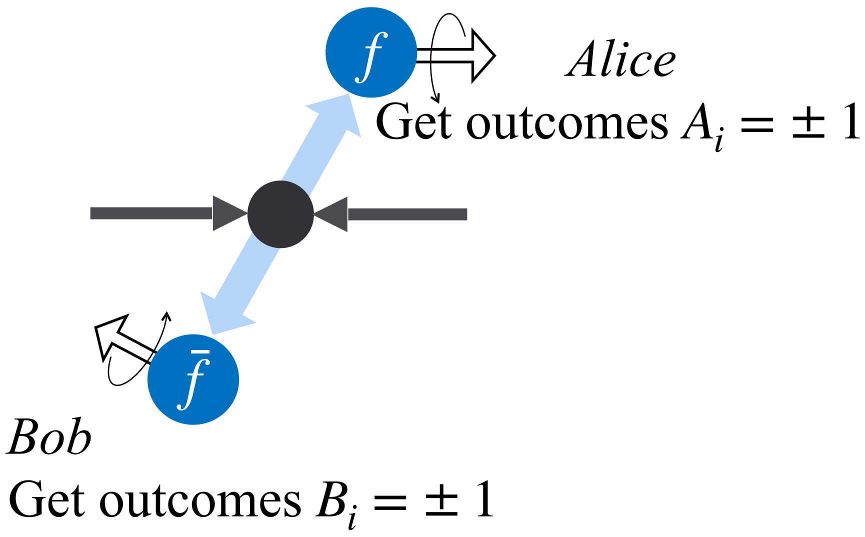
Testing the violation of Bell inequalities in W^+W^- system

arXiv:2307.14895: Q. Bi, Q.-H. Cao, **KC** and H. Zhang

Introduction: Bell inequalities

Bell inequality: constructed from four two-outcome measurements $\hat{A}_{1,2}$ and $\hat{B}_{1,2}$



Alice

Get outcomes $A_i = \pm 1$

Bob

Get outcomes $B_i = \pm 1$

$$\mathcal{I}_2(\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2) = \left| \langle \hat{A}_1 \hat{B}_1 \rangle + \langle \hat{A}_1 \hat{B}_2 \rangle + \langle \hat{A}_2 \hat{B}_1 \rangle - \langle \hat{A}_2 \hat{B}_2 \rangle \right| \leq 2$$

In QM, the two measurements may not commute, $[\hat{A}_1, \hat{A}_2] \neq 0$ and $[\hat{B}_1, \hat{B}_2] \neq 0$ and the inequality can be violated

Bell inequality test at colliders: $t\bar{t}, \tau^+\tau^-, W^+W^-, ZZ, \cdots\cdots$

Test the fundamental principle of QM at extreme high-energy

W^+W^-/ZZ: the only fundamental particles with 3d spin space.

2x2 system, $A_i, B_i = \pm 1$ **3x3 system, $A_i, B_i = -1,0,1.$**

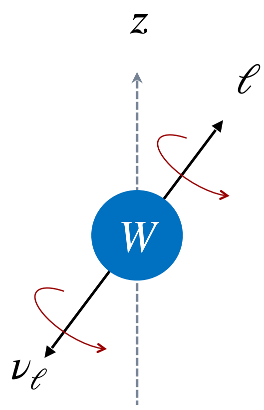
The Bell inequality is generalized to $I_3 \leq 2$:

$$\begin{aligned} \mathcal{I}_3 \equiv &+ \left[P\left(A_1 = B_1\right) + P\left(B_1 = A_2 + 1\right) + P\left(A_2 = B_2\right) + P\left(B_2 = A_1\right) \right] \\ &- \left[P\left(A_1 = B_1 - 1\right) + P\left(B_1 = A_2\right) + P\left(A_2 = B_2 - 1\right) + P\left(B_2 = A_1 - 1\right) \right] \end{aligned}$$

Method and assumptions

t or W^+ rest frame

$\vec{\mathfrak{n}}$: normalized direction of ℓ^+



The spin of top quark or W boson cannot be measured directly, and they are inferred from the distribution of their decay:

A spin-up top quark $t_\uparrow \to \ell^+ \nu b$: $\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta_\ell} \approx \frac{1}{2}(1+\cos\theta_\ell)$

A general density matrix $\rho^t = \frac{1}{2}(I_2 + B_i\sigma_i)$ $\boxed{B_i = 3\langle \mathfrak{n}_i \rangle}$

W boson is similar. E.g., $W^+_\uparrow \to \ell^+\nu$: $\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta_\ell} = \frac{3}{8}(1+\cos\theta_\ell)^2$

