Analytical calculation of multi-loop vacuum Feynman integrals

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Zhang, Feng, GKZ hypergeometric systems of the three-loop vacuum Feynman integrals, arXiv: 2303.02795 [JHEP05(2023)075].

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Zhang, Feng, GKZ hypergeometric systems of the four-loop vacuum Feynman integrals, arXiv: 2403.13025.

V. Summary

I. Introduction

1. Background

- Higher-order corrections are more important, with the increasing precision at the future colliders: CLIC, ILC, CEPC, FCC, HL-LHC, STCF, SKEKB, $\cdots$
- Vacuum integrals are the important subsets of Feynman integrals, which constitute a main building block in asymptotic expansions of Feynman integrals. The calculation of multi-loop vacuum integrals is a good breakthrough window in the calculation of multi-loop Feynman integrals.
- Considering Feynman integrals as the generalized hypergeometric functions, one finds that the $D-$module of a Feynman diagram is isomorphic to Gel'fand-Kapranov-Zelevinsky (GKZ) $D-$module.

I. Introduction

2. Relevant research

- Hypergeometric functions of some Feynman integrals are obtained from Mellin-Barnes representations.

  Feng, Chang, Chen, Gu, Zhang, *NPB* 927(2018)516 [arXiv:1706.08201]
  Feng, Chang, Chen, Zhang, *NPB* 940(2019)130 [arXiv:1809.00295]
  Gu, Zhang, *CPC* 43(2019)083102 [arXiv:1811.10429]
  Gu, Zhang, Feng, *IJMPA* 35(2020)2050089.

- Using GKZ hypergeometric system, we can obtain the fundamental solution systems of Feynman integrals.

  Feng, Chang, Chen, Zhang, *NPB* 953(2020)114952, [arXiv:1912.01726]
  Feng, Zhang, Chang, *PRD* 106(2022)116025 [arXiv: 2206.04224]
  Feng, Zhang, Dong, Zhou, *EPJC* 83(2023)314 [arXiv:2209.15194].
  Zhang, Feng, *JHEP* 05(2023)075 [arXiv: 2303.02795].
  Zhang, Feng, [arXiv: 2403.13025].

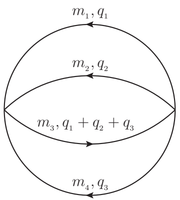
I. Introduction

3. Generally strategy

- We can derive GKZ hypergeometric systems of Feynman integrals, basing on Mellin-Barnes representations and Miller's transformation. We can formulate Feynman integrals as hypergeometric functions through GKZ hypergeometric systems.
- Steps: (1) we write out the GKZ hypergeometric systems satisfied by the Feynman integrals. (2) fundamental solution systems are constructed in neighborhoods of regular singularities of the GKZ hypergeometric systems. The combination coefficients can be determined from Feynman integrals with some special kinematic parameters.

II. 3-loop vacuum integrals

1. 3-loop vacuum with 4 propagates



- Feynman integral of the 3-loop vacuum diagram with 4 propagates is written as

$$U_4 = \left(\Lambda^2_{\rm RE}\right)^{6-\frac{3D}{2}} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D q_3}{(2\pi)^D} \times \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)((q_1 + q_2 + q_3)^2 - m_3^2)(q_3^2 - m_4^2)} . \quad (2.1)$$

II. 3-loop vacuum integrals

- Through Mellin-Barnes transformation

$$U_4 = \frac{\left(\Lambda^2_{\rm RE}\right)^{6-\frac{3D}{2}}}{(2\pi i)^3}\int_{-i\infty}^{+i\infty} ds_1 ds_2 ds_3 \left[\prod_{i=1}^{3}(-m_i^2)^{s_i}\Gamma(-s_i)\Gamma(1+s_i)\right] I_q\ , \quad (2.2)$$

where

$$I_q \equiv \int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\frac{d^Dq_3}{(2\pi)^D}\frac{1}{(q_1^2)^{1+s_1}(q_2^2)^{1+s_2}((q_1+q_2+q_3)^2)^{1+s_3}(q_3^2-m_4^2)}\ . \quad (2.3)$$

II. 3-loop vacuum integrals

- Using Feynman parametrization and Beta function,

$$B(m,n) = \int_0^1 dx\, x^{m-1}(1-x)^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \,, \qquad (2.4)$$

one can have

$$I_q = \frac{-i}{(4\pi)^{\frac{3D}{2}}}(-)^{\sum_{i=1}^{3} s_i}\left(\frac{1}{m_4^2}\right)^{4-\frac{3D}{2}+\sum_{i=1}^{3} s_i}\left[\prod_{i=1}^{3}\Gamma(\frac{D}{2}-1-s_i)\Gamma(1+s_i)^{-1}\right]$$

$$\times\Gamma(3-D+\sum_{i=1}^{3} s_i)\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{3} s_i) \,. \qquad (2.5)$$

