广州大学

Analytic N^3LO QCD corrections to top quark and semileptonic $b\to u$ decays

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2024-1-22

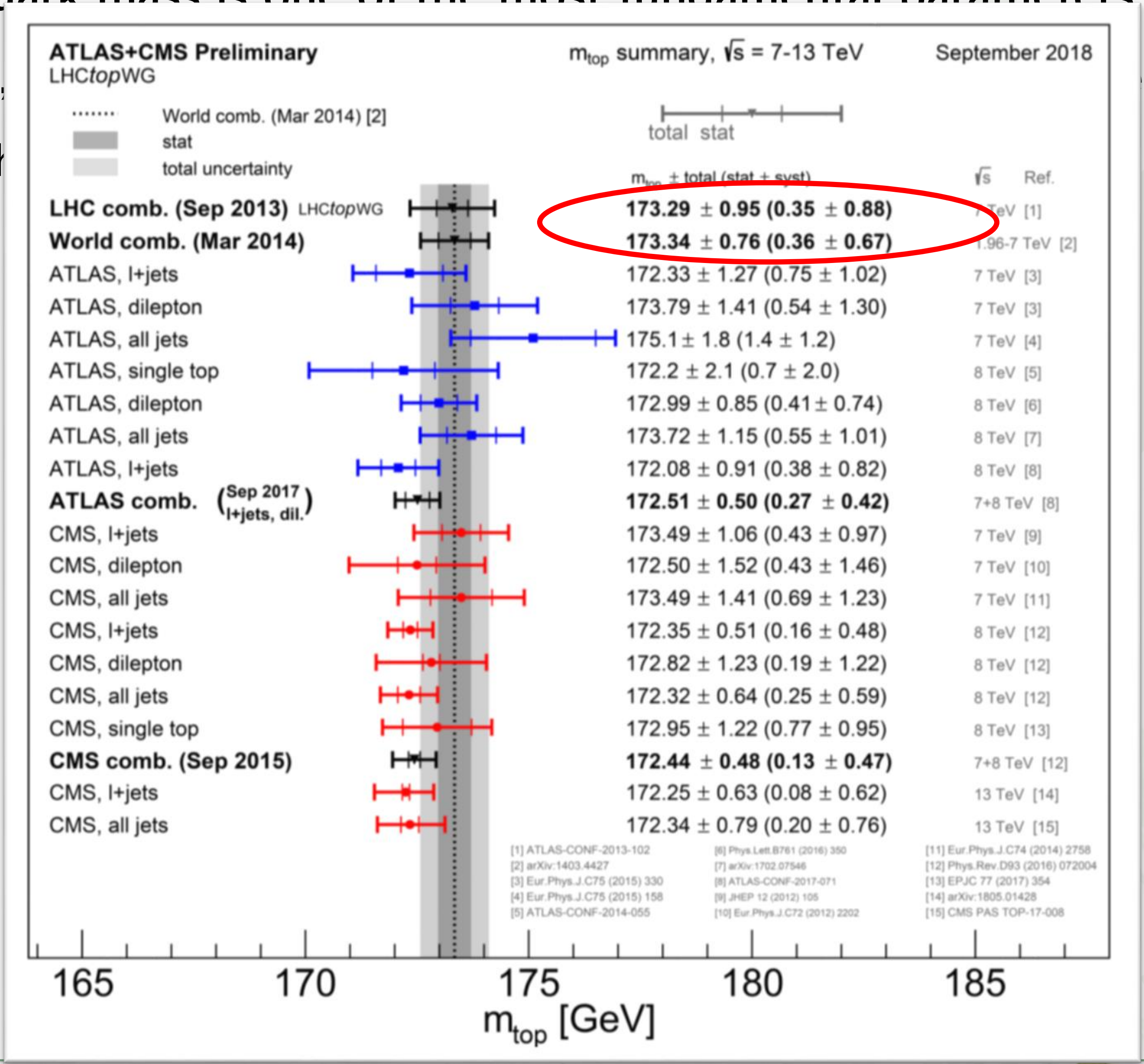
Based on arXiv:2212.06341(Phys.Rev.D 108.054003)
and arXiv:2309.00762

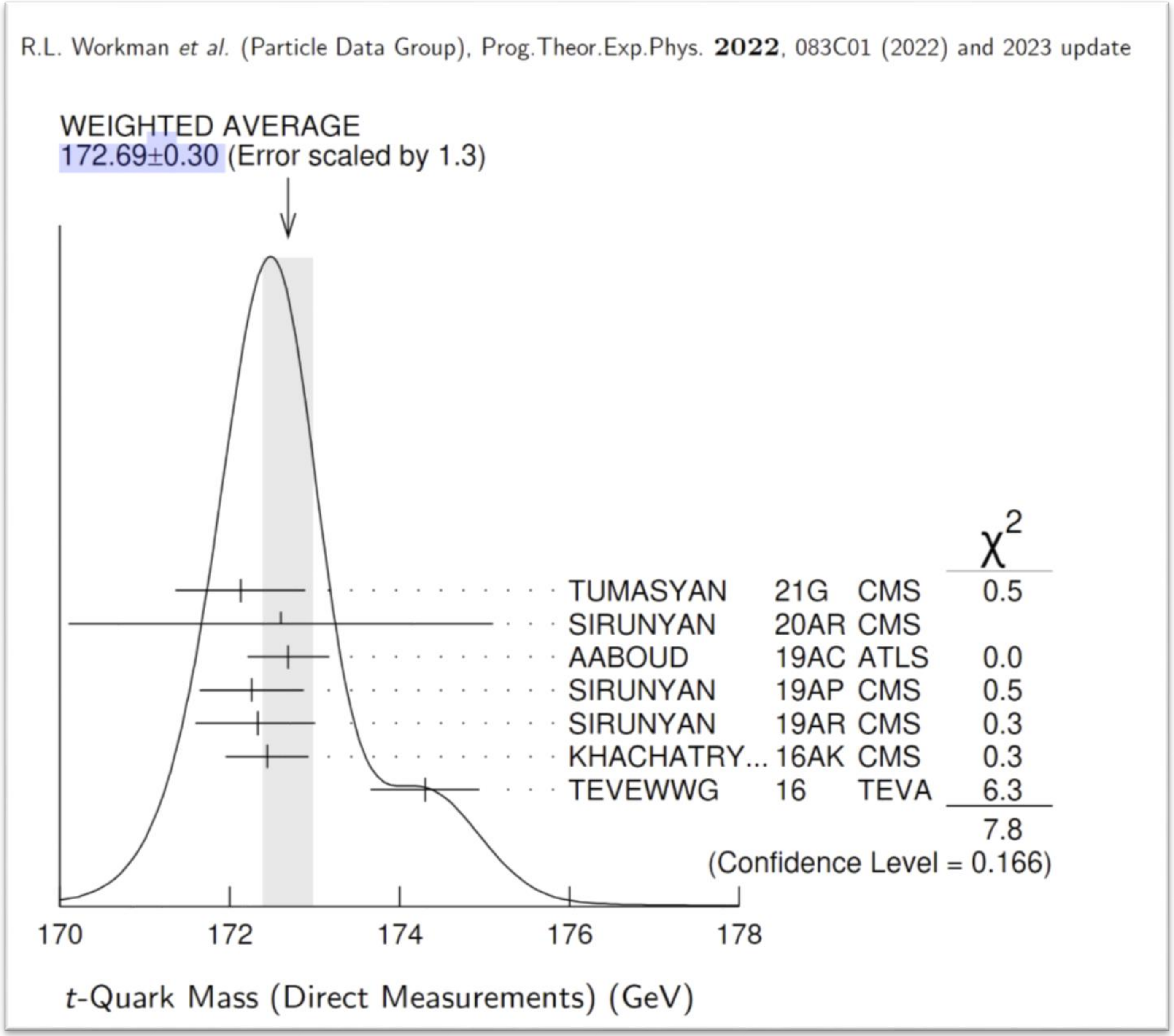
In collaborations with H.T.Li, Z.Li, J.Wang, Y.F.Wang, Q.F. Wu

01

Motivation

- Top quark mass is one of the most fundamental parameters in Standard model, … ns doors to new ph…





