Modular Grand Unification Theories

Gui-Jun Ding

University of Science and Technology of China

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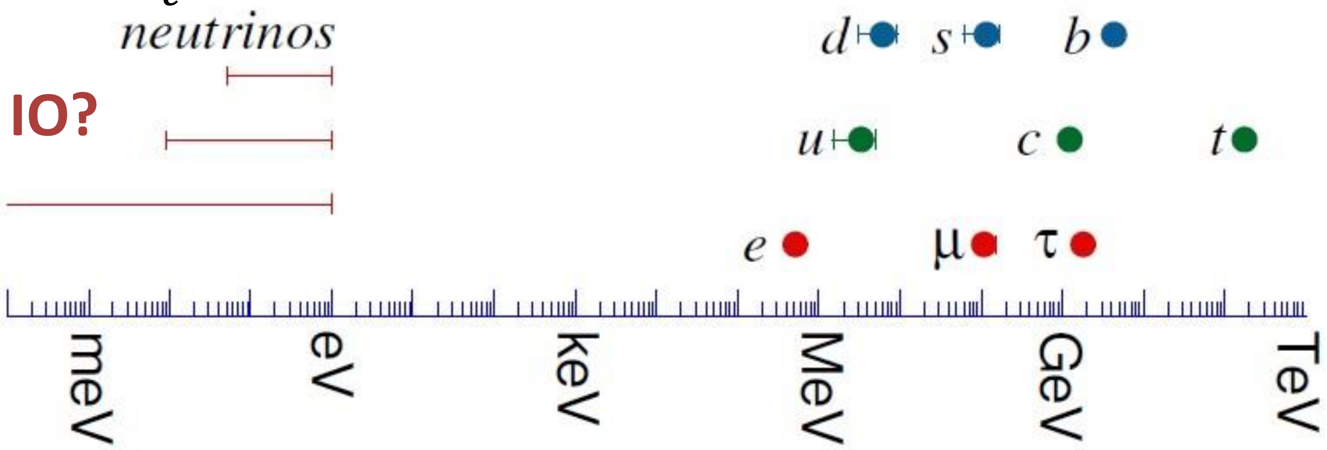
Flavor puzzles in SM

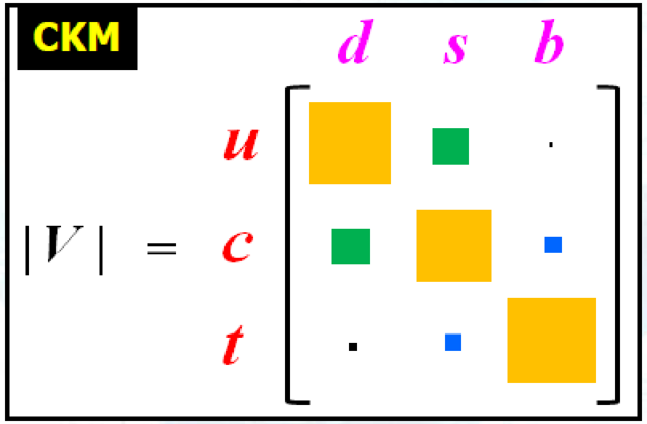
- **Hierarchical masses, e.g.** $\frac{m_t}{m_e} = \mathcal{O}(10^5)$

NO or IO?

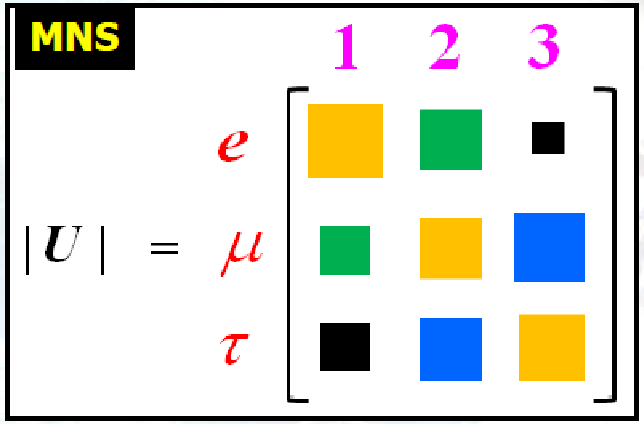


- **Quark mixing vs lepton mixing**

Quark mixings are small



Lepton mixings are large



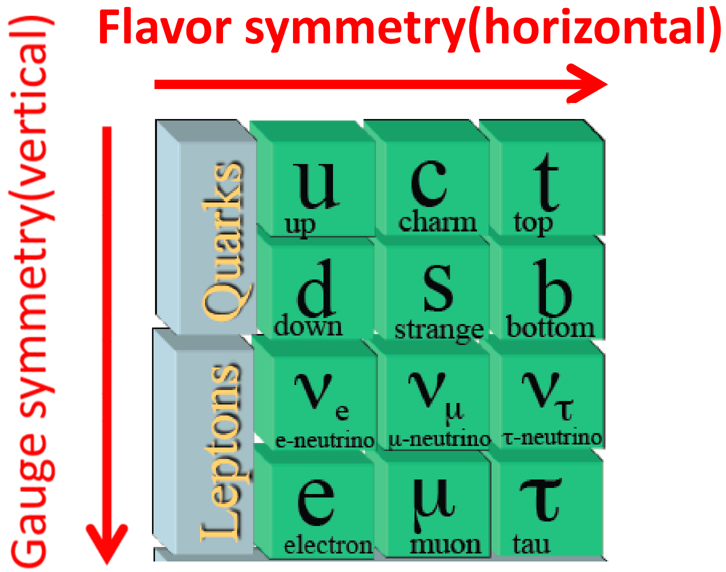
Quark and lepton mixing matrices have **distinctive** structures!

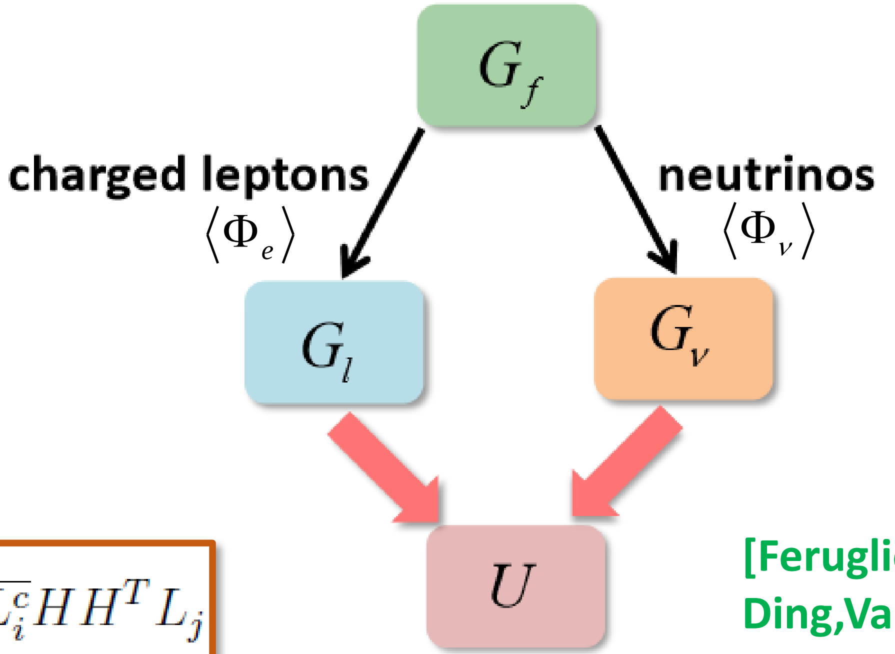
In SM, the fermion masses and flavor mixing are determined by Yukawa coupling constants which are unconstrained by gauge symmetry.

Symmetry as a guiding principle to flavor puzzle

The fundamental principle of fermion masses and flavor mixing structure is unkonwn so far. **Symmetry can help to reduce the number of free parameters in the Yukawa coupling.**

- **GUTs:** connecting quarks and leptons leptons ⟺ quarks
- **Flavor symmetry:** relations among three families e-family ⟺ muon-family ⟺ tau-family





Tri-bimaximal mixing:

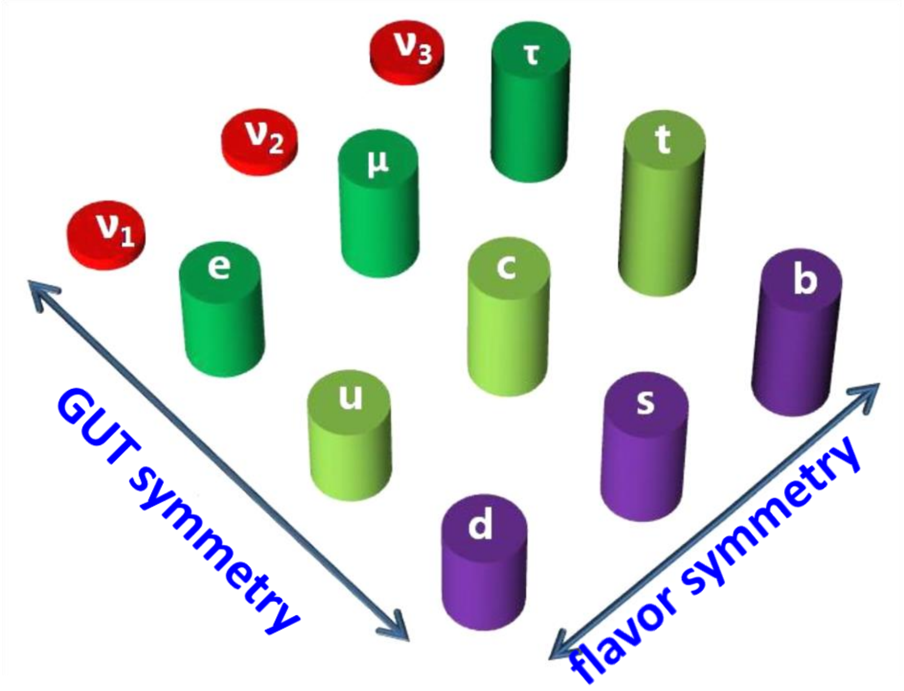
$$\langle \Phi_e \rangle \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \Phi_\nu \rangle \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\boxed{\mathcal{L}_m = -Y_{ij}^e(\langle \Phi_e \rangle)\overline{L_i}He_{Rj} - \frac{1}{2}Y_{ij}^\nu(\langle \Phi_\nu \rangle)\overline{L_i^c}HH^T L_j}$$

