A Beautiful Path to Quantum Chromodynamics

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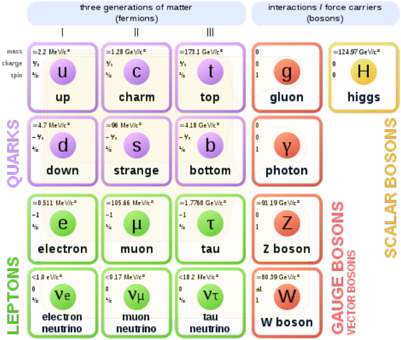
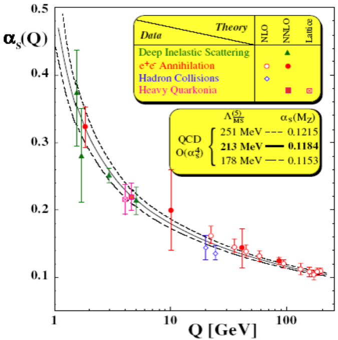
The Standard Model of Particle Physics

- Elementary Particles:



- Asymptotic Freedom:



- The Millennium Prize Problem: Yang-Mills existence and mass gap.
  - *Prove that for any compact simple gauge group G, a non-trivial quantum Yang-Mills theory exists on* $\mathbb{R}^4$ *and has a mass gap* $\Delta > 0$.

Why precision calculations?

- Understanding the general properties of power expansion in EFTs (HQET, SCET, NRQCD). [e.g., convergence (divergence) of this expansion, quark-hadron duality violation, etc.]

- Interesting to understand the strong interaction dynamics of heavy quark decays.
  - Factorization properties of the subleading-power amplitudes.
  - Perturbative properties of the (higher-twist) B-meson DAs (soft functions).
  - Interplay of different QCD techniques.

- Precision determinations of the CKM matrix elements $|V_{ub}|$ and $|V_{cb}|$.
  Power corrections, QED corrections, BSM physics.

- Crucial to understand the CP violation in B-meson decays.
  Strong phase of $\mathscr{A}(B \to M_1 M_2)$ @ m_b scale in the leading power.

- Indispensable for understanding the flavour puzzles (continuously updated).
  - P_5' and $R_{K^{(*)}}$ anomalies in $B \to K^{(*)} \ell^+ \ell^-$.
  - $R_{D^{(*)}}$ anomalies in $B \to D^{(*)} \ell \bar{\nu}_\ell$.
  - Color suppressed hadronic B-meson decays.
  - Polarization fractions of penguin dominated $B_{(s)} \to VV$ decays.

Theory tools for precision flavor physics

New Physics: $\mathscr{L}_{NP}$

$\downarrow$

EW scale (m_W): $\mathscr{L}_{SM} + \mathscr{L}_{D>4}$

$\downarrow$

Heavy-quark scale (m_b):

$$\mathscr{L}_{eff} = -\frac{G_F}{\sqrt{2}} \sum_i C_i \, Q_i + \mathscr{L}_{eff,D>6}$$

$\downarrow$

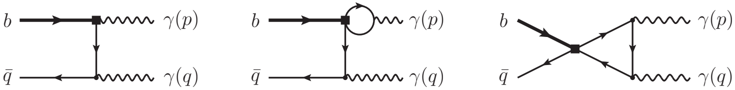
QCD scale ($\Lambda_{\rm QCD}$)

- Aim: $\langle f | Q_i | \bar{B} \rangle = ?$
- QCD factorization [Diagrammatic approach].
- SCET factorization [Operator formalism].
- TMD factorization.
- (Light-cone) QCD sum rules [Khodjamirian, Melić, YMW, arXiv: 2311.08700].
- Lattice QCD.

- QCD is complicated (non)-perturbatively (Factorization, Resummation, Multi-loop techniques).

Double Radiative B_q-Meson Decays

- Leading-order contributions at $\mathcal{O}(\alpha_s^0)$:



Kinematics:

$$p_\mu = \frac{n \cdot p}{2}\, \bar{n}_\mu \equiv \frac{m_{B_q}}{2}\, \bar{n}_\mu \,, \qquad q_\mu = \frac{\bar{n} \cdot q}{2}\, n_\mu \equiv \frac{m_{B_q}}{2}\, n_\mu \,.$$

Interplay of the $\underbrace{\text{soft}}_{B_q\text{-meson}}$ and $\underbrace{\text{collinear}}_{\text{photons}}$ QCD dynamics!

- Decay amplitude:

$$\bar{\mathcal{A}}(\bar{B}_q \to \gamma\gamma) = -\frac{4\, G_F}{\sqrt{2}}\, \frac{\alpha_{\rm em}}{4\pi}\, \varepsilon_1^{*\alpha}(p)\, \varepsilon_2^{*\beta}(q) \sum_{p=u,c} V_{pb} V_{pq}^{*} \sum_{i=1}^{8} C_i\, T_{i,\alpha\beta}^{(p)} \,.$$

Hadronic tensors:

$$\begin{aligned} T_{7,\alpha\beta} &= 2\overline{m}_b(\nu) \int d^4x\, e^{iq\cdot x}\, \langle 0|\mathrm{T}\left\{ j_\beta^{\rm em}(x), \bar{q}_L(0)\sigma_{\mu\alpha} p^\mu b_R(0) \right\} |\bar{B}_q(p+q)\rangle \\ &\quad + [p \leftrightarrow q, \alpha \leftrightarrow \beta] \,, \\ T_{i,\alpha\beta}^{(p)} &= -(4\,\pi)^2 \int d^4x \int d^4y\, e^{ip\cdot x}\, e^{iq\cdot y}\, \langle 0|\mathrm{T}\left\{ j_\alpha^{\rm em}(x), j_\beta^{\rm em}(y), P_i^{(p)}(0) \right\} |\bar{B}_q(p+q)\rangle \,, \\ &\quad (i = 1, \ldots 6, 8) \,. \end{aligned}$$

Main task: Construct the SCET factorization formulae beyond the leading power.

General aspects of $B_q \to \gamma\gamma$

- Parametrization:

$$T^{(p)}_{i,\alpha\beta} = i m^3_{B_q} \left[\left(g^{\perp}_{\alpha\beta} - i\,\varepsilon^{\perp}_{\alpha\beta} \right) F^{(p)}_{i,L} - \left(g^{\perp}_{\alpha\beta} + i\,\varepsilon^{\perp}_{\alpha\beta} \right) F^{(p)}_{i,R} \right] .$$

  - Only two helicity form factors due to the Ward identities and the transversity conditions.
  - Similar decomposition for the radiative leptonic B-meson decay amplitude.

- Hierarchy structure due to the chiral weak interactions and helicity conservation:

$$F^{(p)}_{i,L} : F^{(p)}_{i,R} = 1 : \left(\frac{\Lambda_{\rm QCD}}{m_b} \right) .$$

In analogy to the two-body nonleptonic $B \to VV$ decays [Beneke, Rohrer, Yang, 2007].

- Transversity form factors:

$$F^{(p)}_{i,\parallel} = F^{(p)}_{i,L} - F^{(p)}_{i,R} \,, \qquad F^{(p)}_{i,\perp} = F^{(p)}_{i,L} + F^{(p)}_{i,R} \,.$$

The two-photon final states as the CP eigenstates with the eigenvalues $+1$ and -1.