A general density matrix

$$\rho_W = \frac{I_3}{3} + \sum_{i=1}^{3} d_i S_i + \sum_{i,j=1}^{3} q_{ij}\{S_i, S_j\} \qquad \boxed{d_i = \langle \mathfrak{n}_i \rangle, \quad q_{ij} = \frac{5}{2}\left\langle \mathfrak{n}_i\mathfrak{n}_j - \frac{\delta_{ij}}{3} \right\rangle}$$

Top quark: $\langle \mathfrak{n}_i \rangle \Longrightarrow$ angular momentum of top quark

W boson: $\langle \mathfrak{n}_i \rangle \Longrightarrow$ angular momentum, $\langle \mathfrak{n}_i\mathfrak{n}_j - \delta_{ij}/3 \rangle \Longrightarrow$ **linear polarization**

Measure the $\ell_i^\pm$ distribution from $W^\pm$ decay

Construct the density matrix

Alice and Bob perform some measurements on this density matrix

Theory assumption of the decay mechanism

Four measurements: $\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2$

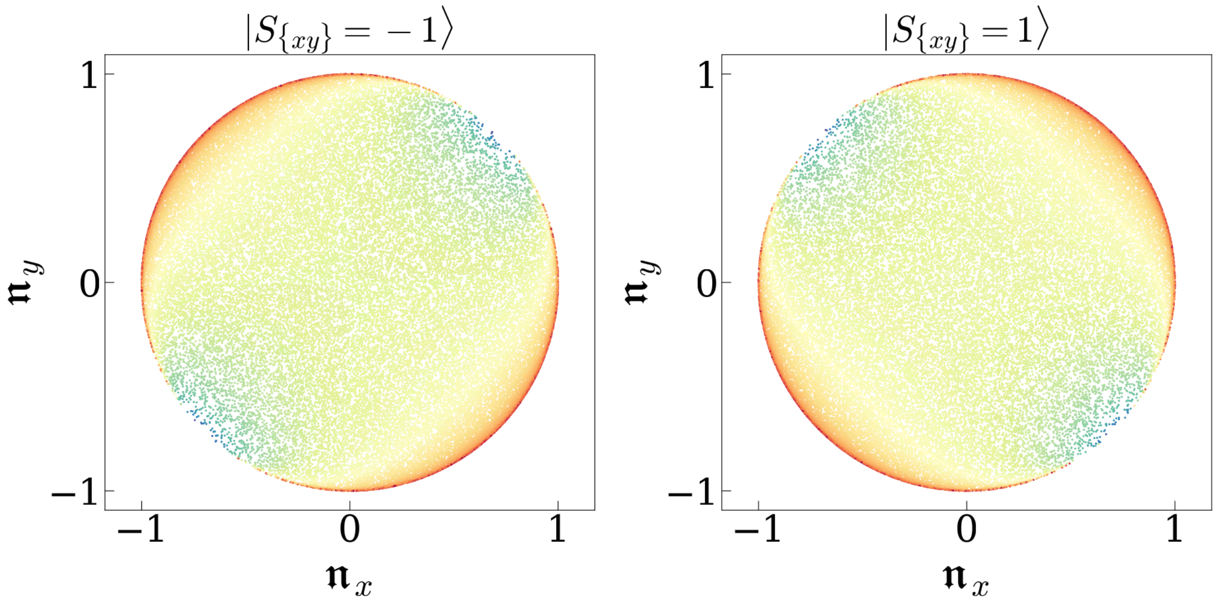
W-boson density matrix from its decayed product distribution

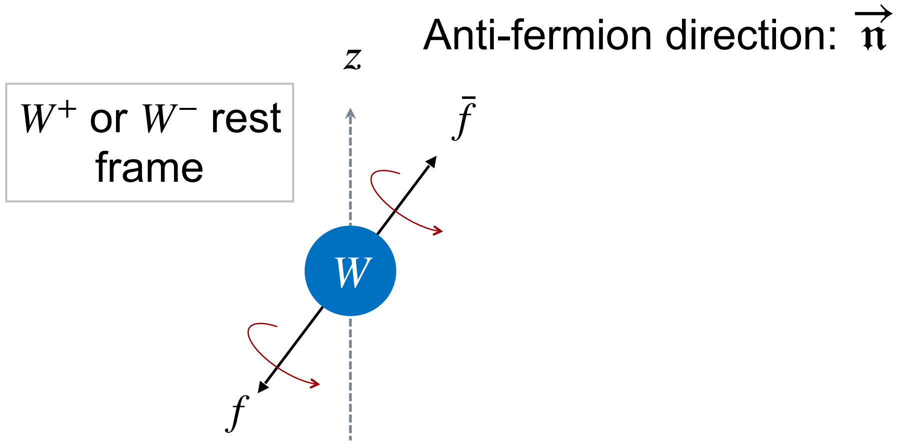
$$\rho_W = \frac{I_3}{3} + \sum_{i=1}^{3} d_i S_i + \sum_{i,j=1}^{3} q_{ij} S_{\{ij\}}. \qquad S_{\{ij\}} = S_i S_j + S_j S_i$$