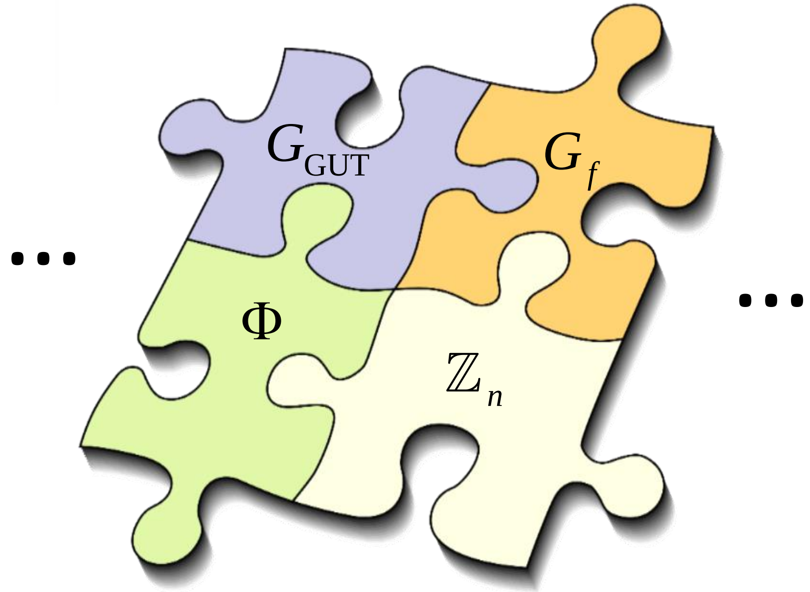
[Feruglio, Romanino, 1912.06028; Ding,Valle, 2402.16963]

- To make the Lagrangian invariant under flavor symmetry, Higgs-like fields "flavons" Φ_e, Φ_ν are needed
- Structure of Yukawa couplings arises from the vacuum alignment of flavons

Combining flavor symmetry with GUT: $G_{GUT} \times G_f$



[talk by Ning Chen for alternative]



- Flavor symmetry unifies three families
- GUT symmetry unifies the gauge coupling constants as well as the SM matter fields in a generation
- Flavons and the vacuum alignment make the models **quite complicated**

An example of Pati-Salam GUT model with traditional flavor symmetry

	name	field	$SU(4)_C \times SU(2)_L \times SU(2)_R$	A_4	Z_5	R
Matter fields	Quarks and leptons	F	$(4,2,1)$	3	1	1
		$F^c_{1,2,3}$	$(\overline{4},1,2)$	1	α,α^3,1	1
GUT symmetry breaking sector	PS Higgs	$\overline{H^c}$, H^c	$(4,1,2)$, $(\overline{4},1,2)$	1	1	0
	Higgs bidoublets	h_3	$(1,2,2)$	3	1	0
		h_u	$(1,2,2)$	$1''$	α	0
		h_d,h^d_{15}	$(1,2,2)$, $(15,2,2)$	$1'$	α^3,α^4	0
		h^u_{15}	$(15,2,2)$	1	α	0
	Dynamical masses	Σ_u	$(1,1,1)$	$1''$	α	0
		Σ_d,Σ^d_{15}	$(1,1,1)$, $(15,1,1)$	$1'$	α^3,α^2	0
flavor symmetry breaking sector	A_4 triplet flavons	$\phi^u_{1,2}$	$(1,1,1)$	3	α^4, α^2	0
		$\phi^d_{1,2}$	$(1,1,1)$	3	α^3, α	0
	Majoron	ξ	$(1,1,1)$	1	α^4	0
	Fermion Messengers	$X_{F''_{1,3}}$	$(4,2,1)$	$1''$	α,α^3	1
		$X_{F'_{1,3}}$	$(4,2,1)$	$1'$	α,α^3	1
		$X_{\overline{F_i}}$	$(\overline{4},2,1)$	1	α^i	1
		X_{ξ_i}	$(1,1,1)$	1	α^i	1

- extra symmetry Z_5 and R-symmetry
- **complicated dynamics of** aligning flavon VEVs

$$\begin{cases} \langle\phi_1^u\rangle \propto (0,1,1)^T, \langle\phi_2^u\rangle \propto (1,4,2)^T, \\ \langle\phi_1^d\rangle \propto (1,0,0)^T, \langle\phi_2^d\rangle \propto (0,1,0)^T, \end{cases}$$

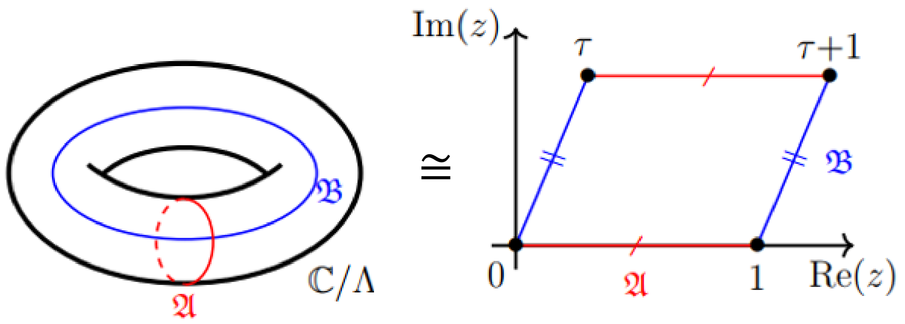
- higher dimensional operators reduce predictive power

Modular symmetry

- Modular action **[Feruglio, 1706.08749, Ding, King, 2311.09282 for review]** **[talks by Hajime Otsuka, Arsenii Titov]**

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

$$SL(2,Z) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$



$SL(2,Z)$ on torus T^2

- The field transformation (**non-linear**)

$$\psi \to \boxed{(c\tau + d)^{-k}}\ \boxed{\rho(\gamma)}\,\psi$$

weight $k \in \mathbb{Z}$ ρ is a unitary representation of Γ_N or Γ'_N

finite modular groups

$$\begin{cases} \Gamma_N \equiv SL(2,Z)/\pm\Gamma(N) \\ \Gamma'_N \equiv SL(2,Z)/\Gamma(N) \end{cases}$$

- Superpotential

$$\mathcal{W} = \sum Y_{I_1 I_2 \ldots I_n}(\tau)\, \psi_{I_1}\psi_{I_2}\ldots\psi_{I_n}$$

Modular invariance requires

$$Y_{I_1 I_2 \ldots I_n}(\tau) \to Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$

$$k_Y = k_{I_1} + k_{I_2} + \ldots + k_{I_n},\quad \rho_Y \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_n} \supset 1$$

Principal congruence subgroup of level N:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ N) \right\}$$

Yukawa couplings are modular forms $Y_{I_1 I_2 \ldots I_n}(\tau)$

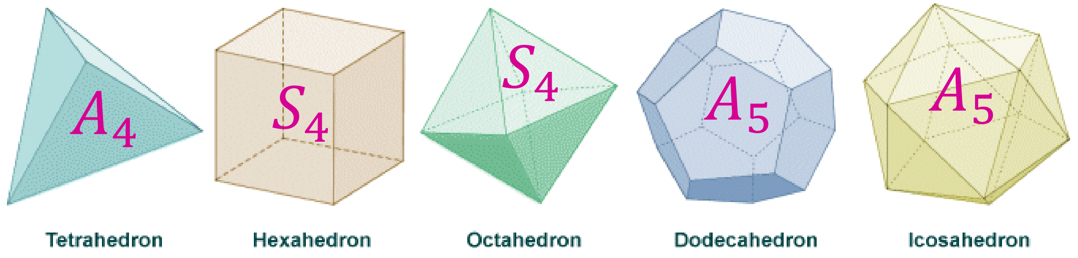
- Γ_N and Γ'_N play the role of flavor symmetry groups