Top quark mass

VALUE (GeV)
172.69± 0.30

Motivation

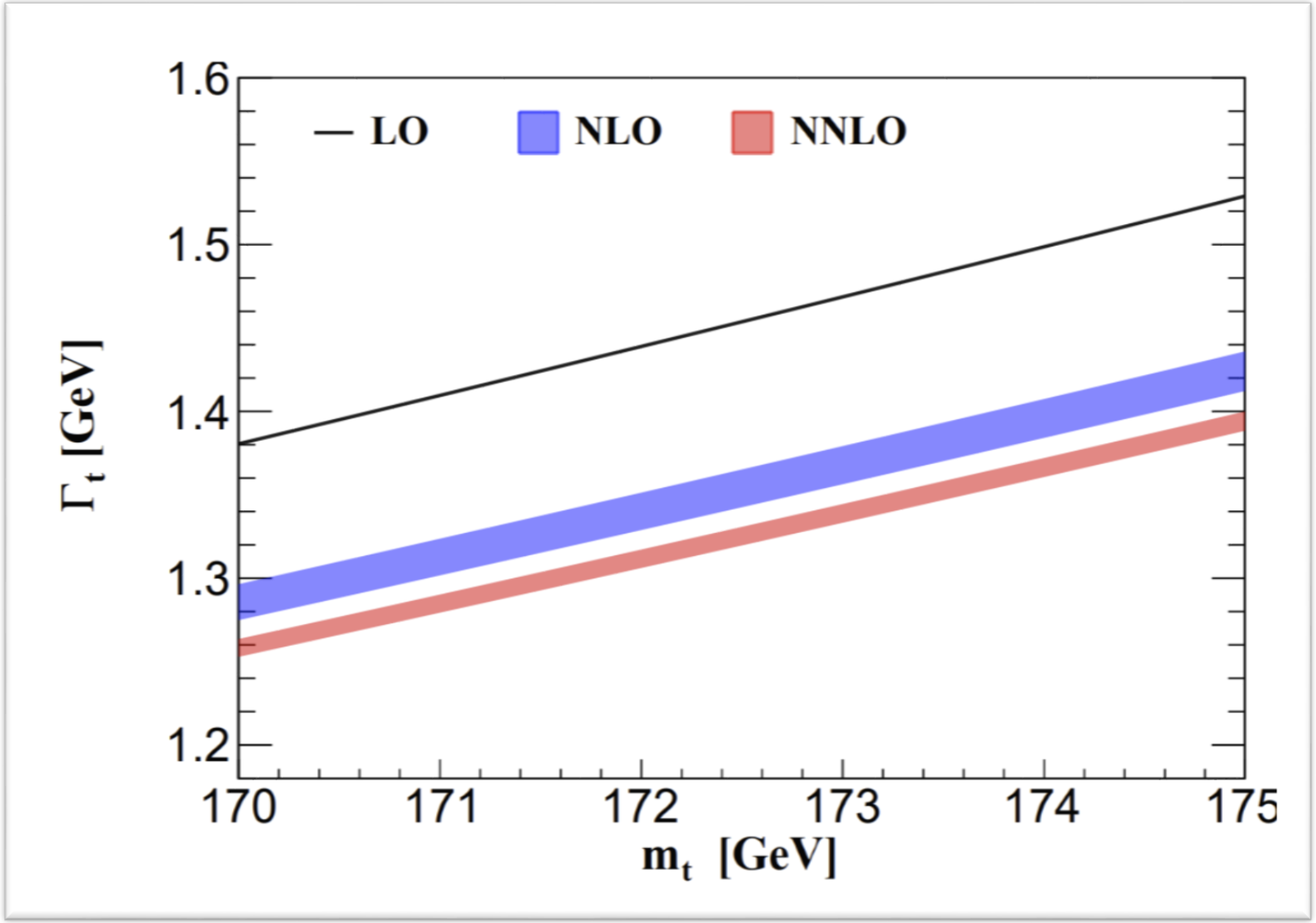
- Top decay width Γ_t is the fundamental properties of top-quark.
- Γ_t is large due to the large mass of top quark(Γ_t > 1GeV).
- The measurement of Γ_t may hint at new-physics.
- The study of b → u decay is helpful for the determination of the CKM matrix element V_{ub}. (See Yanxi Zhang's talk)

Motivation

- Top decay decays almost exclusively to W b. $\Gamma_t = \Gamma_t(t \to W\ b)$.
- At LHC, indirect techniques are precise but model dependent.
- The most precise measurement is $\Gamma_t = 1.36 \pm 0.02(stat.)^{+0.14}_{-0.11}(syst.)$GeV [CMS, 2014].
- Direct techniques are less precise but model independent.
- Direct result by ATLAS is $\Gamma_t = 1.9 \pm 0.5$GeV [ATLAS, 2019].
- Γ_t cab be measured with an uncertainty of 30 MeV in future electron-positron collider [Martinez, Miquel 2019].

Top quark decay width up to NNLO

Motivation



$\mu \in [mt/2, 2mt]$

NNNLO QCD corrections is need

Motivation

- To obtain more precise theoretical predictions, a crucial step is to calculate multi-loop Feynman integrals.
- Analytic calculation of the high order corrections have many benefits. We can obtain the numerical effect of high order corrections very fast, expand them at any kinematic point, and integrate them to obtain more interesting results.

Reduction

Integration-By-Parts (IBP)

$$F(a) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^a}.$$

↓

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2 - m^2)^a} = 0,$$

↓

$$(d - 2a) F(a) - 2am^2 F(a+1) = 0.$$

↓

$$F(a) = \frac{d - 2a + 2}{2(a-1)m^2} F(a-1).$$

→

$$F(q_1, \ldots, q_n; a_1, \ldots, a_N; d) = \int \cdots \int \prod_{i=1}^{h} \mathrm{d}^d k_i \frac{1}{\prod_{j=1}^{N} E_j^{a_j}},$$

$$\int \cdots \int \prod_{i'=1}^{h} \mathrm{d}^d k_{i'} \frac{\partial}{\partial k_i} \left(p_j \prod_{j'=1}^{N} E_j^{-a_{j'}} \right) = 0$$

$$\sum \alpha_i F(a_1 + b_{i,1}, \ldots, a_N + b_{i,N}) = 0.$$

K.G. Chetyrkin, F.V. Tkachov, Nucl. Phys. B **192**, 159 (1981)

Packages:
FIRE, Kira, Blade…

Calculation Method for master integrals: Differential Equations (DE)

$$\frac{d}{dx}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} A_{11} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{n1} & \dots & A_{n,n} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

x are Lorentz invariant kinematics

A. V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett.* **B254** (1991) 158–164.

A. V. Kotikov, *Differential equation method: The Calculation of N point Feynman diagrams*, *Phys. Lett.* **B267** (1991) 123–127. [Erratum: Phys. Lett.B295,409(1992)].

Multiloop integrals in dimensional regularization made simple

Johannes M. Henn (Princeton, Inst. Advanced Study) (Apr 5, 2013)

Published in: *Phys.Rev.Lett.* 110 (2013) 251601 • e-Print: 1304.1806 [hep-th]

Choosing canonical(UT) basis

For random basis g， we may have:

$$\partial_x \vec{g}(x;\epsilon) = B(x,\epsilon)\,\vec{g}(x;\epsilon)$$

We can choose new basis f:

$$\vec{f} = T\vec{g},$$

space time dimension d=4-2ϵ

$$d\,\vec{f}(x,\epsilon) = \epsilon\,\left(d\,\tilde{A}\right)\,\vec{f}(x;\epsilon)$$

$$\tilde{A} = \left[\sum_k A_k \log \alpha_k(x)\right] .$$

Top quark inclusive decay

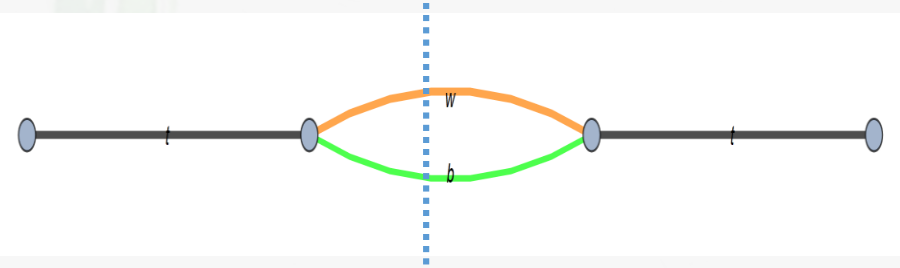
02

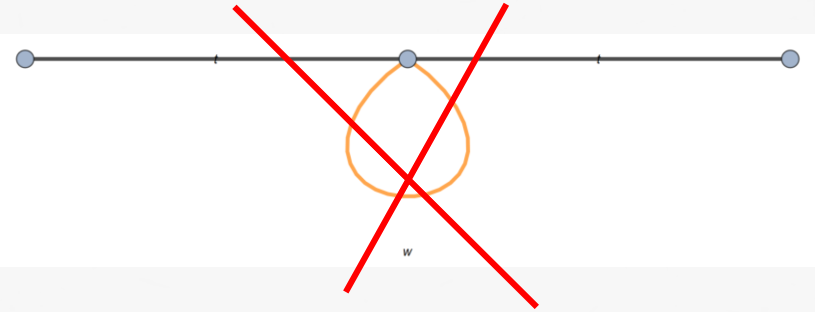
$$\Gamma_t = \frac{\mathrm{Im}[\mathcal{M}(t \to Wb \to t)]}{m_t} .$$

Optical Theorem

1) Leading-order

Calculate the imaginary parts of propagator type integrals





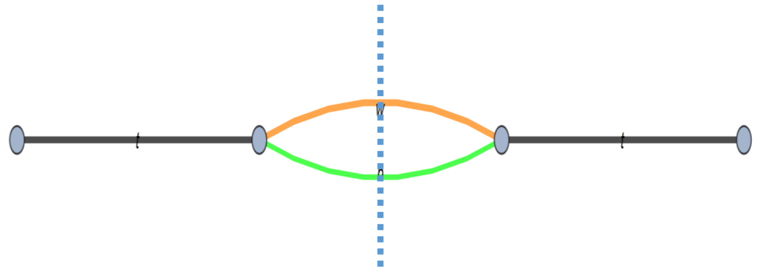
Calculate the following Integral

$$\frac{1}{\pi}\,\mathrm{Im}\left[(1-2\,\epsilon)\,\frac{B_0[m_t\,\hat{}\,2,\,0,\,m_W\,\hat{}\,2]}{\left(\frac{m_W}{m_t}\right)\hat{}\,2-1}\right]$$

Calculated by Integrating the Feyman parameters

Analytic Results：

$$\frac{e^{\gamma\,\epsilon}\,\epsilon\,\Gamma(-\epsilon)\left(1-\frac{m_W^2}{m_t^2}\right)^{-2\,\epsilon}}{\Gamma(1-2\,\epsilon)}$$



Expansion in ∈

$$-1+2\,\epsilon\log\left(1-\frac{m_W^2}{m_t^2}\right)+\frac{1}{4}\,\epsilon^2\left(\pi^2-8\log^2\left(1-\frac{m_W^2}{m_t^2}\right)\right)+\frac{1}{6}\,\epsilon^3\left(8\log^3\left(1-\frac{m_W^2}{m_t^2}\right)-3\,\pi^2\log\left(1-\frac{m_W^2}{m_t^2}\right)+14\,\zeta(3)\right)+O(\epsilon^4)$$

