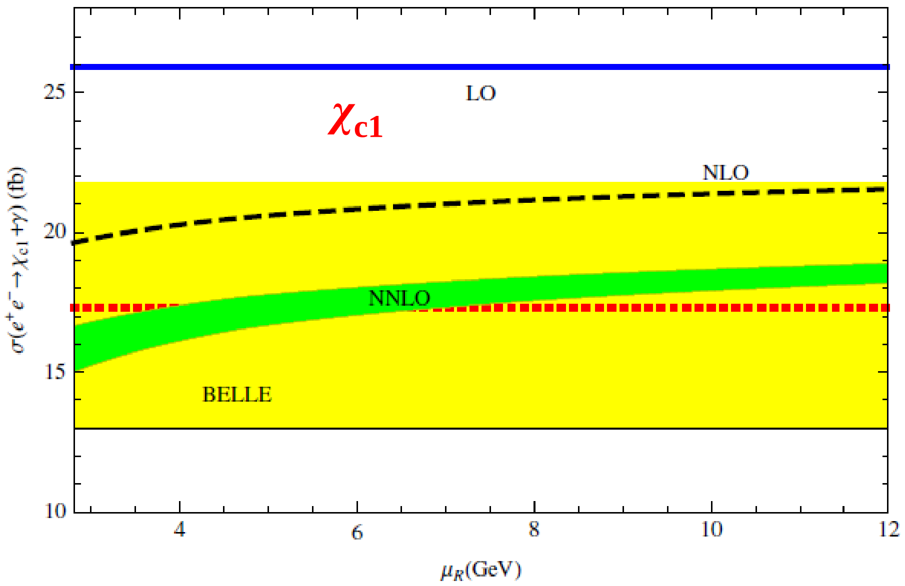
The results explain why the other two states are not observed !

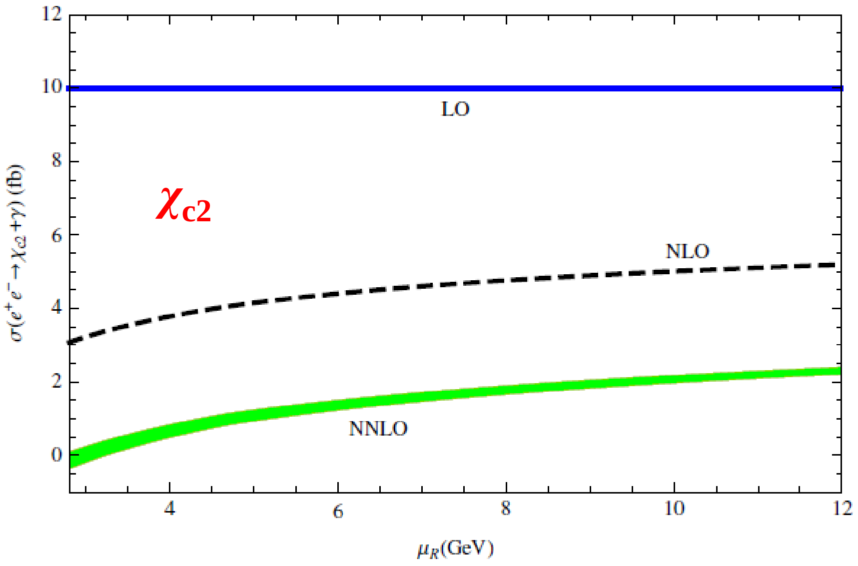
2. Charmonium production at B factory $e^+e^- \to \chi_{cJ} + \gamma$

SWL, Feng, Jia, JHEP(2020)



NRQCD predictions for the cross sections of $\chi_{cJ}+\gamma$ as a function of μ_R at various levels of accuracy in α_s with ***m*=1.4** GeV





The uncertainty in the theoretical prediction corresponds to the change of μ_Λ from 1 GeV to ***m***. We did not consider the uncertainties from the input parameters.

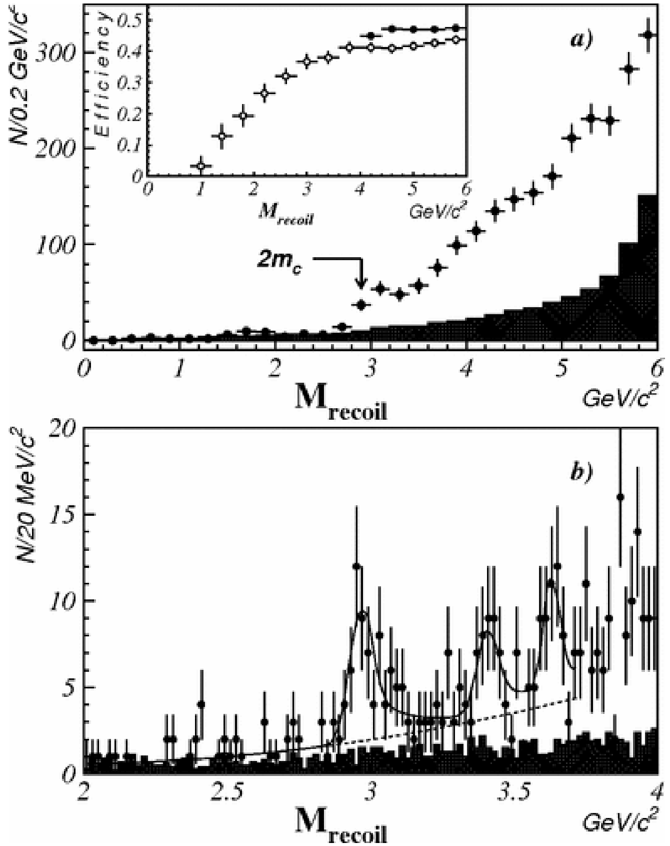
2. Charmonium production at B factory $e^+e^- \to J/\psi + \eta_c$

$$\sigma(e^+e^- \to J/\psi + \eta_c) \times \mathcal{B}_{>4} = 33^{+7}_{-6} \pm 9 \text{ fb} \quad \text{@BELLE},$$ PRL(2002), BELLE

$$\sigma(e^+e^- \to J/\psi + \eta_c) \times \mathcal{B}_{>2} = 25.6 \pm 2.8 \pm 3.4 \text{ fb} \quad \text{@BELLE},$$ PRD(2004), BELLE

$$\sigma(e^+e^- \to J/\psi + \eta_c) \times \mathcal{B}_{>2} = 17.6 \pm 2.8^{+1.5}_{-2.1} \text{ fb} \quad \text{@BABAR},$$ PRD(2005), BABAR

Braaten, Lee, PRD(2003)
Liu, He, Chao, PLB(2003)
Zhang, Gao, Chao, PRL(2006)
He, Fan, Chao, PRD(2007)
Bodwin, Lee, Yu, PRD(2008)
Gong, Wang, PRD(2008),
Dong, Feng, Jia, PRD(2012)
… …