Symmetry groups of modular GUTs

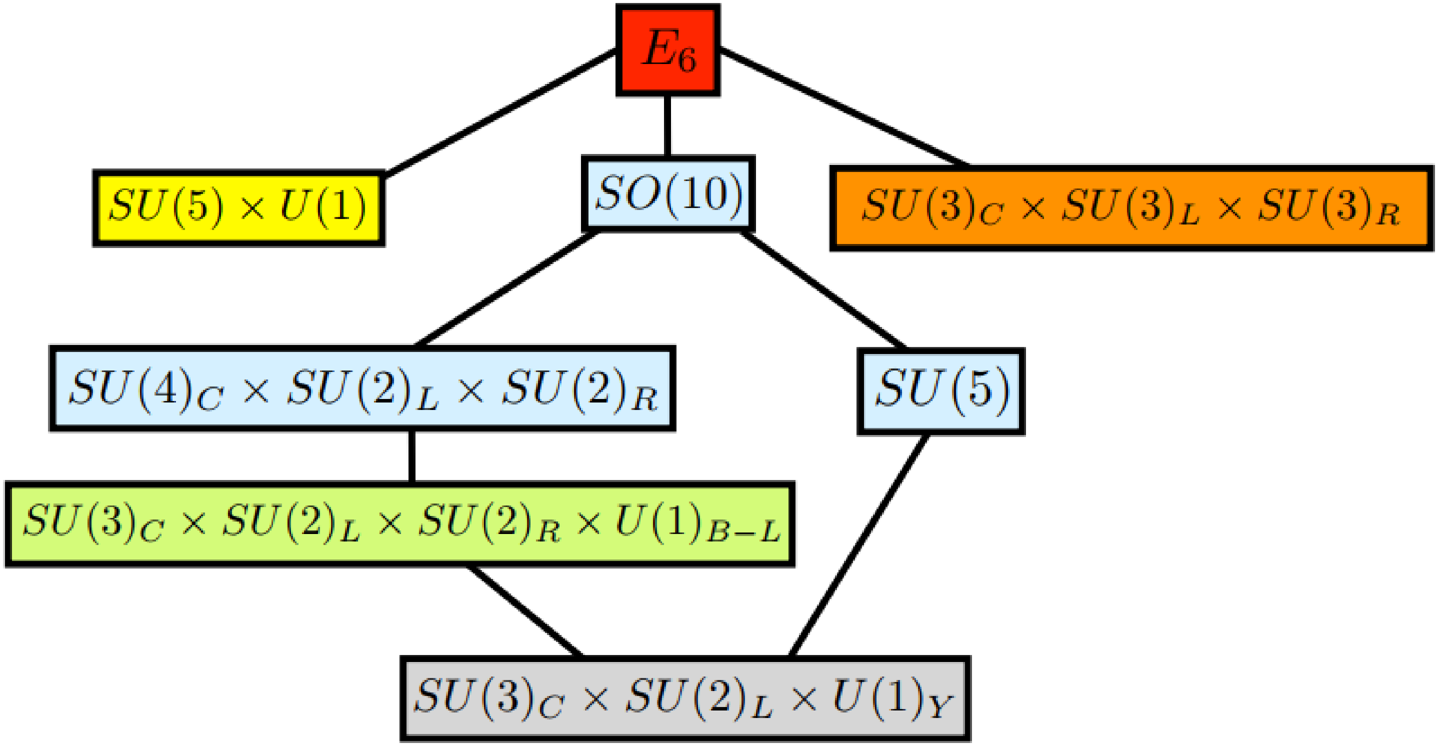
The symmetry group of modular GUT is $G_{\text{GUT}} \times \Gamma_N$ or $G_{\text{GUT}} \times \Gamma'_N$

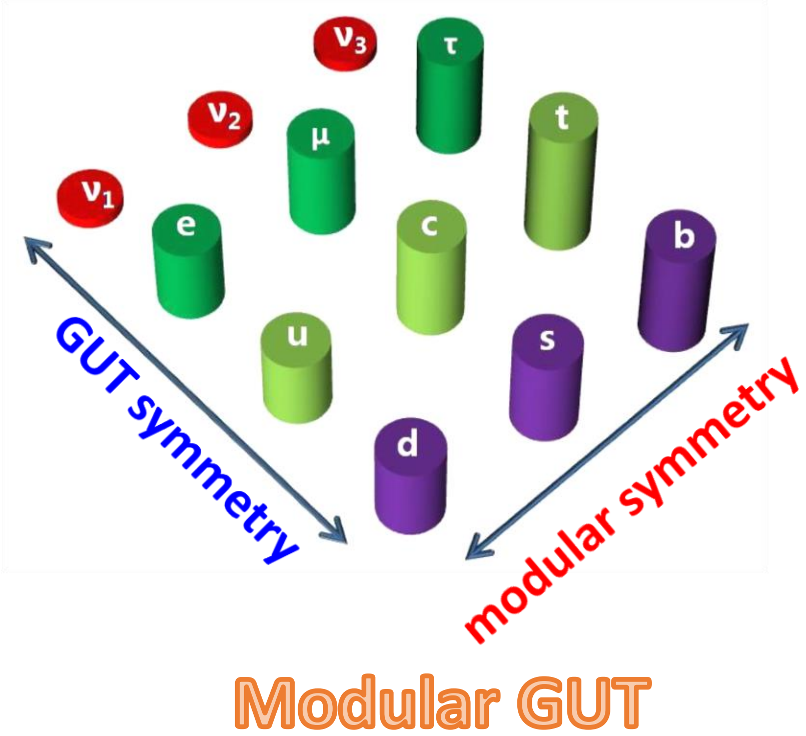
- Finite modular groups Γ_N and Γ'_N

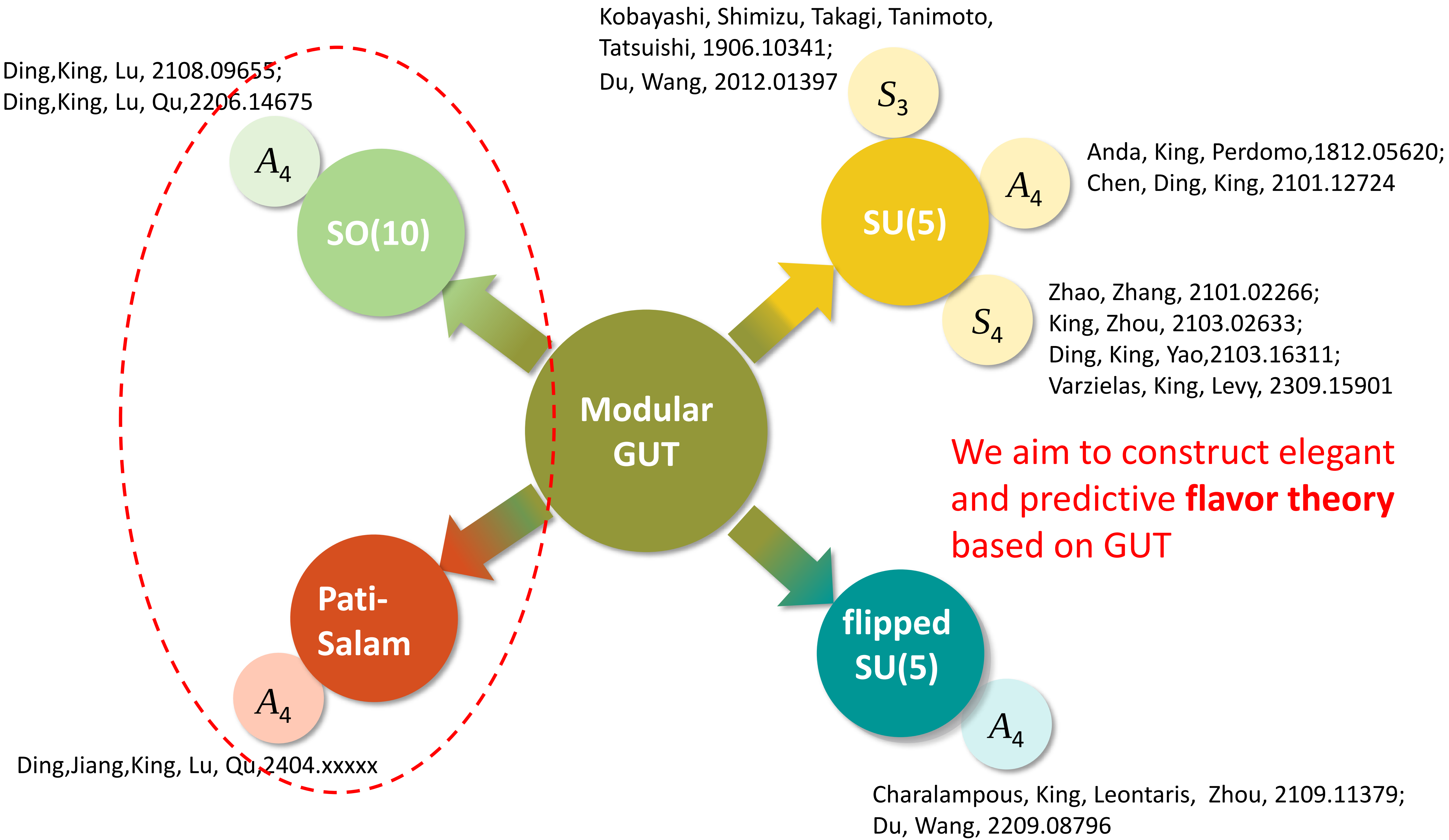
N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 = T'$	$S'_4 \cong SL(2,\mathbb{Z}_4)$	$A'_5 \cong SL(2,\mathbb{Z}_5)$



