Generating Function of One-loop Integrals

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Contents

1. Motivation

As the bridge between the theoretical frame and the experiment data, the computation of scattering amplitudes is extremely important.

With current accuracy of experimental data, the one-loop correction is necessary. For some cases, even higher loops are needed.

However, directly evaluating each Feynmna diagram is not smart, because they share some common features. A well used and nicer method is the reduction method.

- The key idea is that any Feynman integral can be written as

$$I = \sum_i c_i I_i$$

- Thus the evaluation of integrals has been divided into two parts: the evaluation of basis and the computation of reduction coefficients c_i

Thus the computation of loop corrections can be roughly divided into three parts:

- The construction of integrand
- The computation of master integrals
- The computation of reduction coefficients

The improvement in each part will lead the improvement of the loop computations.

My focus in the talk will be the third part, i.e., the reduction.

The reduction method can be classified from different points of view. One classification is:

- Reduction at the integrand level:
  It has been solved, in principle, by the computational algebraic geometry
  [Ossola, Papadopoulos, Pittau, 2006] [Mastrolia, Ossola, Papadopoulos, Pittau, 2011] [Badger, Frellesvig, Zhang, 2012] [Zhang, 2012]
- Reduction at the integrand level: There are various proposals.

Some widely mentioned methods of integral level reduction include

- Passarino-Veltman(PV)-reduction [Passarino and Veltman, 1979]
- Integration-by-Part (IBP) method [Chetyrkin and Tkachov, 1981; Tkachov, 1981; Laporta, 2001]
- Unitarity cut method [Bern, Dixon, Dunbar and Kosower, hep-ph/9403226, hep-ph/9409265] [Britto, Cachazo and Feng, hep-th/0412103]
- Intersection theory [Mastrolia and Mizera, 1810.03818] [Frellesvig, Gasparotto, Mandal, Mastrolia, Mattiazzi and Mizera, 1907.02000]

No matter which method, a notable common feature in the reduction procedure is the appearance of an iterative structure. For example, the tadpole

$$I_1(a) = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - M^2)^a}. \quad (1)$$

it is easy to find the IBP relation

$$I_1(a) = \frac{D - 2a + 2}{2(a-1)M^2} I_1(a-1), \quad (2)$$

From it we can solve the reduction coefficient as

$$C_1(a) = \frac{(-1)^{a-1}(1 - \frac{D}{2})_{a-1}}{(a-1)!(M^2)^{a-1}}. \quad (3)$$

When encountering an iterative structure, a very useful approach is to consider the corresponding **generating function**. In many examples, it is much easier to find the generating function rather than to find expansion coefficients of each order. For example, For example, the Hermite Polynomial $H_n(x)$ of eigenvectors of harmonic oscillator can be read out from the generation function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \tag{4}$$

Thus it is natural to conisder the **generating function for the reduction**!!!

As far as we know, some early work are:

- Ablinger etc introduced the generating function for an operator insertion on an l-leg vertex , which facilitates the computation of operator matrix elements of higher loop contributions. [Ablinger, Blumlein, Raab, Schneider and Wissbrock, 1403.1137]
- Kosower introduced generating functions for a specific tensor structure of some two-loop integrals, which provide explicit reductions of that tensor structure with arbitrary power. [Kosower, 1804.00131]

Our aim is to solve the reduction of general loop integrals, i.e., with arbitrary polynomial of loop momenta in the numerator and arbitrary powers of propagators in the denominator:

- To deal with numerator, a key gredient is the introduction of **auxiliary vector** R, i.e.,

$$\int d\ell \frac{\ell_{\mu_1}\cdots\ell_{\mu_m}}{\prod D_i} \Longrightarrow \frac{\partial}{\partial R_{\mu_1}}\cdots\frac{\partial}{\partial R_{\mu_m}} \int d\ell \frac{(\ell\cdot R)^m}{\prod D_i} \tag{5}$$

  Thus we have our first generating function

$$\boxed{I_{n+1}^R \equiv \int \frac{d^D\ell}{i\pi^{D/2}} \frac{e^{2\ell\cdot R}}{\prod_{i=0}^n((\ell-K_i)^2-M_i^2)} = \boldsymbol{J}_{n+1}\cdot\vec{\alpha}^{gen}(R)}, \tag{6}$$

