“Workshop on Grand Unified Theories: Phenomenology and Cosmology”

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$\mathcal{F}$-theory GUTs:

Prospects and Challenges

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GREECE

Outline of the Talk

- Introductory remarks
- $\mathcal{F}$-Theory basics
- Building $\mathcal{F}$-Theory GUTs
- $\mathcal{F}$-GUTs and Discrete Symmetries
- Concluding Remarks

$\mathcal{A}$

A few remarks

F-theory is an exciting reformulation of String Theory in a twelve dimensional space

It involves a number of fascinating mathematical concepts such as: **Topology, Algebraic Geometry and Elliptic Fibrations**

The aim of this talk is to describe the methodology in building effective unified theories (GUTs) and discuss possible phenomenological predictions

$\mathcal{B}$

The **Defining Features** of **F-theory** (C. Vafa, hep-th/9602022)

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i) **Non-perturbative formulation of Type II-B string compactifications**

-

ii) **Presence of 7-branes which backreact on the geometry**

-

in particular

iii) **D7 branes are magnetic sources for the RR axion C_0.**

-

iv) **Inherits $SL(2, Z)$ invariance from Type II-B**

$SL(2,Z)$-invariance

1. The dilaton ϕ determines the *string coupling:*

$$g_{IIB} = e^{\phi}$$

2. *The RR axion C_0, and the dilaton ϕ are combined to one modulus, the axio-dilaton field:*

$$\tau = C_0 + i\, e^{-\phi} \to C_0 + \frac{i}{g_{IIB}}$$

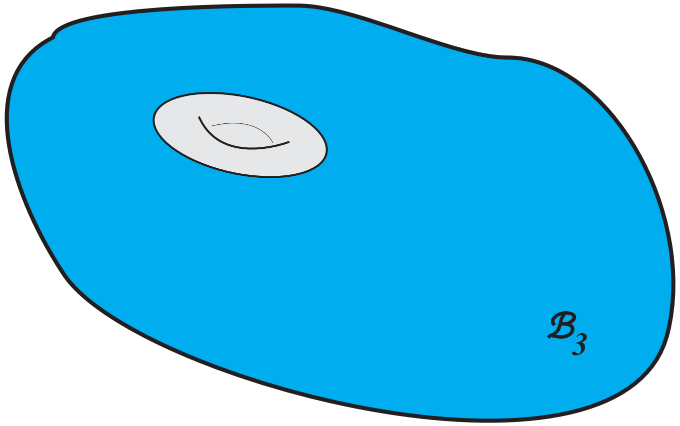
3. The importance of τ is that it can be used to write the type **IIB** action in an $SL(2,Z)$ invariant way

$$S_{IIB} \quad \propto \quad \int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}\frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{(\mathrm{Im}\tau)^2} - \frac{1}{2}\frac{|G_3|^2}{\mathrm{Im}\tau} - \frac{1}{4}|F_5|^2\right)$$

$$-\frac{i}{4}\int\frac{1}{\mathrm{Im}\tau}C_4 + G_3\wedge\tilde{G}_3$$

Elliptic Curves & Elliptic Fibration

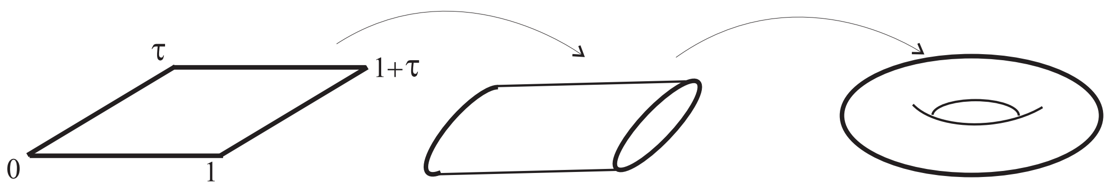
An extremely important implication of the variation of the axio-dilaton τ *is that it gives rise to an elliptic fibration over the physical space-time. In order to see this, let's start with II-B theory which is defined in 10-d space described by:* $\mathcal{R}^{3,1} \times \mathcal{B}_3$



- $\mathcal{R}^{3,1}$ is the usual 4-d space-time
- $\mathcal{B}_3$ Calabi-Yau (CY) manifold of 3 complex dimensions (3-fold)

▲ ▲ **F-theory** is compactified on an **elliptically fibered manifold** where $\mathcal{B}_3$ is the base of the fibration.