Current status of $B_q \to \gamma\gamma$

- QCD factorization at leading power in Λ/m_b and at NLO in α_s [Descotes-Genon, Sachrajda, 2003].
  - No collinear strong interaction dynamics at LP ($\Rightarrow$ no need of the hadronic photon LCDAs).
  - The two-loop $b \to q\gamma$ matrix elements of QCD penguin operators NOT included.
    $\Rightarrow$ A complete factorization-scale independence at NLO is absent!
- Subleading power contributions from the weak annihilation diagrams [Bosch, Buchalla, 2002].
  - Complex perturbative hard functions evaluated at one loop.
  - Diagrammatic factorization established at two loops.
- Previous model-dependent calculations introduce additional systematic uncertainties.
- The new (technical)-ingredients from [Shen, YMW, Wei, 2020]:
  - A complete NLL calculation for the LP contribution $\Rightarrow$ 2-loop evolution of ϕ_B^+.
  - The NLP factorization for the energetic photon radiation off the light quark.
    The so-called soft form factor introduced by Beneke and Rohrwild is calculable!
  - The NLP factorization for the light-quark mass effect.
  - The NLP factorization for the SCET current $J^{(A2)} \supset (\bar{\xi}_{\overline{\rm hc}} W_{\overline{\rm hc}}) \gamma_\alpha^\perp P_L \left(\frac{i \overrightarrow{\not{D}}_\top}{2 m_b} \right) h_v$.
  - The NLP factorization for the subleading twist B-meson LCDAs (2-particle $\oplus$ 3-particle).
  - The resolved photon contribution with the dispersion technique.
- QCD factorization for the long-distance penguin contribution to $B_{d,s} \to \gamma\gamma$ decays by introducing a novel generalized bottom-meson distribution amplitude [Qin, Shen, Wang, YMW, 2023].

SCET factorization at leading power

- QCD $\to$ SCET$_{\rm I}$ matching at LP:

$$\sum_{i=1}^{8} C_i\, T^{(p)}_{i,\alpha\beta} = \sum_{i=1}^{8} C_i\, H_i^{(p)} \left\{ \int d^4x\, e^{iq\cdot x} \langle 0|{\rm T}\left\{ j^{\rm em}_{\beta,\,{\rm SCET_I}}(x),\ \left[(\bar{\xi}_{\overline{\rm hc}} W_{\overline{\rm hc}}) \gamma_\alpha^{\perp} P_L\, h_v \right](0) \right\} |\bar{B}_q\rangle \right.$$
$$\left. + [p \leftrightarrow q, \alpha \leftrightarrow \beta] \right\}.$$

Universal SCET$_{\rm I}$ correlation function independent of the weak-operator index!

- Perturbative matching coefficients at NLO:

$$\sum_{i=1}^{8} C_i\, H_i^{(p)} = (-2\,i)\, \overline{m}_b(\nu)\, m_{B_q}\, V^{(p)}_{7,\rm eff}\,,$$

$$V^{(p)}_{7,\rm eff} = C_7^{\rm eff}\, C_{T_1}^{(\rm A0)} + \sum_{i=1,\ldots,6,8} \frac{\alpha_s(\mu)}{4\pi}\, C_i^{\rm eff}\, F^{(p)}_{i,7}\,.$$

  - The hard function $C_{T_1}^{(\rm A0)}$ from matching the heavy-light tensor current onto SCET$_{\rm I}$.
  - The hard functions $F^{(p)}_{i,7}$ $(i = 1,..6,8)$ from perturbative matching of the $b \to q\gamma$ matrix elements [Buras, Czarnecki, Misiak, Urban, 2002].
  - $V^{(p)}_{7,\rm eff}$ due to both the electro-weak scale and the heavy-quark mass scale fluctuations.

SCET factorization at leading power

- $\mathrm{SCET_I} \to \mathrm{SCET_{II}}$ matching in coordinate space:

$$\mathscr{T}_{\alpha\beta} = \int d^4x\, e^{iq\cdot x} \langle 0|\mathrm{T}\left\{ j^{\mathrm{em}}_{\beta,\mathrm{SCET_I}}(x),\ \left[(\bar{\xi}_{\overline{\mathrm{hc}}} W_{\overline{\mathrm{hc}}})\, \gamma^{\perp}_{\alpha} P_L\, h_v \right](0) \right\} |\bar{B}_q\rangle$$
$$= \int dt\, \mathscr{J}\left(\frac{\bar{n}\cdot q}{\mu^2\, t} \right) \langle 0|(\bar{q}_s Y_s)(tn)\gamma^{\perp}_{\beta}\, \not{n}\, \gamma^{\perp}_{\alpha} P_L\, (Y_s^{\dagger} h_v)(0)|\bar{B}_q\rangle\,.$$

  - The explicit form of the $\mathrm{SCET_I}$ electromagnetic current [Lunghi, Pirjol, Wyler, 2003]:

$$j^{\mu}_{q,\mathrm{SCET_I}}(x) = \sum_q Q_q \left[\overline{q}_{\mathrm{hc}} \left(\gamma^{\mu}_{\perp} \frac{1}{in_+ D_{\mathrm{hc}}} i\not{D}_{\mathrm{hc}\perp} + i\not{D}_{\mathrm{hc}\perp} \frac{1}{in_+ D_{\mathrm{hc}}} \gamma^{\mu}_{\perp} \right) \frac{\not{n}_+}{2} q_{\mathrm{hc}} \right](x)$$
$$+ \sum_q Q_q \left[\overline{q}_s(x_-)\, \gamma^{\mu}_{\perp}\, q_{\mathrm{hc}}(x) \right].$$

  - Only need the one-loop jet function [2-loop result by Liu and Neubert, 2020].

$$J = 1 + \frac{\alpha_s(\mu)\, C_F}{4\pi} \left[\ln^2\left(\frac{\mu^2}{m_b\, \omega} \right) - \frac{\pi^2}{6} - 1 \right] + \mathscr{O}(\alpha_s^2)\,.$$

  - The soft dynamics encoded in the twist-two HQET B-meson LCDA [Grozin, Neubert, 1997].
  - The resulting LP factorization formula:

$$\bar{\mathscr{A}}_{\mathrm{LP}}(\bar{B}_q \to \gamma\gamma) = i\, \frac{4\, G_F}{\sqrt{2}}\, \frac{\alpha_{\mathrm{em}}}{4\pi}\, \varepsilon_1^{*\alpha}(p)\, \varepsilon_2^{*\beta}(q) \left[g^{\perp}_{\alpha\beta} - i\varepsilon^{\perp}_{\alpha\beta} \right] e_q f_{B_q} m^2_{B_q} K^{-1}(m_b,\mu)$$
$$\left[\sum_{p=u,c} V_{pb} V^*_{pq}\, \overline{m}_b(\nu)\, V^{(p)}_{7,\mathrm{eff}}(m_b,\mu,\nu) \right] \int_0^{\infty} \frac{d\omega}{\omega}\, \phi_B^{+}(\omega,\mu)\, J(m_b,\omega,\mu)\,.$$

SCET factorization at leading power

- No common choice of the factorization scale to avoid the parametrically large logarithms.
  $\Rightarrow$ QCD rsummation for the enhanced logarithms with the standard RG formalism.

- RG evolution functions for the hard functions at NLL [Beneke, Rohrwild, 2011]:

$$\begin{aligned} V_{7,\mathrm{eff}}^{(p)}(m_b,\mu,\nu) &= \hat{U}_1(m_b,\mu_\mathrm{h},\mu)\, V_{7,\mathrm{eff}}^{(p)}(m_b,\mu_\mathrm{h},\nu)\,, \\ K^{-1}(m_b,\mu) &= \hat{U}_2(m_b,\mu_\mathrm{h},\mu)\, K^{-1}(m_b,\mu_\mathrm{h})\,. \end{aligned}$$

  [The NNLL evolution functions are actually also available.]

- Taking the factorization scale of order $\sqrt{m_b \Lambda_{\mathrm{QCD}}}$ $\Rightarrow$ No resummation for the jet function.