Angular momentum measurements

Linear polarization measurements

E.g. $$\begin{cases} \vec{\epsilon}\,\big|_{S_z=1\rangle} = (1, i, 0)/\sqrt{2} \\ \vec{\epsilon}\,\big|_{S_{\{xy\}}=1\rangle} = (1, -1, 0)/\sqrt{2} \end{cases}$$





$$p(\vec{\mathfrak{n}}; \rho_W) = \frac{3}{4\pi} \mathrm{Tr}\left[\rho_W \hat{\Pi}_{|S_{\mathfrak{n}}=1\rangle}\right]$$

Decay product distribution of purely linear polarized W boson

If we want to measure the angular momentum of $W^{\pm}$, we need the direction of $\bar{f}$: $\vec{\mathfrak{n}}^{\pm}$
If we want to measure the linear polarization of $W^{\pm}$, we need the quadruple distribution

chengkun@pku.edu.cn Kun Cheng

Distinguishing fermion from anti-fermion in W decay (flavor tagging)

$W^+ \to \ell^+\nu$ or $W^+ \to u\bar{d}$

Flavor tagging:
use leptonic decay, or perform jet charge tagging

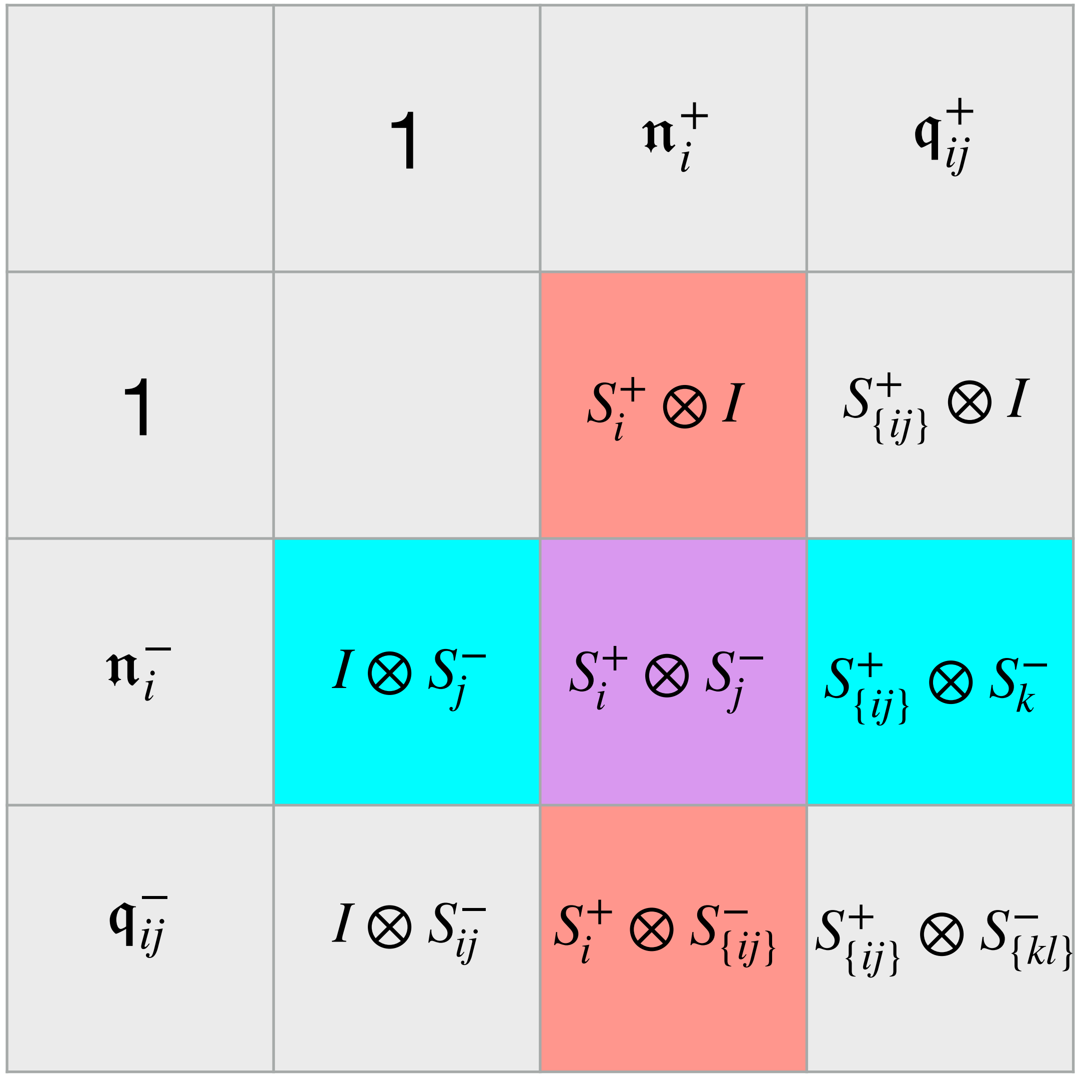
🟥 : Flavor tagging in W^+ decay

🟦 : Flavor tagging in W^- decay

🟪 : Flavor tagging in both decay

$\mathfrak{n}_i$:	**normalized direction of anti-fermion in W rest frame, need flavor tagging.**
$\mathfrak{q}_{ij} = \mathfrak{n}_i \mathfrak{n}_j - \frac{\delta_{ij}}{3}$:	**do not need flavor tagging**