- Candidates of GUT gauge group G_{GUT}



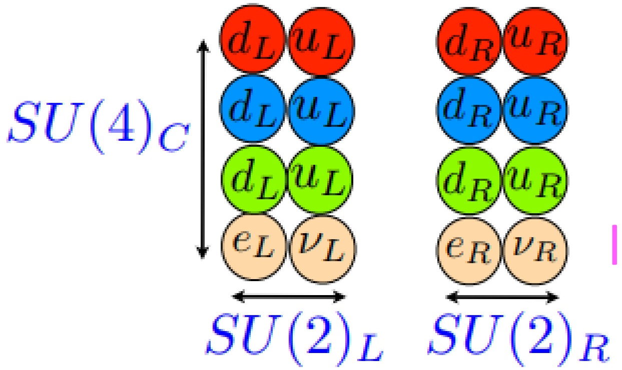




We aim to construct elegant and predictive **flavor theory** based on GUT

Minimal Supersymmetric Pati-Salam theory

- Pati-Salam Gauge group: $SU(4)_C \times SU(2)_L \times SU(2)_R$ [Pati, Salam,Phys. Rev. D 10 (1974)]



lepton is the 4th color

left-right symmetric, ν_R is predicted

symmetry breaking chain

$$SU(4)_C \times SU(2)_L \times SU(2)_R$$

$$\Big\downarrow A = (\mathbf{15}, \mathbf{1}, \mathbf{1})$$

$$SU(3)_C \times U(1)_{B-L} \times SU(2)_L \times SU(2)_R$$

$$\Big\downarrow \Delta_R = (\mathbf{10}, \mathbf{1}, \mathbf{3}) \quad \overline{\Delta}_R = (\overline{\mathbf{10}}, \mathbf{1}, \mathbf{3})$$

$$SU(3)_C \times SU(2)_L \times U(1)_Y$$

$$\Big\downarrow \Phi = (\mathbf{1}, \mathbf{2}, \mathbf{2}) \quad \Sigma = (\mathbf{15}, \mathbf{2}, \mathbf{2})$$

$$SU(3)_C \times U(1)_{EM}$$

- The quarks and leptons are unified in **two** PS representations

$$F_i = (\mathbf{4}, \mathbf{2}, \mathbf{1})_i = \begin{pmatrix} u_r & u_g & u_b & \nu \\ d_r & d_g & d_b & e \end{pmatrix}_i \to (Q_i, L_i)\,,$$

$$F_i^c = (\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})_i = \begin{pmatrix} d_r^c & d_g^c & d_b^c & e^c \\ -u_r^c & -u_g^c & -u_b^c & -\nu^c \end{pmatrix}_i \to (u_i^c, d_i^c, \nu_i^c, e_i^c)$$

- Quark and lepton masses, neutrino masses generated by type I seesaw

$$\mathcal{W}_Y = \mathcal{Y}_{ij}^1 F_i^c F_j \Phi + \mathcal{Y}_{ij}^{15} F_i^c F_j \Sigma + \mathcal{Y}_{ij}^{10_R} F_i^c F_j^c \Delta_R$$

[see talk by Tianjun Li for top-down construction]

$$\begin{cases} M_u = (\mathcal{Y}^1 + r_2 \mathcal{Y}^{15})\, v_u\,, & M_d = r_1 (\mathcal{Y}^1 + \mathcal{Y}^{15})\, v_d\,, \\ M_{\nu_D} = (\mathcal{Y}^1 - 3 r_2 \mathcal{Y}^{15})\, v_u\,, & M_e = r_1 (\mathcal{Y}^1 - 3\mathcal{Y}^{15})\, v_d\,, \quad M_{\nu_R} = \mathcal{Y}^{10_R} v_R \end{cases}$$

$$r_1 (3M_u + M_{\nu_D}) = (3M_d + M_e)\tan\beta, \qquad r_1 (M_u - M_{\nu_D}) = r_2 (M_d - M_e)\tan\beta$$

Minimal A_4 x PS modular models

- Yukawa couplings are level 3 modular forms related by $\mathbf{A_4}$ modular symmetry, modular forms replace "flavons" so that the resulting model is much simpler

$$\mathcal{W}_Y = \sum_{\{\boldsymbol{r}_a, \boldsymbol{r}'_a\}} \alpha_a \left((F^c F \Phi)_{\boldsymbol{r}'_a} Y_{\boldsymbol{r}_a}^{(k_F + k_{F^c} + k_\Phi)}(\tau) \right)_{\mathbf{1}} + \sum_{\{\boldsymbol{r}_b, \boldsymbol{r}'_b\}} \beta_b \left((F^c F \Sigma)_{\boldsymbol{r}'_b} Y_{\boldsymbol{r}_b}^{(k_F + k_{F^c} + k_\Sigma)}(\tau) \right)_{\mathbf{1}}$$
$$+ \sum_{\{\boldsymbol{r}_d, \boldsymbol{r}'_d\}} \gamma_d \left((F^c F^c \Delta_R)_{\boldsymbol{r}'_d} Y_{\boldsymbol{r}_d}^{(2k_{F^c} + k_{\Delta_R})}(\tau) \right)_{\mathbf{1}}$$

$$\boxed{Y_{\mathbf{3}}^{(2)}(\tau) = \begin{pmatrix} 1 + 12q + \ldots \\ -6q^{1/3}(1 + 7q + \ldots) \\ -18q^{2/3}(1 + 2q + \ldots) \end{pmatrix}, \quad q \equiv e^{2\pi i \tau}}$$

- Classifying A_4xPS modular models

	$F \sim \mathbf{3}$	$F_{1,2,3} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}'')$
$F^c \sim \mathbf{3}$	✘	✔
$F^c_{1,2,3} \sim \mathbf{1}(\mathbf{1}', \mathbf{1}'')$	✔	✘

- A benchmark model: $(F^c, F_1, F_2, F_3) \sim (\mathbf{3}, \mathbf{1}', \mathbf{1}, \mathbf{1}'), (k_{F^c}, k_{F_1}, k_{F_2}, k_{F_3}, k_\Sigma) = (0, 2, 6, 8, -2)$

$$\mathcal{W}_\Phi = \left[\alpha_1 (F^c F_1)_{\mathbf{3}} Y_{\mathbf{3}}^{(2)} + \beta_1 (F^c F_2)_{\mathbf{3}} Y_{\mathbf{3}I}^{(6)} + \beta_2 (F^c F_2)_{\mathbf{3}} Y_{\mathbf{3}II}^{(6)} + \gamma_1 (F^c F_3)_{\mathbf{3}} Y_{\mathbf{3}I}^{(8)} + \gamma_2 (F^c F_3)_{\mathbf{3}} Y_{\mathbf{3}II}^{(8)} \right] \Phi \,,$$
$$\mathcal{W}_\Sigma = \left[\hat{\beta}_1 (F^c F_2)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} + \hat{\gamma}_1 (F^c F_3)_{\mathbf{3}} Y_{\mathbf{3}I}^{(6)} + \hat{\gamma}_2 (F^c F_3)_{\mathbf{3}} Y_{\mathbf{3}II}^{(6)} \right] \Sigma \,,$$
$$\mathcal{W}_{\Delta_R} = \alpha_{R1} (F^c F^c)_{\mathbf{1}} \Delta_R \,.$$

- Resulting mass matrices

[Ding,Jiang,King, Lu, Qu,2404.xxxxx]

$$\mathcal{Y}^1 = \begin{pmatrix} \alpha_1 Y^{(2)}_{\mathbf{3},3} & \beta_1 Y^{(6)}_{\mathbf{3}I,1} + \beta_2 Y^{(6)}_{\mathbf{3}II,1} & \gamma_1 Y^{(8)}_{\mathbf{3}I,3} + \gamma_2 Y^{(8)}_{\mathbf{3}II,3} \\ \alpha_1 Y^{(2)}_{\mathbf{3},2} & \beta_1 Y^{(6)}_{\mathbf{3}I,3} + \beta_2 Y^{(6)}_{\mathbf{3}II,3} & \gamma_1 Y^{(8)}_{\mathbf{3}I,2} + \gamma_2 Y^{(8)}_{\mathbf{3}II,2} \\ \alpha_1 Y^{(2)}_{\mathbf{3},1} & \beta_1 Y^{(6)}_{\mathbf{3}I,2} + \beta_2 Y^{(6)}_{\mathbf{3}II,2} & \gamma_1 Y^{(8)}_{\mathbf{3}I,1} + \gamma_2 Y^{(8)}_{\mathbf{3}II,1} \end{pmatrix}$$

$$\mathcal{Y}^{15} = \begin{pmatrix} 0 & \hat\beta_1 Y^{(4)}_{\mathbf{3},1} & \hat\gamma_1 Y^{(6)}_{\mathbf{3}I,3} + \hat\gamma_2 Y^{(6)}_{\mathbf{3}II,3} \\ 0 & \hat\beta_1 Y^{(4)}_{\mathbf{3},3} & \hat\gamma_1 Y^{(6)}_{\mathbf{3}I,2} + \hat\gamma_2 Y^{(6)}_{\mathbf{3}II,2} \\ 0 & \hat\beta_1 Y^{(4)}_{\mathbf{3},2} & \hat\gamma_1 Y^{(6)}_{\mathbf{3}I,1} + \hat\gamma_2 Y^{(6)}_{\mathbf{3}II,1} \end{pmatrix} \qquad \mathcal{Y}^{10_R} = \alpha_{R1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- χ^2 analysis to estimate the goodness of fit

$$\chi^2 = \sum_i \left(\frac{P_i(x) - O_i}{\sigma_i} \right)^2$$

Input parameters: $x \in \{\tau,\ \alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2, \hat\beta_1, \hat\gamma_1, \hat\gamma_2, \alpha_{R1}, r_1, r_2\}$

Observables: $O_i \in \begin{cases} \text{quark sector:}\, m_u, m_c, m_t, m_d, m_s, m_b, \theta^q_{12}, \theta^q_{13}, \theta^q_{23}, \delta^q_{CP} \\ \text{lepton sector:}\, m_e, m_\mu, m_\tau, \Delta m^2_{21}, \Delta m^2_{31}, \theta^\ell_{12}, \theta^\ell_{13}, \theta^\ell_{23}, \delta^\ell_{CP}, \end{cases}$