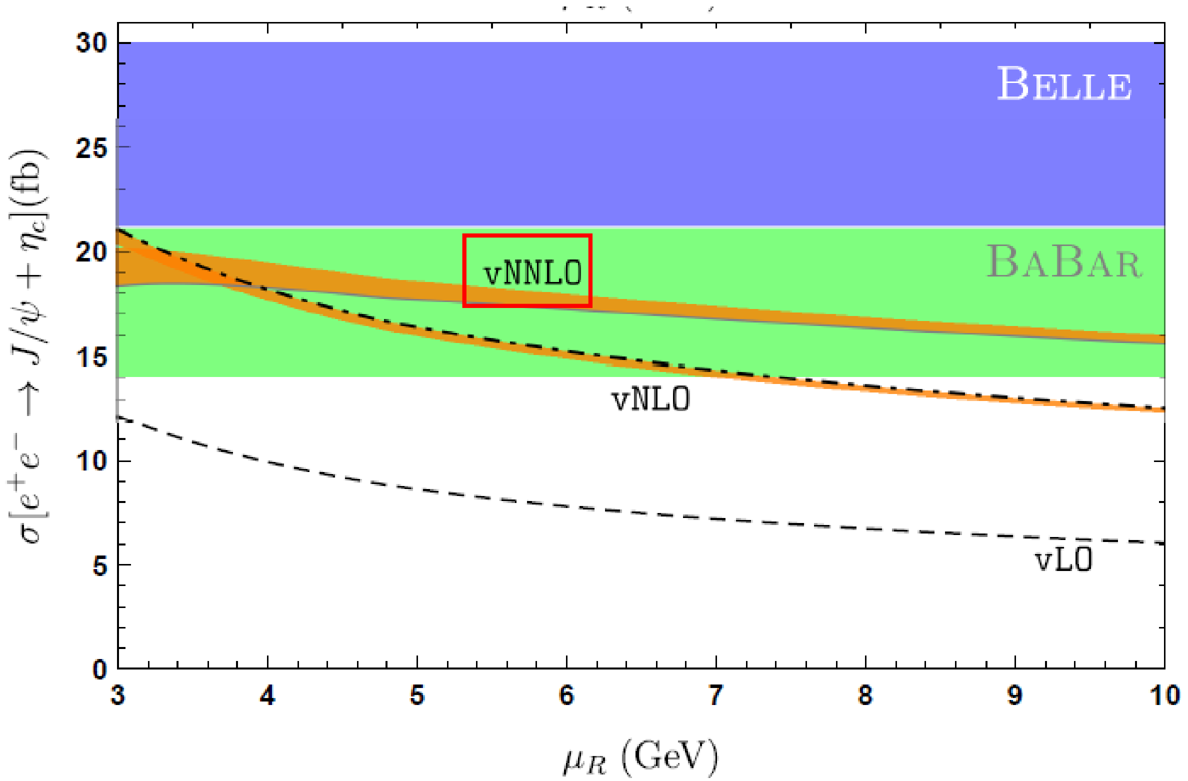
BELLE, PRL(2002)

2. Charmonium production at B factory $e^+e^- \to J/\psi + \eta_c$

Feng, Jia, Mo, **SWL**, Zhang, arXiv: 1901.08447

TABLE I: Individual contributions to the predicted $\sigma[e^+e^- \to J/\psi + \eta_c]$ at $\sqrt{s} = 10.58$ GeV. Each column is labeled by the powers of α_s and v, and given in units of fb. We fix $\mu_\Lambda = m$, and consider $\mu_R = 2m$ and $\sqrt{s}/2$. The two upper rows and the two lower rows correspond to $m = 1.4$ GeV and $m = 1.68$ GeV, respectively.

μ_R	LO	$\mathcal{O}(v^2)$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s v^2)$	$\mathcal{O}(\alpha_s^2)$	Total
$2m$	8.48	4.36	8.64	0.34	$-3.7(5)$	18.1(5)
$\frac{\sqrt{s}}{2}$	5.52	2.84	6.48	1.18	1.6(2)	17.6(2)
$2m$	5.59	1.44	4.71	-0.33	$-1.4(4)$	10.0(4)
$\frac{\sqrt{s}}{2}$	4.16	1.07	4.08	0.06	0.7(2)	10.1(2)



微扰展开收敛性较好！与实验也比较吻合

The two-loop correction is confirmed by
Huang, Gong, Wang, arXiv: 2212.03631, JHEP(2023)

2. Charmonium production at B factory $e^+e^- \to J/\psi + \chi_{cJ}$

Experiment:

$\sigma(e^+e^- \to J/\psi + \chi_{c0}) \times \mathcal{B}_{>2} = 6.4 \pm 1.7 \pm 1.0$ fb @BELLE, **PRD(2004), BELLE**

$\sigma(e^+e^- \to J/\psi + \chi_{c0}) \times \mathcal{B}_{>2} = 10.3 \pm 2.5^{+1.4}_{-1.8}$ fb @BABAR, **PRD(2005), BABAR**

$[\sigma(e^+e^- \to J/\psi + \chi_{c1}) + \sigma(e^+e^- \to J/\psi + \chi_{c2})] \times \mathcal{B}_{>2} < 5.3$ fb at 90%C.L.@BELLE, **PRD(2004), BELLE**

Theory:

Braaten, Lee, **PRD2003**;
Liu, He, Hagiwara, Kou, Qiao, **PLB2003**;
He, Chao, **PLB 2003**
Zhang, Ma, Chao, **PRD2008**;
Wang, Ma, Chao, **PRD2011**;
Dong, Feng, Jia, **JHEP 2011**.
Jiang, Sun, **EPJC 2018**.
Sun, **JHEP 2021**
… …

2. Charmonium production at B factory $e^+e^- \to J/\psi + \chi_{cJ}$

SWL, Feng, Jia, Mo, Zhang, PLB(2023),arXiv:2202.11615

Table 3

Comparison between our predictions to the unpolarized cross sections and the measurements in two B factories (in units of fb). The sources of theoretical uncertainties are the same as in Table 2, respectively. The experimental data are the double charmonium cross sections multiplied by the branching fractions of χ_{cJ} decay into more than 2 charged tracks. The Belle data for $e^+e^- \to \psi(2S) + \chi_{cJ}$ production correspond to χ_{cJ} decay into at least 1 charged track [56].