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Next-to Leading Order

Jeabek, Kuhn 1989,Czarnecki 1990, Li, Oakes, Yuan 1991

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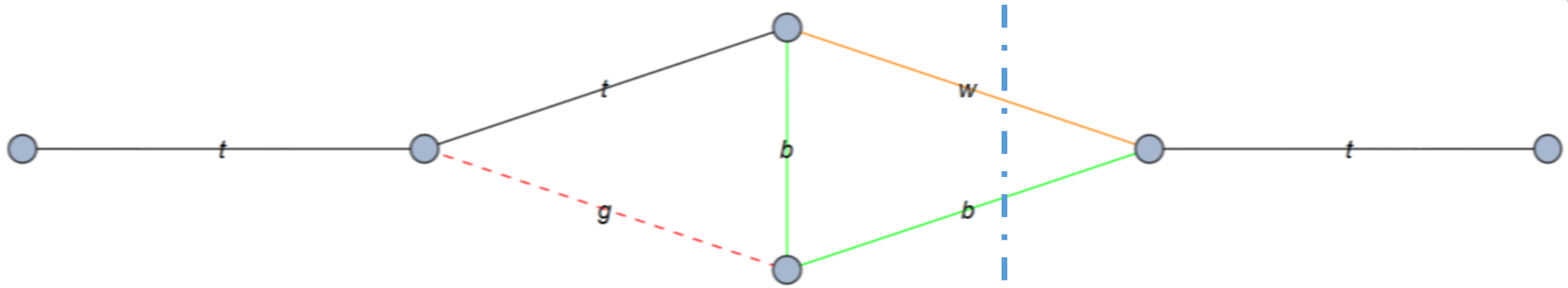
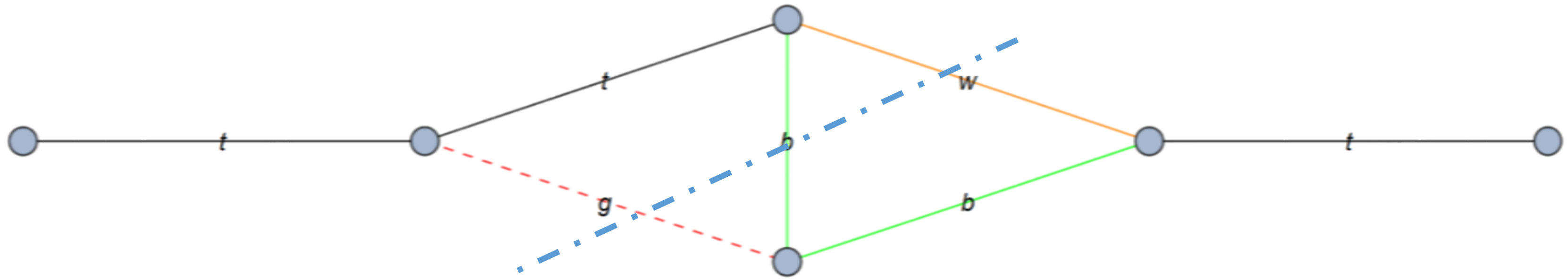
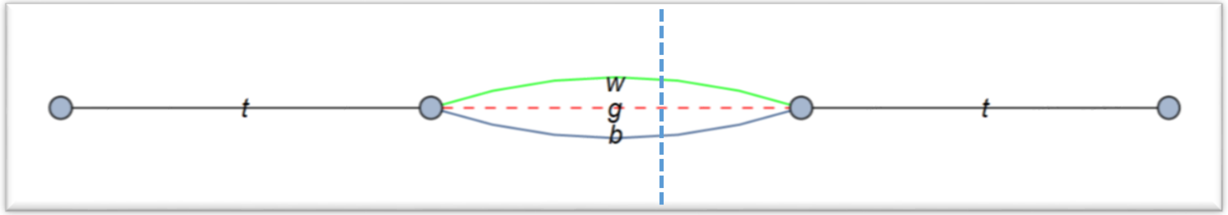
NLO master Integrals: ("kite integrals")

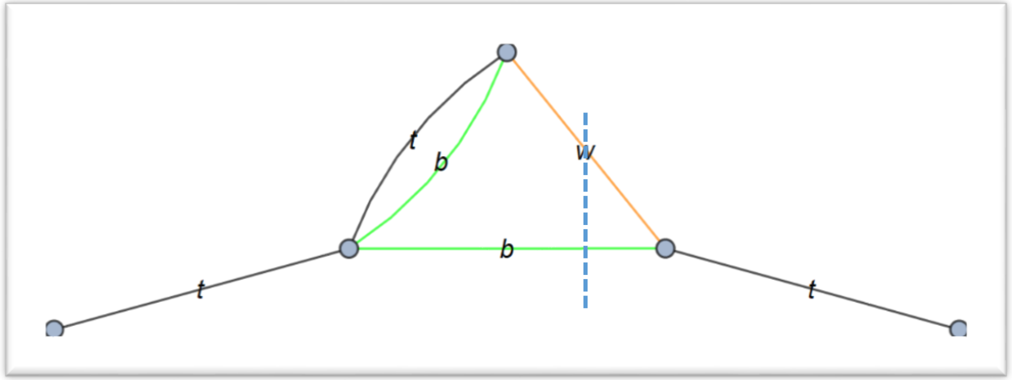
$$\mathrm{I}_{n_1,n_2,n_3,n_4,n_5} = \frac{1}{\pi}\mathrm{Im}\int \frac{\mathcal{D}^d k_1\ \mathcal{D}^d k_2}{[k_1^2-m_W^2]^{n_1}\ [(k_1-p)^2]^{n_2}\ [k_2^2-m_t^2]^{n_3}\ [(k_2+p)^2]^{n_4}\ [(k_1+k_2)^2]^{n_5}}$$

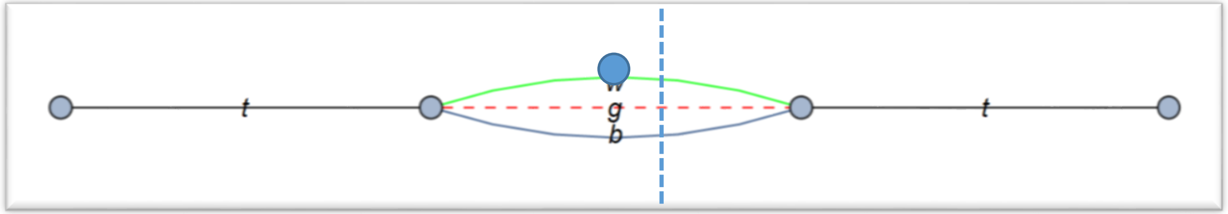


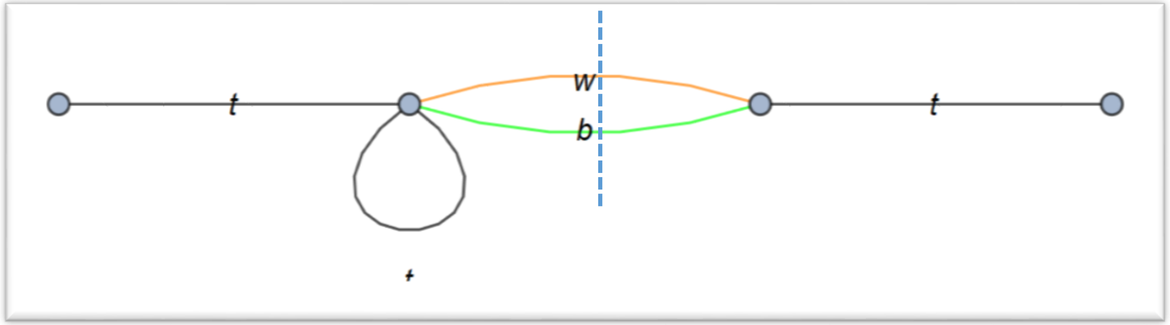


IBP produces four Master Integrals:









UT basis for NLO

Choosing UT Basis

UT-basis :

$$
\begin{aligned}
\mathrm{B}_1 &= \epsilon^2 m_t^2 \mathrm{I}_{2,0,0,2,1}, \\
\mathrm{B}_2 &= \epsilon^2 (m_W^2 - m_t^2)\mathrm{I}_{1,0,0,2,2} + 2\epsilon^2 m_W^2 \mathrm{I}_{2,0,0,2,1}, \\
\mathrm{B}_3 &= \epsilon^2 m_t^2 \mathrm{I}_{2,1,2,0,0}, \\
\mathrm{B}_4 &= \frac{m_t^2}{m_W^2 + m_t^2}\left(\epsilon^2 (1-\epsilon) m_W^2 \mathrm{I}_{1,1,1,0,2} - \epsilon^2 m_t^2 \mathrm{I}_{2,1,2,0,0}\right).
\end{aligned}
$$

Construct UT basis:

Libra 2012.00279

CANONICA 1705.06252

DlogBasis 2002.09492

….