- RG evolution of $\phi_B^+(\omega,\mu)$ at two loops [Braun, Ji, Manashov, 2019]:

$$\begin{aligned} \frac{d\phi_B^+(\omega,\mu)}{d\ln\mu} = \left[\Gamma_{\mathrm{cusp}}(\alpha_s)\ln\frac{\omega}{\mu} - \gamma_\eta(\alpha_s)\right]\phi_B^+(\omega,\mu) + \Gamma_{\mathrm{cusp}}(\alpha_s)\int_0^\infty dx\,\Gamma(1,x)\,\phi_B^+(\omega/x,\mu) \\ + \left(\frac{\alpha_s}{2\pi}\right)^2 C_F \int_0^1 \frac{dx}{1-x}\,h(x)\,\phi_B^+(\omega/x,\mu)\,. \end{aligned}$$

  The last missing element for the NLL predictions of exclusive B-meson decay observables!

SCET factorization at leading power

- Solving the integro-differential equation governing the ϕ_B^+ evolution nontrivial but interesting.
  - One-loop eigenfunctions first derived in [Bell, Feldmann, YMW, Yip, 2013].
  - Collinear conformal symmetry for the Lange-Neubert kernel at one loop
    $\Rightarrow$ Commutation relations [Braun, Manashov, 2014].
  - The two-loop eigenfunctions depend on the strong coupling α_s [Braun, Ji, Manashov, 2019]
  - The explicit solution to the momentum-space evolution equation at 2-loops [Galda, Neubert, Wang, 2022].

- Applying the Laplace transformation of the LCDA [Galda, Neubert, 2020]

$$\tilde{\phi}_B^+(\eta,\mu) = \int_0^\infty \frac{d\omega}{\omega}\, \phi_B^+(\omega,\mu) \left(\frac{\omega}{\bar{\omega}}\right)^{-\eta},$$

$\Rightarrow$ the general solution to the two-loop RGE of $\phi_B^+(\omega,\mu)$

$$\begin{aligned}\tilde{\phi}_B^+(\eta,\mu) \quad &= \quad \exp\left[S(\mu_0,\mu)+a_\gamma(\mu_0,\mu)+2\,\gamma_E\, a_\Gamma(\mu_0,\mu)\right]\left(\frac{\bar{\omega}}{\mu_0}\right)^{-a_\Gamma(\mu_0,\mu)} \\ &\times \frac{\Gamma(1+\eta+a_\Gamma(\mu_0,\mu))\,\Gamma(1-\eta)}{\Gamma(1-\eta-a_\Gamma(\mu_0,\mu))\,\Gamma(1+\eta)} \exp\left[\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)}\, \mathscr{G}(\eta+a_\Gamma(\mu_\alpha,\mu),\alpha)\right] \\ &\times \tilde{\phi}_B^+(\eta+a_\Gamma(\mu_0,\mu),\mu_0)\,.\end{aligned}$$

$\Rightarrow$ The NLL resummation improved expression for the LP decay amplitude.

Historical remarks on the B-meson LCDA

Renormalization group evolution of the B meson light cone distribution amplitude

Bjorn O. Lange (Cornell U., LNS), Matthias Neubert (Cornell U., LNS) (Mar, 2003)

Published in: *Phys.Rev.Lett.* 91 (2003) 102001 • e-Print: hep-ph/0303082 [hep-ph]

pdf DOI cite claim

reference search 172 citations

Light-Cone Distribution Amplitudes for Heavy-Quark Hadrons

Guido Bell (Oxford U., Theor. Phys.), Thorsten Feldmann (Siegen U.), Yu-Ming Wang (RWTH Aachen U. and Munich, Tech. U.), Matthew W Y Yip (Durham U., IPPP) (Aug 28, 2013)

Published in: *JHEP* 11 (2013) 191 • e-Print: 1308.6114 [hep-ph]

pdf DOI cite claim

reference search 105 citations

Conformal symmetry of the Lange-Neubert evolution equation

V.M. Braun (Regensburg U.), A.N. Manashov (Regensburg U. and St. Petersburg State U.) (Feb 24, 2014)

Published in: *Phys.Lett.B* 731 (2014) 316-319 • e-Print: 1402.5822 [hep-ph]

pdf DOI cite claim

reference search 34 citations

Two-loop evolution equation for the B-meson distribution amplitude

V.M. Braun (Regensburg U.), Yao Ji (Regensburg U.), A.N. Manashov (Regensburg U. and Hamburg U., Inst. Theor. Phys. II and Steklov Math. Inst., St. Petersburg) (May 11, 2019)

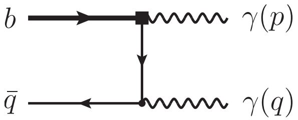
Published in: *Phys.Rev.D* 100 (2019) 1, 014023 • e-Print: 1905.04498 [hep-ph]

pdf DOI cite claim

reference search 33 citations

QCD factorization beyond leading power

- The NLP factorization from the hard-collinear propagator:



Heavy quark expansion (the soft quark momentum k):

$$\frac{i\,(\not{q}-\not{k})}{(q-k)^2} = \frac{i\not{q}}{(q-k)^2} - \underbrace{\frac{i\not{k}}{(q-k)^2}}_{\text{power suppressed}}.$$

- The yielding NLP amplitude:

$$T^{\rm hc,NLP}_{7,\alpha\beta} = 2\,e_q\,\overline{m}_b(\nu)\,m_{B_q}\int d^4x\int\frac{d^4\ell}{(2\pi)^4}\exp\left[i\,(q-\ell)\cdot x\right]\frac{1}{\ell^2+i0}$$
$$\times\,\langle 0|\bar{q}(x)\,\gamma_\beta^{\perp}\,(\not{q}-\not{\ell})\,\not{\bar{n}}\,\gamma_\alpha^{\perp}\,P_R\,h_v(0)|\bar{B}_q\rangle\,.$$

  - Light-cone decomposition of the four-momentum $(\not{q}-\not{\ell})$:

$$\not{q}-\not{\ell} = \underbrace{\frac{n\cdot(q-\ell)}{2}\,\not{\bar{n}}}_{\text{vanishes obviously}} + \underbrace{\frac{\bar{n}\cdot(q-\ell)}{2}\,\not{n}}_{\text{the only relevant term}} + (\not{q}_\perp-\not{\ell}_\perp)\,.$$

    The third term does not contribute to the matrix element (hint: symmetry with $\alpha\leftrightarrow\beta$).

  - Applying the IBP reduction leads to

$$T^{\rm hc,NLP}_{7,\alpha\beta} = \left[i\,e_q\,\overline{m}_b(\nu)\,m_{B_q}\right]\int d^4x\int\frac{d^4\ell}{(2\pi)^4}\exp\left[i\,(q-\ell)\cdot x\right]\frac{\bar{n}_\mu}{\ell^2+i0}$$
$$\times\,\frac{\partial}{\partial x_\mu}\,\langle 0|\bar{q}(x)\,\gamma_\beta^{\perp}\,\not{n}\,\not{\bar{n}}\,\gamma_\alpha^{\perp}\,P_R\,h_v(0)|\bar{B}_q\rangle\,.$$

QCD factorization beyond leading power

- Performing the integration over ℓ leads to (note that $q_\mu = (\bar{n} \cdot q/2)\, n_\mu$)

$$T^{\rm hc, NLP}_{7,\alpha\beta} = \left[-\frac{e_q\, \overline{m}_b(\nu)\, m_{B_q}}{4\,\pi^2} \right] \int d^4x\, \frac{e^{i q\cdot x}}{x^2}\, (2\, v_\mu - n_\mu) \\ \times \frac{\partial}{\partial x_\mu}\, \langle 0 | \bar{q}(x)\, \gamma_\beta^\perp\, \not{n}\, \not{\bar{n}}\, \gamma_\alpha^\perp\, P_R\, h_v(0) | \bar{B}_q \rangle \,.$$

The second term $n_\mu \left(\partial / \partial x_\mu \right) [...] = 2\, (\partial / \partial \bar{n} \cdot x)\, [...]$ can be computed directly.