Fibration *is implemented by the axio-dilaton modulus* $\tau = C_0 + \imath\, e^{-\phi}$ *which can be thought as describing a torus*

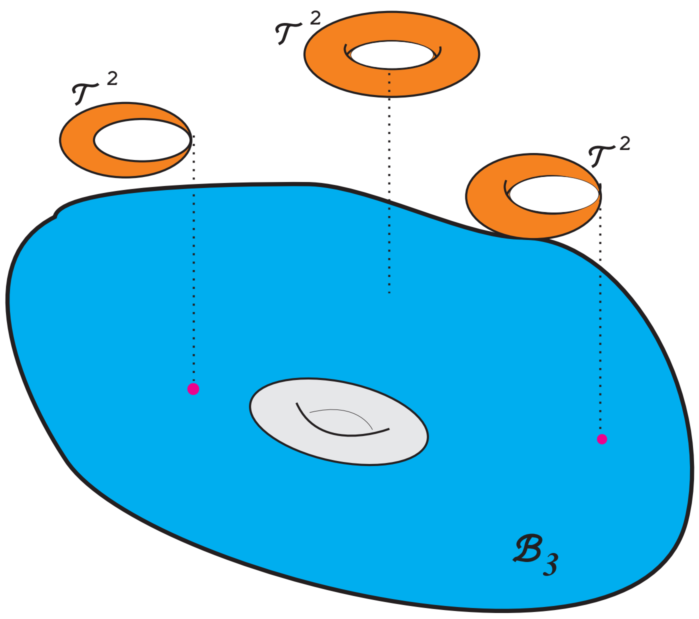


More precisely, we make a continuous mapping of τ to the points of the base B_3. Thus, we say that:

▲ F-theory is defined on $\mathcal{R}^{3,1} \times \mathcal{X}$ ▲

where $\mathcal{X}$, elliptically fibered **CY** 4-fold over the base B_3

This is depicted below where τ-tori are associated with points of B_3. Red points correspond to possible geometric singularities of the fiber



Mathematically, the **Elliptic Fibration** is described by the vanishing locus of the $\mathcal{W}$eierstraß $\mathcal{E}$quation

$$y^2 = x^3 + f(z)\, xw^4 + g(z)w^6$$

1. $f(z)$, $g(z) \to 8^{th}$ and 12^{th} degree polynomials.
2. Equivalence relations of homogeneous (projective) coordinates $(x, y, w, z) \simeq (\lambda^2 x, \lambda^3 y, \lambda w, z)$ and $(x, y, z, w) \simeq (\lambda^4 x, \lambda^6 y, \lambda z, w)$
3. The zero section σ_0 is described by the intersection $w = 0$ which marks the point $[x : y : w] \to [1 : 1 : 0]$.
4. The elliptic fibration is a CY, as long as $f(z)$ and $g(z)$ are holomorphic sections of line bundles[a] $\mathcal{O}(K_B^{-4})$ and $\mathcal{O}(K_B^{-6})$ respectively.

---

[a] K_B is the canonical class of the base B_3.

Two important quantities characterise the fibration:

▲ The discriminant: $(24^{th}$*-degree in* $z)$

$$\Delta(z) = 4\, f(z)^3 + 27 g(z)^2$$

▲ The zeros of the discriminant determine the fiber singularities:

$$\Delta = \prod_{i=1}^{24} (z - z_i) = 0$$

$$\Downarrow$$

24 roots z_i

▲▲ the j-invariant:

$$j(\tau) = \frac{4(24f(z))^3}{\Delta(z)}$$

▲▲ The j-invariant provides a relation between the modulus τ and the *coordinate* z:[a]

$$j(\tau(z)) = 4\,\frac{(24f(z))^3}{\Delta(z)} \propto e^{-2\pi i\tau} + \cdots \qquad (1)$$

---

[a] $j(\tau) \sim e^{-2\pi i\tau} + 744 + \mathcal{O}(e^{2\pi i\tau}) \sim e^{2\pi/g_s} e^{-2\pi i C_0} + 744 + \mathcal{O}(e^{-2\pi/g_s})$.

Its solution determines the axio-dilaton τ around the zeros z_i of Δ:

$$\tau \approx \frac{1}{2\pi i} \log(z - z_i)$$

Now recall that the log is a multivalued function

▲ Hence, while Encircling a root z_i, the real component of τ shifts:

$$\tau \to \tau + 1 \ \Rightarrow C_0 \to C_0 + 1$$

In other words, τ and thus C_0 undergo Monodromy.

▲ The **Interpretation** of this picture is that at each root

$$z = z_i$$

there is a source of RR-flux which is associated with a $D7$-brane **perpendicular** to the “tangent plane”.