The χ^2 function is minimized: $\chi^2_{\min} = 7.0272 \rightarrow$ excellent agreement with data

- Unmeasured observables as predictions

  The lightest neutrino mass: $m_1 = 4.986$ meV, normal ordering

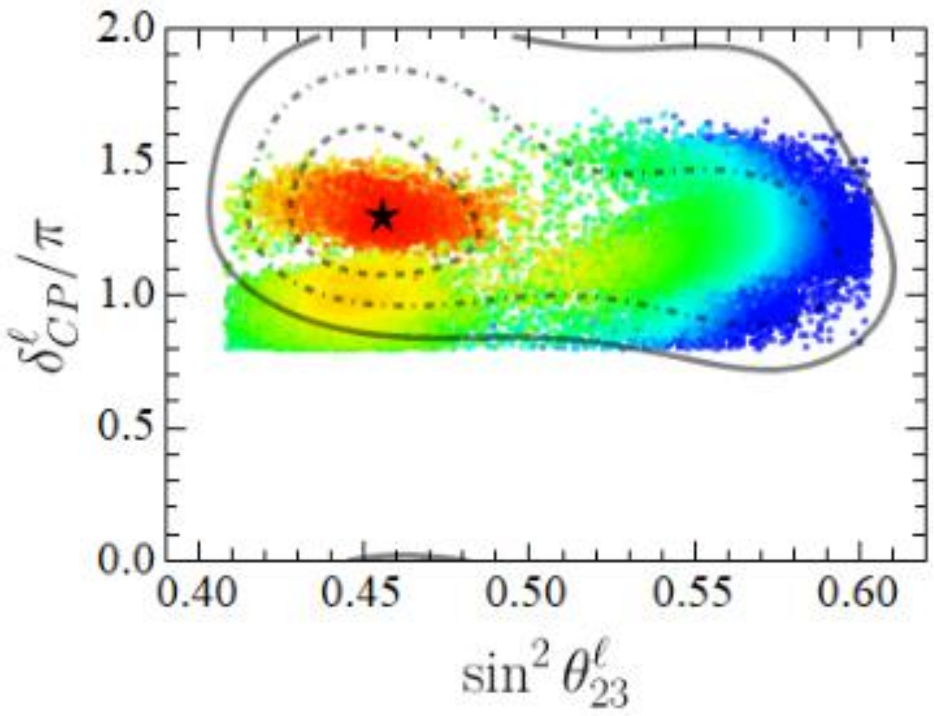
  Majorana CP violation phases : $\alpha_{21} = 0.962\pi,\ \alpha_{31} = 0.643\pi$

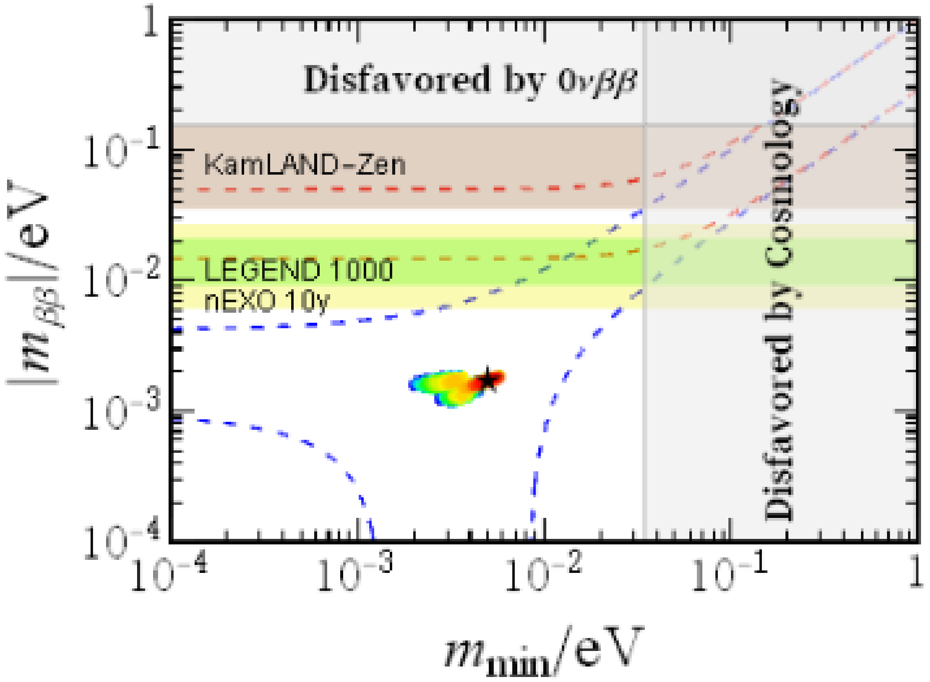
  Masses of the heavy right-handed neutrinos: $M_1 = M_2 = M_3 = 3.945 \times 10^{11}$GeV

  Effective $0\nu\beta\beta$ decay mass: $|m_{\beta\beta}| = 1.686$ meV

[Ding,Jiang,King, Lu, Qu,2404.xxxxx]

➢ Scan the parameter space with the algorithm of nested sampling





Testing δ^{ℓ}_{CP} and θ^{ℓ}_{23} at DUNE and T2HK

$0\nu\beta\beta$ decay is below the sensitivities of future tonn-scale experiments such as LEGEND 1000 and nEXO 10y.

Renormalizable SO(10) GUT

- The quarks and leptons plus a right-handed neutrino are embedded into a single 16 spinor representation of SO(10)

u	u	u	$\lvert ++--+\rangle, \lvert +-+-+\rangle, \lvert -++-+\rangle$
d	d	d	$\lvert ++-+-\rangle, \lvert +-++-\rangle, \lvert -+++-\rangle$
u^c	u^c	u^c	$\lvert --+++\rangle, \lvert -+-++\rangle, \lvert +--++\rangle$
d^c	d^c	d^c	$\lvert --+--\rangle, \lvert -+---\rangle, \lvert +----\rangle$
		ν_e	$\lvert ----+\rangle$
		e	$\lvert ---+-\rangle$
		e^c	$\lvert +++--\rangle$
		ν^c	$\lvert +++++\rangle$ →right-handed neutrino ⇒ seesaw mechanism

- Renormalizable Yukawa couplings: fermion× fermion× Higgs

fermion× fermion: $16 \otimes 16 = 10_S \oplus 120_A \oplus 126_S$ 10 and 126 **symmetric**, 120 **antisymmetric**

Higgs: $H \sim 10,\ \overline{\Delta} \sim \overline{126},\ \Sigma \sim 120$ Minimal SO(10) contains only $H \sim 10,\ \overline{\Delta} \sim \overline{126}$

Yukawa superpotential: $$\boxed{\mathcal{W}_Y = \mathcal{Y}_{ab}^{10}\psi_a\psi_b H + \mathcal{Y}_{ab}^{\overline{126}}\psi_a\psi_b\overline{\Delta} + \mathcal{Y}_{ab}^{120}\psi_a\psi_b\Sigma}$$

decomposition of Higgs fields under the SM gauge symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$10 \supset (\mathbf{1}, \mathbf{2}, 1/2) \oplus (\mathbf{1}, \mathbf{2}, -1/2) \equiv \Phi_d^{10} \oplus \Phi_u^{10}$$

$$120 \supset (\mathbf{1}, \mathbf{2}, 1/2) \oplus (\mathbf{1}, \mathbf{2}, -1/2) \oplus (\mathbf{1}, \mathbf{2}, 1/2) \oplus (\mathbf{1}, \mathbf{2}, -1/2) \equiv \Phi_d^{120} \oplus \Phi_u^{120} \oplus \Phi_d'^{120} \oplus \Phi_u'^{120}$$

$$\overline{126} \supset (\mathbf{1}, \mathbf{2}, 1/2) \oplus (\mathbf{1}, \mathbf{2}, -1/2) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{3}, 1) \equiv \Phi_d^{\overline{126}} \oplus \Phi_u^{\overline{126}} \oplus \overline{\Delta}_R \oplus \overline{\Delta}_L$$

RH neutrino Majorana mass, type-I seesaw

LH neutrino Majorana mass, type-II seesaw

➢ Fermion mass matrices

quarks: $M_u = \left(\mathcal{Y}^{10} + r_2 \mathcal{Y}^{\overline{126}} + r_3 \mathcal{Y}^{120}\right) v_u \,, \quad M_d = r_1 \left(\mathcal{Y}^{10} + \mathcal{Y}^{\overline{126}} + \mathcal{Y}^{120}\right) v_d$

charged leptons: $M_e = r_1 \left(\mathcal{Y}^{10} - 3\mathcal{Y}^{\overline{126}} + c_e \mathcal{Y}^{120}\right) v_d$

neutrinos: $M_{\nu_D} = \left(\mathcal{Y}^{10} - 3r_2 \mathcal{Y}^{\overline{126}} + c_\nu \mathcal{Y}^{120}\right) v_u \,, \quad M_{\nu_R} = v_R \mathcal{Y}^{\overline{126}} \,, \quad M_L = v_L \mathcal{Y}^{\overline{126}}$

$$\Rightarrow \quad M_\nu = M_L - M_{\nu_D}^T M_{\nu_R}^{-1} M_{\nu_D}$$

type-II seesaw type-I seesaw

[see talks by Jessica Turner, Bowen Fu]