- The equation of motion for the non-local operator [Kawamura, Kodaira, Qiao, Tanaka, 2001]:

$$v_\mu\, \frac{\partial}{\partial x_\mu}\, [\bar{q}(x)\, \Gamma\, h_v(0)] = i \int_0^1 du\, \bar{u}\, \bar{q}(x)\, g_s\, G_{\alpha\beta}(ux)\, x^\alpha\, v^\beta\, \Gamma\, h_v(0) + (v \cdot \partial)\, [\bar{q}(x)\, \Gamma\, h_v(0)] \,.$$

  - Translation invariance of the HQET matrix element (note the total translation operator ∂_α)

$$(v \cdot \partial)\, \langle 0 |\, [\bar{q}(x)\, \Gamma\, h_v(0)]\, | \bar{B}_q(v) \rangle = -i\, \bar{\Lambda}\, \langle 0 |\, [\bar{q}(x)\, \Gamma\, h_v(0)]\, | \bar{B}_q(v) \rangle \,.$$

  - The yielding NLP factorization formula (preserving the large-recoil symmetry)

$$T^{\rm hc, NLP}_{7,\alpha\beta} = \left[-2\, i\, e_q\, \overline{m}_b(\nu) f_{B_q}\, m_{B_q} \right] \left[g_{\alpha\beta}^\perp - i\, \varepsilon_{\alpha\beta}^\perp \right] \left\{ \frac{1}{2} - \left(\frac{\bar{\Lambda}}{\lambda_{B_q}} \right) \right. \\ \left. + \int_0^\infty d\omega_1 \int_0^\infty d\omega_2\, \frac{1}{\omega_2} \left[\frac{1}{\omega_2}\, \ln \frac{\omega_1}{\omega_1 + \omega_2} + \frac{1}{\omega_1} \right] \underbrace{\Psi_4(\omega_1, \omega_2, \mu)}_{\propto\, \omega_1\, \omega_2 \text{ at small } \omega_i} \right\} .$$

QCD factorization beyond leading power

- The NLP correction from the non-vanishing quark mass:

$$T_{7,\alpha\beta}^{m_q,\text{NLP}} = \left[-i\, e_q\, \overline{m}_b(\nu)\, m_q f_{B_q}\, m_{B_q}\right] \left[g_{\alpha\beta}^{\perp} - i\,\varepsilon_{\alpha\beta}^{\perp}\right] \int_0^\infty d\omega\, \frac{\phi_B^-(\omega,\mu)}{\omega}\,.$$

  The asymptotic behaviour of $\phi_B^-(\omega,\mu_0) \to$ constant at small ω.

- Rapidity divergence implies the breakdown of the naive soft-collinear factorization.

- Nonperturbative parametrization of the convolution integral [BBNS, 2001]:

$$\int_0^\infty d\omega\, \frac{\phi_B^-(\omega,\mu)}{\omega} = \left[\int_0^{\Lambda_{\rm UV}} + \int_{\Lambda_{\rm UV}}^\infty\right] d\omega\, \frac{\phi_B^-(\omega,\mu)}{\omega}$$

$$= \left[\phi_B^-(0,\mu)\, X_{\rm NLP} + \int_0^{\Lambda_{\rm UV}} d\omega\, \frac{\phi_B^-(\omega,\mu) - \phi_B^-(0,\mu)}{\omega}\right] + \int_{\Lambda_{\rm UV}}^\infty d\omega\, \frac{\phi_B^-(\omega,\mu)}{\omega}\,.$$

$\Downarrow$ UV and IR Finite!

$$X_{\rm NLP} = \left[1 + \rho_S \exp(i\,\varphi_S)\right] \ln\left(\frac{\Lambda_{\rm UV}}{\Lambda_{\rm h}}\right).$$

  - Alternative estimate from the LCSR method also possible [Li^2, Lü, Shen, 2022].
  - The complete SCET factorization in demand but rather challenging.
    Refactorization-based subtraction scheme? [Beneke, Neubert, Feldmann, Liu, Wang^2, ...]

QCD factorization beyond leading power

- The NLP contribution from the subleading SCET current [Beneke, Feldmann, 2002]

$$J^{(\mathrm{A2})} \supset (\bar{\xi}_{\overline{\mathrm{hc}}} W_{\overline{\mathrm{hc}}})\, \gamma_\alpha^\perp \, P_L \left(\frac{i \overrightarrow{\not{D}}_\top}{2\, m_b} \right) h_v + \ldots , \qquad D_\top^\mu \equiv D^\mu - (v \cdot D)\, v^\mu \,.$$

Arise from the HQET representation of the QCD b-quark field.

- The resulting non-local hadronic matrix element

$$T^{A2,\mathrm{NLP}}_{7,\alpha\beta} = \left[-\frac{i\, e_q\, m_{B_q}^2}{2} \right] \int d^4x \int \frac{d^4\ell}{(2\pi)^4} \exp\left[i\,(q-\ell)\cdot x\right] \frac{1}{\ell^2 + i0}$$
$$\times \langle 0 | \bar{q}(x)\, \gamma_\beta^\perp \, \not{n}\, \not{\bar{n}}\, \gamma_\alpha^\perp \, \overrightarrow{\not{D}}\, P_L\, h_v(0) | \bar{B}_q \rangle \,.$$

$\Rightarrow$ Applying the HQET equation of motion at the the classical level

$$\bar{q}(x)\, \Gamma\, \overrightarrow{D}_\rho\, h_v(0) = \partial_\rho \left[\bar{q}(x)\, \Gamma\, h_v(0) \right] + i \int_0^1 du\, \bar{u}\, \bar{q}(x)\, g_s\, G_{\lambda\rho}(ux)\, x^\lambda\, \Gamma\, h_v(0) - \frac{\partial}{\partial x^\rho} \bar{q}(x)\, \Gamma\, h_v(0) \,,$$

leads to the soft-collinear factorization formula

$$T^{A2,\mathrm{NLP}}_{7,\alpha\beta} = \left[i e_q f_{B_q} m_{B_q}^2 \right] \left[g_{\alpha\beta}^\perp - i\, \varepsilon_{\alpha\beta}^\perp \right] \left\{ \frac{1}{2} \left(\frac{\bar{\Lambda}}{\lambda_{B_q}} \right) - 1 \int_0^\infty d\omega_1 \int_0^\infty d\omega_2\, \frac{\Phi_3(\omega_1, \omega_2, \mu)}{\omega_1\, (\omega_1 + \omega_2)} \right\} .$$

  - The convolution integral free of the end-point divergence (because $\Phi_3 \sim \omega_1\, \omega_2^2$ at small ω_i).
  - In agreement with the counterpart contribution to $B \to \gamma \ell \nu$ [Beneke, Braun, Ji, Wei, 2018].

QCD factorization beyond leading power

- The two-particle higher-twist B-meson LCDAs up to the $\mathcal{O}(x^2)$ accuracy:

$$\langle 0|(\bar{q}_s Y_s)_\beta(x)\ (Y_s^\dagger h_v)_\alpha(0)|\bar{B}_q\rangle$$
$$= -\frac{i\tilde{f}_{B_q}(\mu)\, m_{B_q}}{4} \int_0^\infty d\omega\, e^{-i\omega v\cdot x} \left[\frac{1+\not{v}}{2}\left\{2\left[\phi_B^+(\omega,\mu) + x^2\, g_B^+(\omega,\mu)\right]\right.\right.$$
$$\left.\left. -\frac{1}{v\cdot x}\left[(\phi_B^+(\omega,\mu)-\phi_B^-(\omega,\mu)) + x^2\,(g_B^+(\omega,\mu)-g_B^-(\omega,\mu))\right]\not{x}\right\}\gamma_5\right]_{\alpha\beta}.$$

- The three-particle higher-twist B-meson LCDAs up to twist-6 [Braun, Ji, Manashov, 2017].
  - The EOM relation between the two-particle and three-particle LCDAs.

$$-2\,\frac{d^2}{d\omega^2}\, g_B^-(\omega,\mu) = \left[\frac{3}{2} + (\omega-\bar{\Lambda})\,\frac{d}{d\omega}\right]\phi_B^-(\omega,\mu) - \frac{1}{2}\,\phi_B^+(\omega,\mu)$$
$$+\int_0^\infty \frac{d\omega_2}{\omega_2}\left[\frac{d}{d\omega} - \frac{1}{\omega_2}\right]\Psi_5(\omega,\omega_2,\mu) + \int_0^\omega \frac{d\omega_2}{\omega_2^2}\,\Psi_5(\omega-\omega_2,\omega_2,\mu).$$

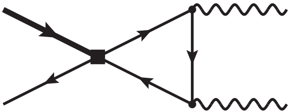
  - The systematic parametrization requires 8 independent LCDAs.