$$\begin{aligned}\rho_{WW} =& \frac{I_9}{9} + \frac{1}{3} d_i^+ S_i \otimes I_3 + \frac{1}{3} q_{ij}^+ S_{\{ij\}} \otimes I_3 \\ &+ \frac{1}{3} d_i^- I_3 \otimes S_i + \frac{1}{3} q_{ij}^- I_3 \otimes S_{\{ij\}} \\ &+ C_{ij}^d S_i \otimes S_j + C_{ij,kl}^q S_{\{ij\}} \otimes S_{\{kl\}} \\ &+ C_{i,jk}^{dq} S_i \otimes S_{\{jk\}} + C_{ij,k}^{qd} S_{\{ij\}} \otimes S_k\end{aligned}$$



	1	$\mathfrak{n}_i^+$	$\mathfrak{q}_{ij}^+$
1		$S_i^+ \otimes I$	$S_{\{ij\}}^+ \otimes I$
$\mathfrak{n}_i^-$	$I \otimes S_j^-$	$S_i^+ \otimes S_j^-$	$S_{\{ij\}}^+ \otimes S_k^-$
$\mathfrak{q}_{ij}^-$	$I \otimes S_{ij}^-$	$S_i^+ \otimes S_{\{ij\}}^-$	$S_{\{ij\}}^+ \otimes S_{\{kl\}}^-$

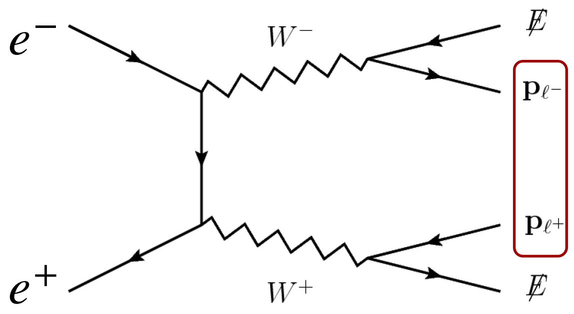
Two fold ambiguity in the neutrino solution

$$\mathcal{I}_3 \equiv +\left[P\left(A_1 = B_1\right) + P\left(B_1 = A_2 + 1\right) + P\left(A_2 = B_2\right) + P\left(B_2 = A_1\right)\right]$$
$$-\left[P\left(A_1 = B_1 - 1\right) + P\left(B_1 = A_2\right) + P\left(A_2 = B_2 - 1\right) + P\left(B_2 = A_1 - 1\right)\right]$$

It is a usual practice to test the Bell inequality by measuring angular momentum

$$\mathcal{I}_3^{(S)} \equiv \mathcal{I}_3(\hat{S}_{\vec{a}_1}, \hat{S}_{\vec{a}_2}; \hat{S}_{\vec{b}_1}, \hat{S}_{\vec{b}_2})$$

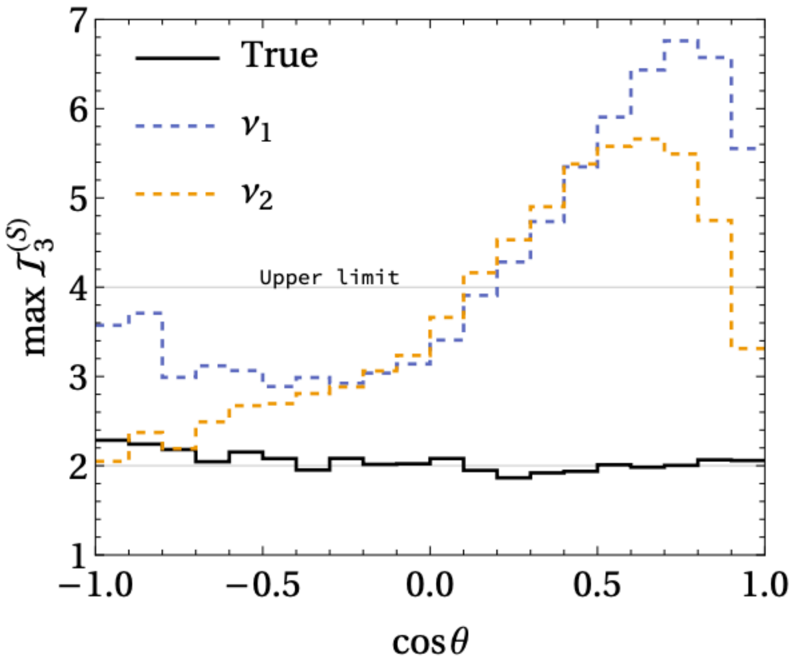
Then the di-lepton decay mode should be used.