agree with experimental measurements

	LO	NLO	NNLO	Belle $\sigma \times \mathcal{B}_{>2(0)}$ [56]	BaBar $\sigma \times \mathcal{B}_{>2}$ [2]
$\sigma(J/\psi + \chi_{c0})$	$3.35^{+1.14}_{-0.99}$	$6.05^{+1.13}_{-1.17}$	$10.45^{+0.11+2.60}_{-0.58-3.87}$	$6.4 \pm 1.7 \pm 1.0$	$10.3 \pm 2.5^{+1.4}_{-1.8}$
$\sigma(J/\psi + \chi_{c1})$	$0.503^{+0.172}_{-0.148}$	$0.63^{+0.08}_{-0.09}$	$0.867^{+0.006+0.188}_{-0.005-0.291}$	–	–
$\sigma(J/\psi + \chi_{c2})$	$0.64^{+0.22}_{-0.19}$	$0.72^{+0.04}_{-0.07}$	$0.728^{+0.123+0.121}_{-0.161-0.212}$	–	–
$\sigma(J/\psi + \chi_{c1}) + \sigma(J/\psi + \chi_{c2})$	$1.15^{+0.39}_{-0.34}$	$1.36^{+0.12}_{-0.16}$	$1.60^{+0.12+0.31}_{-0.17-0.50}$	<5.3 at 90% C.L.	–
$\sigma(\psi(2S) + \chi_{c0})$	$2.19^{+0.75}_{-0.64}$	$3.95^{+0.74}_{-0.76}$	$6.82^{+0.07+1.70}_{-0.38-2.52}$	$12.5 \pm 3.8 \pm 3.1$	–
$\sigma(\psi(2S) + \chi_{c1})$	$0.328^{+0.112}_{-0.097}$	$0.413^{+0.052}_{-0.058}$	$0.566^{+0.004+0.123}_{-0.003-0.190}$	–	–
$\sigma(\psi(2S) + \chi_{c2})$	$0.420^{+0.144}_{-0.124}$	$0.473^{+0.026}_{-0.048}$	$0.476^{+0.080+0.079}_{-0.105-0.14}$	–	–
$\sigma(\psi(2S) + \chi_{c1}) + \sigma(\psi(2S) + \chi_{c2})$	$0.75^{+0.26}_{-0.22}$	$0.89^{+0.08}_{-0.11}$	$1.04^{+0.08+0.20}_{-0.11-0.33}$	<8.6 at 90% C.L.	–

agree with experimental measurement, albeit large uncertainties.

2. Charmonium production at B factory $e^+e^- \to J/\psi + \chi_{cJ}$

Predictions for the angular distribution parameter.

$$\frac{d\sigma(e^+e^- \to J/\psi + \chi_{cJ})}{d\cos\theta} = A_J \left(1 + \alpha_J \cos^2\theta\right), \qquad J = 0, 1, 2$$

It is worth noting that the value of α_J is insensitive to choice of the NRQCD matrix elements.

Table 1:

The uncertainties are very small!

	LO	NLO	NNLO	Belle
$J/\psi + \chi_{c0}$	0.252	$0.260^{+0.005}_{-0.004}$	$0.291^{+0.014+0.002}_{-0.012-0.002}$	$-1.01^{+0.38}_{-0.33}$
$J/\psi + \chi_{c1}$	0.697	$0.739^{+0.028}_{-0.027}$	$0.880^{+0.054+0.004}_{-0.060-0.008}$	—
$J/\psi + \chi_{c2}$	-0.197	$-0.075^{+0.012}_{-0.014}$	$0.025^{+0.070+0.005}_{-0.047-0.006}$	—

2. Charmonium production at B factory $e^+e^- \to J/\psi + J/\psi$

$\sigma(e^+e^- \to J/\psi + J/\psi) \times \mathcal{B}_{>2} \quad < \quad 9.1\text{fb} \quad \text{@BELLE},$ **PRD(RC) (2004), BELLE**

			Cross Section
2002:	Bodwin, Lee, Braaten	NRQCD LO	**8.7** fb
2003:	Bodwin, Lee, Braaten	NRQCD LO	**6.65** fb
2006:	Davier, Peskin, Snyder	VMD	**2.38** fb
2006:	Bodwin, Braaten, Lee, Yu	fragmentation+nonfragmentation	**1.69±0.35** fb
2008:	Gong, Wang	NRQCD NLO	**-3.4—2.3** fb
2013:	Fan, Lee, Yu	NRQCD NLO in α_s and v^2	**1—1.5** fb

1. How about the **perturbative convergence**? NNLO correction?
2. To provide useful guidance for experimentalists to search for this channel, it is crucial to present the **precise theoretical prediction**.

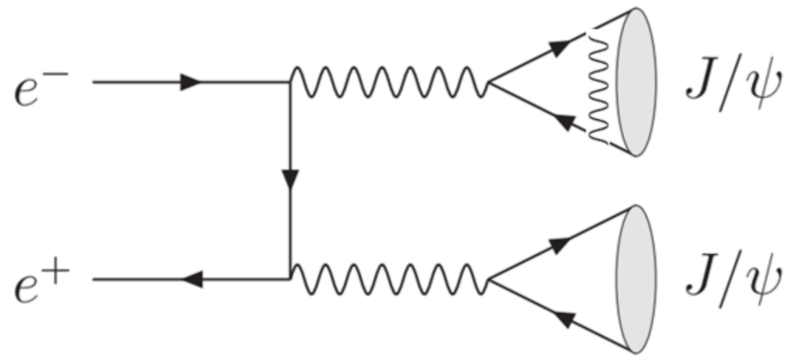
2. Charmonium production at B factory

$e^+e^- \to J/\psi + J/\psi$

Gong, Wang PRL(2008)

m_c (GeV)	μ	$\alpha_s(\mu)$	$\sigma_{\rm LO}$ (fb)	$\sigma_{\rm NLO}$ (fb)	$\sigma_{\rm NLO}/\sigma_{\rm LO}$
1.5	m_c	0.369	7.409	−2.327	−0.314
1.5	$2m_c$	0.259	7.409	0.570	0.077
1.5	$\sqrt{s}/2$	0.211	7.409	1.836	0.248
1.4	m_c	0.386	9.137	−3.350	−0.367
1.4	$2m_c$	0.267	9.137	0.517	0.057
1.4	$\sqrt{s}/2$	0.211	9.137	2.312	0.253

The main contribution comes from the fragmentation diagrams.



$$\langle J/\psi|\bar{c}\gamma^\mu c|0\rangle = -f_{J/\psi} M_{J/\psi} \varepsilon^{*\mu}_{J/\psi}$$