- The higher-twist factorization formula at tree level:

$$T_{7,\alpha\beta}^{\mathrm{HT,NLP}} \simeq \left[-i\,e_q\,\overline{m}_b(\nu) f_{B_q}\, m_{B_q}\right]\left[g_{\alpha\beta}^{\perp} - i\,\varepsilon_{\alpha\beta}^{\perp}\right]\left\{-1+2\int_0^\infty d\omega\,\ln\omega\,\underbrace{\Delta\phi_B^-(\omega,\mu)}\right.$$
$$\left. -2\int_0^\infty d\omega_2\,\frac{1}{\omega_2}\,\Phi_4(\omega_1=0,\omega_2,\mu)\right\}. \qquad \equiv \left[-\omega\,\frac{d}{d\omega}\,\phi_B^-(\omega,\mu) - \phi_B^+(\omega,\mu)\right]$$

QCD factorization beyond leading power

- The NLP contribution from the weak annihilation diagram:



The resulting helicity form factors:

$$\sum_{i=1}^{6} C_i \, F_{i,L}^{(p),\,\mathrm{WA,NLP}} = \frac{f_{B_q}}{m_{B_q}} \left[\mathscr{F}_V^{(p),\,\mathrm{WA}} - \mathscr{F}_A^{(p),\,\mathrm{WA}} \right],$$

$$\sum_{i=1}^{6} C_i \, F_{i,R}^{(p),\,\mathrm{WA,NLP}} = \frac{f_{B_q}}{m_{B_q}} \left[\mathscr{F}_V^{(p),\,\mathrm{WA}} + \mathscr{F}_A^{(p),\,\mathrm{WA}} \right].$$

  - The weak-annihilation effect will spoil the large-recoil symmetry (non-vanishing $F_{i,R}^{(p)}$).
  - Massive quark loops generate the non-trivial strong phase.
    $\Leftrightarrow$ Dual to the final-state rescattering $\bar{B}_q \to H_c \, H'_{\bar{c}} \to \gamma\gamma$ at hadronic level.
  - Tree-operator contributions consistent with [Bosch, Buchalla, 2002].

- The NLP contribution from the (anti)-collinear photon radiation off the bottom-quark:

$$T_{7,\,\alpha\beta}^{e_b,\,\mathrm{NLP}} = \left[-i \, e_q f_{B_q} \, m_{B_q}^2 \right] \left[g_{\alpha\beta}^{\perp} - i \, \varepsilon_{\alpha\beta}^{\perp} \right].$$

  - Local correction preserves the large-recoil symmetry!
  - In analogy to the P_7 contribution to $B_q \to \gamma \ell \bar{\ell}$ with the B-type insertion [Beneke, Bobeth, YMW, 2020].

The resolved photon contribution

- The NLP contribution from the "hadronic" component of the on-shell photon state [Ball, Braun, Kivel, 2003].

- The dispersion technique [Khodjamirian, 1999; Braun, Khodjamirian, 2013; YMW, 2016]:

$$\begin{aligned}\tilde{T}_{7,\alpha\beta} &= 2\overline{m}_b(\nu)\int d^4x\, e^{iq\cdot x}\,\langle 0|\mathrm{T}\left\{j^{\mathrm{em}}_\beta(x), \bar{q}_L(0)\sigma_{\mu\alpha}p^\mu b_R(0)\right\}|\bar{B}_q(p+q)\rangle\Big|_{q^2<0} \\ &\quad + [p\leftrightarrow q, \alpha\leftrightarrow\beta] \\ &= -\frac{i\,e_q\,\overline{m}_b(\nu)\,m^2_{B_q}}{2}\left\{\left(g^\perp_{\alpha\beta} - i\,\varepsilon^\perp_{\alpha\beta}\right)\tilde{F}_{7,L}(\bar{n}\cdot q, n\cdot q) + [p\leftrightarrow q, \alpha\leftrightarrow\beta]\right\}.\end{aligned}$$

  Power countering scheme for the four-momentum q:

$$\bar{n}\cdot q \sim \mathscr{O}(m_b)\,, \qquad n\cdot q \sim \mathscr{O}(\Lambda)\,.$$

- The hadronic dispersion relation:

$$\begin{aligned}\tilde{T}_{7,\alpha\beta} &= -i\,\overline{m}_b(\nu)\,m^2_{B_q}\left(g^\perp_{\alpha\beta} - i\,\varepsilon^\perp_{\alpha\beta}\right)\left\{\sum_V \frac{c_V f_V m_V T_1^{B_q\to V}(0)}{\bar{n}\cdot q\left(m_V^2/\bar{n}\cdot q - n\cdot q - i0\right)}\right. \\ &\quad \left. + \int_{\omega_s}^\infty d\omega'\,\frac{\rho^{\mathrm{had}}(\bar{n}\cdot q, \omega')}{\omega' - n\cdot q - i0}\right\}.\end{aligned}$$

  The constant c_V determined by the flavour factor and the electric charge of the QED quark-current.

- LCSR for the tensor $B\to V$ form factors:

$$\sum_V \frac{c_V f_V m_V}{\bar{n}\cdot q}\exp\left[-\frac{m_V^2}{\bar{n}\cdot q\,\omega_M}\right] T_1^{B_q\to V}(0) = \frac{e_q}{2}\,\frac{1}{\pi}\int_0^{\omega_s} d\omega'\, e^{-\omega'/\omega_M}\,\mathrm{Im}_{\omega'}\,\tilde{F}_{7,L}(\bar{n}\cdot q, \omega')\,.$$

The resolved photon contribution

- Improved dispersion relations (setting $n \cdot q = 0$) [Master formula]:

$$T_{7,\alpha\beta} = -\frac{i\, e_q\, \overline{m}_b(\nu)\, m_{B_q}^2}{2} \left(g_{\alpha\beta}^{\perp} - i\,\varepsilon_{\alpha\beta}^{\perp}\right) \Bigg\{ \underbrace{\frac{1}{\pi} \int_0^{\infty} \frac{d\omega'}{\omega'}\, \mathrm{Im}_{\omega'}\, \tilde{F}_{7,L}(\bar{n}\cdot q, \omega')}_{\text{LP factorization formulae}}$$

$$+ \underbrace{\frac{1}{\pi} \int_0^{\omega_s} d\omega' \left[\frac{\bar{n}\cdot q}{m_V^2} \exp\left(\frac{m_V^2 - \bar{n}\cdot q\, \omega'}{\bar{n}\cdot q\, \omega_M} \right) - \frac{1}{\omega'} \right] \mathrm{Im}_{\omega'}\, \tilde{F}_{7,L}(\bar{n}\cdot q, \omega')}_{\text{nonperturbative modification}} \Bigg\} + [p \leftrightarrow q, \alpha \leftrightarrow \beta]\,.$$

- Power counting scheme for the sum-rule parameters:

$$\omega_s = \frac{s_0}{\bar{n}\cdot q} \sim \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{m_{B_q}}\right), \qquad \omega_M = \frac{M^2}{\bar{n}\cdot q} \sim \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{m_{B_q}}\right).$$

$\Rightarrow$ Nonperturbative modification yields the soft non-factorizable contribution.