The basis is the scalar integrals

$$J_{n+1;\widehat{S}} = \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{\prod_{i=0,i\notin S}^{n}((\ell - K_i)^2 - M_i^2)}, \tag{7}$$

where $\widehat{S}$ is the list of removed propagators, then the components of row vector $\boldsymbol{J}_{n+1}$ will be ordered as

$$\begin{aligned} \boldsymbol{J} \;\; = \;\; & \Big\{ J_{n+1}; J_{n+1;\widehat{0}}, J_{n+1;\widehat{1}}, ..., J_{n+1;\widehat{n}}; J_{n+1;\widehat{01}}, J_{n+1;\widehat{02}}, \\ & ..., J_{n+1;\widehat{(n-1)n}}; ...; J_{n+1;\widehat{01...(n-1)}}, .. \Big\} \; . \end{aligned} \tag{8}$$

To deal with denominator with arbitrary powers

$$I^R_{n+1;\{\boldsymbol{a}\}} \equiv \int \frac{d^D\ell}{i\pi^{D/2}} \frac{e^{2\ell\cdot R}}{\prod_{i=0}^n ((\ell - K_i)^2 - M_i^2)^{a_i}} = \boldsymbol{J}_{n+1} \cdot \vec{\alpha}^{gen}_{\boldsymbol{a}} \qquad (9)$$

there are two approaches:

- The first one is to use

$$I^R_{n+1;\{\boldsymbol{a}\}} = \left(\prod_{j=0}^{n} \frac{1}{(a_j - 1)!} \frac{\partial^{a_j - 1}}{\partial (M_j^2)^{a_j - 1}} \right) I^R_{n+1} \,. \qquad (10)$$

  Using the result

$$\begin{aligned} \frac{\partial}{\partial(M_j^2)} I^R_{n+1} &= \left(\frac{\partial}{\partial(M_j^2)} \boldsymbol{J}_{n+1} \right) \cdot \vec{\alpha}^{gen} + \boldsymbol{J}_{n+1} \cdot \left(\frac{\partial}{\partial(M_j^2)} \vec{\alpha}^{gen} \right) \\ &\equiv \boldsymbol{J}_{n+1} \cdot \mathcal{D}_{n+1;j} \vec{\alpha}^{gen} \,, \qquad (11) \end{aligned}$$

where we have rewritten the differential action over basis as

$$\boxed{\frac{\partial}{\partial(M_j^2)}\boldsymbol{J}_{n+1} \equiv \boldsymbol{J}_{n+1}\mathcal{H}_{n+1;j}} \tag{12}$$

and defined the "covariant derivative" as

$$\mathcal{D}_{n+1;j} \equiv \frac{\partial}{\partial(M_j^2)} + \mathcal{H}_{n+1;j} \ , \tag{13}$$

Thus we can obtain the reduction of general integrals in (9) as

$$I^R_{n+1;\{\boldsymbol{a}\}} = \boldsymbol{J}_{n+1} \cdot \left\{ \left(\prod_{j=0}^{n} \frac{1}{(a_j-1)!}\mathcal{D}_{n+1;j}^{a_j-1} \right) \vec{\alpha}^{gen} \right\} \ . \tag{14}$$

The second approach to solve (9) is to sum over all $\boldsymbol{a}$ as

$$\sum_{a_0,\cdots,a_n=1}^{\infty} t_0^{a_0-1} \cdot t_1^{a_1-1} \cdot t_2^{a_2-1} \cdot \cdots \cdot t_n^{a_n-1} I_{n+1;\{\boldsymbol{a}\}}^{R} \tag{15}$$

to reach

$$\boxed{I_{n+1}^{R}(\boldsymbol{t}) \equiv \int \frac{d^D\ell}{i\pi^{D/2}} \frac{e^{2\ell\cdot R}}{\prod_{i=0}^{n}((\ell-K_i)^2-M_i^2-t_i)}}. \tag{16}$$

which is the most general generating function for one loop reduction we are looking for.