II. 3-loop vacuum integrals

- Mellin-Barnes representation of the Feynman integral:

$$U_4 = \frac{-im_4^4}{(2\pi i)^3(4\pi)^6}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_4^2}\right)^{6-\frac{3D}{2}}\int_{-i\infty}^{+i\infty} ds_1 ds_2 ds_3 \Big[\prod_{i=1}^{3}\left(\frac{m_i^2}{m_4^2}\right)^{s_i}\Gamma(-s_i)\Big]$$
$$\times\Big[\prod_{i=1}^{3}\Gamma(\frac{D}{2}-1-s_i)\Big]\Gamma(3-D+\sum_{i=1}^{3}s_i)\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{3}s_i)\ . \quad (2.6)$$

- It is well known that negative integers and zero are simple poles of the function $\Gamma(z)$. As all s_i contours are closed to the right in corresponding complex planes, one finds that the analytic expression of the the three-loop vacuum integral can be written as the linear combination of generalized hypergeometric functions.

II. 3-loop vacuum integrals

- Taking the residue of the pole of $\Gamma(-s_i),\ (i=1,\ 2,\ 3)$, we can derive one linear independent term:

$$U_4 \ni \frac{-im_4^4}{(4\pi)^6}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_4^2}\right)^{6-\frac{3D}{2}}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}(-)^{\sum_{i=1}^{3}n_i}x_1^{n_1}x_2^{n_2}x_3^{n_3}$$
$$\times\left[\prod_{i=1}^{3}\Gamma(\frac{D}{2}-1-n_i)(n_i!)^{-1}\right]\Gamma(3-D+\sum_{i=1}^{3}n_i)$$
$$\times\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{3}n_i)\ , \qquad (2.7)$$

with $x_i=\frac{m_i^2}{m_4^2},\ (i=1,\ 2,\ 3)$.

II. 3-loop vacuum integrals

$$U_4 \ni \frac{im_4^4}{(4\pi)^6}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_4^2}\right)^{6-\frac{3D}{2}} \frac{\pi^3}{\sin^3\frac{\pi D}{2}} T_4(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x})\ , \tag{2.8}$$

with

$$T_4(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x}) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} A_{n_1 n_2 n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}\ , \tag{2.9}$$

$$A_{n_1 n_2 n_3} = \frac{\Gamma(a_1 + \sum\limits_{i=1}^{3} n_i)\Gamma(a_2 + \sum\limits_{i=1}^{3} n_i)}{n_1!n_2!n_3!\Gamma(b_1+n_1)\Gamma(b_2+n_2)\Gamma(b_3+n_3)}\ . \tag{2.10}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2, b_3)$ with

$$a_1 = 3 - D,\ a_2 = 4 - \frac{3D}{2},\ b_1 = b_2 = b_3 = 2 - \frac{D}{2}\ . \tag{2.11}$$

II. 3-loop vacuum integrals

- We can define auxiliary function

$$\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}_3}T_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x})\ . \tag{2.12}$$

Through Miller's transformation,

$$\begin{aligned} \vartheta_{u_j}\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) &= a_j\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ , \\ \vartheta_{v_k}\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) &= (b_k-1)\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ , \end{aligned} \tag{2.13}$$

which naturally induces the notion of GKZ hypergeometric system. Euler operators: $\vartheta_{x_k} = x_k\partial_{x_k}$.

II. 3-loop vacuum integrals

- Through the transformation

$$z_j = \frac{1}{u_j}, \quad z_{2+k} = v_k, \; z_{5+k} = \frac{x_k}{u_1 u_2 v_k}, \tag{2.14}$$

we have GKZ hypergeometric system for the integral

$$\mathbf{A_4} \cdot \vec{\vartheta}_4 \Phi_4 = \mathbf{B_4} \Phi_4 \; , \tag{2.15}$$

$$\mathbf{A_4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}_{5\times 8} ,$$

$$\vec{\vartheta}_4^{\,T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_8}) \, ,$$

$$\mathbf{B_4}^T = (-a_1, \; -a_2, \; b_1 - 1, \; b_2 - 1, \; b_3 - 1) \, . \tag{2.16}$$

II. 3-loop vacuum integrals

- Defining the combined variables

$$y_1 = \frac{z_3 z_6}{z_1 z_2}\;, \quad y_2 = \frac{z_4 z_7}{z_1 z_2}\;, \quad y_3 = \frac{z_5 z_8}{z_1 z_2}\;, \qquad (2.17)$$

we write the solutions as

$$\Phi_4(\mathbf{z}) = \Big(\prod_{i=1}^{8} z_i^{\alpha_i}\Big)\varphi_4(y_1,\, y_2,\, y_3)\;. \qquad (2.18)$$

Here $\vec{\alpha}^{\,T} = (\alpha_1,\, \alpha_2,\, \cdots\,,\, \alpha_8)$ denotes a sequence of complex number such that

$$\mathbf{A_4} \cdot \vec{\alpha} = \mathbf{B_4}\;, \qquad (2.19)$$

namely,

$$\alpha_1 + \alpha_6 + \alpha_7 + \alpha_8 = -a_1\;, \quad \alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 = -a_2\;,$$
$$\alpha_3 - \alpha_6 = b_1 - 1\;, \quad \alpha_4 - \alpha_7 = b_2 - 1\;, \quad \alpha_5 - \alpha_8 = b_3 - 1. \qquad (2.20)$$