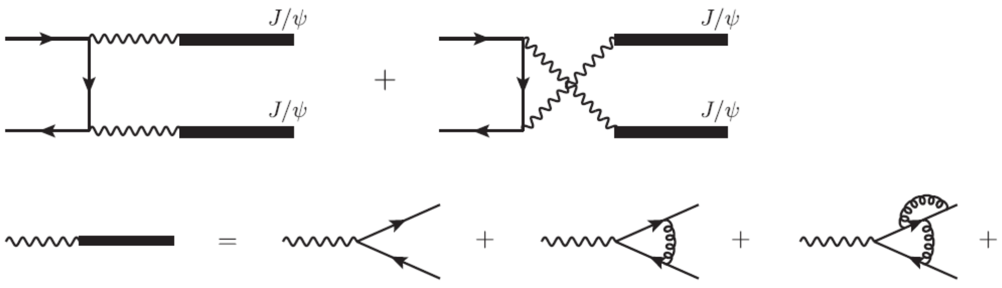
$$f_{J/\psi} = \sqrt{\frac{2\langle \mathcal{O}\rangle_{J/\psi}}{M_{J/\psi}}}\left(1 - 2C_F\frac{\alpha_s}{\pi} + f^{(2)}\frac{\alpha_s^2}{\pi^2} + \cdots\right)$$

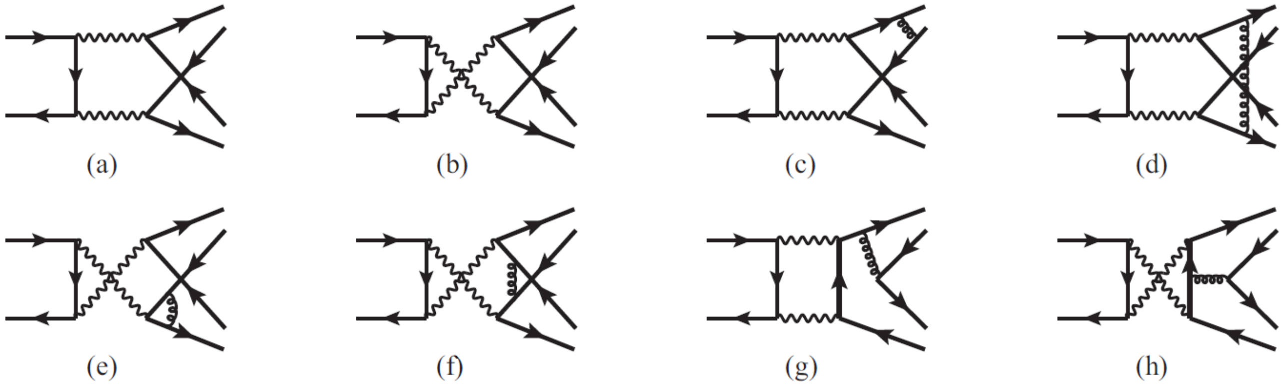
$$\sim (1 - 2C_F\tfrac{\alpha_s}{\pi})^4 \approx 1 - 11\tfrac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \approx 1 - 0.72$$

$$f^{(2)} = -43.33$$

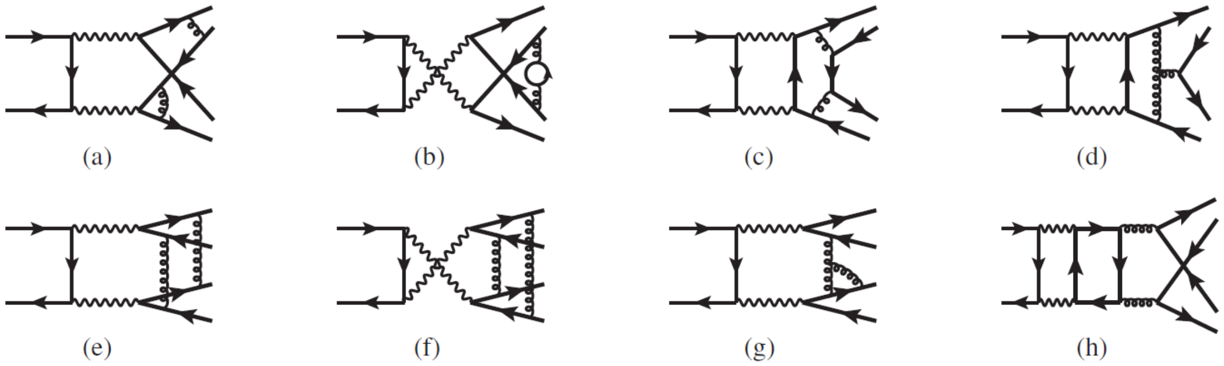
2. Charmonium production at B factory $e^+e^- \to J/\psi + J/\psi$

SWL, Feng, Jia, Mo, Pan, Zhang, PRL(2023),arXiv:2306.11538





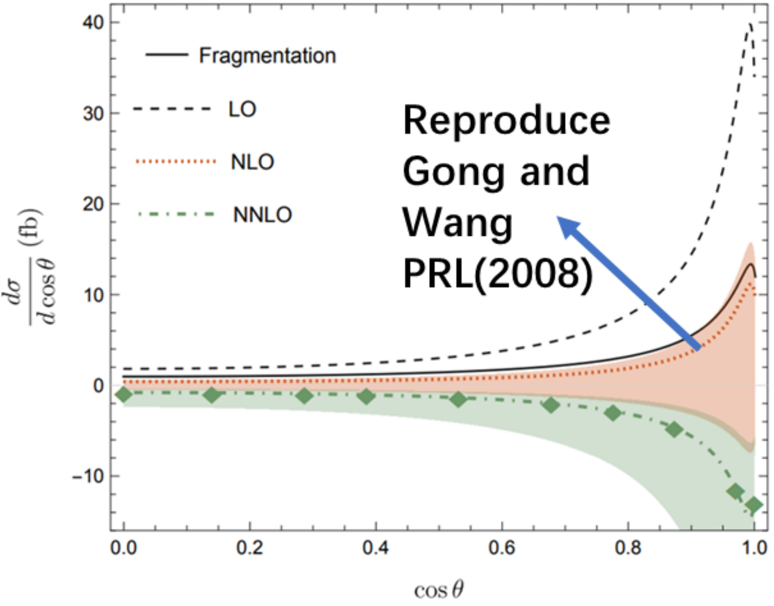
$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\frac{\beta}{16\pi}\frac{1}{4}\sum_{\rm spin}\left|\mathcal{M}_{\rm fr} + \mathcal{M}_{\rm nfr}\right|^2 .$$