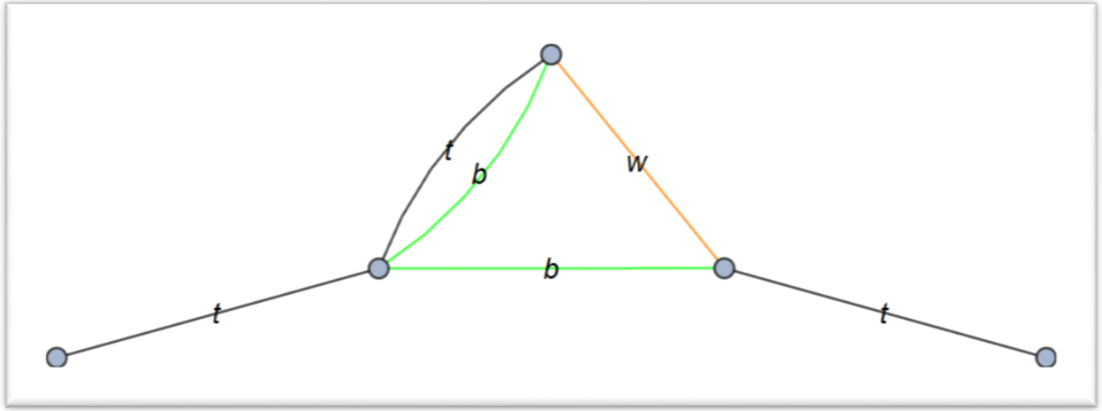
The differential equations is very simple

Set $w = \frac{m_W^2}{m_t^2}$, We have

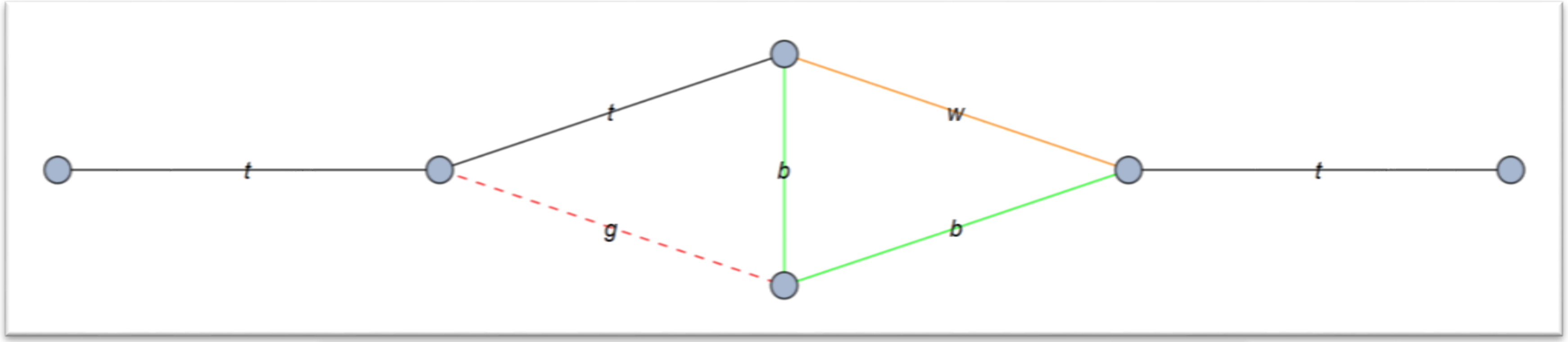
$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}w} = \epsilon\left(\frac{\mathbf{P}}{w} + \frac{\mathbf{N}}{w-1}\right)\mathbf{B}\,.$$

$$\mathbf{P} = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \mathbf{N} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Chosing alternative basis



Replaced with



$$\mathrm{B}_4^{\mathrm{new}} = \epsilon^3(1-\epsilon)m_t^2\mathrm{I}_{1,1,1,1,1}.$$

Make the UT basis more simple.

Calculate the UT basis order by order in ∈ expansion

$$B = \mathbf{B}^0 + \epsilon \mathbf{B}^1 + \epsilon^2 \mathbf{B}^2 + \epsilon^2 \mathbf{B}^3 + \epsilon^4 \mathbf{B}^4 + \dots$$

$$\mathbf{B}^i = \int \left(\frac{\mathbf{P}}{w} + \frac{\mathbf{N}}{w-1} \right) \mathbf{B}^{i-1} \mathrm{d}w + \mathbf{C}^i$$

Integrate Iteratively

Determination of Integration Constants (a more convenient way)

	Weigth-0	Weigth-1	Weigth-2	Weigth-3	Weigth-4
Integration constants C^i	1	0	π^2	$\zeta(3)$	π^4

Combing analytic calculation with AMFLOW, PSLQ,
C can be obtained in a more easy way.

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -\frac{3\pi^2}{2} & -\frac{56\zeta(3)}{3} & \frac{71\pi^4}{360} & \cdots \\ -2 & 0 & \frac{5\pi^2}{3} & \frac{64\zeta(3)}{3} & -\frac{31\pi^4}{180} & \cdots \\ -1 & 0 & \frac{\pi^2}{6} & \frac{8\zeta(3)}{3} & \frac{\pi^4}{40} & \cdots \\ 1 & 0 & -\frac{\pi^2}{6} & -\frac{8\zeta(3)}{3} & -\frac{\pi^4}{40} & \cdots \end{pmatrix}$$

Elegant results

Results of UT basis (Harmonic Polylogarithms hep-ph/9905237):

$$
\mathbf{B} = \left(\begin{array}{c|c|c|c}
\text{weight-0} & \text{weight-1} & \text{weight-2} & \dots \\
\hline
1 & H_0(w)+4H_1(w) & -H_{0,0}(w)+4H_{0,1}(w)+2H_{1,0}(w)+16H_{1,1}(w)-\frac{3\pi^2}{2} & \dots \\
\hline
-2 & -8H_1(w) & -4H_{1,0}(w)-32H_{1,1}(w)+\frac{5\pi^2}{3} & \dots \\
\hline
-1 & -2H_1(w) & \frac{\pi^2}{6}-4H_{1,1}(w) & \dots \\
\hline
1 & 3H_1(w) & H_{0,1}(w)+10H_{1,1}(w)-\frac{\pi^2}{6} & \dots
\end{array}\right)
$$

$$
\begin{aligned}
H_0(w) &= \log(w), \\
H_1(w) &= -\log(1-w), \\
H_{-1}(w) &= \log(1+w), \\
H_{0,1}(w) &= \mathrm{Li}_2(w).
\end{aligned}
$$

Evaluate HPL: hep-ph/0507152

04

Next-to-Next-to Leading Order

Gao, Li, Zhu, 2013, Brucherseifer, Caola, Melnikov 2013

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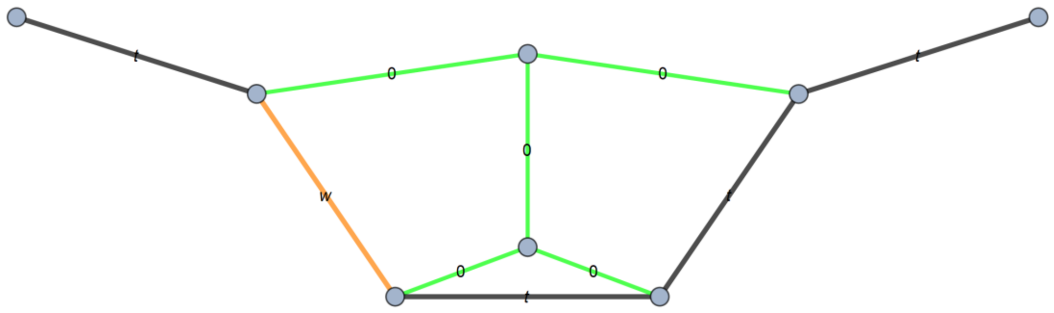
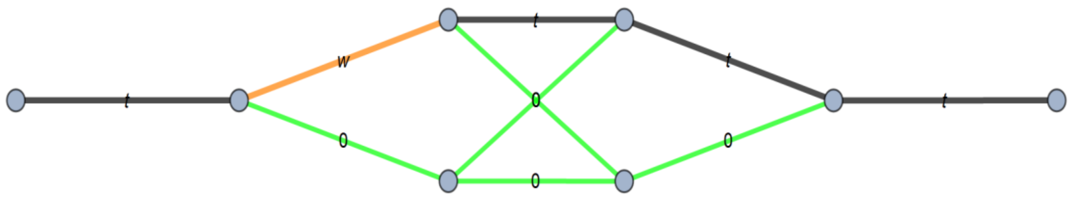
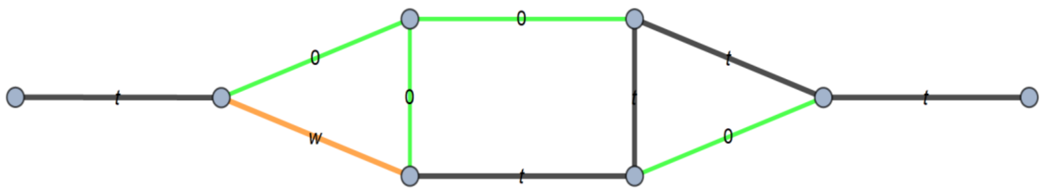
NNLO integral Families



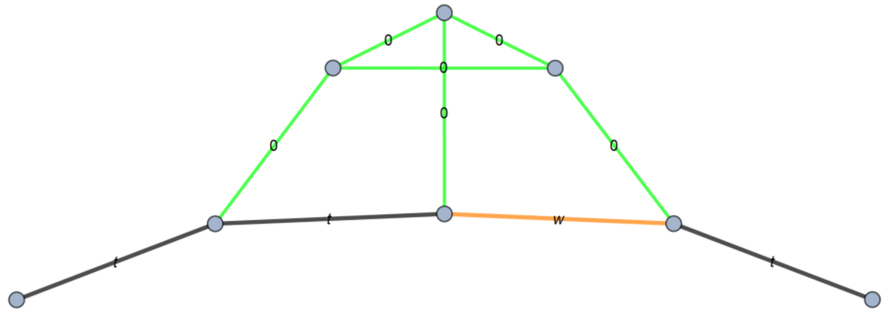
(A)



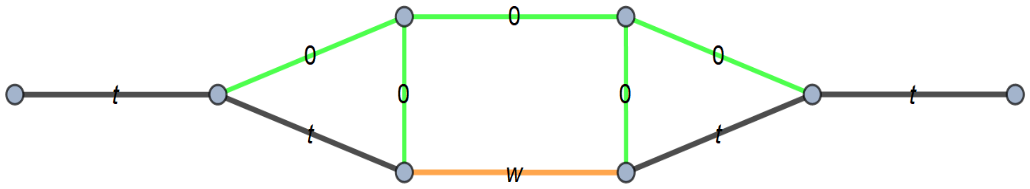
(B)