$D7$ branes are magnetic sources for the RR axion C_0

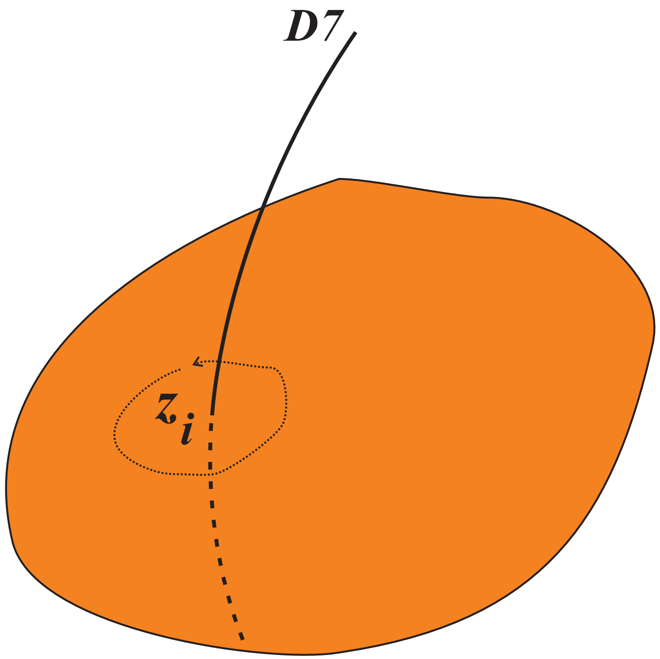


Figure 1: Moving around z_i, $\log(z) \rightarrow \log|z| + i(2\pi + \theta)$ and the modulus shifts by $\tau \rightarrow \tau + 1$

Geometric Singularities

Summarising the analysis so far, the elliptic fibration is represented by the Weierstraß equation (*fixing* $w = 1$):

$$y^2 = x^3 + f(z)\, x + g(z)$$

- At the points where the discriminant $\Delta = 27g^2 + 4f^3$ vanishes, the elliptic fiber **degenerates**.
- The type of Manifold **singularity** is specified by the vanishing order of Δ and the polynomials $f(z)$, $g(z)$ of Weierstraß eqn
- As proved by Kodaira (in '60s), these **geometric singularities** are classified in terms of $\mathcal{ADE}$ Lie groups.

In F-theory these singularities are interpreted as:

$$\Downarrow$$

CY_4-**Singularities** $\rightleftarrows$ **gauge symmetries**

▲ The above description concerns the non-abelian part of the effective theory which according to $\mathcal{ADE}$ classification will result to an effective model with one of the following gauge groups (in standard notation)

$$\text{Non Abelian Gauge Groups} \Rightarrow \begin{cases} SU(n) \\ SO(m) \\ \mathcal{E}_n \end{cases}$$

▲▲ There are also Abelian symmetries associated with the elliptic fibers of the CY_4 and will be discussed shortly

$\textcircled{\mathcal{C}}$

The Non Abelian Sector

Rôle of Geometric Singularities in EFTs

Kodaira classified the type of singularities in terms of the vanishing order of $f(z)$, $g(z)$ and $\Delta(z) = 4f(z)^3 + 27g(z)^2$.

For phenomenological applications in local model building it is more convenient to use

Tate's Algorithm

$$y^2 + a_1 x\, y + a_3 y = x^3 + a_2\, x^2 + a_4 x + a_6$$

All information is encoded in the coefficients a_i

$$a_n = \sum_{\ell=k\geq 0} a_{n,\ell} z^\ell$$

An ADE classification of the Geometric Singularities w.r.t. vanishing order of a_i and Δ is shown in the following Table:

	Group	a_1	a_2	a_3	a_4	a_6	Δ
T	$SU(2)$	1	1	1	1	2	3
a	$SU(3)$	1	1	1	2	3	4
t	$SU(2n)$	1	1	n	n	$2n$	$2n$
e	$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$
A	$SO(4k+1)$	1	1	k	$k+1$	$2k$	$2k+3$
g	$SO(4k+2)$	1	1	k	$k+1$	$2k+1$	$2k+3$
o	$SO(4k+3)$	1	1	$k+1$	$k+1$	$2k+1$	$2k+4$
l	$SU(5)$	0	1	2	3	5	5
i	$SO(10)$	1	1	2	3	5	7
t	$\mathcal{E}_6$	1	2	3	3	5	8
h	$\mathcal{E}_7$	1	2	3	3	5	9
m	$\mathcal{E}_8$	1	2	3	4	5	10

$\mathcal{EXAMPLE}$

Define $b_k = b_{k,0} + b_{k,1} z + \cdots$, $(b_{k,0} \neq 0)$, then choose a_i to be:

$$a_1 = -b_5,\ a_2 = b_4 z,\ a_3 = -b_3 z^2,\ a_4 = b_2 z^3,\ a_6 = b_0 z^5$$

Then, the vanishing orders of each a_n is:

Vanishing	a_1	a_2	a_3	a_4	a_6	Δ	
order	–	z^1	z^2	z^3	z^5	z^5	$\to$ **SU(5)**

$\Rightarrow$ Weierstraß' equation for the $SU(5)$ singularity

$$y^2 = x^3 + b_0 z^5 + b_2 x z^3 + b_3 y z^2 + b_4 x^2 z + b_5 x y \tag{2}$$

$\star$ A useful notion for local model building is the spectral cover obtained by defining homogeneous coordinates $z \to U$, $x \to V^2$, $y \to V^3$ and affine parameter $s = \frac{U}{V}$, so that (2) implies:

$$\mathcal{C}_5 : \boxed{0 \ = \ b_0 s^5 + b_2 s^3 + b_3 s^2 + b_4 s + b_5}$$

$\mathcal{E}$

F-theory Model Building

(*Original papers: Beasley, Heckman, Vafa : 0802.3391, 0806.0102*
Donagi et al 0808.2223, 0904.1218)
Early reviews: 1001.0577, 1203.6277, 1212.0555
Recent: 1806.01854; 2212.07443