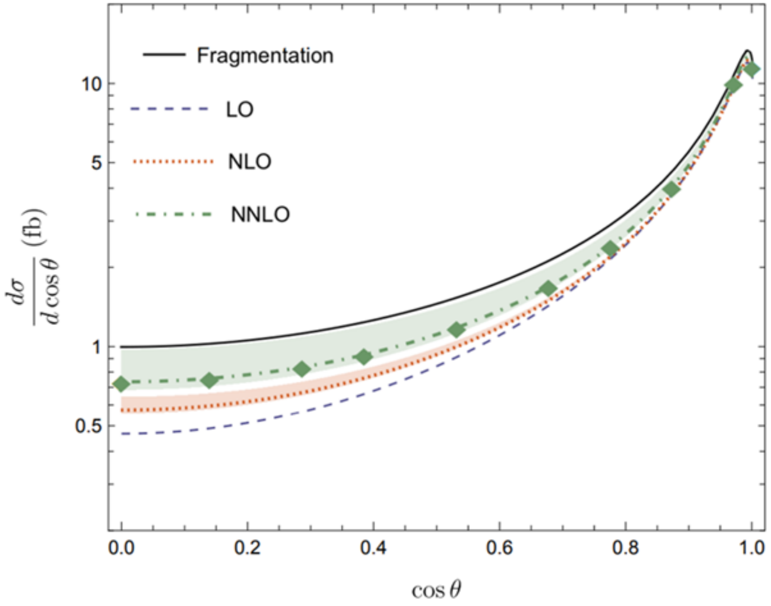
$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2s}\frac{\beta}{16\pi}\frac{e^8 e_c^4}{4}\left[\mathcal{C}_{\rm fr} f_{J/\psi}^4 + \mathcal{C}_{\rm int} f_{J/\psi}^2 \frac{\langle\mathcal{O}\rangle_{J/\psi}}{m_c} + \mathcal{C}_{\rm nfr}\left(\frac{\langle\mathcal{O}\rangle_{J/\psi}}{m_c}\right)^2\right]$$

Optimized NRQCD

2. Charmonium production at B factory $e^+e^- \to J/\psi + J/\psi$



Traditional NRQCD



Optimized NRQCD

Negative and unphysical for differential and total cross section

- Both NLO and NNLO correction is positive!
- Exhibit decent convergence behavior!
- When J/ψ production plane is nearly collinear to $e^+e^-(\theta \to 0)$, fragmentation contribution dominates!
- As θ deviates from 0, corrections from non-fragmentation amplitude start to play non-negligible role!

σ (fb)	Fragmentation	LO	NLO	NNLO
Optimized NRQCD	2.52	1.85	$1.93^{+0.05}_{-0.01}$	$2.13^{+0.30}_{-0.06}$
Traditional NRQCD		6.12	$1.56^{+0.73}_{-2.95}$	$-2.38^{+1.27}_{-5.35}$

See, Huang, Gong, Niu, Wang, arXiv: 2311.04751
Through different treatment

3. Bottomonium decay into double charmonia $\Upsilon \to J/\psi + \eta_c(\chi_{cJ})$

From BELLE's measurements PRD(2014)

$$\mathcal{B}[\Upsilon(1S) \to J/\psi + \chi_{c1}] = 3.90 \pm 1.21\ (\text{stat.})\ \pm 0.23\ (\text{syst.})\ \times 10^{-6}$$

$$\mathcal{B}[\Upsilon(1S) \to J/\psi + \eta_c] < 2.2 \times 10^{-6}, \quad \mathcal{B}[\Upsilon(1S) \to J/\psi + \chi_{c0}] < 3.4 \times 10^{-6}$$

$$\mathcal{B}[\Upsilon(1S) \to J/\psi + \chi_{c2}] < 1.4 \times 10^{-6}, \quad \mathcal{B}[\Upsilon(1S) \to J/\psi + \eta_c(2S)] < 2.2 \times 10^{-6}$$

$$\mathcal{B}[\Upsilon(2S) \to J/\psi + \eta_c(2S)] < 2.5 \times 10^{-6}, \quad \mathcal{B}[\Upsilon(2S) \to J/\psi + \chi_{c0}] < 3.4 \times 10^{-6}$$

$$\mathcal{B}[\Upsilon(2S) \to J/\psi + \chi_{c1}] < 1.2 \times 10^{-6}, \quad \mathcal{B}[\Upsilon(2S) \to J/\psi + \chi_{c2}] < 2.0 \times 10^{-6}$$

$$\mathcal{B}[\Upsilon(2S) \to J/\psi + \eta_c] < 5.4 \times 10^{-6}$$

理论计算 PRD(2007) Jia PRD(2013) Xu, Dong, Feng, Gao, Jia $\Gamma^{\text{LO}} \propto \alpha_s^6$

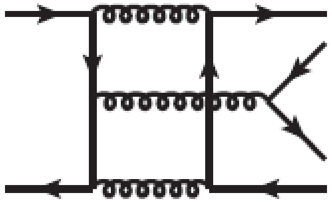
	$\Gamma^{[0]}$ (eV)	$\mathcal{B}^{[0]}$	$\Gamma^{[1]}$ (eV)	$\mathcal{B}^{[1]}$	$\Gamma^{[2]}$(eV)	$\mathcal{B}^{[2]}$
Υ(1S)	0.070	1.3×10^{-6}	0.26	4.9×10^{-6}	0.011	2.0×10^{-7}
Υ(2S)	0.035	1.1×10^{-6}	0.13	4.1×10^{-6}	0.0054	1.7×10^{-7}
Υ(3S)	0.026	8.6×10^{-7}	0.099	3.3×10^{-6}	0.0041	1.3×10^{-7}

large Scale dependence!

3. Bottomonium decay into double charmonia $\Upsilon \to J/\psi + \eta_c(\chi_{cJ})$

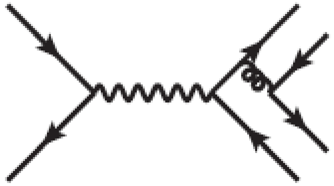
Zhang, **SWL**, Zhang, arXiv:2205.06124, PRL(2022)

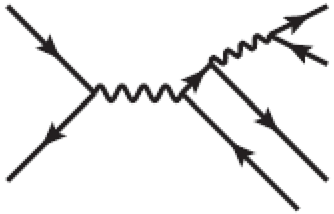
$$\Upsilon \to J/\psi + \eta_c(\chi_{cJ})$$



(i)