- Spectral density at tree level:

$$\frac{1}{\pi}\, \mathrm{Im}_{\omega'}\, \tilde{F}_{7,L}(\bar{n}\cdot q, \omega') \quad = \quad f_{B_q}\, \underbrace{\phi_B^+(\omega', \mu)}_{} + \mathcal{O}(\alpha_s, \Lambda/m_b)\,.$$

of $\mathcal{O}(1/\Lambda)[\mathcal{O}(1/m_b)]$ for $\omega' \sim \mathcal{O}(\Lambda)[\omega' \sim \mathcal{O}(\Lambda^2/m_b)]$

Power suppressed soft contribution!

- Alternative LCSR calculation with the subleading-twist photon LCDAs [Shen, YMW, 2018].

Summary for the helicity amplitudes of $B_q \to \gamma\gamma$

- Final expressions for the factorized NLP corrections:

$$\sum_{i=1}^{8} C_i F_{i,L}^{(p),\,\mathrm{fac,NLP}} = C_7^{\mathrm{eff}} \left[F_{7,L}^{\mathrm{hc,NLP}} + F_{7,L}^{m_q,\mathrm{NLP}} + F_{7,L}^{A2,\mathrm{NLP}} + F_{7,L}^{\mathrm{HT,NLP}} + F_{7,L}^{e_b,\mathrm{NLP}} \right] + \frac{f_{B_q}}{m_{B_q}} \left[\mathscr{F}_V^{(p),\,\mathrm{WA}} - \mathscr{F}_A^{(p),\,\mathrm{WA}} \right],$$

$$\sum_{i=1}^{8} C_i F_{i,R}^{(p),\,\mathrm{fac,NLP}} = \frac{f_{B_q}}{m_{B_q}} \left[\mathscr{F}_V^{(p),\,\mathrm{WA}} + \mathscr{F}_A^{(p),\,\mathrm{WA}} \right].$$

Large-recoil symmetry violation from the weak annihilation correction completely.

- Final expressions for the two helicity amplitudes:

$$\bar{\mathscr{A}}_L = \sum_{p=u,c} V_{pb} V_{pq}^* \sum_{i=1}^{8} C_i \left[F_{i,L}^{(p),\,\mathrm{LP}} + F_{i,L}^{(p),\,\mathrm{fac,NLP}} + F_{i,L}^{(p),\,\mathrm{soft,NLP}} \right],$$

$$\bar{\mathscr{A}}_R = \sum_{p=u,c} V_{pb} V_{pq}^* \sum_{i=1}^{8} C_i \left[F_{i,R}^{(p),\,\mathrm{LP}} + F_{i,R}^{(p),\,\mathrm{fac,NLP}} + F_{i,R}^{(p),\,\mathrm{soft,NLP}} \right].$$

- The fundamental nonperturbative functions: HQET B-meson LCDAs.
  - Key hadronic inputs for exclusive B-meson decay phenomenologies, for instance $B \to \pi \ell \nu$, $B \to D^{(*)} \ell \nu$, $B \to K^* \gamma$, $B \to K^{(*)} \ell\ell$, $B \to \pi\pi$.

B-meson distribution amplitudes in HQET

- The LP contribution depends on λ_{B_q} and the inverse-logarithmic moments.
- Applying the general ansatz of the 2- and 3- particle B-meson LCDAs

$$\phi_B^+(\omega,\mu) = \omega\, f(\omega)\,, \quad \Phi_3(\omega_1,\omega_2,\mu_0) = -\frac{1}{2}\,\kappa(\mu_0)\left[\lambda_E^2(\mu_0)-\lambda_H^2(\mu_0)\right]\,\omega_1\,\omega_2^2\, f'(\omega_1+\omega_2)\,,$$
$$\Phi_4(\omega_1,\omega_2,\mu_0) = \frac{1}{2}\,\kappa(\mu_0)\left[\lambda_E^2(\mu_0)+\lambda_H^2(\mu_0)\right]\,\omega_2^2\, f(\omega_1+\omega_2)\,,$$
$$\Psi_4(\omega_1,\omega_2,\mu_0) = \kappa(\mu_0)\,\lambda_E^2(\mu_0)\,\omega_1\,\omega_2\, f(\omega_1+\omega_2)\,.$$

The factorized NLP corrections can then be parameterized by the local HQET parameters.

- The NLP soft contribution sensitive to the precise shape of $\phi_B^+(\omega,\mu)$.

$$\phi_B^+(\omega,\mu_0) = \int_0^\infty ds\,\sqrt{w\,s}\,J_1(2\sqrt{w\,s})\,\eta_+(s,\mu_0)\,, \qquad \eta_+(s,\mu_0) = {}_1F_1(\alpha;\beta;-s\,\omega_0)\,.$$

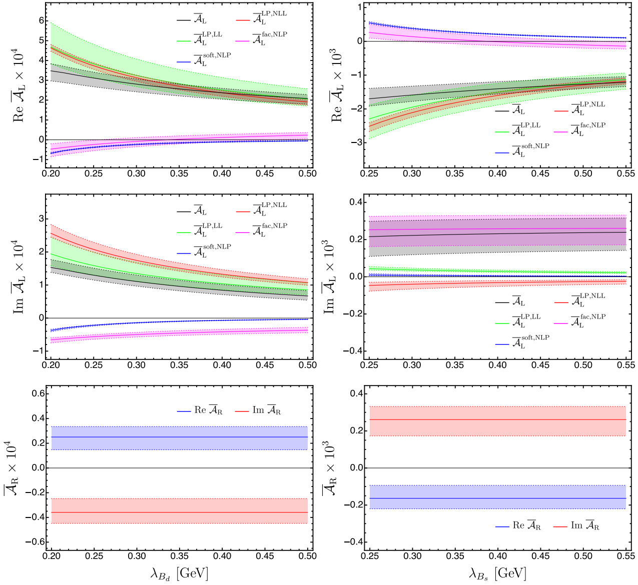
  - Such three-parameter ansatz [Beneke, Braun, Ji, Wei, 2018] is advantageous, since the resulting RG evolution can be done analytically in terms of ${}_2F_2$ functions.
  - An alternative parametrization of the twist-two B-meson DA in Laplace space [Galda, Neubert, Wang, 2022]:

$$\tilde{\phi}_B^+(\eta,\mu) = \int_0^\infty \frac{d\omega}{\omega}\left(\frac{\omega}{\bar{\omega}}\right)^{\eta}\phi_B^+(\omega,\mu) \overset{|\eta|\ll 1}{=} \frac{1}{\lambda_B(\mu)}\left[1+\sum_{n\geq 1}\frac{\eta^n}{n!}\,\sigma_n^B(\mu)\right].$$

  - New parametrization of the momentum-space DA in terms of associated Laguerre polynomials [Feldmann, Lüghausen, van Dyk, 2022; Feldmann, Lüghausen, Seitz, 2023].

Theory predictions for the helicity amplitudes

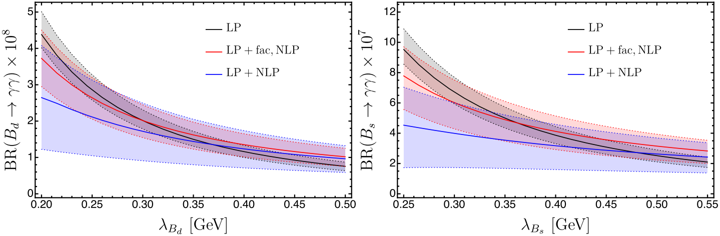
- Breakdown of the various QCD mechanisms:



- NLL effects stabilize the factorization-scale dependence.
- Factorizable NLP effects around $\mathcal{O}(30\%)$.
- Destructive effects from the NLP soft corrections.
- Strong phase from the 2-loop matrix element of P_2^c and the weak annihilation.

