Relativistic second-order spin hydrodynamics from Zubarev's non-equilibrium statistical operator

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第十四届全国粒子物理学术会议 山东 · 青岛

2024 年 8 月 16 日

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Motivation

The experimental measurement of the global polarization of quark matter in heavy ion collisions has confirmed the existence of spin polarization and spin transport phenomena in quark matter, which has greatly enriched people's understanding of the strong interaction. However, there are still several problems that need to be studied in both theoretical and experimental aspects: the difference of $\Lambda/\bar{\Lambda}$ hyperon polarization, the rapidity dependence of global polarization, the local hyperon polarization, the difference and similarity of global polarization of different vector mesons, etc.

Conservation laws

Total angular momentum tensor conservation

$$\partial_\lambda \hat{J}^{\lambda\mu\nu} = \partial_\lambda \hat{S}^{\lambda\mu\nu} + 2\,\hat{T}^{[\mu\nu]} = 0, \quad \hat{J}^{\lambda\mu\nu} = \underbrace{x^\mu \hat{T}^{\lambda\nu} - x^\nu \hat{T}^{\lambda\mu}}_{\text{orbital angular momentum}} + \underbrace{\hat{S}^{\lambda\mu\nu}}_{\text{spin angular momentum}},$$

Energy-momentum tensor conservation

$$\partial_\mu \hat{T}^{\mu\nu} = 0,$$

Charges currents conservation

$$\partial_\mu \hat{N}^\mu_a = 0, \quad a = 1, \ldots, l.$$

The decomposition of the total angular momentum current obtained by choosing a particular form of the energy-momentum tensor and the spin tensor is not unique. A given pair of these currents can always be transformed to another one through the so-called pseudo-gauge.

Conservation laws

We choose the canonical forms of conserved currents with the energy-momentum tensor having both symmetric and antisymmetric parts and the spin tensor being totally antisymmetric in all its Lorentz indices. One can add to energy-momentum tensor an additional totally divergent term: $\hat{\tilde{T}}^{\mu\nu} = \hat{T}^{\mu\nu} + \partial_\lambda(u^\nu \hat{S}^{\mu\lambda})$.

Total angular momentum tensor conservation

$$\partial_\lambda \hat{J}^{\lambda\mu\nu} = 2\hat{\tilde{T}}^{[\mu\nu]} + \partial_\lambda \hat{S}^{\lambda\mu\nu} = 2\hat{T}^{[\mu\nu]} + \partial_\lambda(u^\nu \hat{S}^{\mu\lambda}) - \partial_\lambda(u^\mu \hat{S}^{\nu\lambda}) + \partial_\lambda \hat{S}^{\lambda\mu\nu} = 0,$$

Energy-momentum tensor conservation

$$\partial_\mu \hat{\tilde{T}}^{\mu\nu} = \partial_\mu \hat{T}^{\mu\nu} + \partial_\mu \partial_\lambda(u^\nu \hat{S}^{\mu\lambda}) = \partial_\mu \hat{T}^{\mu\nu} = 0,$$

Charges currents conservation

$$\partial_\mu \hat{N}^\mu_a = 0, \quad a = 1, \ldots, l.$$

Non-equilibrium statistical operator

Thermodynamic relations

$$Tds = d\epsilon - \sum_a \mu_a dn_a - \omega_{\alpha\beta} dS^{\alpha\beta},$$

$$\epsilon + p = Ts + \sum_a \mu_a n_a + \omega_{\alpha\beta} S^{\alpha\beta},$$

$$dp = sdT + \sum_a n_a d\mu_a + S^{\alpha\beta} d\omega_{\alpha\beta}.$$

Zubarev's formalism is based on a generalization of the Gibbs canonical ensemble to non-equilibrium states. The full non-equilibrium statistical operator should be found from the quantum Liouville equation with an infinitesimal source term (memory effects),

$$\hat{\rho}(t) = Q^{-1}(t) \exp\Big\{-\int d^3x \hat{Z}(\boldsymbol{x}, t)\Big\}, \quad Q(t) = \mathrm{Tr} \exp\Big\{-\int d^3x \hat{Z}(\boldsymbol{x}, t)\Big\},$$

$$\hat{Z}(\boldsymbol{x}, t) = \varepsilon \int_{-\infty}^{t} dt_1 e^{\varepsilon(t_1 - t)} \Big[\beta_\nu(\boldsymbol{x}, t_1)\, \hat{T}^{0\nu}(\boldsymbol{x}, t_1) - \sum_a \alpha_a(\boldsymbol{x}, t_1)\, \hat{N}^0_a(\boldsymbol{x}, t_1) - \widetilde{\omega}_{\alpha\beta}(\boldsymbol{x}, t_1)\, \hat{S}^{0\alpha\beta}(\boldsymbol{x}, t_1)\Big].$$