II. 3-loop vacuum integrals

- Correspondingly the dual matrix $\tilde{\mathbf{A}}_4$ of $\mathbf{A}_4$ is

$$\tilde{\mathbf{A}}_4 = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} . \qquad (2.21)$$

  The row vectors of the matrix $\tilde{\mathbf{A}}_4$ induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.
- We denote the submatrix composed of the first, third, and fourth column vectors of the dual matrix of Eq. (2.21) as $\tilde{\mathbf{A}}_{134}$, i.e.

$$\tilde{\mathbf{A}}_{134} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} . \qquad (2.22)$$

II. 3-loop vacuum integrals

- Obviously $\det\tilde{\mathbf{A}}_{134} = -1 \neq 0$, and

$$\mathbf{B}_{134} = \tilde{\mathbf{A}}_{134}^{-1} \cdot \tilde{\mathbf{A}}_{4} = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}. \quad (2.23)$$

Taking 3 row vectors of the matrix $\mathbf{B}_{134}$ as the basis of integer lattice, one constructs the GKZ hypergeometric series solutions in parameter space through choosing the sets of column indices $I_i \subset [1,8]$ $(i = 1, \cdots, 8)$ which are consistent with the basis of integer lattice $\mathbf{B}_{134}$.

II. 3-loop vacuum integrals

- We take the set of column indices $I_1 = [2,5,6,7,8]$, i.e. the implement $J_1 = [1,8] \setminus I_1 = [1,3,4]$. The choice on the set of indices implies the exponent numbers $\alpha_1 = \alpha_3 = \alpha_4 = 0$. Through Eq. (2.20), one can have

$$\begin{aligned} &\alpha_2 = a_1 - a_2,\ \alpha_5 = b_1 + b_2 + b_3 - a_1 - 3, \\ &\alpha_6 = 1 - b_1,\ \alpha_7 = 1 - b_2,\ \alpha_8 = b_1 + b_2 - a_1 - 2\,. \end{aligned} \quad (2.24)$$

Combined with Eq. (2.11), we can have

$$\alpha_2 = \frac{D}{2} - 1,\ \alpha_5 = -\frac{D}{2},\ \alpha_6 = \frac{D}{2} - 1,\ \alpha_7 = \frac{D}{2} - 1,\ \alpha_8 = -1\,. \quad (2.25)$$

II. 3-loop vacuum integrals

- According the basis of integer lattice $\mathbf{B}_{134}$, the corresponding hypergeometric series solution with triple independent variables is written as

$$\Phi^{(1)}_{[134]}(\alpha,z)=\prod_{i=1}^{8}z_i^{\alpha_i}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}c^{(1)}_{[134]}(\alpha,\mathbf{n})\Big(\frac{z_1z_2}{z_5z_8}\Big)^{n_1}\Big(\frac{z_3z_6}{z_5z_8}\Big)^{n_2}\Big(\frac{z_4z_7}{z_5z_8}\Big)^{n_3}$$
$$=\prod_{i=1}^{8}z_i^{\alpha_i}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}c^{(1)}_{[134]}(\alpha,\mathbf{n})\Big(\frac{1}{y_3}\Big)^{n_1}\Big(\frac{y_1}{y_3}\Big)^{n_2}\Big(\frac{y_2}{y_3}\Big)^{n_3}\;, \quad (2.26)$$

with the coefficient is

$$c^{(1)}_{[134]}(\alpha,\mathbf{n})=\Big\{n_1!n_2!n_3!\Gamma(1+\alpha_2+n_1)\Gamma(1+\alpha_5-n_1-n_2-n_3)$$
$$\times\Gamma(1+\alpha_6+n_2)\Gamma(1+\alpha_7+n_3)\Gamma(1+\alpha_8-n_1-n_2-n_3)\Big\}^{-1}\;. \quad (2.27)$$

II. 3-loop vacuum integrals

- And then, through Eq. (2.25), the corresponding hypergeometric series solution can be written as

$$\Phi^{(1)}_{[134]}(\alpha,z)=y_1^{\frac{D}{2}-1}y_2^{\frac{D}{2}-1}y_3^{-1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}c^{(1)}_{[134]}(\alpha,\mathbf{n})\Big(\frac{1}{y_3}\Big)^{n_1}\Big(\frac{y_1}{y_3}\Big)^{n_2}\Big(\frac{y_2}{y_3}\Big)^{n_3}, \tag{2.28}$$

with the coefficient is

$$c^{(1)}_{[134]}(\alpha,\mathbf{n})=\frac{\Gamma(\frac{D}{2}+n_1+n_2+n_3)\Gamma(1+n_1+n_2+n_3)}{n_1!n_2!n_3!\Gamma(\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)}. \tag{2.29}$$

Here, the convergent region is

$$\Xi_{[134]}=\{(y_1,\ y_2,\ y_3)\Big|1<|y_3|,\ |y_1|<|y_3|,\ |y_2|<|y_3|\}, \tag{2.30}$$

which shows that $\Phi^{(1)}_{[134]}(\alpha,z)$ is in neighborhood of regular singularity ∞.