Phenomenological observables for $B_q \to \gamma\gamma$

- Time-integrated branching fraction (for the flavour-tagged measurement):



  - The yielding theory predictions

$$\mathscr{BR}(B_d \to \gamma\gamma) = \left(1.44^{+1.35}_{-0.80}\right) \times 10^{-8}\,, \qquad \mathscr{BR}(B_s \to \gamma\gamma) = \left(3.17^{+1.96}_{-1.74}\right) \times 10^{-7}\,.$$

    with the dominant uncertainties from λ_{B_q}, $\widehat{\sigma}^{(1)}_{B_q}$, $\widehat{\sigma}^{(2)}_{B_q}$ and the QCD renormalization scale ν.
  - Both the factorizable and soft NLP corrections are numerically important.
  - The ratio of the two branching ratios for $B_{d,s} \to \gamma\gamma$

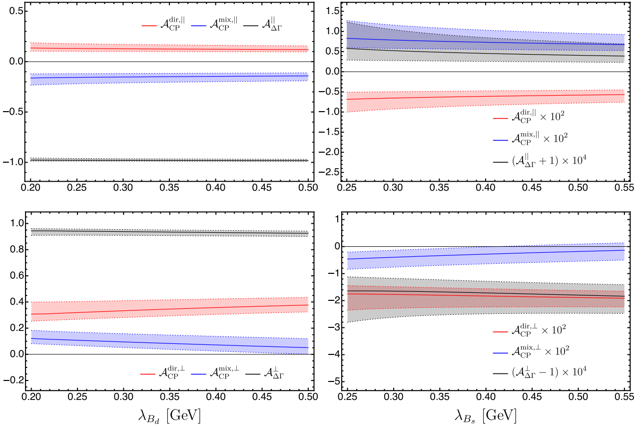
$$\frac{\mathscr{BR}(B_s \to \gamma\gamma)}{\mathscr{BR}(B_d \to \gamma\gamma)} = 33.80 \left(\frac{\lambda_{B_d}}{\lambda_{B_s}}\right)^2 + \mathscr{O}\left(\frac{\Lambda}{m_b}, \alpha_s\right)\,.$$

    The λ_{B_q}-scaling violation effect due to the NLP contributions approximately $(10-20)\,\%$.
  - The ratio $\lambda_{B_s} : \lambda_{B_d} = 1.19 \pm 0.14$ from QCDSR [Khodjamirian, Mandal, Mannel, 2020].

Phenomenological observables for $B_q \to \gamma\gamma$

- Time-dependent CP asymmetries:

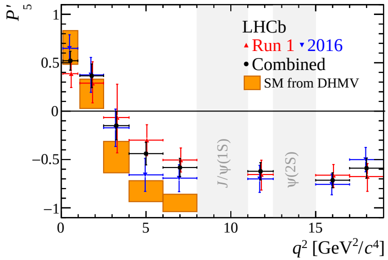
$$A_{\rm CP}^{\chi}(t) = \frac{\bar{\Gamma}^{\chi}(\bar{B}_q(t) \to \gamma\gamma) - \Gamma^{\chi}(B_q(t) \to \gamma\gamma)}{\bar{\Gamma}^{\chi}(\bar{B}_q(t) \to \gamma\gamma) + \Gamma^{\chi}(B_q(t) \to \gamma\gamma)} = -\frac{\mathscr{A}_{\rm CP}^{\rm dir,\,\chi}\cos(\Delta m_q t) + \mathscr{A}_{\rm CP}^{\rm mix,\,\chi}\sin(\Delta m_q t)}{\cosh(\Delta\Gamma_q t/2) + \mathscr{A}_{\Delta\Gamma}^{\chi}\sinh(\Delta\Gamma_q t/2)}\,.$$

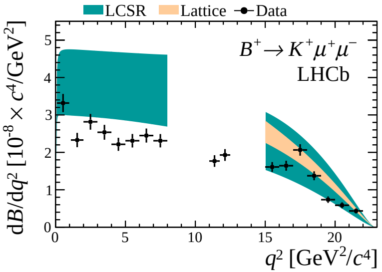


- $|\mathscr{A}_{\rm CP}^{\rm dir,\,\chi}|$ and $|\mathscr{A}_{\rm CP}^{\rm mix,\,\chi}|$ around $(10-40)\%$ for $B_d \to \gamma\gamma$.
- Tiny CP asymmetries for $B_s \to \gamma\gamma$ as expected.
- The difference between $\mathscr{A}_{\rm CP}^{\rm dir,\,\parallel}$ and $\mathscr{A}_{\rm CP}^{\rm dir,\,\perp}$ due to the NLP corrections.

Exclusive $b \to s\ell\bar{\ell}$ anomalies

- Several LHCb measurements derivative from the Standard Model (SM) predictions by $2-3\,\sigma$.
  - Angular observables in $B \to K^* \mu^+ \mu^-$ [JHEP 02 (2016) 104, 951 citations].
  - Branching fractions of $B \to K \mu^+ \mu^-$, $B \to K^* \mu^+ \mu^-$, and $B_s \to \phi \mu^+ \mu^-$.



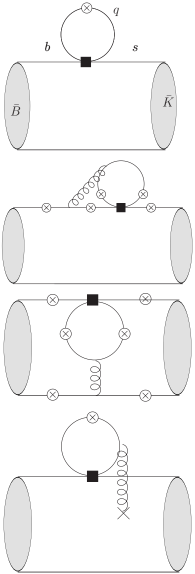


- Better understanding of the local and non-local hadronic effects in high demand.

A longstanding charming-penguin story

- (Non)-factorizable quark loops in $B \to K\ell\ell$:
  - OPE works far below the charm-quark threshold!
  - QCD factorization valid for $1\,\mathrm{GeV}^2 < q^2 \ll 4\,m_c^2$.
  - Factorizable charm loops reduced to $B \to K$ form factors.
  - Charm-loop induced spectator effects calculable in QCD factorization [Beneke, Feldmann, Seidel, 2001, 2005].
  - Soft charm loop computed in the OPE controlled approach with the dispersion relation [Khodjamirian, Mannel, Pivovarov, YMW, 2010].
  - Parametrization of the hadronic-operator effects:

$$\Delta C_9^{(BK)}(q^2) = \frac{16\pi^2 \mathscr{H}^{(BK)}(q^2)}{f_{BK}^{+}(q^2)}\,.$$



A longstanding charming-penguin story

Forward backward asymmetry of dilepton angular distribution in the decay b ---> s l+ l-

Ahmed Ali (DESY), T. Mannel (DESY), T. Morozumi (Rockefeller U.) (Sep, 1991)

Published in: *Phys.Lett.B* 273 (1991) 505-512

pdf DOI cite claim

reference search 369 citations

Lepton polarization in the decays b ---> X(s) mu+ mu- and B ---> X(s) tau+ tau-

F. Kruger (Aachen, Tech. Hochsch.), L.M. Sehgal (Aachen, Tech. Hochsch.) (Mar, 1996)

Published in: *Phys.Lett.B* 380 (1996) 199-204 • e-Print: hep-ph/9603237 [hep-ph]

pdf DOI cite claim

reference search 303 citations

Charm-loop effect in $B \to K^{(*)}\ell^+\ell^-$ and $B \to K^*\gamma$

A. Khodjamirian (Siegen U.), Th. Mannel (Siegen U.), A.A. Pivovarov (Siegen U.), Y.-M. Wang (Siegen U.) (Jun, 2010)

Published in: *JHEP* 09 (2010) 089 • e-Print: 1006.4945 [hep-ph]

pdf DOI cite claim

reference search 486 citations

$B \to K\ell^+\ell^-$ decay at large hadronic recoil

A. Khodjamirian (Siegen U.), Th. Mannel (Siegen U.), Y.M. Wang (Siegen U.) (Nov, 2012)

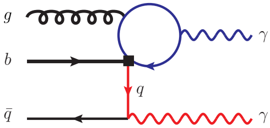
Published in: *JHEP* 02 (2013) 010 • e-Print: 1211.0234 [hep-ph]

pdf DOI cite claim

reference search 196 citations

Charming-penguin effect in $B_q \to \gamma\gamma$

- Soft-gluon radiation off the factorizable quark loop [Qin, Shen, Wang, YMW, 2023]:



The resulting helicity form factors:

$$\sum_{i=1}^{8} C_i F_{i,L}^{(p),\,\text{soft}\,4q} = -\frac{Q_q f_{B_q}}{m_{B_q}} \int_{-\infty}^{+\infty} \frac{d\omega_1}{\omega_1 - i0} \int_{-\infty}^{+\infty} \frac{d\omega_2}{\omega_2 - i0}$$
$$\left\{ \left(C_2 - \frac{C_1}{2N_c} \right) Q_p \left[F(z_p) - 1 \right] + 6\, C_6\, Q_c \left[F(z_c) - 1 \right] \right.$$
$$\left. - \left[\left(C_3 - \frac{C_4}{2N_c} \right) + 16 \left(C_5 - \frac{C_6}{2N_c} \right) \right] Q_q \right\} \Phi_{\rm G}(\omega_1, \omega_2, \mu)\,.$$

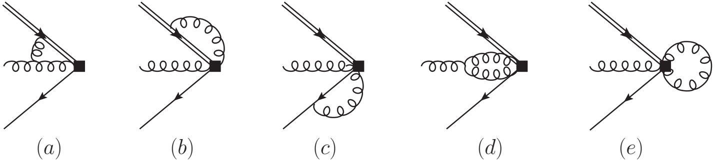
- Adopt the favored power-counting scheme $m_b \gg m_c \sim \mathcal{O}(\sqrt{\Lambda_{\rm QCD}\, m_b}) \gg \Lambda_{\rm QCD}$.
- The long-distance penguin contribution is indeed power-suppressed.
- The subleading distribution amplitude $\Phi_{\rm G}$ defined by the HQET matrix element of the non-local operator with quark-gluon fields localized on distinct light-cone directions:

$$\langle 0 | (\bar{q}_s S_n)(\tau_1 n)\, (S_n^\dagger S_{\bar{n}})(0)\, (S_{\bar{n}}^\dagger\, g_s\, G_{\mu\nu}\, S_{\bar{n}})(\tau_2 \bar{n})\, \bar{n}^\nu \not{n} \gamma_\perp^\mu \gamma_5\, (S_{\bar{n}}^\dagger h_v)(0) | \bar{B}_v \rangle$$
$$= 2 \tilde{f}_B(\mu)\, m_B \int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2 \exp\left[-i(\omega_1 \tau_1 + \omega_2 \tau_2)\right] \Phi_{\rm G}(\omega_1, \omega_2, \mu)\,.$$

$\Longrightarrow$ Non-trivial RG evolution of this soft function in analogy to the QED-generalized bottom-meson distribution amplitude [Beneke, Böer, Toelstede, Vos, 2022].

RG evolution of the B-meson soft function

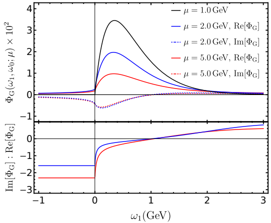
- Sample Feynman diagrams at one loop [Huang, Ji, Shen, Wang, YMW, 2023]:



- Novel feature of the UV renormalization due to the diagram (e):
  - Generate the (unexpected?) mixing from $\omega_i' > 0$ to $\omega_i < 0$, independent of the initial condition of the soft function Φ_G.
  - The Wilson-line structure differs from the one in the QED-generalized B-meson LCDA, on account of the absence of the infinite long Wilson lines.

- An exact solution to the RG equation can be derived with the Laplace transform technique.

Numerics for the RG evolution:



The constant phase of the RG-evolved soft function with the default initial condition.

$B_q \to \gamma\gamma$ versus $B_q \to V\ell\ell$

	$B_q \to V\ell\ell$	$B_q \to \gamma\gamma$
Factorization properties	Incomplete @ LP (FFs, WA)	Complete @ LP
Perturbative corrections	Incomplete @ NLO [penguin amplitude, WA]	Complete @ NLL
Nonperturbative inputs	LCDAs of B_q- and vector-mesons @ LP $B_q \to V$ form factors	B_q-meson LCDA @ LP
NLP improvement	Less studied (LCSR)	better studied (QCDF, LCSR)
NNLO improvement	Rather difficult (?)	In active progress (Misiak et al)
QCD uncertainties	Strong dynamical cancellation @ LP [photon-pole enhancement]	No strong cancellation @ LP

Current status of $B \to \gamma \ell \bar{\nu}_\ell$ (for Belle II)

- Factorization properties at leading power [Korchemsky, Pirjol, Yan, 2000; Descotes-Genon, Sachrajda, 2002; Lunghi, Pirjol, Wyler, 2003; Bosch, Hill, Lange, Neubert, 2003].
- Leading power contributions at NLL and (partial)-subleading power corrections at tree level [Beneke, Rohrwild, 2011].
- Subleading power corrections from the dispersion technique:
  - Soft two-particle correction at tree level [Braun, Khodjamirian, 2013].
  - Soft two-particle correction at one loop [YMW, 2016].
  - Three-particle B-meson DA's contribution at tree level [Beneke, Braun, Ji, Wei, 2018].
  - Subleading effective current and twist-5 and 6 corrections (in the factorization limit) at tree level [Beneke, Braun, Ji, Wei, 2018].
- Subleading power corrections from the direct QCD approach:
  - Hadronic photon corrections at tree level up to the twist-4 accuracy [Khodjamirian, Stoll, Wyler, 1995; Ali, Braun, 1995; Eilam, Halperin, Mendel, 1995].
  - Hadronic photon corrections of twist-two at one loop and of higher-twist at tree level [Ball, Kou, 2003; Wang, Shen, 2018].
- Leading power contributions at NNLL and the updated NLP corrections:
  - Two-loop RG evolution of $\phi_B^+(\omega,\mu)$ derived in [Braun, Ji, Manashov, 2019].
  - Two-loop jet function obtained in [Liu, Neubert, 2020].
  - Updated (approximate) NNLL analysis at LP [Galda, Neubert, Wang, 2022].
  - Further improvement in progress [Cui, Shen, Wang, YMW, Wei, 2024].

Theoretical wishlist

- Systematic understanding of the (higher-twist) B-meson distribution amplitudes.
  - Renormalization properties beyond the one-loop approximation [conformal symmetry].
  - Perturbative constraints at large ω_i [OPE technique].
  - Rapidity/Ultraviolet subtraction to get rid of the radiative tail (short-distance effect).
  - Nonperturbative determinations from the lattice QCD simulation [Wang, YMW, Xu, Zhao, 2020; Zhao, Radyushkin, 2021].

- QCD factorization for the subleading power corrections.
  - SCET analysis for the pion-photon form factor as the first step [operator structures, symmetry constraints, etc].
  - General treatment of the rapidity divergences in the (naïve)-factorization formulae.
  - Rigorous factorization proof taking into account the Glauber gluons.
  - Novel resummation techniques for collecting enhanced logarithms.

- Future phenomenological applications in preparation.
  - Subleading power corrections to the radiative leptonic $B \to \gamma \ell \bar{\nu}_\ell$ decays.
  - Nonfactorizable quark-loop effects for $B \to V \ell \ell$ and $B \to V \gamma$.
  - QED factorization for the exclusive bottom-meson decays.

- Very promising future for QCD aspects of heavy-quark physics!