Non-equilibrium statistical operator

It can also be divided into local equilibrium and non-equilibrium contributions by integrating it by parts and using the conservation laws. The statistical operator is promoted to a non-local functional of the thermodynamic parameters and their space-time derivatives,

$$\hat{\rho}(t) = Q^{-1} e^{-\hat{A}+\hat{B}}, \quad Q = \mathrm{Tr} e^{-\hat{A}+\hat{B}},$$

$$\hat{A}(t) = \int d^3x \Big[\beta_\nu(\boldsymbol{x},t)\,\hat{T}^{0\nu}(\boldsymbol{x},t) - \sum_a \alpha_a(\boldsymbol{x},t)\hat{N}_a^0(\boldsymbol{x},t) - \widetilde{\omega}_{\alpha\beta}(\boldsymbol{x},t)\hat{S}^{0\alpha\beta}(\boldsymbol{x},t)\Big],$$

$$\hat{B}(t) = \int d^3x \int_{-\infty}^{t} dt_1\, e^{\varepsilon(t_1-t)}\,\hat{C}(\boldsymbol{x},t_1),$$

$$\hat{C}(\boldsymbol{x},t) = \hat{T}^{\mu\nu}(\boldsymbol{x},t)\partial_\mu\beta_\nu(\boldsymbol{x},t) - \sum_a \hat{N}_a^\mu(\boldsymbol{x},t)\partial_\mu\alpha_a(\boldsymbol{x},t) - \hat{S}^{\lambda\alpha\beta}(\boldsymbol{x},t)\,\partial_\lambda\widetilde{\omega}_{\alpha\beta}(\boldsymbol{x},t)$$
$$+\, 2\widetilde{\omega}_{\alpha\beta}(\boldsymbol{x},t)\,\hat{T}^{[\alpha\beta]}(\boldsymbol{x},t) + 2u^\alpha\hat{S}^{\beta\lambda}\partial_\lambda\widetilde{\omega}_{\alpha\beta}(\boldsymbol{x},t).$$

Second-order expansion of the statistical operator

The non-equilibrium statistical operator can then be expanded in a series with respect to the thermodynamic forces up to the second order,

$$\hat{\rho} = \hat{\rho}_l + \hat{\rho}_1 + \hat{\rho}_2,$$

$$\hat{\rho}_1 = \int d^4x_1 \int_0^1 d\tau \left[\hat{C}_\tau(x_1) - \left\langle \hat{C}_\tau(x_1) \right\rangle_l \right] \hat{\rho}_l, \quad \hat{X}_\tau = e^{-\tau\hat{A}} \hat{X} e^{\tau\hat{A}}, \quad \int d^4x_1 \equiv \int d^3x_1 \int_{-\infty}^{t} dt_1 e^{\varepsilon(t_1-t)},$$

$$\hat{\rho}_2 = \frac{1}{2} \int d^4x_1 d^4x_2 \int_0^1 d\tau \int_0^1 d\lambda \Big[\widetilde{T}\Big\{ \hat{C}_\lambda(x_1) \hat{C}_\tau(x_2) \Big\} - \Big\langle \widetilde{T}\Big\{ \hat{C}_\lambda(x_1) \hat{C}_\tau(x_2) \Big\} \Big\rangle_l$$
$$- \left\langle \hat{C}_\lambda(x_1) \right\rangle_l \hat{C}_\tau(x_2) - \hat{C}_\lambda(x_1) \left\langle \hat{C}_\tau(x_2) \right\rangle_l + 2 \left\langle \hat{C}_\lambda(x_1) \right\rangle_l \left\langle \hat{C}_\tau(x_2) \right\rangle_l \Big] \hat{\rho}_l.$$

We can now write down the statistical average of an arbitrary operator $\langle \hat{X}(x) \rangle$ as

$$\langle \hat{X}(x) \rangle = \langle \hat{X}(x) \rangle_l + \int d^4x_1 \left(\hat{X}(x), \hat{C}(x_1) \right) + \int d^4x_1 \int d^4x_2 \left(\hat{X}(x), \hat{C}(x_1), \hat{C}(x_2) \right),$$

Second-order expansion of the statistical operator

where we defined a two-point correlation function and the three-point correlation function

$$\left(\hat{X}(x), \hat{Y}(x_1)\right) = \int_0^1 d\tau \left\langle \hat{X}(x) \left[\hat{Y}_\tau(x_1) - \langle \hat{Y}_\tau(x_1)\rangle_l\right]\right\rangle_l,$$

$$\left(\hat{X}(x), \hat{Y}(x_1), \hat{Z}(x_2)\right) = \frac{1}{2}\int_0^1 d\tau \int_0^1 d\lambda \Big\langle \widetilde{T}\Big\{\hat{X}(x)\Big[\hat{Y}_\lambda(x_1)\hat{Z}_\tau(x_2) - \langle \widetilde{T}\hat{Y}_\lambda(x_1)\hat{Z}_\tau(x_2)\rangle_l$$
$$- \langle \hat{Y}_\lambda(x_1)\rangle_l \hat{Z}_\tau(x_2) - \hat{Y}_\lambda(x_1)\langle \hat{Z}_\tau(x_2)\rangle_l + 2\langle \hat{Y}_\lambda(x_1)\rangle_l \langle \hat{Z}_\tau(x_2)\rangle_l\Big]\Big\}\Big\rangle_l.$$