In comparison with SM, the complexity lies in the parameters $r_{1,2,3}$ and $c_{e,\mu}$ which are the mixing parameters relating the SM Higgs doublets to the SO(10) Higgs multiplets

SO(10)xA_4 modular GUT

- Assignment under A_4
  matter fields (unique): $\psi \equiv (\psi_1, \psi_2, \psi_3)^T \sim \mathbf{3}$ Higgs fields: $H,\ \Sigma,\ \overline{\Delta} \sim \mathbf{1}$
- A_4 modular invariant Yukawa superpotential: **completely specified by the weights k_F, $k_{10,120,126}$**

$$\mathcal{W}_Y = \sum_{\mathbf{r}_a} \alpha_a \left((\psi\psi)_{\mathbf{r}'_a} Y^{(2k_F+k_{10})}_{\mathbf{r}_a}(\tau)\right)_{\mathbf{1}} H + \sum_{\mathbf{r}_c} \beta_b \left((\psi\psi)_{\mathbf{r}'_b} Y^{(2k_F+k_{120})}_{\mathbf{r}_b}(\tau)\right)_{\mathbf{1}} \Sigma$$
$$+ \sum_{\mathbf{r}_c} \gamma_c \left((\psi\psi)_{\mathbf{r}'_c} Y^{(2k_F+k_{\overline{126}})}_{\mathbf{r}_c}(\tau)\right)_{\mathbf{1}} \overline{\Delta}$$

$$Y^{(2)}_{\mathbf{3}} = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + \dots \\ -6q^{1/3}(1 + 7q + \dots) \\ -18q^{2/3}(1 + 2q + \dots) \end{pmatrix}$$

- General form of Yukawa matrix: **sum over all contributions of modular forms**

$$\mathcal{Y}^{10}\Big|_{k=2k_F+k_{10}} = \alpha_1 Y^{(k)}_{\mathbf{1}}(\tau) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \alpha_2 Y^{(k)}_{\mathbf{1}'}(\tau) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \alpha_3 Y^{(k)}_{\mathbf{1}''}(\tau) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$+\alpha_4 \begin{pmatrix} 2Y^{(k)}_{\mathbf{3},1}(\tau) & -Y^{(k)}_{\mathbf{3},3}(\tau) & -Y^{(k)}_{\mathbf{3},2}(\tau) \\ -Y^{(k)}_{\mathbf{3},3}(\tau) & 2Y^{(k)}_{\mathbf{3},2}(\tau) & -Y^{(k)}_{\mathbf{3},1}(\tau) \\ -Y^{(k)}_{\mathbf{3},2}(\tau) & -Y^{(k)}_{\mathbf{3},1}(\tau) & 2Y^{(k)}_{\mathbf{3},3}(\tau) \end{pmatrix}$$

symmetric

similarly for $\mathcal{Y}^{\overline{126}}$

$$\mathcal{Y}^{120}\Big|_{k=2k_F+k_{120}} = \beta_1 \begin{pmatrix} 0 & Y^{(k)}_{\mathbf{3},3}(\tau) & -Y^{(k)}_{\mathbf{3},2}(\tau) \\ -Y^{(k)}_{\mathbf{3},3}(\tau) & 0 & Y^{(k)}_{\mathbf{3},1}(\tau) \\ Y^{(k)}_{\mathbf{3},2}(\tau) & -Y^{(k)}_{\mathbf{3},1}(\tau) & 0 \end{pmatrix}$$

antisymmetric

[Ding, King, Lu,2108.09655]

Benchmark models [Ding, King, Lu,2108.09655]

minimal SO(10): $10_H + \overline{126}_H$			
$2k_F + k_{10}$	10	10	10
$2k_F + k_{\overline{126}}$	6	8	10
next-to-minimal SO(10): $10_H + 120_H + \overline{126}_H$			
$2k_F + k_{10}$	4	4	4
$2k_F + k_{120}$	8	2	6
$2k_F + k_{\overline{126}}$	0	4	4

- The Higgs sector of minimal SO(10) models is simpler, while more free parameters in Yukawa than the next-to-minimal SO(10)
- Only the mixture of type-I+II seesaw can be compatible with data in minimal SO(10) .

Minimal model 1: $(2k_F + k_{10}, 2k_F + k_{\overline{126}}) = (10, 6)$

$$\mathcal{W}_Y = \alpha_1 Y_{\mathbf{1}}^{(10)}\psi\psi H + \alpha_2 Y_{\mathbf{1'}}^{(10)}\psi\psi H + \alpha_3 Y_{\mathbf{3}I}^{(10)}\psi\psi H + \alpha_4 Y_{\mathbf{3}II}^{(10)}\psi\psi H + \alpha_5 Y_{\mathbf{3}III}^{(10)}\psi\psi H$$
$$+\gamma_1 Y_{\mathbf{1}}^{(6)}\psi\psi\overline{\Delta} + \gamma_2 Y_{\mathbf{3}I}^{(6)}\psi\psi\overline{\Delta} + \gamma_3 Y_{\mathbf{3}II}^{(6)}\psi\psi\overline{\Delta}$$

8 terms

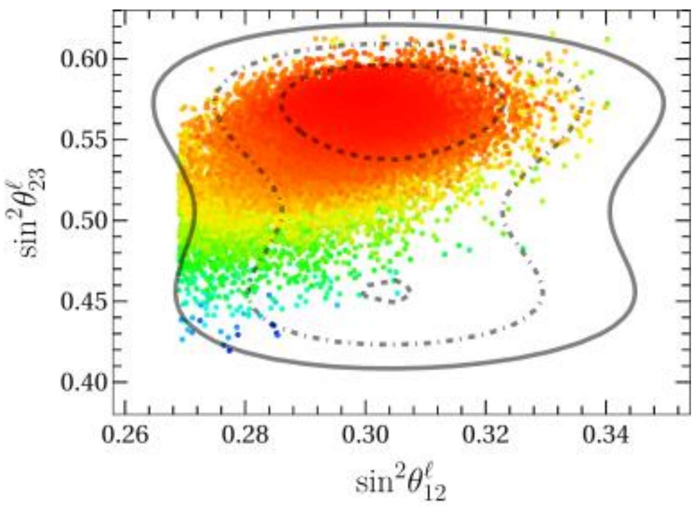
Next-to-minimal model 1: $(2k_F + k_{10}, 2k_F + k_{120}, 2k_F + k_{\overline{126}}) = (4, 8, 0)$

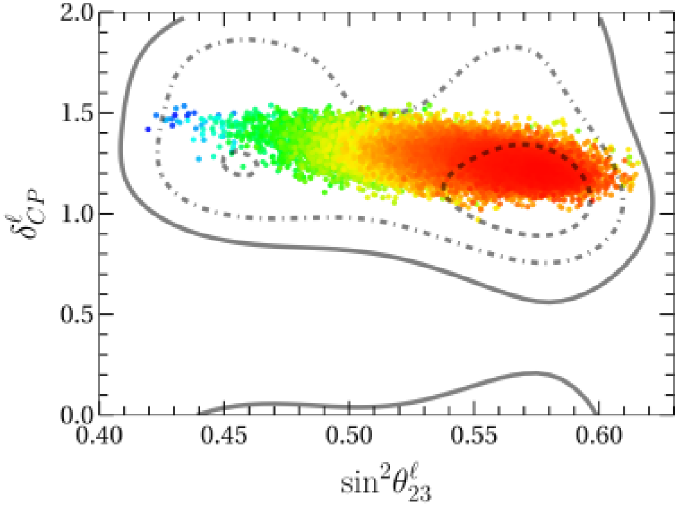
$$\mathcal{W}_Y = \alpha_1 Y_{\mathbf{1}}^{(4)}\psi\psi H + \alpha_2 Y_{\mathbf{1'}}^{(4)}\psi\psi H + \alpha_3 Y_{\mathbf{3}}^{(4)}\psi\psi H + \beta_1 Y_{\mathbf{3}I}^{(8)}\psi\psi\Sigma + \beta_2 Y_{\mathbf{3}II}^{(8)}\psi\psi\Sigma + \gamma_1\psi\psi\overline{\Delta}$$

6 terms

- Numerically fitting the model on fermion masses and mixing angles evolved at the GUT scale

Parameters	type-I seesaw	type-I+II seesaw
τ	$-0.47503+0.897003i$	$0.492883+0.933623i$
r_2	$0.788723\ e^{i0.1\pi}$	$0.79888\ e^{i0.0628905\pi}$
r_3	$0.144759\ e^{i1.97672\pi}$	$0.12476\ e^{i1.56739\pi}$
c_e	$10.543\ e^{i0.380093\pi}$	$13.1899\ e^{i0.304315\pi}$
c_ν	$8.91773\ e^{i0.456123\pi}$	$11.0295\ e^{i0.457549\pi}$
$\alpha_1 v_L(\text{meV})$	—	$39.1648\ e^{i0.000050371\pi}$
α_2/α_1	$9.84853\ e^{i0.0687569\pi}$	$1.93063\ e^{i0.245422\pi}$
α_3/α_1	$9.84418\ e^{i0.0684131\pi}$	$1.93307\ e^{i0.245233\pi}$
β_1/α_1	$0.717383\ e^{i0.480072\pi}$	$0.11335\ e^{i1.05365\pi}$
β_2/α_1	$0.00999426\ e^{i1.26573\pi}$	$0.0306156\ e^{i0.835143\pi}$
γ_1/α_1	$2.45837\ e^{i1.76308\pi}$	$0.571433\ e^{i0.366949\pi}$
$\alpha_1 v_u^2/v_R(\text{meV})$	42.1872	3.20875
$\alpha_1 v_u/\text{GeV}$	1.48805	8.17528
$\alpha_1 r_1 v_d/\text{GeV}$	0.0157628	0.0864676