Solve the neutrino momentum

$$|\vec{p}_\nu|^2 = \ldots$$
$$\vec{p}_\nu \cdot \vec{\ell}^+ = \ldots$$
$$\vec{p}_\nu \cdot \vec{\ell}^- = \ldots$$

Quadratic equation, two fold solution.



240GeV $e^+e^- \rightarrow W^+W^-$

Option a) Try unfolding (as is always needed for $t\bar{t}$), flavor tagging, etc.

Option b) Choose other observables to reduce this difficulty.

Construct Bell observables from linear polarization

For example:
Use the correlation between the ***linear polarization of W^+*** and the ***angular momentum of W^-*** to test Bell inequalities

$$\mathcal{I}_3^{(L,S)} \equiv \mathcal{I}_3(\hat{S}_{\{x_1y_1\}}, \hat{S}_{\{x_2y_2\}}; \hat{S}_{\vec{b}_1}, \hat{S}_{\vec{b}_2})$$

Measurements are done by projecting ρ_{WW} to

$$\left|S_{\{x_iy_i\}} = a\right\rangle \otimes \left|S_{b_i} = b\right\rangle, a, b = -1,0,1$$

This depends on only part of the density matrix

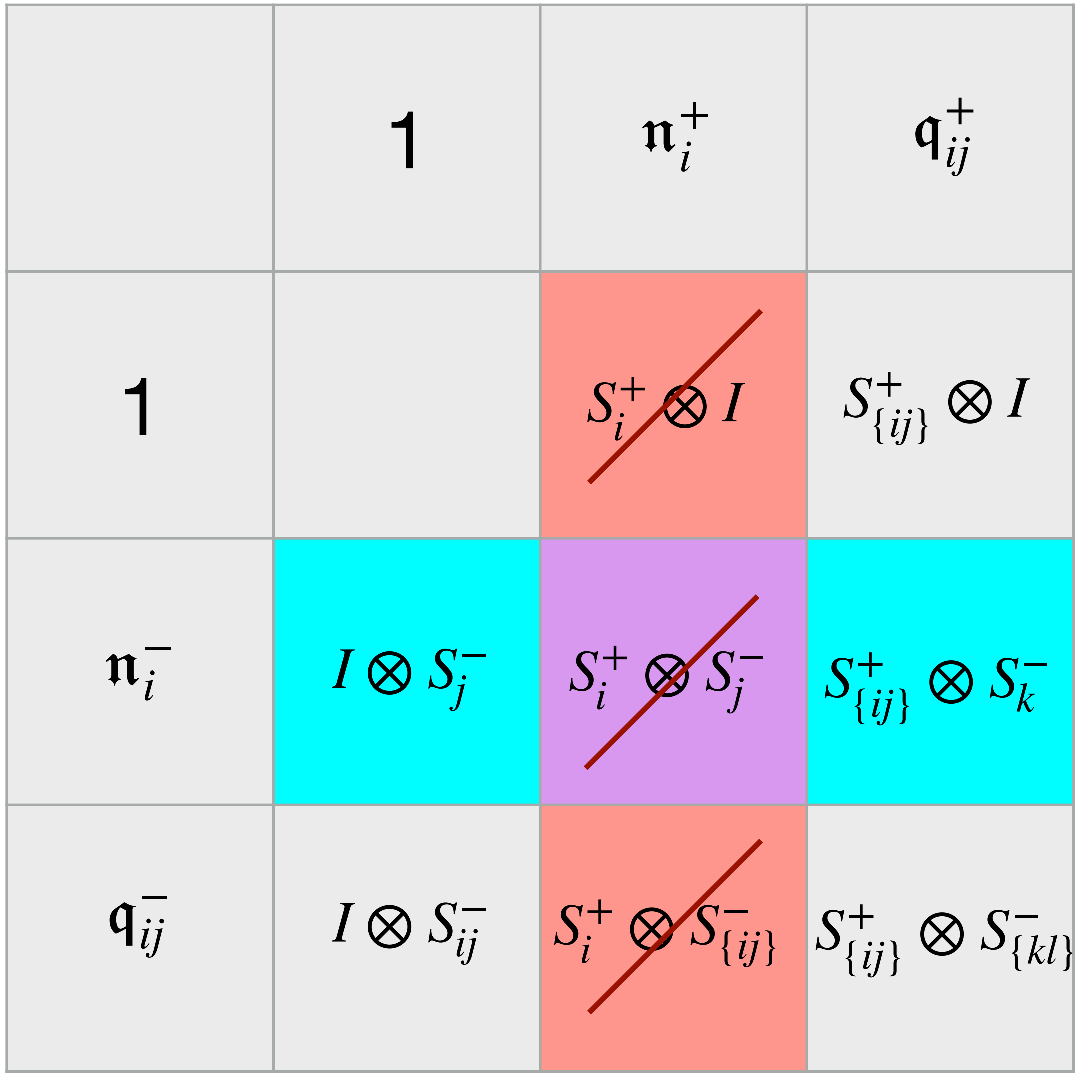
$$\mathrm{Tr}\left[\hat{S}_k \hat{\Pi}_{|S_{\{xy\}}=a\rangle}\right] = 0$$