It is straightforward to find the symmetry relation

$$\int d^4x_1 d^4x_2 \left(\hat{X}(x), \hat{Y}(x_1), \hat{Z}(x_2)\right) = \int d^4x_1 d^4x_2 \left(\hat{X}(x), \hat{Z}(x_1), \hat{Y}(x_2)\right).$$

Hydrodynamic equations

The decompositions of the charge currents, the spin tensor and the energy-momentum tensor in terms of their equilibrium and dissipative parts

$$
\begin{aligned}
\hat{N}_a^\mu &= \hat{n}_a u^\mu + \hat{j}_a^\mu, \\
\hat{S}^{\lambda\mu\nu} &= u^\lambda \hat{S}^{\mu\nu} + u^\mu \hat{S}^{\nu\lambda} + u^\nu \hat{S}^{\lambda\mu} + \hat{\varpi}^{\lambda\mu\nu}, \\
\hat{T}^{\mu\nu} &= \hat{\epsilon} u^\mu u^\nu - \hat{p}\Delta^{\mu\nu} + \hat{h}^\mu u^\nu + \hat{h}^\nu u^\mu + \hat{\pi}^{\mu\nu} + \hat{q}^\mu u^\nu - \hat{q}^\nu u^\mu + \hat{\phi}^{\mu\nu},
\end{aligned}
$$

Hydrodynamic gradient expansion

$$
\begin{aligned}
&\hat{\epsilon} \sim \hat{n}_a \sim \hat{S}^{\mu\nu} \sim u^\mu \sim \mathcal{O}\left(\partial^0\right), \\
&\hat{h}^\mu \sim \hat{\pi}^{\mu\nu} \sim \hat{q}^\mu \sim \hat{\phi}^{\mu\nu} \sim \hat{\varpi}^{\lambda\mu\nu} \sim \omega^{\mu\nu} \sim \mathcal{O}\left(\partial^1\right).
\end{aligned}
$$

Hydrodynamic equations

Note that here we did not separate the equilibrium part of the pressure from the bulk-viscous pressure. The statistical average of the operator $\hat{p}$ gives the actual isotropic (non-equilibrium) pressure, which in general differs from the equilibrium pressure $p(\langle\hat{\epsilon}\rangle, \langle\hat{n}_a\rangle, \langle\hat{S}^{\alpha\beta}\rangle)$ given by the EoS. The bulk-viscous pressure is defined as the difference of these two averages.

Hydrodynamics equations

$$
\begin{aligned}
&Dn_a + n_a\theta + \partial_\mu j_a^\mu = 0,\\
&D\epsilon + (\epsilon + p + \Pi)\theta + \partial_\mu h^\mu - h^\mu Du_\mu - \pi^{\mu\nu}\sigma_{\mu\nu} + \partial_\mu q^\mu + q^\mu Du_\mu - \phi^{\mu\nu}\partial_\mu u_\nu = 0,\\
&(\epsilon + p + \Pi)Du_\alpha - \nabla_\alpha(p + \Pi) + \Delta_{\alpha\mu}Dh^\mu + h^\mu\partial_\mu u_\alpha + h_\alpha\theta + \Delta_{\alpha\nu}\partial_\mu\pi^{\mu\nu}\\
&+ q^\mu\partial_\mu u_\alpha - q_\alpha\theta - \Delta_{\alpha\nu}Dq^\nu + \Delta_{\alpha\nu}\partial_\mu\phi^{\mu\nu} = 0,\\
&DS^{\mu\nu} + \theta S^{\mu\nu} + \partial_\lambda\varpi^{\lambda\mu\nu} + 2q^\mu u^\nu - 2q^\nu u^\mu + 2\phi^{\mu\nu} = 0.
\end{aligned}
$$

Decomposition into different dissipative processes

The final form of the operator $\hat{C}$ to first order in gradients

$$\hat{C} = -\beta\theta\hat{p}^{*} - \sum_a \hat{\mathscr{J}}_a^{\sigma}\nabla_\sigma\alpha_a - \beta\hat{q}^{\mu}\left(Du_\mu + \beta\nabla_\mu T - 4\omega_{\mu\nu}u^{\nu}\right)$$
$$+\beta\hat{\pi}^{\mu\nu}\sigma_{\mu\nu} + \hat{\phi}^{\mu\nu}\left(\Omega_{\mu\nu} + 2\beta\omega_{\langle\mu\rangle\langle\nu\rangle}\right),$$

where

$$\hat{p}^{*} = \hat{p} - \gamma\hat{\epsilon} - \sum_a \delta_a\hat{n}_a - \delta_{\alpha\beta}\hat{S}^{\alpha\beta}, \quad \hat{\mathscr{J}}_a^{\sigma} = \hat{j}_a^{\sigma} - \frac{n_a}{\epsilon + p}\hat{h}^{\sigma},$$

$$\gamma = \left.\frac{\partial p}{\partial \epsilon}\right|_{n_a, S^{\alpha\beta}}, \quad \delta_c = \left.\frac{\partial p}{\partial n_c}\right|_{\epsilon, n_b \neq n_c, S^{\alpha\beta}}, \quad \delta_{\alpha\beta} = \left.\frac{\partial p}{\partial S^{\alpha\beta}}\right|_{\epsilon, n_a}.$$