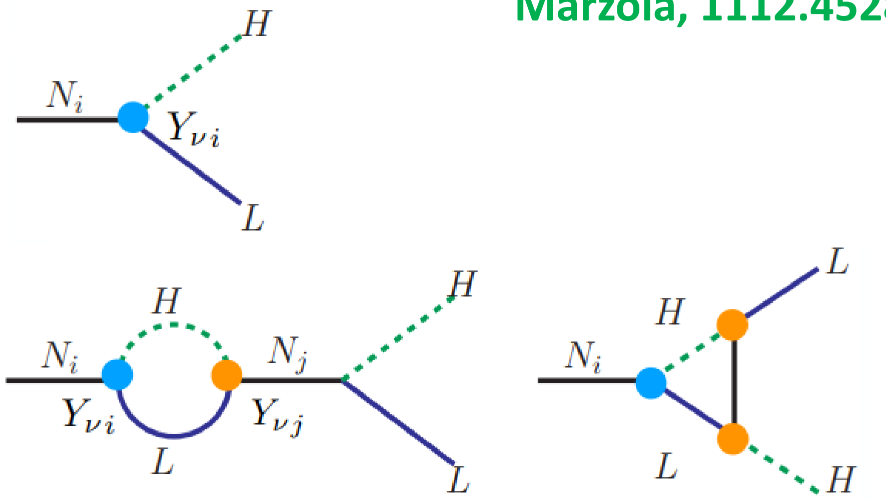
Observables	type-I seesaw	type-I+II seesaw
$\sin^2\theta_{12}^\ell$	0.30062	0.294478
$\sin^2\theta_{13}^\ell$	0.0223223	0.0221286
$\sin^2\theta_{23}^\ell$	0.574905	0.569854
$\delta_{CP}^\ell/^\circ$	221.268	161.915
$\alpha_{21}/^\circ$	193.648	196.185
$\alpha_{31}/^\circ$	223.403	107.988
m_e/m_μ	0.00480896	0.00481101
m_μ/m_τ	0.0591488	0.0613664
m_1/meV	5.52462	9.1019
m_2/meV	10.2333	12.5317
m_3/meV	50.5706	51.0373
$m_{\beta\beta}/\text{meV}$	0.020995	1.92936
M_1/GeV	1.29033×10^{11}	1.19024×10^{13}
M_2/GeV	1.29033×10^{11}	1.19024×10^{13}
M_3/GeV	1.29033×10^{11}	1.19024×10^{13}
$\alpha_1 v_R/\text{GeV}$	5.30122×10^{10}	2.10374×10^{13}
θ_{12}^q	0.228946	0.229003
θ_{13}^q	0.00417186	0.00407841
θ_{23}^q	0.0409926	0.041808
$\delta_{CP}^q/^\circ$	43.6176	49.7097
m_u/m_c	0.00278043	0.00261272
m_c/m_t	0.00248112	0.00253582
m_d/m_s	0.0537242	0.0496242
m_s/m_b	0.0188432	0.0186282
m_b/m_τ	0.820639	0.782847
χ_ℓ^2	0.971826	4.20832
χ_q^2	2.89378	2.67676
$\chi_{b\tau}^2$	9.12829	3.10314
χ^2	12.9939	9.98822

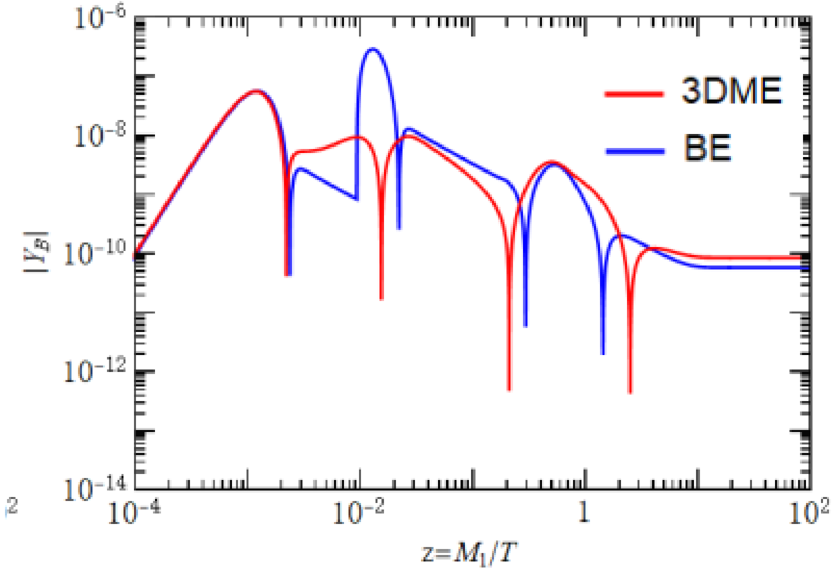
➢ Leptogenesis in modular SO(10) GUT

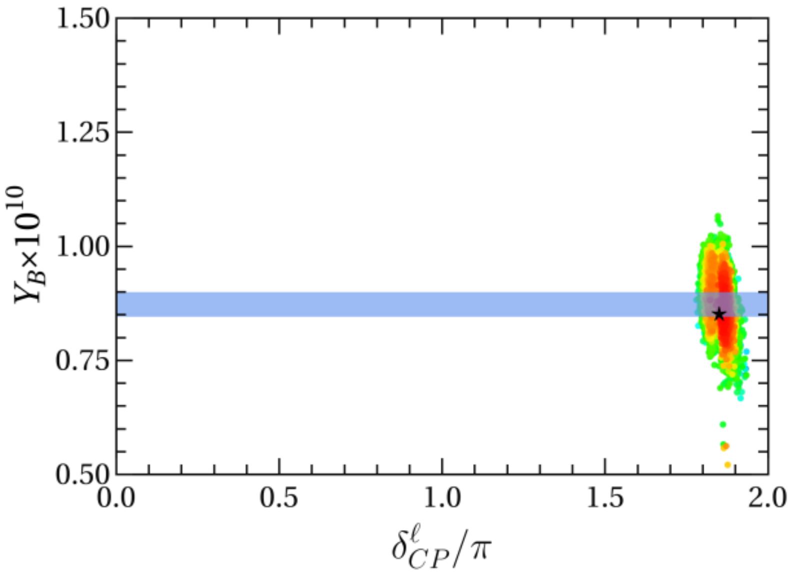
The right-handed neutrino masses and Yukawa couplings are fixed by fermion masses and mixing.

The evolution of baryon asymmetry is described by density matrix equations [Blanchet, Bari, Jones, Marzola, 1112.4528]

$$\frac{dN_{N_i}}{dz} = -D_i(N_{N_i} - N_{N_i}^{\rm eq}),$$
$$\frac{dN_{\alpha\beta}^{B-L}}{dz} = \sum_i \left[\varepsilon_{\alpha\beta}^{(i)} D_i(N_{N_i} - N_{N_i}^{\rm eq}) - \frac{1}{2}W_i\left\{P^{0(i)}, N^{B-L}\right\}_{\alpha\beta}\right]$$
$$- \frac{\Im(\Lambda_\tau)}{Hz}\left(\delta_{\alpha\tau}N_{\tau\beta}^{B-L} + \delta_{\beta\tau}N_{\alpha\tau}^{B-L} - 2\delta_{\alpha\tau}\delta_{\beta\tau}N_{\tau\tau}^{B-L}\right)$$
$$- \frac{\Im(\Lambda_\mu)}{Hz}\left(\delta_{\alpha\mu}N_{\mu\beta}^{B-L} + \delta_{\beta\mu}N_{\alpha\mu}^{B-L} - 2\delta_{\alpha\mu}\delta_{\beta\mu}N_{\mu\mu}^{B-L}\right),$$





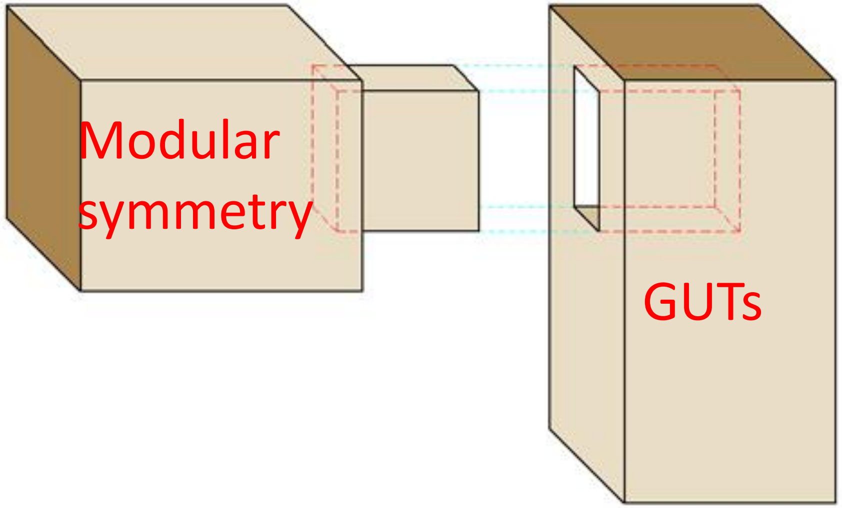


Fermion masses and flavor mixing and baryon asymmetry in a model!