II. 3-loop vacuum integrals

- In a similar way, we can obtain other seven hypergeometric solutions which are consistent with the basis of integer lattice $\mathbf{B}_{134}$, and the convergent region is also $\Xi_{[134]}$.
- The above eight hypergeometric series solutions $\Phi^{(i)}_{[134]}(\alpha, z)$ whose convergent region is $\Xi_{[134]}$ can constitute a fundamental solution system.
- Multiplying one of the row vectors of the matrix $\mathbf{B}_{134}$ by -1, the induced integer matrix can also be chosen as a basis of the integer lattice space of certain hypergeometric series.

II. 3-loop vacuum integrals

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\begin{aligned}\mathbf{B}_{\tilde{1}34} &= \mathrm{diag}(-1,1,1)\cdot\mathbf{B}_{134}\\ &= \begin{pmatrix} -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1\end{pmatrix}\,,\end{aligned} \quad (2.31)$$

one obtains eight hypergeometric series solutions $\Phi^{(i)}_{[\tilde{1}34]}(\alpha,z)$ $(i=1,\cdots,8)$ similarly. The convergent region is

$$\Xi_{[\tilde{1}34]} = \{(y_1,\ y_2,\ y_3)\Big|\,|y_1|<1,\ |y_2|<1,\ |y_3|<1\}\,, \quad (2.32)$$

which shows that $\Phi^{(i)}_{[\tilde{1}34]}(\alpha,z)$ are in neighborhood of regular singularity 0 and can constitute a fundamental solution system.

II. 3-loop vacuum integrals

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\mathbf{B}_{1\tilde{3}4} = \mathrm{diag}(1,-1,1)\cdot\mathbf{B}_{134}$$
$$= \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}, \quad (2.33)$$

one obtains eight hypergeometric series solutions $\Phi^{(i)}_{[1\tilde{3}4]}(\alpha,z)$ $(i=1,\cdots,8)$ similarly. The convergent region is

$$\Xi_{[1\tilde{3}4]} = \{(y_1,\ y_2,\ y_3)\Big|1<|y_1|,\ |y_2|<|y_1|,\ |y_3|<|y_1|\}, \quad (2.34)$$

which shows that $\Phi^{(i)}_{[1\tilde{3}4]}(\alpha,z)$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

II. 3-loop vacuum integrals

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\mathbf{B}_{13\tilde{4}} = \mathrm{diag}(1, 1, -1) \cdot \mathbf{B}_{134} = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}, \quad (2.35)$$

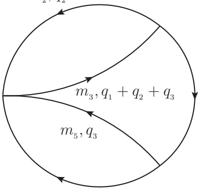
one obtains eight hypergeometric series solutions $\Phi^{(i)}_{[13\tilde{4}]}(\alpha, z)$ $(i = 1, \cdots, 8)$ similarly. The convergent region is

$$\Xi_{[1\tilde{3}4]} = \{(y_1,\ y_2,\ y_3)\Big|1 < |y_2|,\ |y_1| < |y_2|,\ |y_3| < |y_2|\}, \quad (2.36)$$

which shows that $\Phi^{(i)}_{[13\tilde{4}]}(\alpha, z)$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

II. 3-loop vacuum integrals

2. 3-loop vacuum with 5 propagates



- Feynman integral of 3-loop vacuum with 5 propagates:

$$U_5 = \left(\Lambda^2_{\rm RE}\right)^{6-\frac{3D}{2}} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D q_3}{(2\pi)^D}$$
$$\times \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)((q_1 + q_2 + q_3)^2 - m_3^2)((q_1 + q_3)^2 - m_4^2)(q_3^2 - m_5^2)}. \quad (2.37)$$

II. 3-loop vacuum integrals

- One linear independent term of the integral:

$$U_5 \ni \frac{i m_5^2}{(4\pi)^6} \left(\frac{4\pi \Lambda_{\rm RE}^2}{m_5^2} \right)^{6-\frac{3D}{2}} \frac{\pi^3}{\sin^3 \frac{\pi D}{2}} T_5\left(\mathbf{a},\ \mathbf{b} \,\middle|\, \mathbf{x}\right) , \qquad (2.38)$$

$$T_5\left(\mathbf{a},\ \mathbf{b} \,\middle|\, \mathbf{x}\right) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} A_{n_1 n_2 n_3 n_4} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} , \quad (2.39)$$

$$A_{n_1 n_2 n_3 n_4} = \frac{\Gamma(a_1 + \sum_{i=1}^{4} n_i)\Gamma(a_2 + \sum_{i=1}^{4} n_i)\Gamma(a_3 + n_2 + n_3)\Gamma(a_4 + n_2 + n_3)\Gamma(a_5 + n_4)}{n_1! n_2! n_3! n_4! \Gamma(b_1 + n_i)\Gamma(b_2 + n_2)\Gamma(b_3 + n_3)\Gamma(b_4 + \sum_{i=2}^{4} n_i)\Gamma(b_5 + \sum_{i=2}^{4} n_i)}$$

$$a_1 = 4 - D,\ a_2 = 5 - 3D/2,\ a_3 = 2 - D/2,\ a_4 = 3 - D,\ a_5 = 1,$$
$$b_1 = b_2 = b_3 = 2 - D/2,\ b_4 = 3 - D/2,\ b_5 = 4 - D . \qquad (2.40)$$