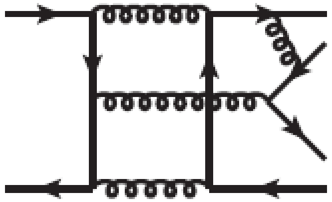
LO

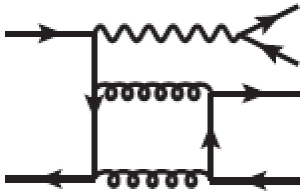


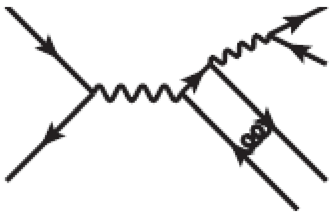


$$\Gamma^{\rm LO} \propto \alpha_s^6$$

$\propto \alpha_s^6$ $\propto \alpha^2\alpha_s^2$ $\propto \alpha^4$



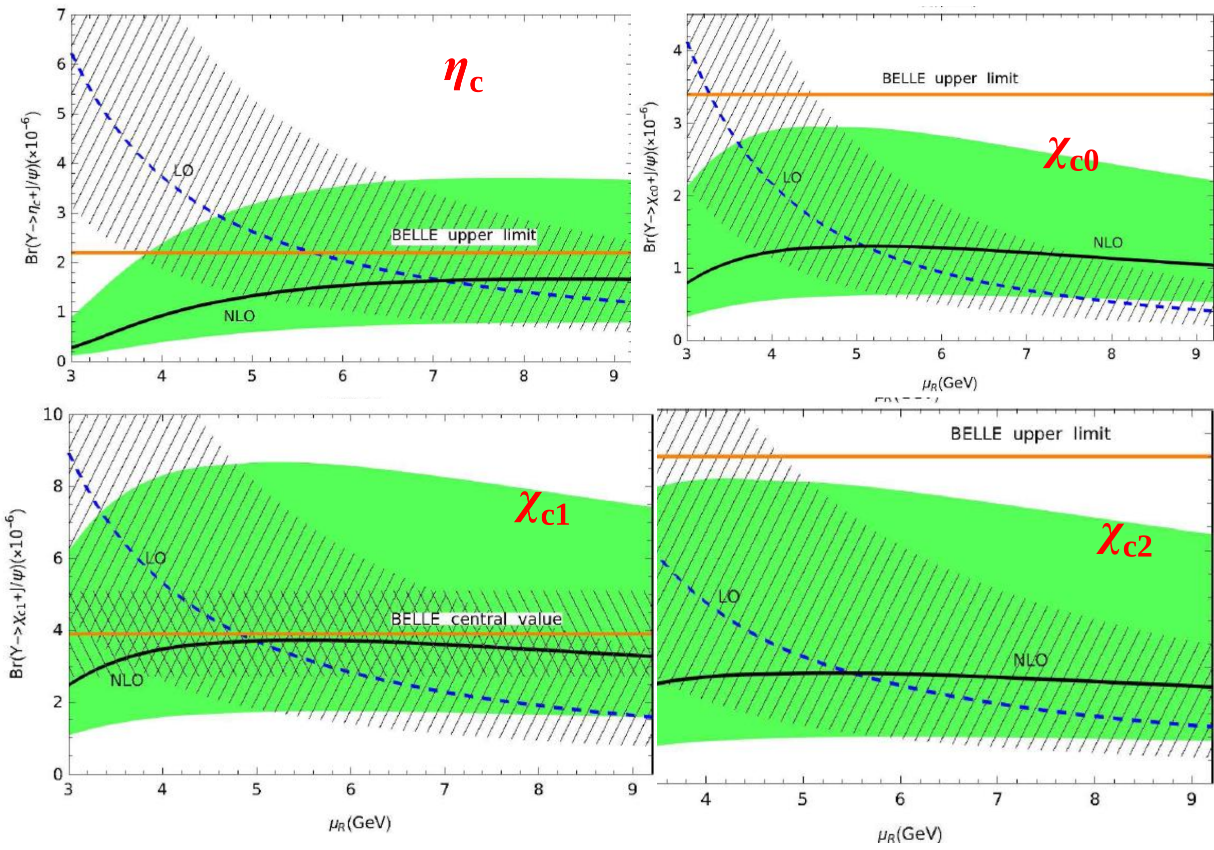




3. Bottomonium decay into double charmonia $\Upsilon \to J/\psi + \eta_c(\chi_{cJ})$

TABLE II: Results of the branching fractions ($\times 10^{-6}$) for $\Upsilon \to J/\psi + \eta_c(\chi_{cJ})$. The two uncertainties in the theoretical predictions are from the choices of the heavy quark mass and renormalization scale. For comparison, the Belle data [14] is juxtaposed in the last row.

Channels	LO	NLO	Belle [14]
$\Upsilon \to J/\psi + \eta_c$	$2.97^{+3.04+3.26}_{-1.48-1.78}$	$1.20^{+1.76+0.47}_{-0.71-0.92}$	< 2.2
$\Upsilon \to J/\psi + \chi_{c0}$	$1.60^{+1.62+2.53}_{-0.76-1.19}$	$1.29^{+1.69+0.02}_{-0.70-0.49}$	< 3.4
$\Upsilon \to J/\psi + \chi_{c1}$	$4.20^{+4.97+4.73}_{-2.15-2.62}$	$3.65^{+4.99+0.12}_{-2.02-1.20}$	$3.9^{+1.21+0.23}_{-1.21-0.23}$
$\Upsilon \to J/\psi + \chi_{c2}$	$0.62^{+0.87+0.68}_{-0.34-0.37}$	$0.47^{+0.84+0.05}_{-0.29-0.14}$	< 1.4
$\Upsilon \to J/\psi + \eta_c(2S)$	$1.94^{+1.99+2.13}_{-0.97-1.16}$	$0.78^{+1.15+0.31}_{-0.46-0.60}$	< 2.2
$\Upsilon(2S) \to J/\psi + \eta_c$	$2.51^{+2.56+2.75}_{-1.25-1.50}$	$1.01^{+1.49+0.39}_{-0.60-0.78}$	< 5.4
$\Upsilon(2S) \to J/\psi + \chi_{c0}$	$1.35^{+1.36+2.13}_{-0.64-1.01}$	$1.08^{+1.43+0.02}_{-0.59-0.41}$	< 3.4
$\Upsilon(2S) \to J/\psi + \chi_{c1}$	$3.54^{+4.19+3.99}_{-1.81-2.21}$	$3.08^{+4.21+0.10}_{-1.70-1.01}$	< 1.2
$\Upsilon(2S) \to J/\psi + \chi_{c2}$	$0.52^{+0.73+0.57}_{-0.28-0.31}$	$0.40^{+0.71+0.04}_{-0.24-0.12}$	< 2.0
$\Upsilon(2S) \to J/\psi + \eta_c(2S)$	$1.64^{+1.67+1.79}_{-0.82-0.98}$	$0.66^{+0.97+0.26}_{-0.39-0.51}$	< 2.5
$\Upsilon(3S) \to J/\psi + \eta_c$	$3.02^{+3.09+3.31}_{-1.51-1.81}$	$1.22^{+1.79+0.47}_{-0.72-0.94}$	–
$\Upsilon(3S) \to J/\psi + \chi_{c0}$	$1.62^{+1.64+2.57}_{-0.77-1.21}$	$1.31^{+1.72+0.02}_{-0.71-0.50}$	–
$\Upsilon(3S) \to J/\psi + \chi_{c1}$	$4.27^{+5.05+4.80}_{-2.19-2.66}$	$3.71^{+5.07+0.12}_{-2.05-1.22}$	–
$\Upsilon(3S) \to J/\psi + \chi_{c2}$	$0.63^{+0.88+0.69}_{-0.34-0.38}$	$0.48^{+0.85+0.05}_{-0.29-0.14}$	–