Expression (16) is very similar to (6) except the mass shifting $M_i^2 \to M_i^2 + t_i$. Thus using (6) we have immediately

$$I_{n+1}^R(\boldsymbol{t}) = \boldsymbol{J}_{n+1}(\boldsymbol{t}) \cdot \vec{\alpha}^{gen}(R, \boldsymbol{t}) \tag{17}$$

where the scalar basis $\boldsymbol{J}_{n+1}(\boldsymbol{t})$ is the one in (7) with the mass shifting $M_i^2 \to M_i^2 + t_i$. To complete the reduction (17), we need to find the reduction

$$\boxed{\boldsymbol{J}_{n+1}(\boldsymbol{t}) = \boldsymbol{J}_{n+1}\, \mathcal{G}(\boldsymbol{t})} \tag{18}$$

where the matrix $\mathcal{G}$ depends on $\boldsymbol{t}$ only, but not R. In other words, to compute $\mathcal{G}(\boldsymbol{t})$, we can start from (16) by setting $R = 0$. Assembling all together we finally have

$$I_{n+1}^R(\boldsymbol{t}) = \boldsymbol{J}_{n+1} \cdot \vec{\gamma}^{gg}(R, \boldsymbol{t}), \qquad \vec{\gamma}^{gg}(R, \boldsymbol{t}) \equiv \mathcal{G}(\boldsymbol{t}) \vec{\alpha}^{gen}(R, \boldsymbol{t})\,. \tag{19}$$

Above two approaches are, in fact, related to each other as following. Using (14) one finds

$$\begin{aligned}
&\sum_{a_0,\cdots,a_n=1}^{\infty} t_0^{a_0-1}\cdot t_1^{a_1-1}\cdot\cdots\cdot t_n^{a_n-1} I^R_{n+1;\{\boldsymbol{a}\}} \\
= & \prod_{j=0}^{n}\left(\sum_{a_j=1}^{\infty}\frac{t_j^{a_j-1}}{(a_j-1)!}\left(\frac{\partial}{\partial M_j^2}\right)^{a_j-1}\right) I^R_{n+1} \\
= & \boldsymbol{J}_{n+1}\left(\prod_{j=0}^{n}\left(\sum_{a_j=1}^{\infty}\frac{t_j^{a_j-1}}{(a_j-1)!}\left(\mathcal{D}_{n+1;j}\right)^{a_j-1}\right)\vec{\alpha}^{gen}\right) \\
= & \boldsymbol{J}_{n+1}\cdot\left(e^{\sum_{j=0}^{n} t_j\mathcal{D}_{n+1;j}}\vec{\alpha}^{gen}\right) \qquad (20)
\end{aligned}$$

thus one gets

$$\vec{\gamma}^{gg}(R,\boldsymbol{t}) = e^{\sum_{j=0}^{n} t_j\mathcal{D}_{n+1;j}}\vec{\alpha}^{gen}(R) \qquad (21)$$

2. Differential equation for generating function

The IBP relations are given by

$$0 = \int \frac{d^D\ell}{i\pi^{D/2}} \frac{d}{d\ell^\mu} \left\{ A^\mu \frac{e^{2\ell\cdot R}}{(\ell^2 - M_0^2)\prod_{i=1}^n((\ell - K_i)^2 - M_i^2)} \right\} \qquad (22)$$

with $A^\mu = \ell^\mu, K_i^\mu$. After simplification we get

$$\boxed{\vec{\mathcal{Y}}_r = \sum_{j=0}^{n} f_{rj}\vec{X}_j}, \quad r = 0, 1..., n \qquad (23)$$

where the **modified Caylay matrix** F with elements

$$f_{ij} = (K_i - K_j)^2 - M_i^2 - M_j^2, \quad i, j = 0, ..., n; \quad K_0 = 0 \qquad (24)$$

and

$$\vec{\mathcal{Y}}_r = -\left(D-2-n-2K_r\cdot R+R\cdot\frac{\partial}{\partial R}\right)\vec{\alpha}^{gen} + \sum_{j=0,j\neq r}^{n}\left(\mathcal{D}_{n+1;j}\vec{\alpha}^{gen}_{n+1;\widehat{r}}\right) \tag{25}$$

and

$$\mathcal{D}_{n+1;i}\vec{\alpha}^{gen}\equiv\vec{X}_i \tag{26}$$

Using $A^\mu = R^\mu$ we get

$$\boxed{0 = 2R^2\vec{\alpha}^{gen} + \sum_{j=0}^{n}\left(\left(2R\cdot K_j - R\cdot\frac{\partial}{\partial R}\right)(F^{-1})_{jr}\vec{\mathcal{Y}}_r\right)} \qquad (27)$$