II. 3-loop vacuum integrals

- Defining the auxiliary function

$$\Phi_5(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}}T_5(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x})\ , \tag{2.41}$$

one can obtain

$$\begin{aligned} \vartheta_{u_j}\Phi_5(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) &= a_j\Phi_5(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ , \\ \vartheta_{v_j}\Phi_5(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) &= (b_j-1)\Phi_5(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ , \end{aligned} \tag{2.42}$$

GKZ hypergeometric system for the 3-loop vacuum integral

$$\mathbf{A_5}\cdot\vec{\vartheta}_5\Phi_5 = \mathbf{B_5}\Phi_5\ , \tag{2.43}$$

II. 3-loop vacuum integrals

$$\mathbf{A_5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \end{pmatrix}_{10\times 14}$$

$$\vec{\vartheta}_5^{\;T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_{14}}) ,$$

$$\mathbf{B_5}^{\;T} = (-a_1, -a_2, -a_3, -a_4, -a_5, b_1 - 1, b_2 - 1, b_3 - 1, b_4 - 1, b_5 - 1) .$$

II. 3-loop vacuum integrals

- Correspondingly the dual matrix $\tilde{\mathbf{A}}_5$ of $\mathbf{A}_5$ is

$$\tilde{\mathbf{A}}_5 = \left(\begin{array}{cccccccccccccc} -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

  The row vectors of the dual matrix $\tilde{\mathbf{A}}_5$ induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.
- Through GKZ hypergeometric system, total 536 hypergeometric functions are obtained in neighborhoods of origin and infinity.
- The fundamental solution systems are composed by 30 linear independent hypergeometric functions.

II. 3-loop vacuum integrals

- **3-loop vacuum with 4 propagates**
  CPU i9-13th, 64GB: FeynGKZ $\sim$ 1 s, FIESTA $\sim$ 50 s

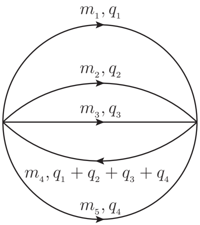
```
SumLim = 15;
ParameterSub = {Dε → 4 - 2 × 0.001, ε → 0.001, a1 → 1,
   a2 → 1, a3 → 1, a4 → 1, m4 → 0.01, m1 → 0.02, m2 → 10 , m3 → 0.04};
NumericalSum[SeriesSolution, ParameterSub, SumLim];
Numerical result = -7.5628 × 10^8
Time Taken 1.16336 seconds
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = -7.56285 × 10^8
Time Taken 53.2344 seconds
```

- **3-loop vacuum with 5 propagates**
  CPU i9-13th, 64GB: FeynGKZ $\sim$ 2 min, FIESTA $\sim$ 10 min

```
SumLim = 15;
ParameterSub = {Dε → 4 - 2 × 0.001, ε → 0.001, a1 → 1, a2 → 1, a3 → 7 / 8,
   a4 → 3 / 4, a5 → 1, m1 → 0.1, m2 → 5, m3 → 0.3, m4 → 0.3, m5 → 100};
NumericalSum[SeriesSolution, ParameterSub, SumLim, RunInParallel → True];
In[46]:=
Numerical result = 1.42136 × 10^7
In[46]:=
Time Taken 120.792 seconds
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = 1.42136 × 10^7
Time Taken 623.497 seconds
```

III. 4-loop vacuum integrals

1. 4-loop vacuum with 5 propagates



- Feynman integral of 4-loop vacuum with 5 propagates:

$$U_5 = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \times \frac{1}{[(q_1 + q_2 + q_3 + q_4)^2 - m_4^2](q_4^2 - m_5^2)}\,, \tag{3.1}$$

III. 4-loop vacuum integrals

- One linear independent term of the integral:

$$U_5 \ni \frac{-m_5^6}{(4\pi)^8}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_5^2}\right)^{8-2D}\frac{\pi^4}{\sin^4\frac{\pi D}{2}}T_5\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right)\ , \tag{3.2}$$

$$T_5\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty} A_{n_1 n_2 n_3 n_4}\, x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}\ , \tag{3.3}$$

$$A_{n_1 n_2 n_3 n_4} = \frac{\Gamma(a_1 + \sum_{i=1}^{4} n_i)\Gamma(a_2 + \sum_{i=1}^{4} n_i)}{\prod_{i=1}^{4} n_i!\Gamma(b_i + n_i)}\ , \tag{3.4}$$

$$a_1 = 4 - \frac{3D}{2},\ \ a_2 = 5 - 2D,\ \ b_1 = b_2 = b_3 = b_4 = 2 - \frac{D}{2}\ . \tag{3.5}$$