Measurable in semi-leptonic decay mode

$$e^+e^- \to W^+W^-$$
$$W^+(\to jj)W^- \to (\ell^-\bar{\nu})$$

Most clear decay channel

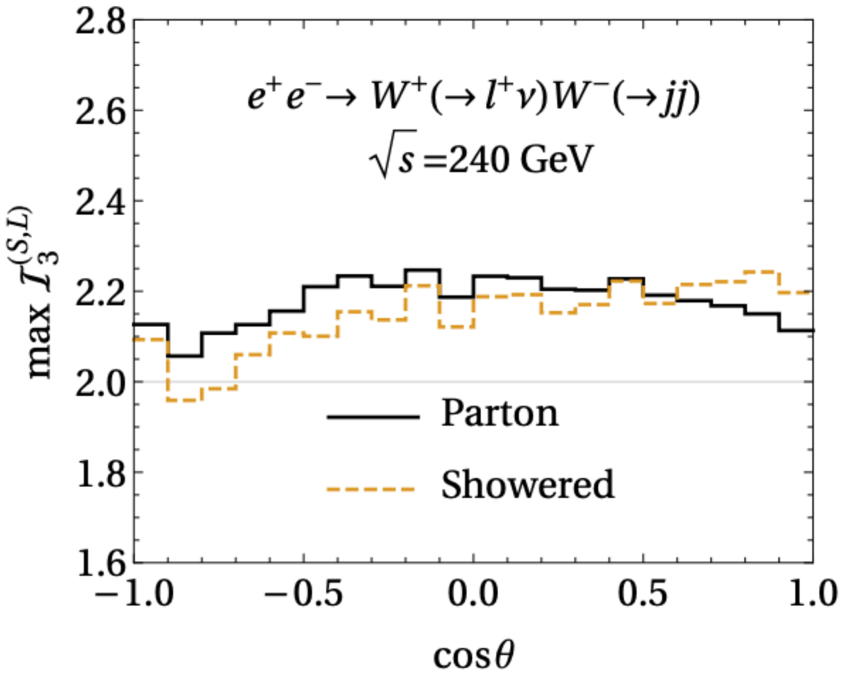
$$\begin{aligned}\rho_{WW} = &\frac{I_9}{9} + \boxed{\cancel{\frac{1}{3}d_i^+ S_i \otimes I_3}} + \frac{1}{3}q_{ij}^+ S_{\{ij\}} \otimes I_3 \\ &+ \frac{1}{3}d_i^- I_3 \otimes S_i + \frac{1}{3}q_{ij}^- I_3 \otimes S_{\{ij\}} \\ &+ \boxed{\cancel{C_{ij}^d S_i \otimes S_j}} + C_{ij,kl}^q S_{\{ij\}} \otimes S_{\{kl\}} \\ &+ \boxed{\cancel{C_{i,jk}^{dq} S_i \otimes S_{\{jk\}}}} + C_{ij,k}^{qd} S_{\{ij\}} \otimes S_k\end{aligned}$$



	1	$\mathfrak{n}_i^+$	$\mathfrak{q}_{ij}^+$
1		$\cancel{S_i^+ \otimes I}$	$S_{\{ij\}}^+ \otimes I$
$\mathfrak{n}_i^-$	$I \otimes S_j^-$	$\cancel{S_i^+ \otimes S_j^-}$	$S_{\{ij\}}^+ \otimes S_k^-$
$\mathfrak{q}_{ij}^-$	$I \otimes S_{ij}^-$	$\cancel{S_i^+ \otimes S_{\{ij\}}^-}$	$S_{\{ij\}}^+ \otimes S_{\{kl\}}^-$