3. Solving $\vec{\alpha}^{gen}$

To simplify notations, let us define

$$B_r = \sum_{j=0, j\neq r}^{n} \left(\mathcal{D}_{n+1;j} \vec{\alpha}^{gen}_{n+1;\widehat{r}} \right) \tag{28}$$

First using (23), we can solve

$$\boxed{\vec{X}_i = (F^{-1})_{ij} \vec{\mathcal{Y}}_j} \tag{29}$$

Putting it back to (27) we get

$$\begin{aligned} & \left\{ 2R^2 - \sum_{j=0}^{n} \left(2R \cdot K_j - R \cdot \frac{\partial}{\partial R} \right) (F^{-1})_{jr} \right. \\ & \left. \left(D - 2 - n - 2K_r \cdot R + R \cdot \frac{\partial}{\partial R} \right) \right\} \vec{\alpha}^{gen} \\ = & - \sum_{j=0}^{n} \left(2R \cdot K_j - R \cdot \frac{\partial}{\partial R} \right) (F^{-1})_{jr} B_r \ . \end{aligned} \tag{30}$$

There are two ways to solve (30). The first one is by the series expansion. Putting

$$\begin{aligned}\vec{\alpha}^{gen} &= \sum_{i_0,...,i_n=0}^{\infty} \vec{\alpha}_{i_0 i_1 ... i_n} (R^2)^{i_0} (2K_1 \cdot R)^{i_1} ... (2K_n \cdot R)^{i_n} \\ B_r &= \sum_{i_0,...,i_n=0}^{\infty} \vec{b}_{r;i_0 i_1 ... i_n} (R^2)^{i_0} (2K_1 \cdot R)^{i_1} ... (2K_n \cdot R)^{i_n} \quad (31)\end{aligned}$$

we get

$$
\begin{aligned}
& \vec{\alpha}_{i_0 i_1 \ldots i_n}(2i_0 + \sum_{t=1}^{n} i_t)(D - 2 - n + (2i_0 + \sum_{t=1}^{n} i_t)) \sum_{j,r=0}^{n} (F^{-1})_{jr} \\
= & -\left[\sum_{j=1}^{n} \sum_{r=0}^{n} (F^{-1})_{jr} \vec{b}_{r;i_0 i_1 \ldots (i_j - 1) \ldots i_n}\right. \\
& \left. -(2i_0 + \sum_{t=1}^{n} i_t) \sum_{j,r=0}^{n} (F^{-1})_{jr} \vec{b}_{r;i_0 i_1 \ldots i_n}\right] \\
& + \sum_{j=1}^{n} \sum_{r=0}^{n} (F^{-1})_{jr} \vec{\alpha}_{i_0 i_1 \ldots (i_j - 1) \ldots i_n}(D - 3 - n + 4i_0 + 2\sum_{t=1}^{n} i_t) \\
& - \sum_{j,r=1}^{n} (F^{-1})_{jr} \vec{\alpha}_{i_0 i_1 \ldots (i_j - 1) \ldots (i_r - 1) \ldots i_n} - 2\vec{\alpha}_{(i_0 - 1) i_1 \ldots i_n} \qquad (32)
\end{aligned}
$$

The second one is to define

$$R = t\widetilde{R} \ . \tag{33}$$

then (30) becomes the second order ordinary differential equation

$$At\frac{d^2}{dt^2}W + (B_0 + B_1 t)\frac{d}{dt}W + (C_0 + C_1 t)W + \mathcal{B}(t) = 0 \ . \tag{34}$$

$$\begin{aligned} A &= \boldsymbol{I}\cdot(F^{-1})\cdot\boldsymbol{I}^T, \qquad B_0 = (D-(n+1))A \\ B_1 &= -2\boldsymbol{I}\cdot(F^{-1})\cdot\mathcal{P}^T, \;\; C_0 = \frac{(D-(n+1))}{2}B_1 \\ C_1 &= 2\widetilde{R}^2 + \mathcal{P}\cdot(F^{-1})\cdot\mathcal{P}^T \ . \end{aligned} \tag{35}$$

with

$$\boldsymbol{I} = (1,1,...,1), \quad \mathcal{P} = (2K_0\cdot\widetilde{R}, 2K_1\cdot\widetilde{R}, ..., 2K_n\cdot\widetilde{R}) \ . \tag{36}$$