A Class of 'semi-local' constructions

The final effective (GUT) model depends on the choice of:

$\Downarrow$

1) Manifold 2) Fluxes 3) Monodromies

$\Downarrow$

▲▼ The manifold: ▲▼

▲ The candidate GUT is embedded in $\mathcal{E}_8$ which is the maximal exceptional group in elliptic fibration.
Thus, we consider a CY with a divisor accommodating our choice while the rest is the symmetry commutant to it.

$$\mathcal{E}_8 \to \mathbf{G_{GUT}} \times \mathcal{C}$$

Example: *Assuming a Manifold with $SU(5)$ divisor*:

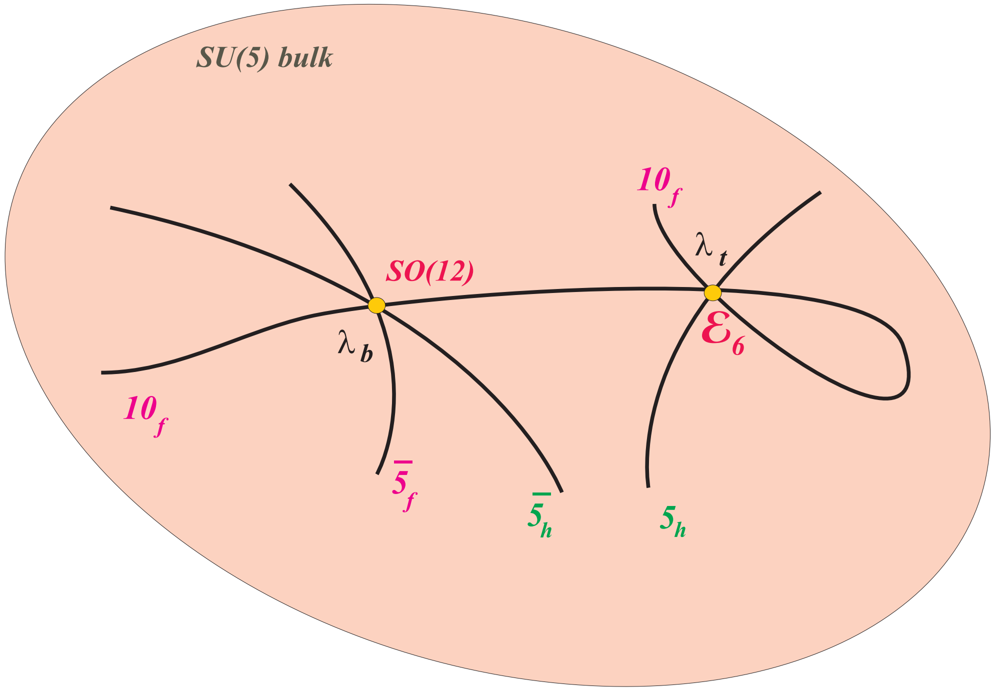
$$\begin{aligned} \mathcal{E}_8 &\to SU(5) \times SU(5)_{\perp} \\ &\to SU(5) \times U(1)_{\perp}^4 \end{aligned}$$

Matter descends from the $\mathcal{E}_8$-Adjoint which decomposes as:

$$248 \to (24,1) + (1,24) + (10,5) + (\overline{5},10) + (\overline{10},\overline{5}) + (5,\overline{10})$$

When branes intersect, the singularity increases and the **gauge symmetry** is further enhanced. Yukawa couplings are formed at tripple intersections . For example, in the $\mathbf{SU(5)}$ case:[a]

$$\lambda_b \, 10 \cdot \bar{5} \cdot \bar{5} \; \in \mathbf{SO(12)}, \;\; \lambda_t \, 10 \cdot 10 \cdot 5 \in \mathbf{E_6}$$



[a]Here we assume that there is a Z_2 monodromy so that λ_t exists.

The fluxes:

Three important implications

- determine $SU(5)$ chirality
- trigger $SU(5)$ Symmetry Breaking

(fluxes act as the surrogate of the Higgs vev)

- Split the $SU(5)$-representations

$SU(5)$ chirality from perpendicular $U(1)_\perp$ Flux

$U(1)_\perp$–Flux on $\in$ **10**'s:

$$\#\mathbf{10} - \#\overline{\mathbf{10}} \quad = \quad M_{10}$$

$U(1)_\perp$– Flux on $\in$ **5**'s:

$$\#\mathbf{5} - \#\overline{\mathbf{5}} \quad = \quad M_5$$

SM chirality form Hypercharge Flux

$U(1)_Y-$**Flux**-splitting of **10**'s:

$$
\begin{aligned}
n_{(3,2)_{\frac{1}{6}}} - n_{(\bar{3},2)_{-\frac{1}{6}}} &= M_{10} \\
n_{(\bar{3},1)_{-\frac{2}{3}}} - n_{(3,1)_{\frac{2}{3}}} &= M_{10} - N_{Y_{10}} \\
n_{(1,1)_{1}} - n_{(1,1)_{-1}} &= M_{10} + N_{Y_{10}}
\end{aligned}
$$