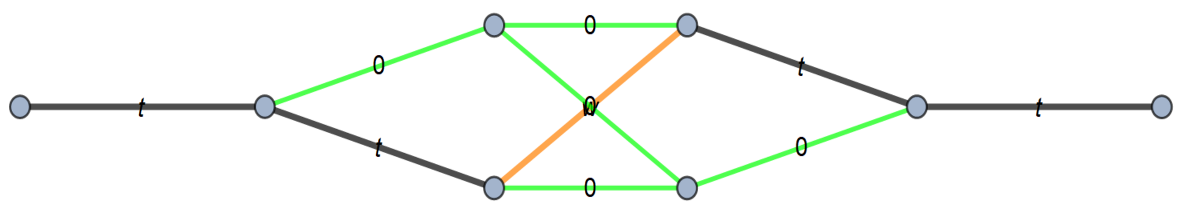
(C)



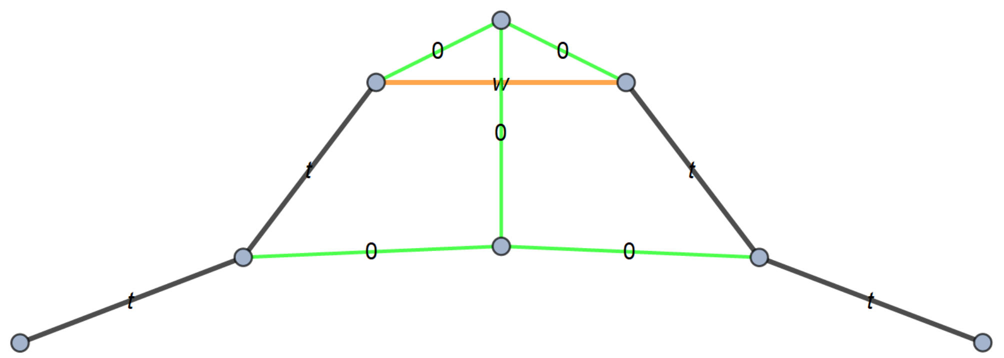
(D)



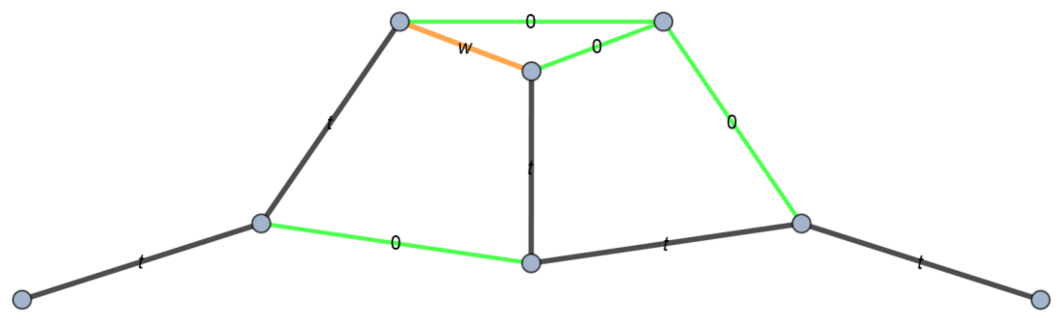
(E)



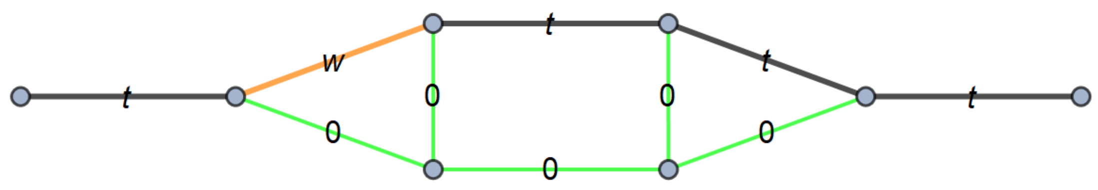
(F)



(G)



(H)



(I)

UT basis $\mathbf{I}(w,\epsilon)$

DEs

$$\mathrm{d}\,\mathbf{I}(w,\epsilon) = \epsilon\,\mathrm{d}\left[\sum_{i=1}^{4} \mathbf{R}_i \log(l_i)\right] \mathbf{I}(w,\epsilon)$$

Letters $l_i \in \{w-2, w-1, w, w+1\}$

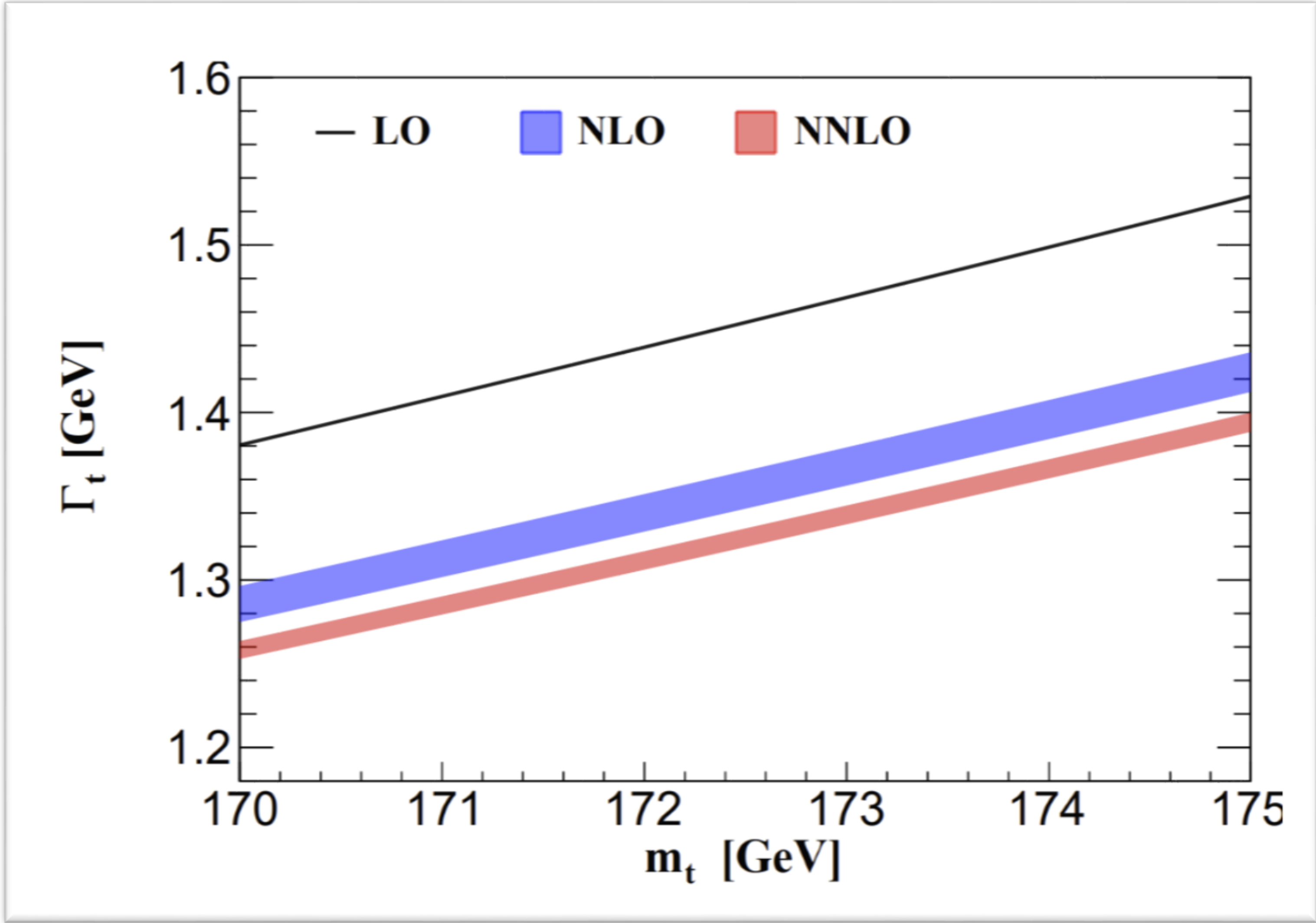
NNLO decay width

$$\Gamma(t \to Wb) = \Gamma_0 \left[X_0 + \frac{\alpha_s}{\pi} X_1 + \left(\frac{\alpha_s}{\pi}\right)^2 X_2 \right]$$

$$X_2 = C_F(T_R n_l X_l + T_R n_h X_h + C_F X_F + C_A X_A).$$

$$\begin{aligned}
X_l &= -\frac{X_0}{3}\left[H_{0,1,0}(w) - H_{0,0,1}(w) - 2H_{0,1,1}(w) + 2H_{1,1,0}(w) - \pi^2 H_1(w) - 3\zeta(3)\right] + g_l(w), \\
X_h &= -\frac{(X_0 - 12w)}{3}\left[\zeta(3) - H_{0,0,1}(w)\right] + g_h(w), \\
X_F &= \frac{1}{12} X_0 \left[-6\left(2H_{0,1,0,1}(w) + 6H_{1,0,0,1}(w) - 3H_{1,0,1,0}(w) - 12\zeta(3)H_1(w)\right) - \pi^2 H_{1,0}(w)\right] \\
&+ (X_0 + 4w)\left(-\frac{1}{6}\pi^2 H_{0,-1}(w) - 2H_{0,-1,0,1}(w)\right) \\
&+ \frac{1}{12}\left(18w^3 - 3w^2 + 76w + 15\right)\pi^2 H_{0,1}(w) - \frac{1}{2}\left(4w^3 - 2w^2 + 4w + 3\right) H_{0,0,0,1}(w) \\
&+ \frac{1}{2}\left(4w^3 - 2w^2 + 16w + 3\right) H_{0,0,1,0}(w) + w\left(2w^2 - 7w - 16\right) H_{0,0,1,1}(w) \\
&- \frac{1}{2}\left(2w^3 - 11w^2 - 28w - 1\right) H_{0,1,1,0}(w) + \frac{1}{720}\pi^4\left(42w^3 - 191w^2 - 328w - 11\right) + g_F(w),
\end{aligned}$$