The solution is given by the **generalized hypergeometric function** with definition

$$ {}_AF_B(a_1,...,a_A;b_1,...,b_B;x)=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_A)_n}{(b_1)_n...(b_B)_n}\frac{x^n}{n!} \tag{37}$$

where the **Pochhammer symbol** is defined by (From the definition one can see that $(x)_{n=0}=1,\forall x$)

$$ (x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}=\prod_{i=1}^{n}(x+(i-1))\ . \tag{38}$$

For our case, the solution is

$$\vec{\widetilde{\alpha}}^{gen}(t,\widetilde{R}) = e^{\frac{-B_1+\sqrt{B_1^2-4C_1}}{2}t}\left({}_1F_1(a_1;b_1;x)\right)\left\{\vec{c}_1+\int_0^x ds\frac{s^{-b_1}e^s}{\left({}_1F_1(a_1;b_1;s)\right)^2}\left(\int_0^s dy g(y) y^{b_1-1}\,{}_1F_1(a_1;b_1;y)e^{-y}\right)\right\} \quad (39)$$

with $\vec{c}_1 = \{1,0,0,...,0\}^T$

$$x = \frac{-t\sqrt{B_1^2-4C_1A}}{A}, \quad b_1 = (D-(n+1)), \quad a_1 = \frac{b_1}{2}$$

$$\vec{g}(x) = \frac{1}{\sqrt{B_1^2-4AC_1}}e^{\frac{-(B_1-\sqrt{B_1^2-4AC_1})x}{2\sqrt{B_1^2-4AC_1}}}\vec{\mathcal{B}}\left(\frac{-xA}{\sqrt{B_1^2-4C_1A}}\right) \quad (40)$$

For the special component

$$
\begin{aligned}
(\vec{\widetilde{\alpha}}^{gen}(t,\widetilde{R}))_{\emptyset} &= e^{\frac{-B_1+\sqrt{B_1^2-4C_1A}}{2A}t} \\
&\quad {}_1F_1\left(\frac{(D-(n+1))}{2};(D-(n+1));\frac{-t\sqrt{B_1^2-4C_1A}}{A}\right) \\
&= e^{\frac{-B_1 t}{2A}}\ {}_0F_1\left(\emptyset;\frac{(D+1-(n+1))}{2};\frac{(B_1^2-4C_1A)t^2}{16A^2}\right) \qquad (41)
\end{aligned}
$$

Example: Massless bubble

The generating function of bubble to bubble is

$$e^{(K\cdot\widetilde{R})t}\ {}_0F_1\left(\emptyset; \frac{(D-1)}{2}; \frac{1}{4}t^2((K\cdot\widetilde{R})^2 - K^2\widetilde{R}^2)\right) \tag{42}$$

One can easily expand (42) and check with known results:

$$\begin{aligned} & 1 + (K\cdot\widetilde{R})t + \frac{(D(K\cdot\widetilde{R})^2 - K^2\widetilde{R}^2)t^2}{2(D-1)} \\ & + \frac{((D+2)(K\cdot\widetilde{R})^3 - 3K^2\widetilde{R}^2(K\cdot\widetilde{R}))t^3}{6(D-1)} \\ & + \frac{((D+2)(D+4)(K\cdot\widetilde{R})^4 - 6(D+2)K^2\widetilde{R}^2(K\cdot\widetilde{R})^2 + 4(K^2\widetilde{R}^2)^2)t^4}{24(D-1)(D+1)} \end{aligned} \tag{43}$$

4. Solving $\mathcal{H}_{n+1;j}$

Using (13), (26), (29) and the boundary condition, i.e., $\vec{\alpha}_{00\ldots0} = \{1,0,0,...,0\}^T$, we arrive

$$\mathcal{H}_{n+1;i} \cdot \{1,0,0,...,0\}^T = \sum_{r=0}^{n} (F_{ir}^{-1}) \left(-(D-2-n)\{1,0,0,...,0\}^T + \vec{b}_{r;00\ldots0}\right) . \quad (44)$$