$U(1)_Y-$ **Flux**-splitting of **5**'s:

$$
\begin{aligned}
n_{(3,1)_{-\frac{1}{3}}} - n_{(\bar{3},1)_{\frac{1}{3}}} &= M_{5} \\
n_{(1,2)_{\frac{1}{2}}} - n_{(1,2)_{-\frac{1}{2}}} &= M_{5} + N_{Y_{5}}
\end{aligned}
$$

The Spectral surface $\mathcal{C}_5 \leftrightarrow U(5)_\perp$ *is described by* 5^{th} *degree equation:*

$$\sum_{k=0}^{5} b_k t^k = 0$$

▲ *Topological properties are encoded* $\in b_k$ *coeffs*

however:

▲ *the description of the EFT model relies on the roots* t_i

Solutions $t_i(b_k)$ induce branch-cuts and a non-trivial monodromy.

▲ *Simplest case*:

Z_2 monodromy reduces the "perpendicular symmetry":

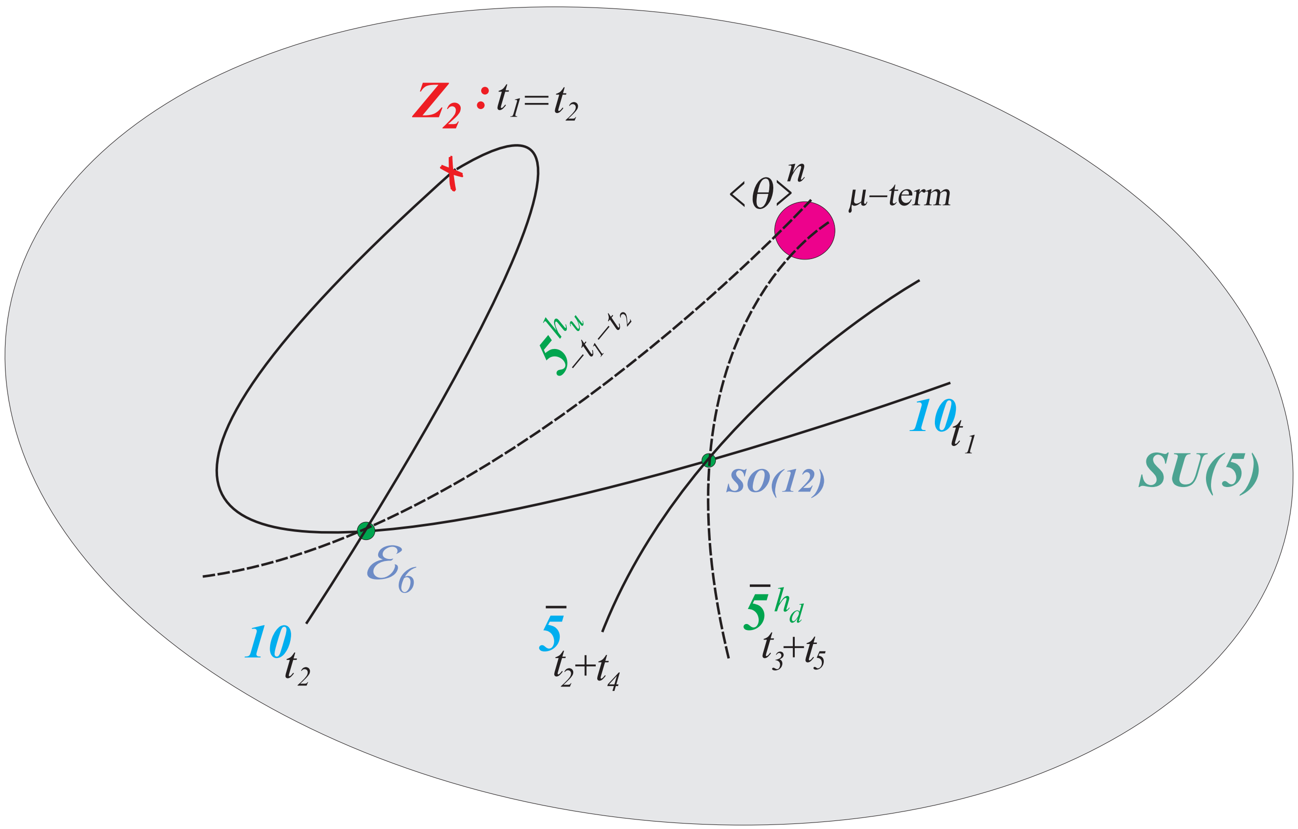
$$Z_2 : \ t_1 \leftrightarrow t_2 \ \Rrightarrow \ U(1)_\perp^4 \to U(1)_\perp^3$$