Computing the dissipative quantities

According to Curie's theorem, in an isotropic medium the correlations between operators of different rank and spatial parity vanish. We obtain the shear-stress tensor to leading order

$$\begin{aligned}\pi_{\mu\nu}(x) \equiv& \langle \hat{\pi}_{\mu\nu}(x)\rangle_1 = \int d^4x_1 \left(\hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1)\right)\beta(x_1)\,\sigma^{\rho\sigma}(x_1) \\ =& \beta(x)\,\sigma^{\rho\sigma}(x)\int d^4x_1 \left(\hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1)\right).\end{aligned}$$

The main contribution to the integrand comes from the range $|x_1 - x| \lesssim \lambda$, where λ is a typical microscopic length scale over which the shear-stress correlation function decays. On the other hand, in the hydrodynamic regime, the thermodynamic parameters and the fluid velocity vary over a macroscopic length scale $L \gg \lambda$.
The isotropy of the medium implies

$$\left(\hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1)\right) = \frac{1}{5}\Delta_{\mu\nu\rho\sigma}(x)\left(\hat{\pi}^{\lambda\eta}(x), \hat{\pi}_{\lambda\eta}(x_1)\right),$$

Computing the dissipative quantities

The required linear relations between the dissipative currents and the thermodynamic forces

$$\pi_{\mu\nu} = 2\eta\sigma_{\mu\nu}, \quad \Pi = -\zeta\theta, \quad \phi^{\mu\nu} = \gamma\left(\Omega^{\mu\nu} + 2\beta\omega^{\langle\mu\rangle\langle\nu\rangle}\right),$$
$$\mathscr{J}_a^\mu = \sum_b \chi_{ab}\nabla^\mu\alpha_b, \quad q^\mu = \lambda\left(Du^\mu + \beta\nabla^\mu T - 4\omega^{\mu\nu}u_\nu\right),$$

$$\eta(x) = \frac{\beta(x)}{10}\int d^4x_1\left(\hat{\pi}_{\mu\nu}(x), \hat{\pi}^{\mu\nu}(x_1)\right) = -\frac{1}{10}\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{\pi}_{\mu\nu}\hat{\pi}^{\mu\nu}}(\omega)\Big|_{\omega=0},$$
$$\zeta(x) = \beta(x)\int d^4x_1\left(\hat{p}^*(x), \hat{p}^*(x_1)\right) = -\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{p}^*\hat{p}^*}(\omega)\Big|_{\omega=0},$$
$$\gamma(x) = \frac{1}{3}\int d^4x_1\left(\hat{\phi}^{\mu\nu}(x), \hat{\phi}_{\mu\nu}(x_1)\right) = -\frac{T}{3}\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{\phi}^{\mu\nu}\hat{\phi}_{\mu\nu}}(\omega)\Big|_{\omega=0},$$
$$\chi_{ab}(x) = -\frac{1}{3}\int d^4x_1\left(\hat{\mathscr{J}}_a^\lambda(x), \hat{\mathscr{J}}_{b\lambda}(x_1)\right) = \frac{T}{3}\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{\mathscr{J}}_a^\lambda\hat{\mathscr{J}}_{b\lambda}}(\omega)\Big|_{\omega=0},$$
$$\lambda(x) = -\frac{\beta(x)}{3}\int d^4x_1\left(\hat{q}^\lambda(x), \hat{q}_\lambda(x_1)\right) = \frac{1}{3}\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{q}^\lambda\hat{q}_\lambda}(\omega)\Big|_{\omega=0}.$$

Decomposing the thermodynamic force up to the second order

The final form of the operator $\hat{C}$ to second order in gradients

$$\hat{C}(x) = \hat{C}_1(x) + \hat{C}_2(x),$$

where $\hat{C}_1$ and $\hat{C}_2$ are the first- and the second-order contributions,