III. 4-loop vacuum integrals

GKZ hypergeometric system for the 4-loop vacuum integral

$$\mathbf{A_5} \cdot \vec{\vartheta}_5 \Phi_5 = \mathbf{B_5} \Phi_5 \ , \tag{3.6}$$

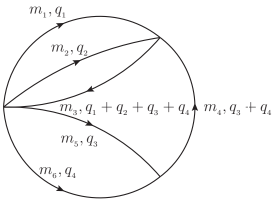
$$\mathbf{A_5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}_{6\times 10} ,$$

$$\vec{\vartheta}_5^{\,T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_{10}}) ,$$

$$\mathbf{B_5}^T = (-a_1, -a_2, b_1 - 1, b_2 - 1, b_3 - 1, b_4 - 1) \ . \tag{3.7}$$

III. 4-loop vacuum integrals

2. 4-loop vacuum with 6 propagates for type A



- Feynman integral of 4-loop vacuum with 6 propagates A:

$$U_{6A} = \left(\Lambda_{\rm RE}^2\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)[(q_1+q_2+q_3+q_4)^2 - m_3^2]} \times \frac{1}{[(q_3+q_4)^2 - m_4^2](q_3^2 - m_5^2)(q_4^2 - m_6^2)}. \quad (3.8)$$

III. 4-loop vacuum integrals

- One linear independent term of the integral:

$$U_{6A} \ni \frac{m_6^4}{(4\pi)^8}\left(\frac{4\pi\Lambda_{RE}^2}{m_6^2}\right)^{8-2D}\frac{\pi^4}{\sin^4\frac{\pi D}{2}}T_{6A}\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right), \quad (3.9)$$

$$T_{6A}\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right) = \sum_{\mathbf{n}=\mathbf{0}}^{\infty} A_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}, \quad (3.10)$$

$$A_{\mathbf{n}} = \frac{\Gamma(a_1+\sum_{i=1}^{5}n_i)\Gamma(a_2+\sum_{i=1}^{5}n_i)\Gamma(a_3+\sum_{i=1}^{3}n_i)\Gamma(a_4+\sum_{i=1}^{3}n_i)\Gamma(a_5+n_4)}{\left[\prod_{i=1}^{5}n_i!\right]\left[\prod_{i=1}^{3}\Gamma(b_i+n_i)\right]\Gamma(b_4+n_5)\Gamma(b_5+\sum_{i=1}^{4}n_i)\Gamma(b_6+\sum_{i=1}^{4}n_i)},$$

$$a_1 = 5-\frac{3D}{2},\ a_2 = 6-2D,\ a_3 = 3-D,\ a_4 = 4-\frac{3D}{2},\ a_5 = 1,$$

$$b_1 = b_2 = b_3 = b_4 = 2-\frac{D}{2},\ b_5 = 4-D,\ b_6 = 5-\frac{3D}{2}. \quad (3.11)$$

III. 4-loop vacuum integrals

GKZ hypergeometric system for the 4-loop vacuum integral A

$$\mathbf{A_{6A}} \cdot \vec{\vartheta}_{6A} \Phi_{6A} = \mathbf{B_{6A}} \Phi_{6A} \;, \tag{3.12}$$

$$\mathbf{A_{6A}} = \begin{pmatrix} \mathbf{I_{11\times 11}} & \mathbf{A_{X6A}} \end{pmatrix}_{11\times 16} \;,$$

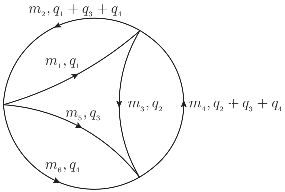
$$\mathbf{A_{X6A}}^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} ,$$

$$\vec{\vartheta}_{6A}^{\,T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_{16}}) \;,$$

$$\mathbf{B_{6A}}^T = (-a_1, \cdots, -a_5, b_1 - 1, \cdots, b_6 - 1) \;. \tag{3.13}$$

III. 4-loop vacuum integrals

3. 4-loop vacuum with 6 propagates for type B



- Feynman integral of 4-loop vacuum with 6 propagates B:

$$U_{6B} = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)[(q_1 + q_3 + q_4)^2 - m_2^2]} \times \frac{1}{(q_2^2 - m_3^2)[(q_2 + q_3 + q_4)^2 - m_4^2](q_3^2 - m_5^2)(q_4^2 - m_6^2)} . \quad (3.14)$$