A simple Z_2 model

(GKL & GG Ross), (GKL & Q. Shafi 1706.08372)

$SU(5), U(1)_i$	SM spectrum	Exotics	R-parity
$10_i, t_i$	Q_i, u_i^c, d_i^c	$-$	$-$
$\bar{5}_1, t_3+t_4$	d_1^c, ℓ_1	$-$	$-$
$\bar{5}_2, t_1+t_3$	d_2^c, ℓ_2	$-$	$-$
$\bar{5}_3, t_1+t_4$	d_3^c, ℓ_3	$-$	$-$
$5_{H_u}, -2t_1$	H_u	D	$+$
$\bar{5}_{H_d}, t_3+t_5$	H_d	$-$	$+$
$5_x, -(t_1+t_5)$	$-$	$(H_{u_i}, D_i)_{i=1,\ldots,n}$	$+$
$\bar{5}_{\bar{x}}, t_4+t_5$	$-$	$D^c + (H_{d_i}, D_i^c)_{i=1,\ldots,n}$	$+$
$\theta_{12,21}$		S (singlet)	$-$

Geometric picture of a generic Z_2 model



$\mathcal{D}$

Origin
of
Abelian and Discrete Symmetries

Our interest in Abelian Groups and other discrete symmetries arises from phenomenological considerations, in particular, of the necessity to
constrain the Yukawa Lagrangian

There are three sources of such symmetries in F-theory

$\mathcal{D}_{\mathcal{A}}$

Abelian Symmetries from Elliptic Fibration

In $\mathcal{F}$-Theory, Abelian gauge symmetries (other than those embedded in E_8) are encoded in rational sections of the Elliptic Fibration and constitute the so called Mordell-Weil group.

Simplest Case (*Morrison-Park: 1208.2695*):

Rank-1 Mordell-Weil

$\Downarrow$

GUT accompanied by new $U(1)$:

$G_{\rm GUT} \times U(1)_{\rm MW}$

but now Tate's coefficients are not all independent!
(Antoniadis, GKL: 1404.6720)

$$y^2 + 2\frac{b_3}{a_2}xyz \pm b_1 a_2 yz^3 = x^3 \pm \left(b_2 - \frac{b_3^2}{a_2^2}\right) x^2 z^2 - b_0 a_2^2 x z^4 - b_0 a_2^2 \left(b_2 - \frac{b_3^2}{a_2^2}\right) z^6$$

Comparing with standard general Tate's form:

$$y^2 + \alpha_1 xyz + \alpha_3 yz^3 = x^3 + \alpha_2 x^2 z^2 - \alpha_4 x z^4 - \alpha_6 z^6$$

we observe

$$\boxed{\alpha_6 = \alpha_2 \alpha_4}$$

$\downarrow$

This eliminates most of the groups in Tate's algorithm!

	Group	a_1	a_2	a_3	a_4	a_6	Δ
T	$SU(2)$	1	1	1	1	2	3
a	$SU(3)$	1	1	1	2	3	4
t	$SU(2n)$	1	1	n	n	$2n$	$2n$
e	$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$
A	$SO(4k+1)$	1	1	k	$k+1$	$2k$	$2k+3$
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Restricted Tate's Algorithm for one $U(1)_{\mathrm{MW}}$

$$y^2 + a_1 x\, y\, z + a_3 y\, z^3 = x^3 + a_2\, x^2 z^2 + a_4 x\, z^4 + a_2 a_4 z^6$$

Group	a_1	a_2	a_3	a_4	a_6	Δ
$SU(2)$	0	1	1	1	2	2
$SU(3)$	0	1	1	2	3	3
$\mathcal{E}_6$	1	2	3	3	5	8
$\mathcal{E}_7$	1	2	3	3	5	9

$\mathcal{D}_{\mathcal{B}}$

Discrete Symmetries from Modular String Symmetries

▲ In String Theories, Dualities imply modular invariance w.r.t. various moduli fields, in particular, the axio-dilaton, Kähler and Complex Structure (CS) moduli. They appear in the fluxed induced superpotential (we restrict here in type-IIB)

$$W_{\rm IIB} \propto \int G_3 \wedge \Omega \equiv \int (F_3 - \tau H_3) \wedge \Omega$$

as well as in the Kähler potential

$$\hat{K} = -\ln(-i(\tau - \bar{\tau})) - 2\log(\mathcal{V}) + \int \Omega \wedge \bar{\Omega} + \cdots$$

where $\mathcal{V} = \frac{1}{6}\kappa_{ijk} t_i t_j t_k$, ($t_i$ =ImT_i)

• Supersymmetric conditions $D_{\tau_i} W_{\rm IIB} = 0$ impose restrictions and reduce the initial $SL(2, Z)$ symmetry to some congruence group (*note that flux parameters are integers*)

• Symmetry may further break down from the Yukawa sector, $W \supset \lambda_{ij}(g_s) f_i f_j h$ unless certain criteria are imposed.