$$\begin{aligned}
\hat{C}_1(x) =& -\beta\theta\hat{p}^* - \sum_a \hat{\mathscr{J}}_a^\sigma \nabla_\sigma \alpha_a - \beta\hat{q}^\mu\left(Du_\mu + \beta\nabla_\mu T - 4\omega_{\mu\nu}u^\nu\right) + \beta\hat{\pi}^{\mu\nu}\sigma_{\mu\nu} \\
&+ \hat{\phi}^{\mu\nu}\left(\Omega_{\mu\nu} + 2\beta\omega_{\langle\mu\rangle\langle\nu\rangle}\right) - \hat{\varpi}^{\lambda\alpha\beta}\Delta_{\{\lambda\alpha\beta\}\{\rho\sigma\delta\}}\nabla^\rho\widetilde{\omega}^{\sigma\delta}, \\
\hat{C}_2(x) =& -\hat{\beta}^*\left(\Pi\theta + \partial_\mu h^\mu - h^\mu Du_\mu - \pi^{\mu\nu}\sigma_{\mu\nu} + \partial_\mu q^\mu + q^\mu Du_\mu - \phi^{\mu\nu}\partial_\mu u_\nu\right) + \sum_a \hat{\alpha}_a^*\partial_\mu j_a^\mu \\
&+ \left(\partial_\lambda\varpi^{\lambda\alpha\beta} + 4q^\alpha u^\beta + 2\phi^{\alpha\beta}\right)\hat{\widetilde{\omega}}^*_{\alpha\beta} - \hat{h}^\sigma\frac{\beta}{\epsilon + p}\left(-TS^{\alpha\beta}\nabla_\sigma\widetilde{\omega}_{\alpha\beta} - \nabla_\sigma\Pi + \Pi Du_\sigma\right. \\
&\left. + Dh_\sigma + h^\mu\partial_\mu u_\sigma + h_\sigma\theta + \partial^\mu\pi_{\mu\sigma} + q^\mu\partial_\mu u_\sigma - q_\sigma\theta - Dq_\sigma + \partial^\mu\phi_{\mu\sigma}\right).
\end{aligned}$$

Decomposing the thermodynamic force up to the second order

with

$$\hat{\beta}^* = \hat{\epsilon}\frac{\partial \beta}{\partial \epsilon}\bigg|_{n_a, S^{\alpha\beta}} + \sum_a \hat{n}_a \frac{\partial \beta}{\partial n_a}\bigg|_{\epsilon, n_b \neq n_a, S^{\alpha\beta}} + \hat{S}^{\alpha\beta}\frac{\partial \beta}{\partial S^{\alpha\beta}}\bigg|_{\epsilon, n_b},$$

$$\hat{\alpha}_a^* = \hat{\epsilon}\frac{\partial \alpha_a}{\partial \epsilon}\bigg|_{n_b, S^{\alpha\beta}} + \sum_c \hat{n}_c \frac{\partial \alpha_a}{\partial n_c}\bigg|_{\epsilon, n_b \neq n_c, S^{\alpha\beta}} + \hat{S}^{\alpha\beta}\frac{\partial \alpha_a}{\partial S^{\alpha\beta}}\bigg|_{\epsilon, n_b},$$

$$\hat{\widetilde{\omega}}^{\alpha\beta *} = \hat{\epsilon}\frac{\partial \widetilde{\omega}_{\alpha\beta}}{\partial \epsilon}\bigg|_{n_b, S^{\alpha\beta}} + \sum_c \hat{n}_c \frac{\partial \widetilde{\omega}_{\alpha\beta}}{\partial n_c}\bigg|_{\epsilon, n_b \neq n_c, S^{\alpha\beta}} + \hat{S}^{\delta\rho}\frac{\partial \widetilde{\omega}_{\alpha\beta}}{\partial S^{\delta\rho}}\bigg|_{\epsilon, n_b, S^{\alpha\beta} \neq S^{\delta\rho}}.$$

We can write the statistical average of an arbitrary operator $\hat{X}(x)$ up to second order

$$\langle \hat{X}(x) \rangle = \langle \hat{X}(x) \rangle_l + \langle \hat{X}(x) \rangle_1 + \langle \hat{X}(x) \rangle_2,$$

Decomposing the thermodynamic force up to the second order

The first-order correction is given by

$$\langle \hat{X}(x) \rangle_1 = \int d^4 x_1 \big(\hat{X}(x) \, , \hat{C}_1 \, (x_1) \big) \big|_{\text{loc}},$$

The second-order correction $\langle \hat{X}(x) \rangle_2$ can be decomposed into three terms,

$$\begin{aligned}
\langle \hat{X}(x) \rangle_2 &= \langle \hat{X}(x) \rangle_2^1 + \langle \hat{X}(x) \rangle_2^2 + \langle \hat{X}(x) \rangle_2^3, \\
\langle \hat{X}(x) \rangle_2^1 &= \int d^4 x_1 \big(\hat{X}(x) \, , \hat{C}_1 \, (x_1) \big) - \langle \hat{X}(x) \rangle_1, \\
\langle \hat{X}(x) \rangle_2^2 &= \int d^4 x_1 \big(\hat{X}(x) \, , \hat{C}_2 \, (x_1) \big), \\
\langle \hat{X}(x) \rangle_2^3 &= \int d^4 x_1 d^4 x_2 \big(\hat{X}(x) \, , \hat{C}_1 \, (x_1) \, , \hat{C}_1 \, (x_2) \big).
\end{aligned}$$