Top quark decay width up to NNLO



$\mu \in [mt/2, 2mt]$

05

NNNLO QCD Corrections to top quark inclusive decay

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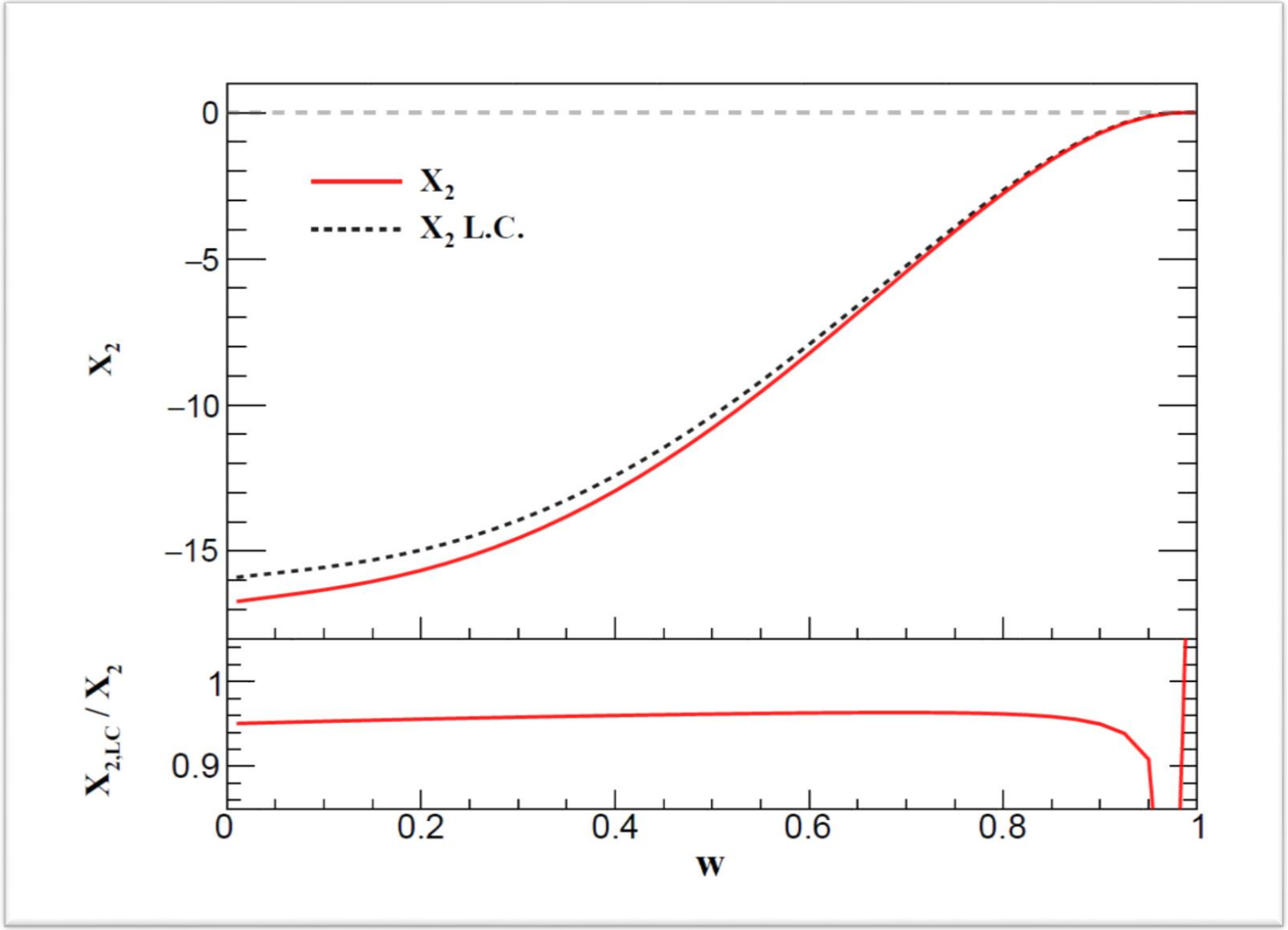
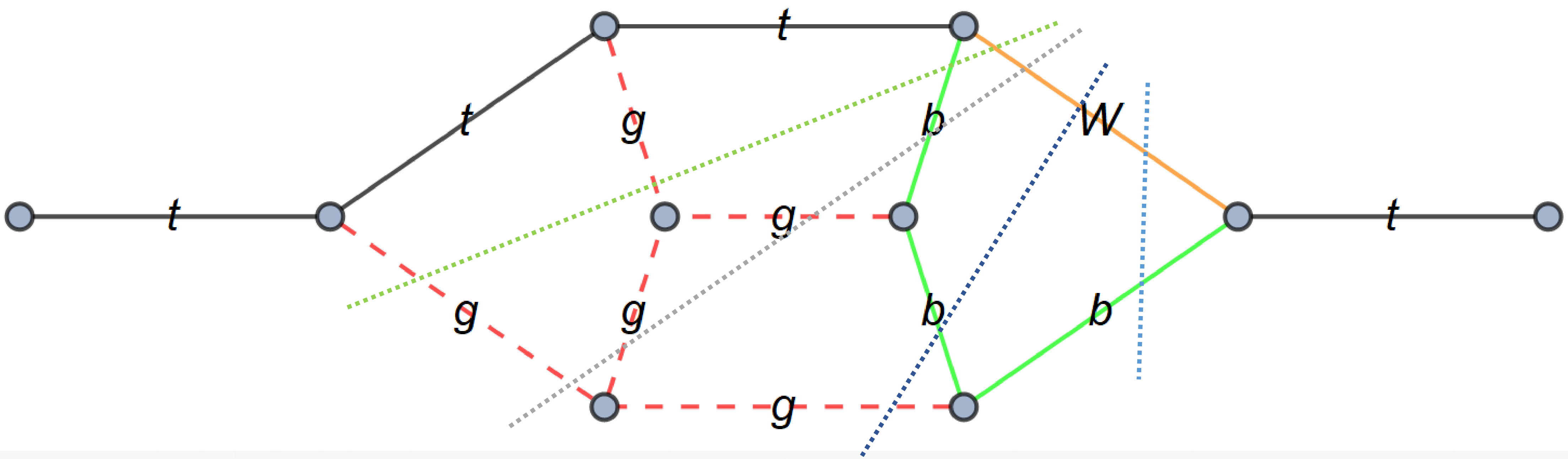
Leading-color contribution is significantly dominant(~95%).

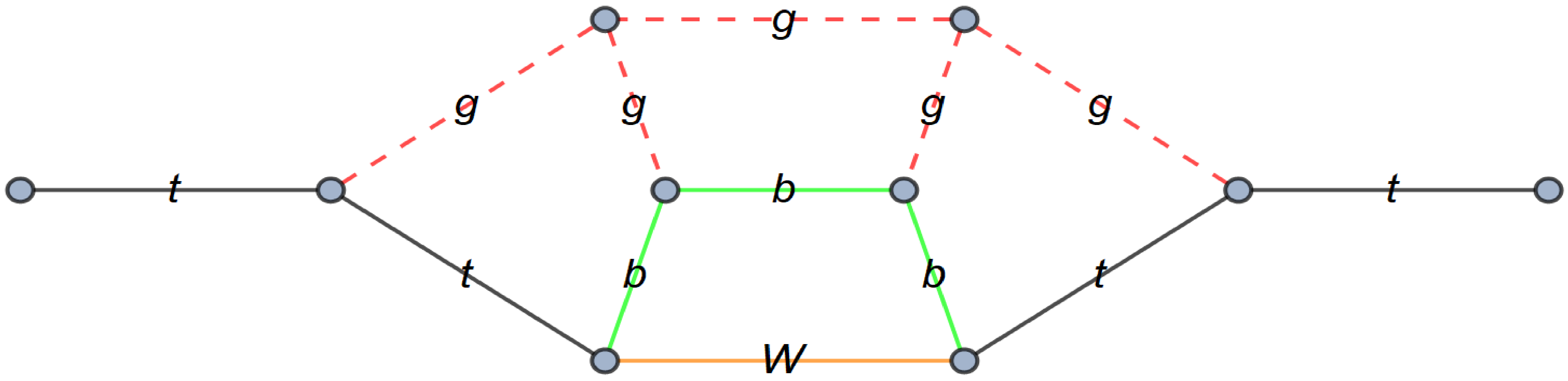
$$
\begin{aligned}
X_2 &= C_F\left[C_F X_F + C_A X_A + T_R n_l X_l + T_R n_h X_h\right] \\
&= C_F\left[N_c\left(X_A + \frac{X_F}{2}\right) + \frac{n_l}{2}X_l \right. \\
&\qquad \left. - \frac{1}{2N_c}X_F + \frac{n_h}{2}X_h\right],
\end{aligned}
$$



NNLO X_2

Two-typical four-loop master integrals for leading color contribution





4×10^4 **integrals(IBP) → 185 master integrals**

DEs for NNNLO 185 UT Basis

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}w} = \epsilon \left(\frac{\mathbf{P}}{w} + \frac{\mathbf{N}}{w-1} \right) \mathbf{B}$$

For any loop, can the letters of leading color(Large Nc limit) integrals still be {w, w-1}?