$\mathcal{A}$). Axio-dilaton τ

Noticing that

$$\mathrm{Im}\tau = \frac{\tau - \bar{\tau}}{2i} = \frac{1}{g_s}$$

we can readily deduce that $e^K \to |c\tau + d|^2 e^K$. Since the gravitino mass $m_{3/2}^2 = e^K |W|^2$ must stay invariant, W must transform as

$$W \to \frac{W}{c\tau + d} \ , \qquad (3)$$

In most common cases the Yukawa couplings are $\lambda \propto {g_s}^{-1/2}$

$$\lambda \propto g_s^{-1/2} \ \to \ \frac{{g_s}^{-1/2}}{|c\tau + d|} \ \to \ \frac{{g_s}^{-1/2}}{|C_0^2 + {g_s}^{-2}|^{1/2}} \sim g_s^{+1/2}$$

($\to$ i.e., strong-weak coupling duality!)

$\mathcal{B}$). Kähler moduli T_i

Let Q^a various fields,

$$K = \hat{K} + \tilde{K}_{a\bar{b}} Q^a \bar{Q}^{\bar{b}} + \cdots, \ \tilde{K}_{a\bar{b}} = \tilde{K}_{a\bar{b}}(T_i)$$

Canonical kinetic terms imply

$$\tilde{K}_a Q^a \bar{Q}^{\bar{a}} = \hat{Q}^a \hat{\bar{Q}}^a, \ \ \hat{\bar{Q}}^a = \sqrt{\tilde{K}_a(T_i)}\, \bar{Q}^a$$

and a redefinition of the Yukawa couplings in the superpotential

$$W = \frac{\lambda_{ijl}}{\sqrt{\tilde{K}_i \tilde{K}_j \tilde{K}_l}} \hat{\bar{Q}}_i \hat{\bar{Q}}_j \hat{\bar{Q}}_l \ \Rightarrow \ \tilde{\lambda}_{ij} = e^{\hat{K}/2} \frac{\lambda_{ij}}{\sqrt{\tilde{K}_i \tilde{K}_j \tilde{K}_h}}$$

$\mathcal{C}$). CS moduli τ_i : similar analysis...

(*Basiouris, Crispin-Romao, King, GKL,* work in progress)

$\mathcal{D}_{\mathcal{C}}$

Discrete Symmetries from E_8

Another origin of **Non-Abelian Discrete Groups** is from the group "perpendicular" to the GUT group, (both $\in E_8$)

$$E_8 \quad \supset \quad SU(5) \times SU(5)_\perp \tag{4}$$

A wide class of Discrete Groups is $PSL_2(p)$, p prime

▲ **Requirements:**

- must be subgroups of $\mathbf{SU(5)_\perp} \rightarrow \mathbf{p \leq 11}$
- must have 3-d representations ($m_\nu \rightarrow 3 \times 3$) $\rightarrow \mathbf{p \leq 7}$

A promising candidate:

$PSL_2(7) \in SU(3)_\perp$

Then, the maximal symmetry embedded in E_8 is

$$\Downarrow$$

$$E_8 \supset E_6 \times SU(3)_\perp \supset E_6 \times PSL_2(7)$$

promising low energy phenomenology! (*see arXiv:1612.06161*)

CONCLUSIONS

F-theory models :

⇓⇓⇓

Provide a Geometric interpretation of GUTs

Calculability, form a handful of topological properties

Predict natural Doublet-Triplet splitting...

May accommodate a **Variety** of new states for New Physics

Discrete symmetries emanate from various sources and can be used to interpret CKM and the Neutrino data

THANK YOU

$\mathcal{E}$

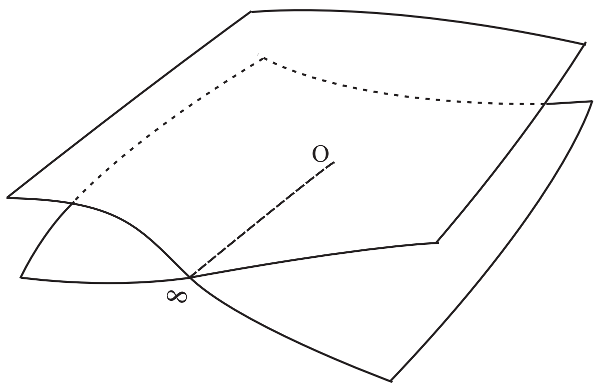
APPENDIX

EXAMPLE ..Simplest monodromy Z_2 : :

$$a_1 + a_2 s + a_3 s^2 = 0 \to s_{1,2} = \frac{-a_2 \pm \sqrt{\Delta}}{2a_3}$$

Under $\theta \to \theta + 2\pi \to \sqrt{\Delta} \to -\sqrt{\Delta}$ branes interchange locations

$$s_1 \leftrightarrow s_2 \text{ or } t_1 \leftrightarrow t_2$$



Two $\mathbf{U(1)}$'s related by monodromies , gauge symmetry reduces to:

$$SU(5) \times U(1)^4 \to \mathbf{SU(5)} \times \mathbf{U(1)^3}$$

▲ Superstring Theories are characterised by dualities associated with the modular group $SL(2,\mathbb{Z})$. The latter is represented by 2×2 matrices
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det A = 1,\ a,b,c,d \in \mathbb{Z}$.