III. 4-loop vacuum integrals

- One linear independent term of the integral:

$$U_{6B} \ni \frac{m_6^4}{(4\pi)^8}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_6^2}\right)^{8-2D}\frac{\pi^4\sin^2\pi D}{\sin^5\frac{\pi D}{2}\sin\frac{3\pi D}{2}}T_{6B}\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right), \quad (3.15)$$

$$T_{6B}\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right) = \sum_{\mathbf{n}=0}^{\infty} A_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}, \quad (3.16)$$

$$A_{\mathbf{n}} = \frac{\Gamma(a_1+\sum_{i=1}^{5}n_i)\Gamma(a_2+\sum_{i=1}^{5}n_i)\Gamma(a_3+\sum_{i=1}^{2}n_i)\Gamma(a_4+\sum_{i=1}^{2}n_i)\Gamma(a_5+\sum_{i=3}^{4}n_i)}{\left[\prod_{i=1}^{5}n_i!\Gamma(b_i+n_i)\right]\Gamma(b_6+\sum_{i=1}^{4}n_i)\Gamma(b_7+\sum_{i=1}^{4}n_i)\left[\Gamma(a_6+\sum_{i=3}^{4}n_i)\right]^{-1}}$$

$$a_1 = 5-\frac{3D}{2},\ a_2 = 6-2D,\ a_3 = a_5 = 2-\frac{D}{2},\ a_4 = a_6 = 3-D\,,$$

$$b_1 = b_2 = b_3 = b_4 = b_5 = 2-\frac{D}{2},\ b_6 = 4-D,\ b_7 = 5-\frac{3D}{2}. \quad (3.17)$$

III. 4-loop vacuum integrals

GKZ hypergeometric system for the 4-loop vacuum integral B

$$\mathbf{A_{6B}} \cdot \vec{\vartheta}_{6B} \Phi_{6B} = \mathbf{B_{6B}} \Phi_{6B} \;, \tag{3.18}$$

$$\mathbf{A_{6B}} = \left(\begin{array}{cc} \mathbf{I_{13\times 13}} & \mathbf{A_{X6B}} \end{array} \right)_{13\times 18} \;,$$

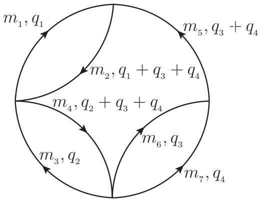
$$\mathbf{A_{X6B}}^T = \left(\begin{array}{ccccccccccccc} 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array} \right),$$

$$\vec{\vartheta}_{6B}^{\,T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_{18}}) \;,$$

$$\mathbf{B_{6B}}^T = (-a_1, \cdots, -a_6, b_1 - 1, \cdots, b_7 - 1) \;. \tag{3.19}$$

III. 4-loop vacuum integrals

4. 4-loop vacuum with 7 propagates for type A



- Feynman integral of 4-loop vacuum with 7 propagates A:

$$U_{7A} = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2-m_1^2)[(q_1+q_3+q_4)^2-m_2^2](q_2^2-m_3^2)} \times \frac{1}{[(q_2+q_3+q_4)^2-m_4^2][(q_3+q_4)^2-m_5^2](q_3^2-m_6^2)(q_4^2-m_7^2)}. \quad (3.20)$$

III. 4-loop vacuum integrals

- One linear independent term of the integral:

$$U_{7A} \ni \frac{-m_7^2}{(4\pi)^8}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_7^2}\right)^{8-2D} \frac{\pi^4 \sin^2 \pi D}{\sin^5 \frac{\pi D}{2} \sin \frac{3\pi D}{2}} T_{7A}\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right), \quad (3.21)$$

$$T_{7A}\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right) = \sum_{\mathbf{n}=\mathbf{0}}^{\infty} A_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}, \quad (3.22)$$

$$A_{\mathbf{n}} = \Gamma(a_1 + \sum_{i=1}^{6} n_i)\Gamma(a_2 + \sum_{i=1}^{6} n_i)\Gamma(a_3 + \sum_{i=1}^{2} n_i)$$
$$\times \frac{\Gamma(a_4 + \sum_{i=1}^{2} n_i)\Gamma(a_5 + \sum_{i=3}^{4} n_i)\Gamma(a_6 + \sum_{i=3}^{4} n_i)\Gamma(a_7 + n_5)}{\left[\prod_{i=1}^{6} n_i!\right]\left[\prod_{i=1}^{4}\Gamma(b_i + n_i)\right]\Gamma(b_5 + n_6)\Gamma(b_6 + \sum_{i=1}^{5} n_i)\Gamma(b_7 + \sum_{i=1}^{5} n_i)}. \quad (3.23)$$

III. 4-loop vacuum integrals

GKZ hypergeometric system for the 4-loop vacuum integral

$$\mathbf{A_{7A}} \cdot \vec{\vartheta}_{7A} \Phi_{7A} = \mathbf{B_{7A}} \Phi_{7A} \,, \tag{3.24}$$

$$\mathbf{A_{7A}} = \begin{pmatrix} \mathbf{I_{14\times 14}} & \mathbf{A_{X7A}} \end{pmatrix}_{14\times 20} \,,$$

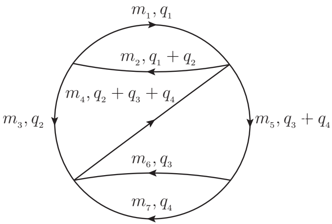
$$\mathbf{A_{X7A}}^T = \left(\begin{array}{cccccccccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right),$$

$$\vec{\vartheta}_{7A}^{\,T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_{20}}) \,,$$

$$\mathbf{B_{7A}}^T = (-a_1, \cdots, -a_7, b_1 - 1, \cdots, b_7 - 1) \,. \tag{3.25}$$