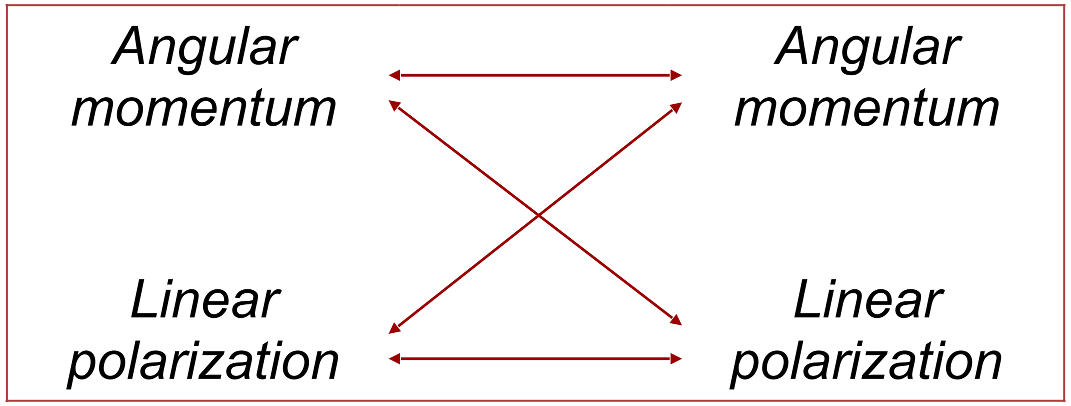
Simulation results

$e^+e^- \to W^+W^-,\ \sqrt{s} = 240\ \text{GeV},$



Good consistence with the parton level result. The reconstruction is much easier.

Many possible choices of measurements for $W^\pm$ pair


Angular momentum — *Angular momentum*

Linear polarization — *Linear polarization*

$\mathcal{I}_3^{(S)} \equiv \mathcal{I}_3(\hat{S}_{\vec{a}_1}, \hat{S}_{\vec{a}_2}; \hat{S}_{\vec{b}_1}, \hat{S}_{\vec{b}_2})$
- *Di-lepton*

$\mathcal{I}_3^{(L,S)} \equiv \mathcal{I}_3(\hat{S}_{\{x_1y_1\}}, \hat{S}_{\{x_2y_2\}}; \hat{S}_{\vec{b}_1}, \hat{S}_{\vec{b}_2})$
- *Di-lepton, semi-leptonic*

$\mathcal{I}_3^{(S,L)} \equiv \mathcal{I}_3(\hat{S}_{\vec{a}_1}, \hat{S}_{\vec{a}_2}; \hat{S}_{\{x_3y_3\}}, \hat{S}_{\{x_4y_4\}})$
- *Di-lepton, semi-leptonic*

$\mathcal{I}_3^{(L)} \equiv \mathcal{I}_3(\hat{S}_{\{x_1y_1\}}, \hat{S}_{\{x_2y_2\}}; \hat{S}_{\{x_3y_3\}}, \hat{S}_{\{x_4y_4\}})$
- *Di-lepton, semi-leptonic, Full hadronic*

➡ Cleanest channel
240GeV, 180 $\text{fb}^{-1} \Longrightarrow 5\sigma$

Summary

- The Hilbert space of W pair to study is 3x3 dimension, for top anti-top its only 2x2.
  - The expression of Bell inequality is more complicated
  - But we have more flexible choices of observables

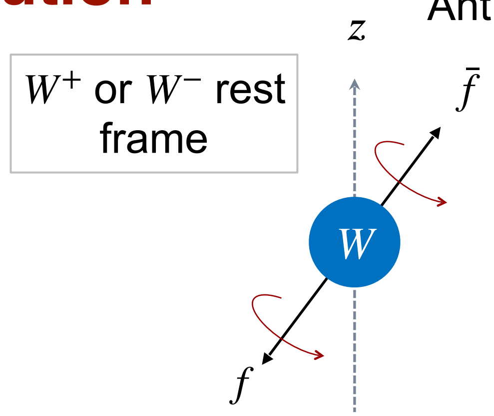
- Correlation between angular momentum:
  - Di-lepton decay mode
- Correlation between circular and linearly polarized state
  - Semi-leptonic decay mode
  - Larger cross section and easier to reconstruct

- Lepton colliders: no need to reconstruct
- Hadron colliders: less experimental difficulties