Using the definition of $\vec{b}$ we get

$$(\mathcal{H}_{n+1;i})_{\widehat{S},\widehat{\emptyset}} = -\left\{(D-2-n)\sum_{r=0}^{n}(F_{ir}^{-1})\right\}\delta_{S,\emptyset} + \sum_{r=0}^{n}(F_{ir}^{-1}) \sum_{j=0,j\neq r}^{n} (\mathcal{H}_{n+1;j})_{\widehat{S},\widehat{r}} , \quad (45)$$

Now based on (45), we can write down the recursive algorithm for all components of the matrix $\mathcal{H}_{n+1;i}$:

- (1) Let us consider the arbitrary element $(\mathcal{H}_{n+1;i})_{\widehat{\mathcal{R}},\widehat{\mathcal{C}}}$ where the removed lists $\mathcal{R},\mathcal{C}$ indicate the corresponding row and column. The first obvious result is when $i\in\mathcal{C}$, $(\mathcal{H}_{n+1;i})_{\widehat{\mathcal{R}},\widehat{\mathcal{C}}}=0,\ \ \forall\mathcal{R}$. Now we will assume that $i\notin\mathcal{C}$. The second obvious result is that when $\mathcal{C}\not\subseteq\mathcal{R}$, the $(\mathcal{H}_{n+1;i})_{\widehat{\mathcal{R}},\widehat{\mathcal{C}}}=0$.
- (2) Now we define some notations for later convenience. First we use $\mathcal{P}=\{0,1,...,n\}$ to represent the complete list of propagators. Given $\mathcal{C}$ we use $n_{\mathcal{C}}$ to denote the number of elements in the list and define the reduced modified Caylay matrix $F_{\mathcal{P}\backslash\mathcal{C}}$ by removing the corresponding rows and columns indicated by the list (see (24)).

- (3) When $\mathcal{R}=\mathcal{C}$, we have

$$(\mathcal{H}_{n+1;i})_{\widehat{\mathcal{C}},\widehat{\mathcal{C}}} = -(D-1-(n+1-n_{\mathcal{C}}))\sum_{r\in\mathcal{P}\backslash\mathcal{C}}(F_{\mathcal{P}\backslash\mathcal{C}})^{-1}_{ir}\,. \quad (46)$$

- (4) When $\mathcal{C}\subset\mathcal{R}$, we have

$$(\mathcal{H}_{n+1;i})_{\widehat{\mathcal{R}},\widehat{\mathcal{C}}} = \sum_{r\in\mathcal{P}\backslash\mathcal{C}}(F_{\mathcal{P}\backslash\mathcal{C}})^{-1}_{ir}\sum_{j\in\mathcal{P}\backslash(\mathcal{C}\bigcup\{r\})}(\mathcal{H}_{n+1;j})_{\widehat{\mathcal{R}},\mathcal{C}\bigcup\{r\}}\,. \quad (47)$$

- (5) Equation (47) shows explicitly the recursive structure. With fixed row list $\mathcal{R}$, the column list at the right hand side is larger than the left hand side by one element. Iteratively using (47) will reach two possible terminations: either $\mathcal{R}=\mathcal{C}$ with the expression (46) and either $\mathcal{C}\not\subseteq\mathcal{R}$ with zero contribution.

5. Solving $\mathcal{G}_{\widehat{S}_1,\emptyset}(\boldsymbol{t})$

For generating function

$$J_n(\boldsymbol{t}) \equiv \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{\prod_{i=1}^{n}((\ell - K_i)^2 - M_i^2 - t_i)} . \quad (48)$$

Using the IBP relations (29) with $R = 0$, we get partial differential equations

$$\frac{\partial}{\partial t_i}\mathcal{G}_{\widehat{S}_1,\emptyset}(\boldsymbol{t}) = \sum_j (F^{-1})_{ij}(\boldsymbol{t}) \left\{ -(D-1-n)\mathcal{G}_{\widehat{S}_1,\emptyset}(\boldsymbol{t}) + \sum_{r\neq j} \frac{\partial}{\partial t_r}\mathcal{G}_{\widehat{S}_1,\widehat{j}}(\boldsymbol{t}) \right\} . \quad (49)$$

with $F_{ij}(\boldsymbol{t}) = (K_i - K_j)^2 - M_i^2 - t_i - M_j^2 - t_j$

For the reduction from n-gon to n-gon, we have

$$\frac{\partial}{\partial t_i}\mathcal{G}_{\emptyset,\emptyset}(\boldsymbol{t}) = -\sum_j (F^{-1})_{ij}(\boldsymbol{t})(D-1-n)\mathcal{G}_{\emptyset,\emptyset}(\boldsymbol{t}) \; . \qquad (50)$$