[Ding, King, Lu, Qu, 2206.14675]

Summary and outlook

- The modular flavor symmetry is a promising approach to address the flavor structure, it unifies the three generations of fermion, Yukawa couplings are modular forms which are holomorphic functions of τ. The structure of flavor models are significantly simplified.
- GUTs unify quarks and leptons as well as gauge coupling constants.
- Modular symmetry + GUTs allows to construct highly constrained and predictive flavor theory testable at neutrino oscillation and $0\nu\beta\beta$ decay experiments.
- GUT symmetry breaking and the interplay of modular symmetry with proton decay, GW etc should be carefully studied for a complete theory of flavor.



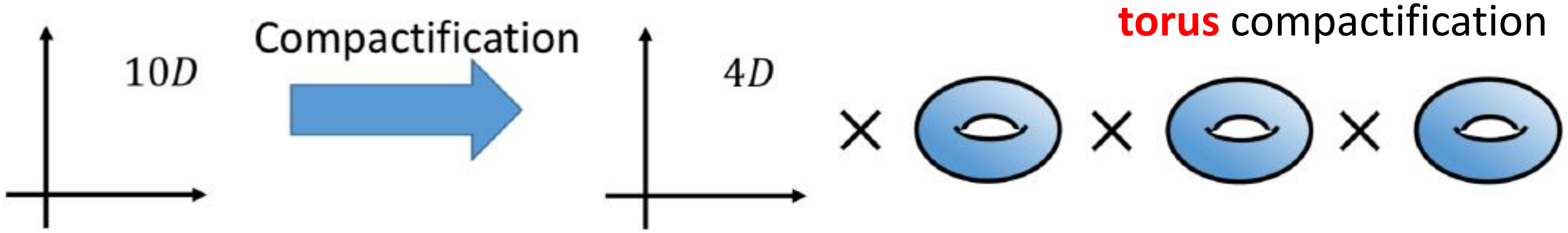
Thank you for your attention!

Backup

Modular invariance as flavor symmetry

Modular invariance is motivated by more fundamental theory such as string theory at high energy scale

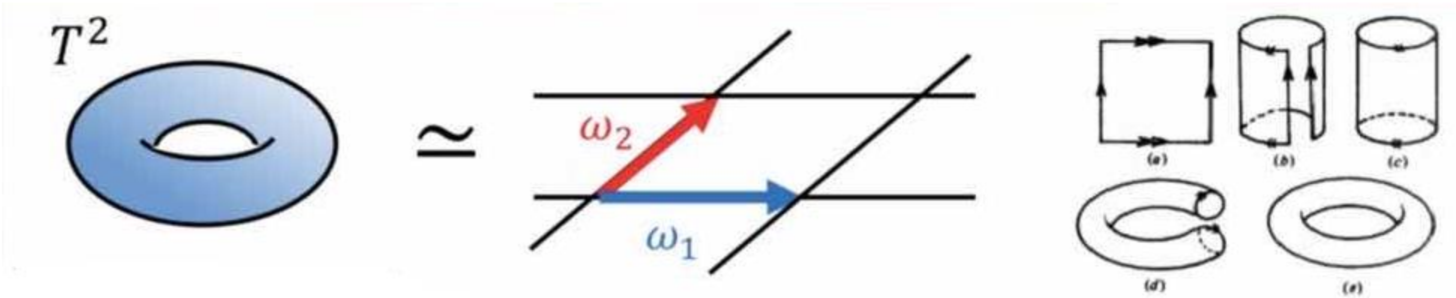
torus compactification

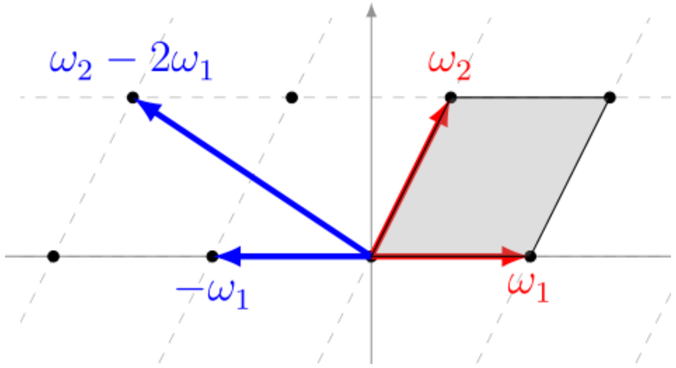


[Ferrara et al, 1989; Feruglio, 1706.08749]

4D effective Lagrangian : $S = \int d^4x d^6y \, \mathcal{L}_{10D} \Rightarrow \int d^4x \, \mathcal{L}_{\text{eff}}(\varphi, \tau_i)$

The shape of a torus T^2 is characterized by a modulus $\tau = \omega_2 / \omega_1, \ \text{Im}\ \tau > 0$





The torus (lattice) is left **invariant** by modular transformations

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \Longrightarrow \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

$SL(2,Z):\ ad - bc = 1,\ \ a, b, c, d$ integers

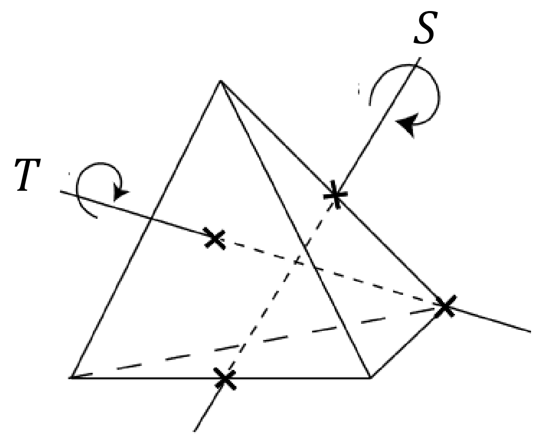
$\mathcal{L}_{\text{eff}}(\tau, \Phi) \Longrightarrow \mathcal{L}_{\text{eff}}$ **modular invariant**

- Finite modular groups as G_f: the quotient over the principal congruence subgroups $\Gamma(N)$

$$\Gamma_N \equiv SL(2,Z) / \pm\Gamma(N), \quad \Gamma_N' \equiv SL(2,Z) / \Gamma(N)$$

A_4 modular group and modular forms of level 3

A_4 is the symmetry group of the tetrahedron $\quad A_4\colon S^2 = T^3 = (ST)^3 = 1$



- A_4 has only 4 irreducible representations: **1, 1', 1'',3**

singlets $\begin{cases} \mathbf{1}: & S=1, \quad T=1 \\ \mathbf{1}': & S=1, \quad T=\omega \\ \mathbf{1}'': & S=1, \quad T=\omega^2 \end{cases}$

triplets $\mathbf{3}: \quad S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

- tensor product

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}_{\mathbf{3}} = (\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)_{\mathbf{1}} \oplus (\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1)_{\mathbf{1}'} \oplus (\alpha_2\beta_2 + \alpha_1\beta_3 + \alpha_3\beta_1)_{\mathbf{1}''}$$

$$\oplus \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \end{pmatrix}_{\mathbf{3}_S} \oplus \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}_{\mathbf{3}_A} .$$

- modular forms at level 3 $\quad$ tensor product→ higher weight modular forms

$$Y_{\mathbf{3}}^{(2)}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} \varepsilon^2(\tau) \\ \sqrt{2}\,\vartheta(\tau)\varepsilon(\tau) \\ -\vartheta^2(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \ldots \\ -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + \ldots) \\ -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + \ldots) \end{pmatrix} \qquad q \equiv e^{2\pi i\tau}$$

weight 1 modular forms: $\vartheta(\tau) = 3\sqrt{2}\,\eta^3(3\tau)/\eta(\tau), \quad \varepsilon(\tau) = -\left[3\eta^3(3\tau) + \eta^3(\tau/3)\right]/\eta(\tau) \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$

- The integral weight modular forms are homogeneous polynomials of the lowest weight one modular forms which can be constructed from the Dedekind eta function and Klein forms.

N=3: $$\mathcal{M}_k(\Gamma(3)) = \bigoplus_{a+b=k,\, a,b\geq 0} \mathbb{C}\frac{\eta^{3a}(3\tau)\eta^{3b}(\tau/3)}{\eta^k(\tau)}$$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,, \quad q \equiv e^{2\pi i\tau}$$

N=4: $$\mathcal{M}_k(\Gamma(4)) = \bigoplus_{a+b=2k,\, a,b\geq 0} \mathbb{C}\frac{\eta^{2b-2a}(4\tau)\eta^{5a-b}(2\tau)}{\eta^{2a}(\tau)}$$

[Ding, King, 2311.09282]

N=5: $$\mathcal{M}_k(\Gamma(5)) = \bigoplus_{a+b=5k,\, a,b\geq 0} \mathbb{C}\frac{\eta^{15k}(5\tau)}{\eta^{3k}(\tau)}\mathfrak{k}^a_{\frac{1}{5},\frac{0}{5}}(5\tau)\mathfrak{k}^b_{\frac{2}{5},\frac{0}{5}}(5\tau)$$

N	$\mathtt{dim}\mathcal{M}_k(\Gamma(N))$	$\Gamma'_N \equiv \Gamma/\Gamma(N)$	$\lvert\Gamma'_N\rvert$	Irreps of lowest weight
2	$k/2+1\,(k\ \ \mathtt{even})$	S_3	6	—
3	$k+1$	T'	24	$\mathbf{2}$
4	$2k+1$	S'_4	48	$\hat{\mathbf{3}}'$
5	$5k+1$	A'_5	120	$\mathbf{6}$