▲ $SL(2,Z)$ describes the equivalence class of diffeomorphisms of the torus and as such it is related to toroidal compactifications.

▲ Because the action of A and $-A$ on the modulus is the same, we define the projective group $\bar{\Gamma} = PSL(2,Z) \equiv SL(2,Z)/\{I,-I\}$.

▲ The **principal congruence subgroup of level** N is defined by the subset of matrices $\Gamma(N) \in SL(2,Z)$ which are equal to identity matrix mod$N$. Identification of positive and negative unit matrices results to $\bar{\Gamma}(N)$.
In Physical applications we deal with the quotient (finite) groups

$$\Gamma_N = PSL(2,Z)/\bar{\Gamma}(N),\ \ S^2 = (ST)^3 = T^N = 1$$

▲ *Construction of 3-d. irreducible representation of* $PSL_2(7)$
(*E.G.Floratos, GKL arXiv:1511.01875*)

$$\mathfrak{a}^2 = \mathfrak{b}^3 = (\mathfrak{a}\mathfrak{b})^7 = ([\mathfrak{a}, \mathfrak{b}])^4 = I$$

Method: use of Weil's Metaplectic Representation
(based on *Balian & Itzykson* Acad. Dc. Paris 303 (1986).)

Defining $\eta = e^{2\pi i/7}$, we generators are found to be:

$$\mathfrak{a} \to A^{[3]} \quad = \quad \frac{i}{\sqrt{7}} \begin{pmatrix} \eta^2 - \eta^5 & \eta^6 - \eta & \eta^3 - \eta^4 \\ \eta^6 - \eta & \eta^4 - \eta^3 & \eta^2 - \eta^5 \\ \eta^3 - \eta^4 & \eta^2 - \eta^5 & \eta - \eta^6 \end{pmatrix}$$

and

$$\mathfrak{b} \to B^{[3]} \quad = \quad \frac{i}{\sqrt{7}} \begin{pmatrix} \eta - \eta^4 & \eta^4 - \eta^6 & \eta^6 - 1 \\ \eta^5 - 1 & \eta^2 - \eta & \eta^5 - \eta \\ \eta^2 - \eta^3 & 1 - \eta^3 & \eta^4 - \eta^2 \end{pmatrix}$$

Application to neutrino mixing:

Invariance of M_ν *under* $PSL_2(7)$ *(sub)group* A_i

$$[M, A_i] = 0$$

$\Rightarrow$ common eigenvectors, $\rightarrow$ mixing matrix.

Observation: $PSL_2(7)$ generators have Latin square structure:

$$U \quad \propto \quad \begin{pmatrix} r_1 & r_2 & r_3 \\ r_2 & r_3 & r_1 \\ r_3 & r_1 & r_2 \end{pmatrix}$$

Imposing conditions: orthogonality, unitarity , . . . , roots satisfy:

$$x^3 + x^2 - r_1 r_2 r_3 = 0$$

for $PSL_2(7)$, $r_1 r_2 r_3 = \frac{1}{7}$

classification of all 168 elements : (*Aliferis, GKL Vlachos*)

Example: *The following elements give the correct mixing* (*commuting with* $[M_\nu, U_1] = 0$, $[M_\ell, U_2] = 0]$ *respectively*)

$$U_1 = \begin{pmatrix} r_3 & -r_1 & -r_2 \\ -r_1 & r_2 & r_3 \\ -r_2 & r_3 & r_1 \end{pmatrix}, \; U_2 = \begin{pmatrix} 0 & 0 & -e^{\frac{6\pi i}{7}} \\ e^{-\frac{2\pi i}{7}} & 0 & 0 \\ 0 & e^{-\frac{4\pi i}{7}} & 0 \end{pmatrix}$$

$$U_\nu = \begin{pmatrix} 0.802e^{0.57i} & 0.577e^{2.39i} & 0.153e^{-1.27i} \\ 0.366e^{0.1065i} & 0.577e^{-0.87i} & 0.729e^{-0.35i} \\ 0.471e^{-1.66i} & 0.577e^{3.05i} & 0.667e^{0.64i} \end{pmatrix}$$

Comparison with experimental data:

▲ $\theta_{12}, \theta_{23}, \theta_{13}$ in agreement with experimental values.

▲ θ_{13} *automatically* non-zero (*see arXiv:1612.06161*)