Renormalization scale dependence

For most channels:

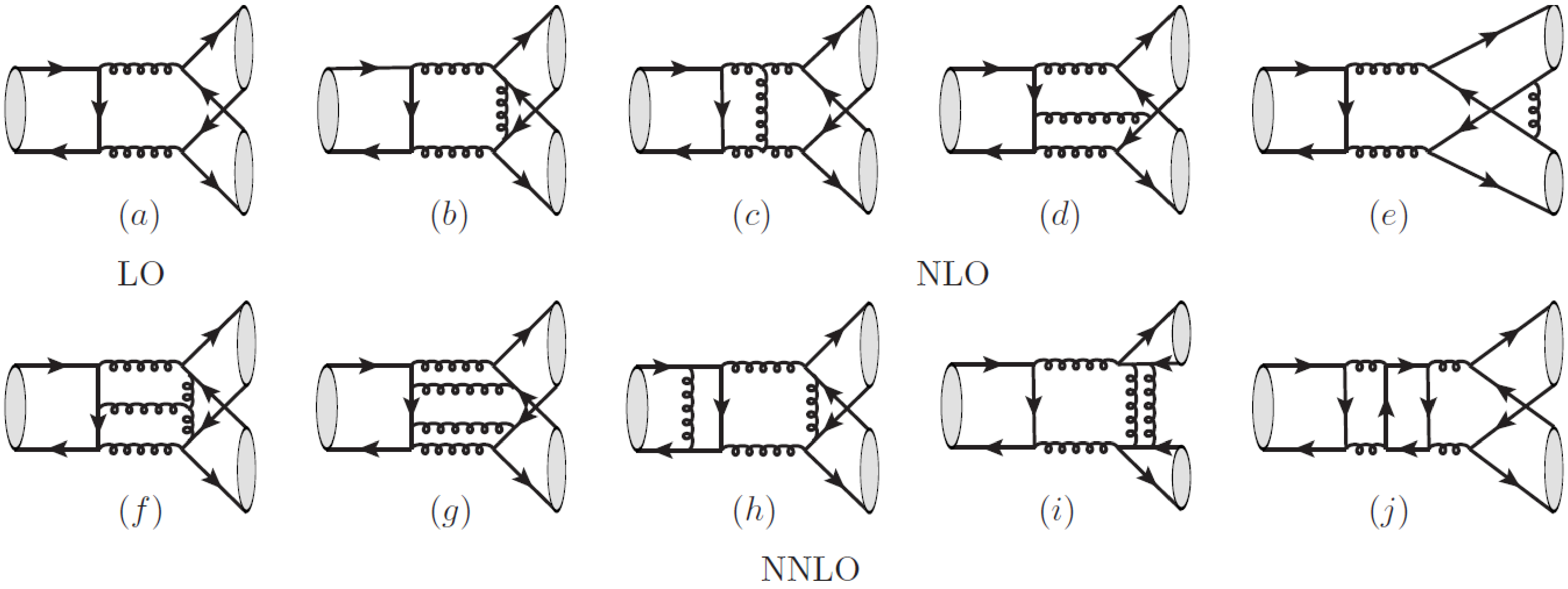
1. Uncertainty from scale is significantly reduced
2. Theoretical predictions are consistent with experiment

3. Bottomonium decay into double charmonia $\eta_b(\chi_{bJ}) \to J/\psi J/\psi$

PRD(2012), BELLE

$$\mathcal{B}(\chi_{b0} \to J/\psi J/\psi) < 7.1 \times 10^{-5}, \quad \mathcal{B}(\chi_{b0} \to J/\psi\psi') < 1.2 \times 10^{-4}, \quad \mathcal{B}(\chi_{b0} \to \psi'\psi') < 3.1 \times 10^{-5},$$
$$\mathcal{B}(\chi_{b1} \to J/\psi J/\psi) < 2.7 \times 10^{-5}, \quad \mathcal{B}(\chi_{b1} \to J/\psi\psi') < 1.7 \times 10^{-5}, \quad \mathcal{B}(\chi_{b1} \to \psi'\psi') < 6.2 \times 10^{-5},$$
$$\mathcal{B}(\chi_{b2} \to J/\psi J/\psi) < 4.5 \times 10^{-5}, \quad \mathcal{B}(\chi_{b2} \to J/\psi\psi') < 4.9 \times 10^{-5}, \quad \mathcal{B}(\chi_{b2} \to \psi'\psi') < 1.6 \times 10^{-5},$$

Representative Feynman diagrams



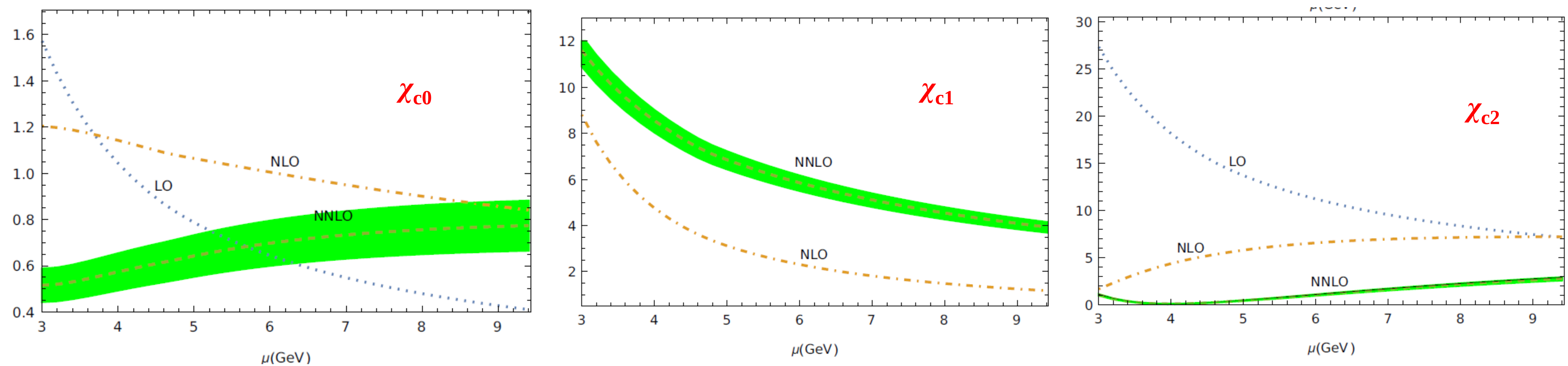
3. Bottomonium decay into double charmonia $\eta_b(\chi_{bJ}) \to J/\psi J/\psi$

Zhang, Bai, Feng, **SWL**, Zhou, arXiv:2310.07453

Table 1: Theoretical predictions on branching fractions ($\times 10^{-5}$) at various level of perturbative accuracy.