06

Physical Results

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Top quark decay width

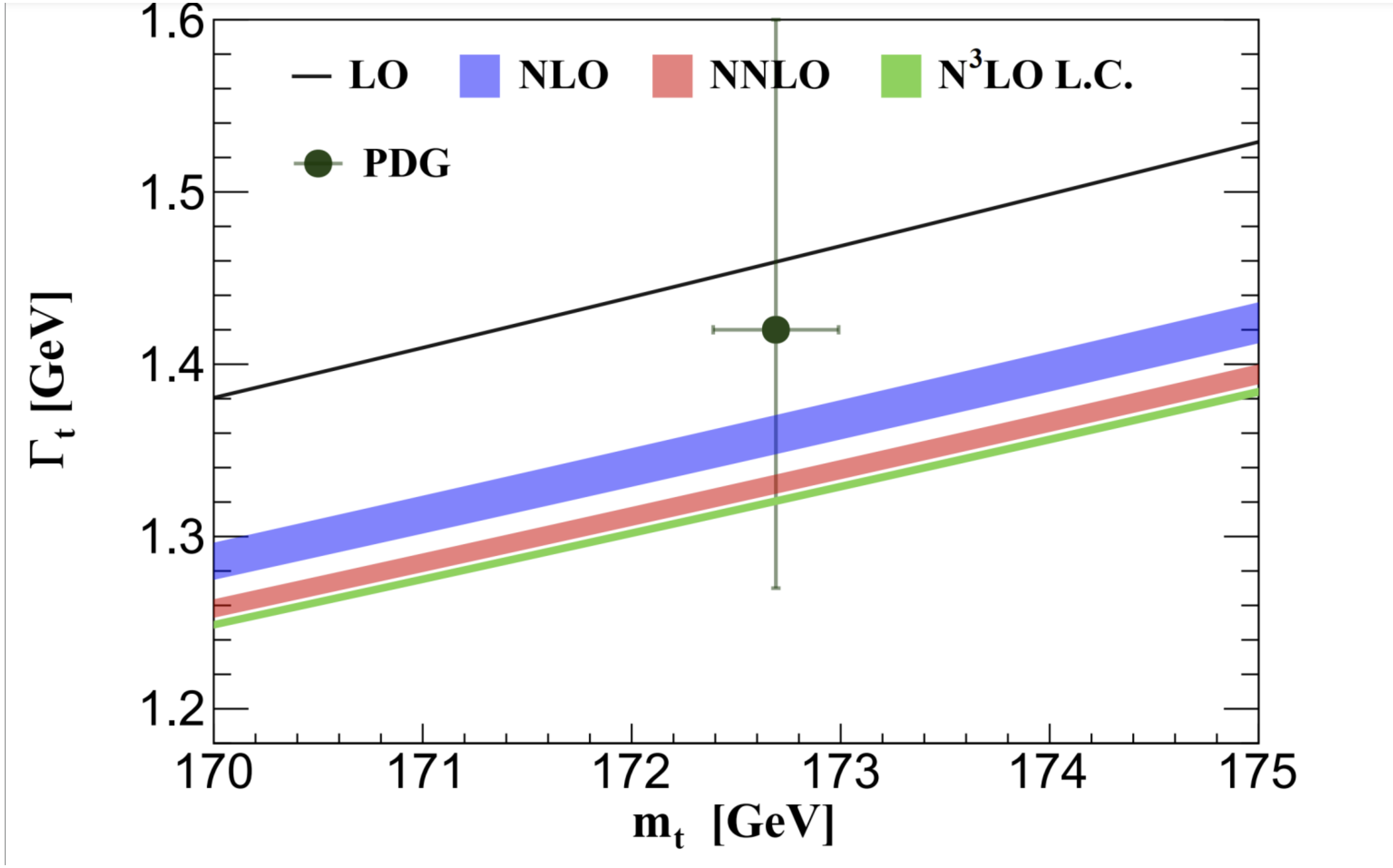
$$\Gamma_t = \Gamma_0 \left[X_0 + \frac{\alpha_s}{\pi} X_1 + \left(\frac{\alpha_s}{\pi}\right)^2 X_2 + \left(\frac{\alpha_s}{\pi}\right)^3 X_3 \right]$$

$$X_3 = C_F \left[N_c^2 Y_A + \widetilde{Y}_A + \frac{\overline{Y}_A}{N_c^2} + n_l n_h Y_{lh} + n_l \left(N_c Y_l + \frac{\widetilde{Y}_l}{N_c} \right) + n_l^2 Y_{l2} + n_h \left(N_c Y_h + \frac{\widetilde{Y}_h}{N_c} \right) + n_h^2 Y_{h2} \right].$$

Expansion of analytic results:

$$Y_A = \left[\frac{203185}{41472} - \frac{12695\pi^2}{1944} - \frac{4525\zeta(3)}{576} - \frac{1109\pi^4}{25920} + \frac{37\pi^2\zeta(3)}{36} + \frac{1145\zeta(5)}{96} + \frac{47\pi^6}{2835} - \frac{3\zeta(3)^2}{4}\right]$$
$$+ w\left[-\frac{157939}{2304} + \frac{140863\pi^2}{20736} + \frac{5073\zeta(3)}{64} - \frac{14743\pi^4}{6480} - \frac{169\pi^2\zeta(3)}{72} - \frac{45\zeta(5)}{16} + \frac{3953\pi^6}{22680} - \frac{15\zeta(3)^2}{4}\right]$$
$$+ w^2\left[\log(w)\left(\frac{851099}{27648} - \frac{5875\pi^2}{2304} - \frac{33\zeta(3)}{8} + \frac{\pi^4}{10}\right) - \frac{82610233}{331776} + \frac{799511\pi^2}{27648}\right.$$
$$\left. + \frac{4093\zeta(3)}{32} - \frac{5987\pi^4}{2880} - \frac{91\pi^2\zeta(3)}{16} - \frac{275\zeta(5)}{8} + \frac{347\pi^6}{3024} - \frac{9\zeta(3)^2}{8}\right] + \mathcal{O}(w^3),$$
$$Y_l = \left[\frac{18209}{20736} + \frac{60025\pi^2}{31104} - \frac{197\zeta(3)}{288} - \frac{14\pi^4}{405} + \frac{5\pi^2\zeta(3)}{36} - \frac{25\zeta(5)}{12}\right]$$
$$+ w\left[-\frac{179}{1152} - \frac{3709\pi^2}{2592} - \frac{73\zeta(3)}{6} + \frac{46\pi^4}{405} + \frac{19\pi^2\zeta(3)}{18} + \frac{5\zeta(5)}{2}\right]$$
$$+ w^2\left[\log(w)\left(-\frac{11077}{1152} + \frac{37\pi^2}{96} + \frac{3\zeta(3)}{8}\right) + \frac{49097}{648} - \frac{817\pi^2}{128} - \frac{2651\zeta(3)}{96} + \frac{17\pi^4}{270} + \frac{5\pi^2\zeta(3)}{6} + \frac{25\zeta(5)}{4}\right]$$
$$Y_{l2} = \left[-\frac{695}{2592} - \frac{91\pi^2}{972} + \frac{11\zeta(3)}{36} - \frac{2\pi^4}{405}\right] + w\left[\frac{245}{144} - \frac{73\pi^2}{648} - \frac{\zeta(3)}{3}\right]$$
$$+ w^2\left[\log(w)\left(\frac{245}{432} - \frac{\pi^2}{72}\right) - \frac{791}{162} + \frac{85\pi^2}{432} + \frac{3\zeta(3)}{4} + \frac{2\pi^4}{135}\right] + \mathcal{O}(w^3),$$

Top-quark width at different values of m_t



	$\delta_b^{(i)}$	$\delta_W^{(i)}$	$\delta_{\mathrm{EW}}^{(i)}$	$\delta_{\mathrm{QCD}}^{(i)}$	Γ_t [GeV]
LO	-0.273	-1.544	–	–	1.459
NLO	0.126	0.132	1.683	-8.575	$1.361^{+0.0091}_{-0.0130}$
NNLO	$*$	0.030	$*$	-2.070	$1.331^{+0.0055}_{-0.0051}$
$\mathrm{N^3LO}$	$*$	0.009	$*$	-0.667	$1.321^{+0.0025}_{-0.0021}$

%

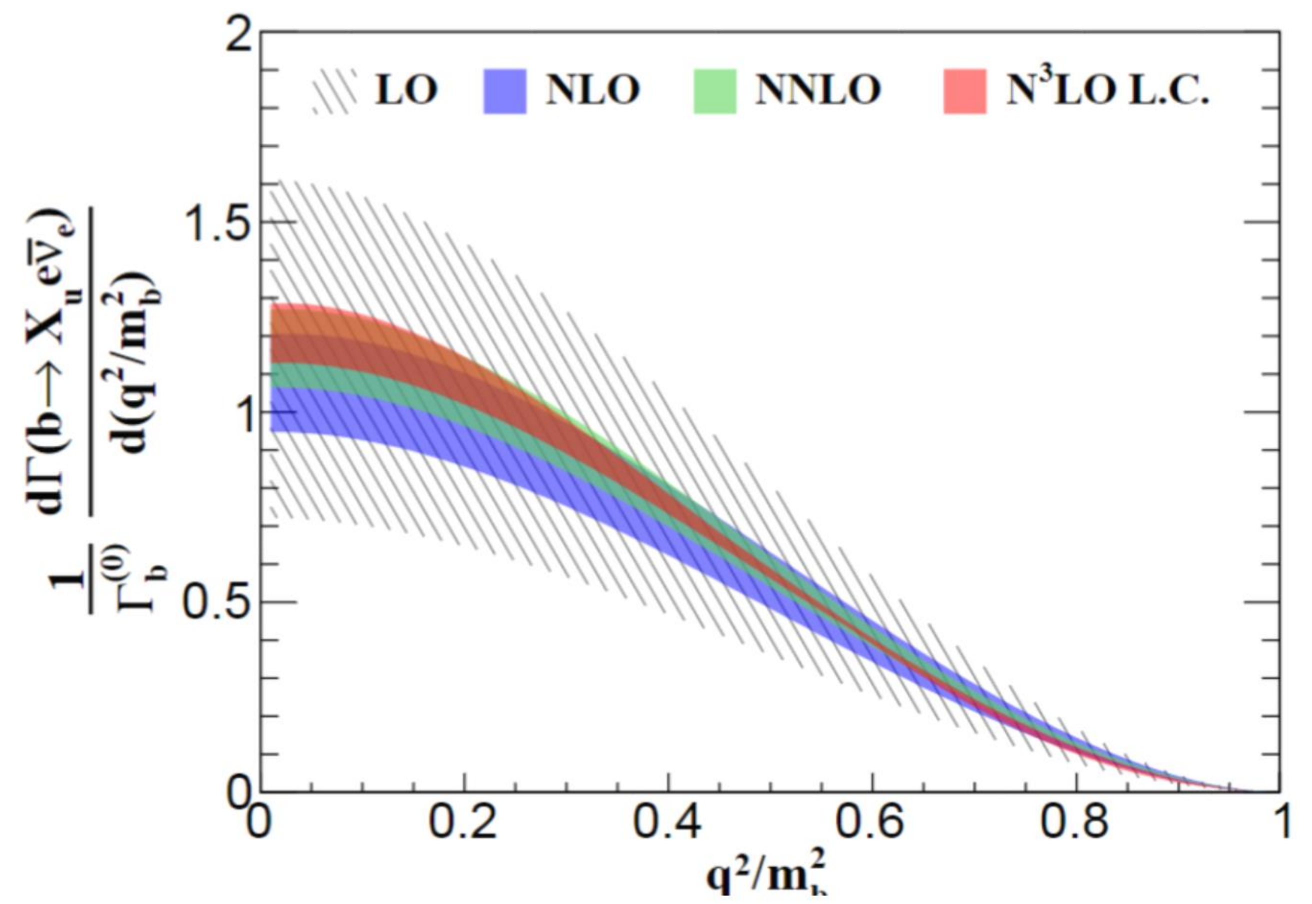
Best fit relation:

$$\Gamma_t(m_t) = 0.027037 \times m_t - 3.34801\ \mathrm{GeV}\,.$$

The dilepton invariant mass spectrum for $b \to X_u e\bar{\nu}_e$ up to NNNLO

$$\frac{\mathrm{d}\Gamma(b \to X_u e\bar{\nu}_e)}{\mathrm{d}q^2} = \Gamma_b^{(0)} \sum_{i=0}^{3} \left(\frac{\alpha_s}{\pi}\right)^i X_i\left(\frac{q^2}{m_b^2}\right).$$

$$\Gamma_b^{(0)} = G_F^2 |V_{ub}|^2 m_b^3 / 96\pi^3.$$



The q^2/m_b^2 distribution for $b \to X_u e \bar{\nu}_e$

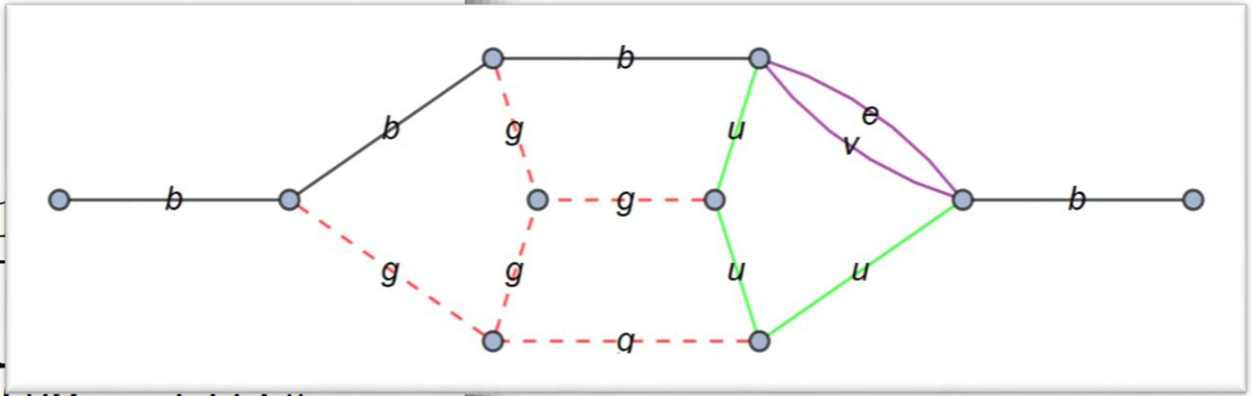
$$|V_{ub}| = (4.19 \pm 0.17) \times 10^{-3}$$

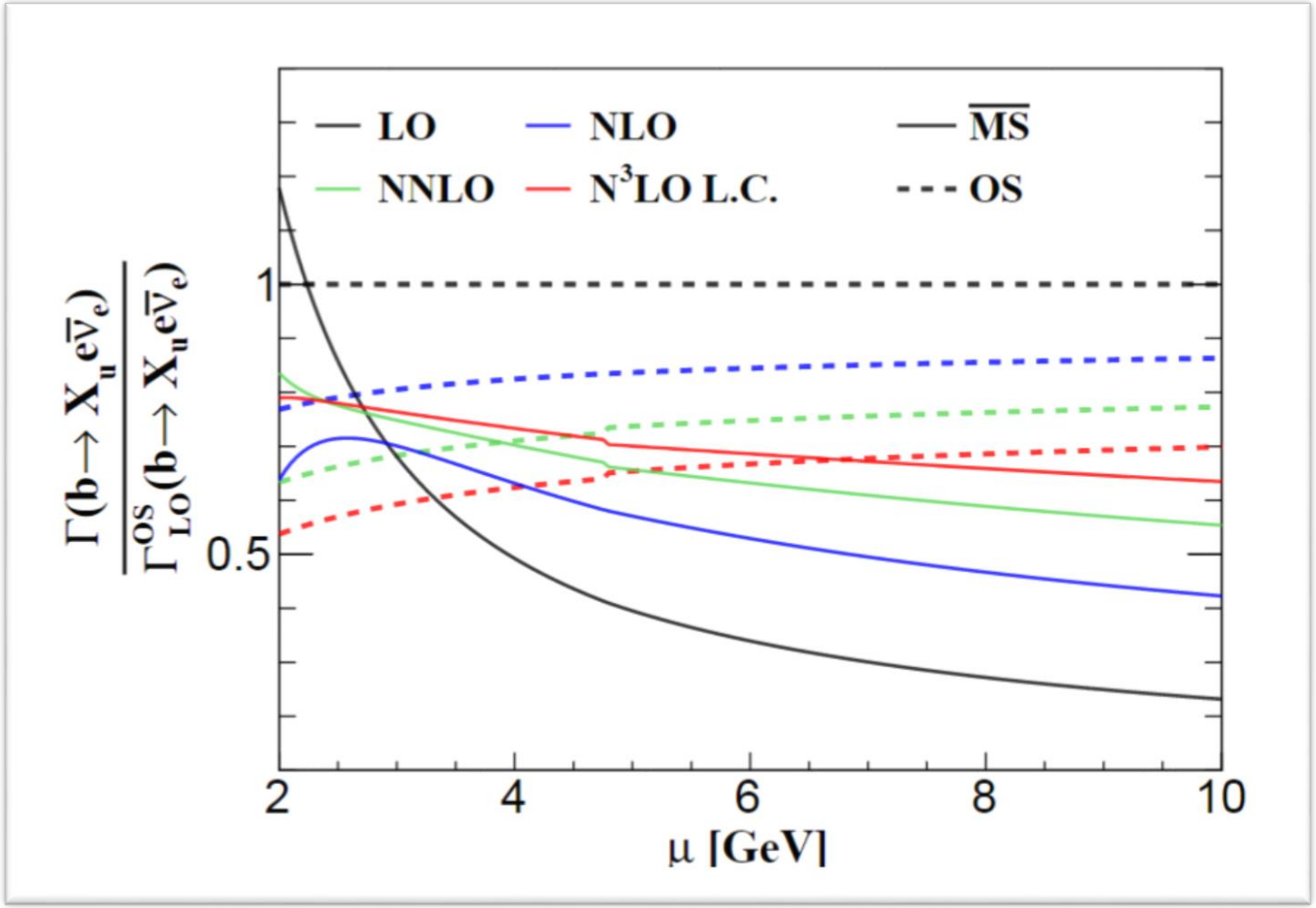
The scale uncertainties are around $\pm$12%, $\pm$9%, and $\pm$6% at NLO, NNLO and N3LO, respectively, at $\frac{q^2}{m_b^2}$ = 0.2.

The semileptonic decay width of b-quark

$$\Gamma(b \to X_u e \bar{\nu}_e) = \frac{G_F^2 |V_{ub}|^2 m_b^5}{192\pi^3} \left[1 + \sum_{i=1} \left(\frac{\alpha_s}{\pi} \right)^i b_i \right]$$

$$\begin{aligned} b_3 \; = \; & C_F \Bigg[N_c^2 \Bigg(\frac{9651283}{82944} - \frac{1051339\pi^2}{62208} - \frac{67189\zeta(3)}{864} \\ & + \frac{4363\pi^4}{6480} + \frac{59\pi^2\zeta(3)}{32} + \frac{3655\zeta(5)}{96} \\ & + n_l N_c \Bigg(-\frac{729695}{27648} + \frac{48403\pi^2}{15552} + \\ & - \frac{13\pi^2\zeta(3)}{72} - \frac{125\zeta(5)}{24} \Bigg) + n_l^2 \Bigg(\\ & - \frac{37\zeta(3)}{216} - \frac{121\pi^4}{6480} \Bigg) + \text{subleading color} \Bigg] \end{aligned}$$





Scale dependence of the b-quark semi-leptonic decay width normalized by the LO width in the on-shell mass scheme. The solid and dashed lines represent the results in the MSbar and on-shell mass schemes, respectively.

Conclusion

The top quark inclusive decay width have been calculated analytically up to N^3LO.

N^3LO corrections for top quark decay width is about -0.667%, with Γ_t = 1.321GeV, the convergence of perturbation series works well.

We also obtained the N^3LO QCD corrections to b quark semileptonic decay, N^3LO QCD is important.

Explore analytic differential decay rates and helicity width.

Thank You!

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