Second-order corrections to the shear-stress tensor

Relaxation equation for the shear stress tensor

$$\tau_\pi \dot{\pi}_{\mu\nu} + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} + \widetilde{\eta}\theta\pi_{\mu\nu} + \overline{\eta}\theta\sigma_{\mu\nu} + \sum_{ab} \widetilde{\eta}_2^{ab} \nabla_{\langle\mu}\alpha_a \nabla_{\nu\rangle}\alpha_b + \sum_a \widetilde{\eta}_3^a \nabla_{\langle\mu}\alpha_a \left(Du_{\nu\rangle} + \beta\nabla_{\nu\rangle} T - 4\omega_{\nu\rangle\beta} u^\beta\right)$$
$$+ \widetilde{\eta}_4 \left(Du_{\langle\mu} + \beta\nabla_{\langle\mu} T - 4\omega_{\langle\mu\sigma} u^\sigma\right)\left(Du_{\nu\rangle} + \beta\nabla_{\nu\rangle} T - 4\omega_{\nu\rangle\beta} u^\beta\right) + \widetilde{\eta}_5 \sigma_{\alpha\langle\mu}\sigma^\alpha_{\nu\rangle}$$
$$+ \widetilde{\eta}_6 \sigma_{(\mu\alpha}\left(\Omega^\alpha{}_{\nu)} + 2\beta\omega^{\langle\alpha\rangle}{}_{\langle\nu\rangle)}\right) + \widetilde{\eta}_7 \left(\Omega_{\langle\nu\alpha} + 2\beta\omega_{\langle\langle\nu\rangle\langle\alpha\rangle}\right)\left(\Omega^\alpha{}_{\mu\rangle} + 2\beta\omega^{\langle\alpha\rangle}{}_{\langle\mu\rangle\rangle}\right).$$

$$\overline{\eta} = \widetilde{\eta}_1 - 2\gamma\eta\tau_\pi, \quad \widetilde{\eta} = \tau_\pi \beta\eta^{-1}\left(\gamma\frac{\partial\eta}{\partial\beta} - \sum_a \delta_a \frac{\partial\eta}{\partial\alpha_a} - \delta_{\alpha\beta}\frac{\partial\eta}{\partial\widetilde{\omega}_{\alpha\beta}}\right),$$

$$\eta\tau_\pi = -i\frac{d}{d\omega}\eta(\omega)\Big|_{\omega=0} = \frac{1}{20}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{\pi}_{\mu\nu}\hat{\pi}^{\mu\nu}}(\omega)\Big|_{\omega=0}, \quad \widetilde{\eta}_1 = -\frac{2}{5}\beta^2\int d^4x_1 d^4x_2 \left(\hat{\pi}_{\gamma\delta}(x), \hat{p}^*(x_1), \hat{\pi}^{\gamma\delta}(x_2)\right),$$

$$\widetilde{\eta}_2^{ab} = \frac{1}{5}\int d^4x_1 d^4x_2 \left(\hat{\pi}_{\gamma\delta}(x), \hat{\mathscr{J}}_a^\gamma(x_1), \hat{\mathscr{J}}_b^\delta(x_2)\right), \quad \widetilde{\eta}_3^a = \frac{2}{5}\beta\int d^4x_1 d^4x_2 \left(\hat{\pi}_{\gamma\delta}(x), \hat{\mathscr{J}}_a^\gamma(x_1), \hat{q}^\delta(x_2)\right),$$

$$\widetilde{\eta}_4 = \frac{1}{5}\beta^2\int d^4x_1 d^4x_2 \left(\hat{\pi}_{\gamma\delta}(x), \hat{q}^\gamma(x_1), \hat{q}^\delta(x_2)\right), \quad \widetilde{\eta}_5 = \frac{12}{35}\beta^2\int d^4x_1 d^4x_2 \left(\hat{\pi}_\gamma^\delta(x), \hat{\pi}_\delta^\lambda(x_1), \hat{\pi}_\lambda^\gamma(x_2)\right),$$

$$\widetilde{\eta}_6 = \frac{8}{15}\beta\int d^4x_1 d^4x_2 \left(\hat{\pi}_\gamma{}^\delta(x), \hat{\pi}_\delta{}^\lambda(x_1), \hat{\phi}_\lambda{}^\gamma(x_2)\right), \quad \widetilde{\eta}_7 = \frac{4}{5}\int d^4x_1 d^4x_2 \left(\hat{\pi}_\gamma{}^\delta(x), \hat{\phi}_\delta{}^\lambda(x_1), \hat{\phi}_\lambda{}^\gamma(x_2)\right),$$