Backup

W-boson density matrix from its decayed product distribution

Anti-fermion direction: $\vec{\mathfrak{n}}$



$$p(\vec{\mathfrak{n}};\rho_W)=\frac{3}{4\pi}\,\mathrm{Tr}\left[\rho_W\hat{\Pi}_{|S_{\mathfrak{n}}=1\rangle}\right]$$

$$\rho_W=\frac{I_3}{3}+\sum_{i=1}^{3}d_iS_i+\sum_{i,j=1}^{3}q_{ij}S_{\{ij\}}.$$

Angular momentum measurements

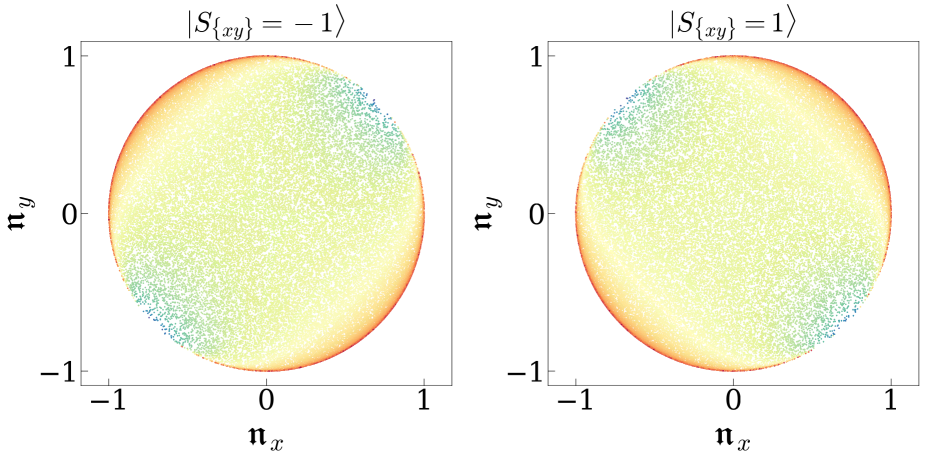
Linear polarization measurements

$$\langle\mathfrak{n}_i\rangle_{\mathrm{av}}\equiv\int\mathfrak{n}_i\;p(\vec{\mathfrak{n}};\rho_W)\,\mathrm{d}\Omega=d_i\qquad\langle\mathfrak{q}_{ij}\rangle_{\mathrm{av}}\equiv\int\mathfrak{q}_{ij}\;p(\vec{\mathfrak{n}};\rho_W)\,\mathrm{d}\Omega=\frac{5}{2}q_{ij}$$

$$\mathfrak{q}_{ij}\equiv\mathfrak{n}_i\mathfrak{n}_j-\frac{1}{3}\delta_{ij}$$

Dipole distribution

Quadrupole distribution



$$\rho_{WW}=\frac{I}{9}+(g_a^+\lambda_a\otimes I+g_a^-I\otimes\lambda_a)+h_{ab}\lambda_a\otimes\lambda_b$$

$$\begin{aligned}\rho_{WW}=&\frac{I_9}{9}+\frac{1}{3}d_i^+S_i\otimes I_3+\frac{1}{3}q_{ij}^+S_{\{ij\}}\otimes I_3\\&+\frac{1}{3}d_i^-I_3\otimes S_i+\frac{1}{3}q_{ij}^-I_3\otimes S_{\{ij\}}\\&+C_{ij}^dS_i\otimes S_j+C_{ij,kl}^qS_{\{ij\}}\otimes S_{\{kl\}}\\&+C_{i,jk}^{dq}S_i\otimes S_{\{jk\}}+C_{ij,k}^{qd}S_{\{ij\}}\otimes S_k\end{aligned}$$

Individual measurements on $W^{\pm}$ decayed product distribution

$$d_i^{\pm}=\langle\mathfrak{n}_i^{\pm}\rangle_{\mathrm{av}}\,,\qquad q_{ij}^{\pm}=\frac{5}{2}\langle\mathfrak{q}_{ij}^{\pm}\rangle_{\mathrm{av}}$$

Correlations

$$C_{ij}^d=\langle\mathfrak{n}_i^+\mathfrak{n}_j^-\rangle_{\mathrm{av}}\,,\qquad C_{i,jk}^{dq}=\frac{5}{2}\left\langle\mathfrak{n}_i^+\mathfrak{q}_{jk}^-\right\rangle_{\mathrm{av}}$$

$$C_{ij,kl}^q=\frac{25}{4}\langle\mathfrak{q}_{ij}^+\mathfrak{q}_{kl}^-\rangle_{\mathrm{av}}\qquad C_{ij,k}^{qd}=\frac{5}{2}\langle\mathfrak{q}_{ij}^+\mathfrak{n}_k^-\rangle_{\mathrm{av}}\,.$$

If we want to measure the angular momentum of $W^{\pm}$, we need the direction of $\bar{f}$: $\vec{\mathfrak{n}}^{\pm}$
If we want to measure the linear polarization of $W^{\pm}$, we need the quadruple distribution