III. 4-loop vacuum integrals

5. 4-loop vacuum with 7 propagates for type B



- Feynman integral of 4-loop vacuum with 7 propagates B:

$$U_{7B} = \left(\Lambda_{\rm RE}^2\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2-m_1^2)[(q_1+q_2)^2-m_2^2](q_2^2-m_3^2)} \times \frac{1}{[(q_2+q_3+q_4)^2-m_4^2][(q_3+q_4)^2-m_5^2](q_3^2-m_6^2)(q_4^2-m_7^2)}. \quad (3.26)$$

III. 4-loop vacuum integrals

- One linear independent term of the integral:

$$U_{7B} \ni \frac{m_7^2}{(4\pi)^8}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_7^2}\right)^{8-2D} \frac{\pi^4}{\sin^4\frac{\pi D}{2}} T_{7B}\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right) , \tag{3.27}$$

$$T_{7B}\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right) = \sum_{\mathbf{n}=\mathbf{0}}^{\infty} A_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}, \tag{3.28}$$

$$A_{\mathbf{n}} = \frac{\Gamma(a_1+\sum_{i=1}^{6} n_i)\Gamma(a_2+\sum_{i=1}^{6} n_i)\Gamma(a_3+\sum_{i=1}^{4} n_i)\Gamma(a_4+\sum_{i=1}^{4} n_i)}{\left[\prod_{i=1}^{6} n_i!\right]\Gamma(b_1+n_1)\Gamma(b_2+n_2)\Gamma(b_3+n_4)\Gamma(b_4+n_6)}$$

$$\times \frac{\Gamma(a_5+\sum_{i=1}^{2} n_i)\Gamma(a_6+\sum_{i=1}^{2} n_i)\Gamma(a_7+n_3)\Gamma(a_8+n_5)}{\Gamma(b_5+\sum_{i=1}^{3} n_i)\Gamma(b_6+\sum_{i=1}^{3} n_i)\Gamma(b_7+\sum_{i=1}^{5} n_i)\Gamma(b_8+\sum_{i=1}^{5} n_i)} . \tag{3.29}$$

III. 4-loop vacuum integrals

GKZ hypergeometric system for the 4-loop vacuum integral

$$\mathbf{A_{7B}} \cdot \vec{\vartheta}_{7B} \Phi_{7B} = \mathbf{B_{7B}} \Phi_{7B} \ , \qquad (3.30)$$

$$\mathbf{A_{7B}} = \begin{pmatrix} \mathbf{I_{16\times16}} & \mathbf{A_{X7B}} \end{pmatrix}_{16\times22} \ ,$$

$$\mathbf{A_{X7B}}^T =$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$\vec{\vartheta}_{7B}^{\,T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_{22}}) \ ,$$

$$\mathbf{B_{7B}}^T = (-a_1, \cdots, -a_8, b_1 - 1, \cdots, b_8 - 1) \ . \qquad (3.31)$$

III. 4-loop vacuum integrals

4-loop vacuum with 6 propagates

- Type A: i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 1500 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.01, m2 → 0.1 , m6 → 10};
NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

Numerical result = 1.2624×10^{11}

Time Taken 0.059701 seconds

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.01, m2 → 0.1 , m6 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
```

FIESTA Value = 1.26241×10^{11}

Time Taken 1525.73 seconds

- Type B: i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 500 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.1, m2 → 0.2 , m6 → 10};
NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

Numerical result = -4.07562×10^{11}

Time Taken 0.092314 seconds

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.1, m2 → 0.2 , m6 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
```

FIESTA Value = -4.07552×10^{11}

Time Taken 570.975 seconds

III. 4-loop vacuum integrals

4-loop vacuum with 7 propagates

- Type A: i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 500 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1 , m7 → 10 };
NumericalSum[SeriesSolution, ParameterSub, SumLim];
Numerical result = -8.23731×10^12
Time Taken 0.082945 seconds
```

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1 , m7 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = -8.23727×10^12
Time Taken 587.565 seconds
```

- Type B: i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 6000 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 9 / 10,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1, m7 → 10 };
NumericalSum[SeriesSolution, ParameterSub, SumLim];
Numerical result = 1.43756×10^8
Time Taken 0.094408 seconds
```

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 9 / 10,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1, m7 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = 1.43754×10^8
Time Taken 6370.08 seconds
```

IV. Summary

- Using Mellin-Barnes representation and Miller's transformation, we derive GKZ hypergeometric systems of 3-loop and 4-loop vacuum Feynman integrals.
- In the neighborhoods of origin 0 including infinity ∞, we can obtain analytical hypergeometric series solutions through GKZ hypergeometric systems.
- One can see that the computing time using the GKZ hypergeometric series solutions is less than that using numerical program FIESTA.
- In order to derive the fundamental solution system in neighborhoods of all possible regular singularities, next we will embed the vacuum integrals in Grassmannian manifold.

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河北大学物理科学与技术学院 张海斌

THANKS!