Second-order corrections to the bulk viscous pressure

Relaxation equation for the bulk viscous pressure

$$
\begin{aligned}
\tau_\Pi \dot{\Pi} + \Pi = & - \zeta\theta + \varsigma\theta^2 + \widetilde{\zeta}\theta\Pi + \sum_a \zeta_{\alpha_a}\partial_\mu \mathscr{J}_a^\mu - \zeta_\beta\left(\Pi\theta - \pi^{\mu\nu}\sigma_{\mu\nu} + \partial_\mu q^\mu\right) - \widetilde{\zeta}_\beta \partial_\mu h^\mu \\
& + h^\mu \left[\zeta_\beta D u_\mu + \sum_a \zeta_{\alpha_a}\nabla_\mu\left(\frac{n_a}{\epsilon + p}\right)\right] - q^\mu\left(\zeta_\beta D u_\mu - 4\zeta_{\omega\mu\nu}u^\nu\right) + \phi^{\mu\nu}\left(\zeta_\beta\partial_\mu u_\nu + 2\zeta_{\omega\mu\nu}\right) \\
& + \zeta_{\omega\alpha\beta}\partial_\lambda\varpi^{\lambda\alpha\beta} + \sum_{ab}\widetilde{\zeta}_2\nabla^\alpha\alpha_a\nabla_\alpha\alpha_b + \widetilde{\zeta}_3\left(Du_\alpha + \beta\nabla_\alpha T - 4\omega_{\alpha\beta}u^\beta\right)\sum_a\nabla^\alpha\alpha_a \\
& + \widetilde{\zeta}_4\left(Du^\alpha + \beta\nabla^\alpha T - 4\omega^{\alpha\nu}u_\nu\right)\left(Du_\alpha + \beta\nabla_\alpha T - 4\omega_{\alpha\beta}u^\beta\right) + \widetilde{\zeta}_5\sigma_{\alpha\beta}\sigma^{\alpha\beta} \\
& + \widetilde{\zeta}_6\left(\Omega^{\alpha\beta} + 2\beta\omega^{\langle\alpha\rangle\langle\beta\rangle}\right)\left(\Omega_{\alpha\beta} + 2\beta\omega_{\langle\alpha\rangle\langle\beta\rangle}\right) + \widetilde{\zeta}_7\Delta_{\{abc\}\{def\}}\nabla^a\widetilde{\omega}^{bc}\nabla^d\widetilde{\omega}^{ef}.
\end{aligned}
$$

$$
\varsigma = \widetilde{\zeta}_1 + \zeta^* + \psi_{\epsilon\epsilon}\zeta_\epsilon^2 + 2\zeta_\epsilon\sum_a\psi_{\epsilon a}\zeta_a + \sum_{ab}\psi_{ab}\zeta_a\zeta_b + 2\sum_a\psi_{a\alpha\beta}\zeta_a\zeta_{\alpha\beta} + \psi_{\alpha\beta\rho\sigma}\zeta_{\alpha\beta}\zeta_{\rho\sigma} + 2\psi_{\epsilon\alpha\beta}\zeta_\epsilon\zeta_{\alpha\beta},
$$

$$
\widetilde{\zeta} = \tau_\Pi\beta\zeta^{-1}\left(\gamma\frac{\partial\zeta}{\partial\beta} - \sum_a\delta_a\frac{\partial\zeta}{\partial\alpha_a} - \delta_{\alpha\beta}\frac{\partial\zeta}{\partial\widetilde{\omega}_{\alpha\beta}}\right), \quad \widetilde{\zeta}_1 = \beta^2\int d^4x_1 d^4x_2\left(\hat{p}^*(x), \hat{p}^*(x_1), \hat{p}^*(x_2)\right),
$$

Second-order corrections to the bulk viscous pressure

$$\widetilde{\zeta}_2^{ab} = \frac{1}{3}\int d^4x_1 d^4x_2 \left(\hat{p}^*(x), \hat{\mathscr{J}}_{a\gamma}(x_1), \hat{\mathscr{J}}_b^{\gamma}(x_2)\right), \quad \widetilde{\zeta}_3^{a} = \frac{2}{3}\beta\int d^4x_1 d^4x_2 \left(\hat{p}^*(x), \hat{\mathscr{J}}_{a\gamma}(x_1), \hat{q}^{\gamma}(x_2)\right),$$

$$\widetilde{\zeta}_4 = \frac{1}{3}\beta^2\int d^4x_1 d^4x_2 \left(\hat{p}^*(x), \hat{q}_{\gamma}(x_1), \hat{q}^{\gamma}(x_2)\right), \quad \widetilde{\zeta}_5 = \frac{1}{5}\beta^2\int d^4x_1 d^4x_2 \left(\hat{p}^*(x), \hat{\pi}_{\gamma\delta}(x_1), \hat{\pi}^{\gamma\delta}(x_2)\right),$$

$$\widetilde{\zeta}_6 = \frac{1}{3}\int d^4x_1 d^4x_2 \left(\hat{p}^*(x), \hat{\phi}_{\gamma\delta}(x_1), \hat{\phi}^{\gamma\delta}(x_2)\right), \quad \widetilde{\zeta}_7 = \int d^4x_1 d^4x_2 \left(\hat{p}^*(x), \hat{\varpi}^{\rho\gamma\delta}(x_1), \hat{\varpi}_{\rho\gamma\delta}(x_2)\right),$$