We can combine (50) to give

$$\begin{aligned}\sum_i \left(\frac{\partial}{\partial t_i}\mathcal{G}_{\emptyset,\emptyset}(\boldsymbol{t})\right) dt_i &= -(D-1-n)\mathcal{G}_{\emptyset,\emptyset}(\boldsymbol{t}) \sum_{i,j}(F^{-1})_{ij}(\boldsymbol{t})dt_i \\ &= -\frac{(D-1-n)}{2}\mathcal{G}_{\emptyset,\emptyset}(\boldsymbol{t})\sum_{i,j}(F^{-1})_{ij}(\boldsymbol{t})(dt_i+dt_j) \\ &= \frac{(D-1-n)}{2}\mathcal{G}_{\emptyset,\emptyset}(\boldsymbol{t})\sum_{i,j}(F^{-1})_{ij}(\boldsymbol{t})dF_{ij}(\boldsymbol{t}) \; , \qquad (51)\end{aligned}$$

Using

$$dg = g\left(g^{-1}\right)^{\mu\nu} dg_{\mu\nu}\ , g = |g_{\mu\nu}| \tag{52}$$

(51) becomes

$$d\mathcal{G}_{\emptyset,\emptyset}(\boldsymbol{t}) = \frac{(D-1-n)}{2}\mathcal{G}_{\emptyset,\emptyset}(\boldsymbol{t})\frac{d|F|}{|F|}\ , \tag{53}$$

which can be solved immediately

$$\mathcal{G}_{\emptyset,\emptyset}(\boldsymbol{t}) = \left(\frac{|F|(\boldsymbol{t})}{|F|(\boldsymbol{t}=0)}\right)^{\frac{(D-1-n)}{2}}\ . \tag{54}$$

For general case we have

$$
\begin{aligned}
d\mathcal{G}_{\widehat{S}_1,\emptyset}(\boldsymbol{t}) &= \frac{(D-1-n)}{2}\mathcal{G}_{\widehat{S}_1,\emptyset}(\boldsymbol{t})d\ln|F| + \sum_i \mathcal{B}_{i;\widehat{S}_1}(\boldsymbol{t})dt_i \\
\mathcal{B}_{i;\widehat{S}_1}(\boldsymbol{t}) &= \sum_j (F^{-1})_{ij}(\boldsymbol{t}) \sum_{r\neq j} \frac{\partial}{\partial t_r}\mathcal{G}_{\widehat{S}_1,\widehat{j}}(\boldsymbol{t})\,. \qquad (55)
\end{aligned}
$$

By the constant variation method we write

$$
\mathcal{G}_{\widehat{S}_1,\emptyset}(\boldsymbol{t}) = \left(\frac{|F|(\boldsymbol{t})}{|F|(\boldsymbol{t}=0)}\right)^{\frac{(D-1-n)}{2}} \widetilde{g}_{\widehat{S}_1,\emptyset}(\boldsymbol{t})\,. \qquad (56)
$$

with

$$\widetilde{g}_{\widehat{S}_1,\emptyset}(\boldsymbol{t}) - \widetilde{g}_{\widehat{S}_1,\emptyset}(\boldsymbol{t}=0) = \int_0^1 d\lambda \left(\frac{|F|(\lambda \boldsymbol{t})}{|F|(\boldsymbol{t}=0)} \right)^{-\frac{(D-1-n)}{2}} \sum_i \mathcal{B}_{i;\widehat{S}_1}(\lambda \boldsymbol{t}) t_i \quad (57)$$

where the boundary condition is that

$$\widetilde{g}_{\widehat{S}_1,\emptyset}(\boldsymbol{t}=0) = \begin{cases} 1, & S_1 = \emptyset \\ 0, & S_1 \neq \emptyset \end{cases} \quad (58)$$

6. Conclusion

Some future works:

- How to use these results to get more information of one-loop integrals?
- How to generalize to higher loops?

Thanks a lot of your attention !