H	LO	NLO	NNLO	$\mathrm{Br}_{\mathrm{exp}}$
η_b	–	$0.022^{+0.033+0.014}_{-0.014-0.007}$	$0.082^{+0.080+0.054}_{-0.045-0.027}$	–
χ_{b0}	$0.844^{+0.727+0.117}_{-0.434-0.118}$	$1.081^{+0.122+0.150}_{-0.240-0.151}$	$0.621^{+0.152+0.086}_{-0.106-0.086}$	<7.1
χ_{b1}	–	$0.035^{+0.053+0.002}_{-0.023-0.002}$	$0.072^{+0.044+0.004}_{-0.033-0.004}$	<2.7
χ_{b2}	$14.669^{+12.636+0.884}_{-7.545-0.789}$	$5.424^{+1.768+0.327}_{-3.781-0.292}$	$0.265^{+2.450+0.016}_{-0.205-0.014}$	<4.5

consistent with BELLE!



4. B_c decay constant at three loop

Feng, Jia, Mo, Pan, SWL, Zhang, arXiv:2208.04302
SWL, Zhang, Zhou, arXiv:2210.02979, PLB(2023)

Definitions of the decay constants

$$\langle 0|\bar{b}\gamma^\mu\gamma_5 c|B_c\rangle = i f_{B_c} P^\mu \qquad \langle 0|\bar{b}\gamma^\mu c|B_c^*\rangle = -f_{B_c^*} m_{B_c^*}\epsilon^\mu$$

According to the NRQCD factorization

$$f_{B_c} = \sqrt{\frac{2}{M_{B_c}}}\,\mathcal{C}(m_b, m_c, \mu_\Lambda)\,\langle 0|\chi_b^\dagger\psi_c(\mu_\Lambda)|B_c\rangle + \mathcal{O}(v^2)$$

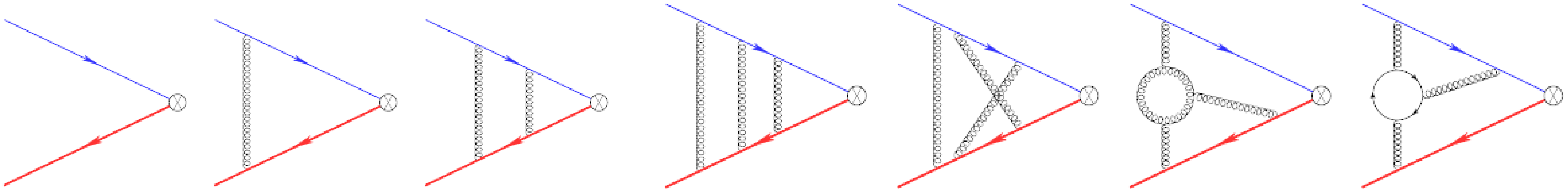
The SDC can be expanded in powers of α_s

Chen, Qiao, PLB(2015)
Tao, Zhu, Xiao, PRD(2022)

$$\begin{aligned}\mathcal{C}(m_b, m_c, \mu_\Lambda, \mu_R) &= 1 + \frac{\alpha_s^{(n_f)}(\mu_R)}{\pi}\mathcal{C}^{(1)}(x) + \left(\frac{\alpha_s^{(n_f)}(\mu_R)}{\pi}\right)^2\left(\mathcal{C}^{(1)}(x)\frac{\beta_0}{4}\ln\frac{\mu_R^2}{m_M^2} + \gamma^{(2)}(x)\ln\frac{\mu_\Lambda^2}{m_M^2} + \mathcal{C}^{(2)}(x)\right)\\ &+ \left(\frac{\alpha_s^{(n_f)}(\mu_R)}{\pi}\right)^3\left\{\left(\frac{\mathcal{C}^{(1)}(x)}{16}\beta_1 + \frac{\mathcal{C}^{(2)}(x)}{2}\beta_0\right)\ln\frac{\mu_R^2}{m_M^2} + \frac{\mathcal{C}^{(1)}(x)}{16}\beta_0^2\ln^2\frac{\mu_R^2}{m_M^2} + \frac{1}{4}\left(2\frac{d\gamma^{(3)}(x,\mu_\Lambda)}{d\ln\mu_\Lambda^2} - \beta_0\gamma^{(2)}(x)\right)\ln^2\frac{\mu_\Lambda^2}{m_M^2}\right.\\ &\left.+ \left(\mathcal{C}^{(1)}(x)\gamma^{(2)}(x) + \gamma^{(3)}(x, m_M)\right)\ln\frac{\mu_\Lambda^2}{m_M^2} + \frac{\beta_0}{2}\gamma^{(2)}(x)\ln\frac{\mu_\Lambda^2}{m_M^2}\ln\frac{\mu_R^2}{m_M^2} + \mathcal{C}^{(3)}(x)\right\} + \mathcal{O}(\alpha_s^4)\end{aligned}$$

We obtain the analytic expressions

4. B_c decay constant at three loop



$\mathcal{C}^{(1)}(m_b/m_c)$ and $\mathcal{C}^{(2)}(m_b/m_c)$ are analytically obtained.

$\mathcal{C}^{(3)}(m_b/m_c)$ is merely numerically obtained at some special values of m_b/m_c.

For B_c^*, we are suprised to find $\mathcal{C}^{(3)}$ can be well approximated with a linear function

$$\mathcal{C}^{(3)} = -1649.17 - 68.42 \times \frac{m_b}{m_c}$$

at $\frac{m_b}{m_c} \in (2.1, 4.0)$ with n_l=3, n_c= n_b=1

$$\mathcal{C}(\tfrac{4.98}{2.04}) = 1 - 2.29\left(\frac{\alpha_s^{(n_l)}}{\pi}\right) - 35.36\left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^2 - 1686.80\left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^3$$

poor convergence!!

4. B_c decay constant at three loop

See the more systematic and elegant works for B_c family

Tao, Zhu, Xiao, EPJC(2023), arXiv:2301.00220
Tao, Xiao, Zhu, JHEP(2023), arXiv:2303.02692, 2303.07220
Tao, Xiao, arXiv:2310.11649
Tao, Xiao, arXiv:2310.17500

5. Summary

1. Perturbative convergence is not bad for quarkonia production, and the theoretical prediction can describe the experiment, notwithstanding large uncertainties.

2. Perturbative convergence is very poor for quarkonium EW decay!

3. The NRQCD factorization is nontrivially testified.

4. The theoretical predictions suffer large uncertainties from heavy quark mass.

5. How about the size of relativistic corrections?

Thank you for your attention！