$$\widetilde{\zeta}_\beta = \zeta_\beta - \frac{1}{\epsilon + p}\sum_a n_a \zeta_{\alpha_a}, \quad \zeta_\beta = T\frac{\partial\beta}{\partial\epsilon}\zeta_\epsilon + \sum_c T\frac{\partial\beta}{\partial n_c}\zeta_c + T\frac{\partial\beta}{\partial S^{\alpha\beta}}\zeta_{\alpha\beta},$$

$$\zeta_{\alpha_a} = T\frac{\partial\alpha_a}{\partial\epsilon}\zeta_\epsilon + \sum_c T\frac{\partial\alpha_a}{\partial n_c}\zeta_c + T\frac{\partial\alpha_a}{\partial S^{\alpha\beta}}\zeta_{\alpha\beta}, \quad \zeta_{\omega\alpha\beta} = \frac{\partial\widetilde{\omega}_{\alpha\beta}}{\partial\epsilon}T\zeta_\epsilon + \sum_c \frac{\partial\widetilde{\omega}_{\alpha\beta}}{\partial n_c}T\zeta_c + \frac{\partial\widetilde{\omega}_{\alpha\beta}}{\partial S^{\delta\rho}}T\zeta_{\alpha\beta},$$

$$\begin{aligned}\zeta^* =& \gamma\zeta\tau_\Pi + 2\zeta_\epsilon\tau_\epsilon\Big(\psi_{\epsilon\epsilon}(\epsilon+p) + \sum_a n_a\psi_{\epsilon a} + \psi_{\epsilon\alpha\beta}S^{\alpha\beta}\Big) + 2\sum_a \zeta_a\tau_a\Big(\psi_{\epsilon a}(\epsilon+p) + \sum_b \psi_{ab}n_b + \psi_{a\alpha\beta}S^{\alpha\beta}\Big)\\ &+ 2\zeta_{\alpha\beta}\tau_{\alpha\beta}\Big(\psi_{\epsilon\alpha\beta}(\epsilon+p) + \sum_a \psi_{a\alpha\beta}n_a + \psi_{\alpha\beta\rho\sigma}S^{\rho\sigma}\Big) - 2\theta^{-1}\delta_{\alpha\beta}u^\alpha Du_\lambda\zeta_{\beta\lambda}\tau_{\beta\lambda},\end{aligned}$$

Second-order corrections to the bulk viscous pressure

$$\zeta\tau_\Pi = -i\frac{d}{d\omega}\zeta(\omega)\Big|_{\omega=0} = \frac{1}{2}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{p}^*\hat{p}^*}(\omega)\Big|_{\omega=0}, \quad \zeta_\epsilon\tau_\epsilon = -i\frac{d}{d\omega}\zeta_\epsilon(\omega)\Big|_{\omega=0} = \frac{1}{2}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{p}^*\hat{\epsilon}}(\omega)\Big|_{\omega=0},$$

$$\zeta_a\tau_a = -i\frac{d}{d\omega}\zeta_a(\omega)\Big|_{\omega=0} = \frac{1}{2}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{p}^*\hat{n}_a}(\omega)\Big|_{\omega=0}, \quad \zeta_{\alpha\beta}\tau_{\alpha\beta} = -i\frac{d}{d\omega}\zeta_{\alpha\beta}(\omega)\Big|_{\omega=0} = \frac{1}{2}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{p}^*\hat{S}^{\alpha\beta}}(\omega)\Big|_{\omega=0},$$

$$\zeta_\epsilon = \beta\int d^4x_1\left(\hat{\epsilon}(x), \hat{p}^*(x_1)\right) = -\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{\epsilon}\hat{p}^*}(\omega)\Big|_{\omega=0}, \zeta_a = \beta\int d^4x_1\left(\hat{n}_a(x), \hat{p}^*(x_1)\right) = -\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{n}_a\hat{p}^*}(\omega)\Big|_{\omega=0},$$

$$\zeta_{\alpha\beta} = \beta\int d^4x_1\left(\hat{S}^{\alpha\beta}(x), \hat{p}^*(x_1)\right) = -\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{S}^{\alpha\beta}\hat{p}^*}(\omega)\Big|_{\omega=0}, \quad \psi_{\epsilon\epsilon} = \frac{1}{2}\frac{\partial^2 p}{\partial\epsilon^2}, \psi_{\epsilon a} = \frac{1}{2}\frac{\partial^2 p}{\partial\epsilon\partial n_a},$$

$$\psi_{ab} = \frac{1}{2}\frac{\partial^2 p}{\partial n_a\partial n_b}, \psi_{a\alpha\beta} = \frac{1}{2}\frac{\partial^2 p}{\partial n_a\partial S^{\alpha\beta}}, \psi_{\alpha\beta\rho\sigma} = \frac{1}{2}\frac{\partial^2 p}{\partial S^{\alpha\beta}\partial S^{\rho\sigma}}, \psi_{\epsilon\alpha\beta} = \frac{1}{2}\frac{\partial^2 p}{\partial\epsilon\partial S^{\alpha\beta}}.$$

Second-order corrections to the diffusion currents

Relaxation equation for the diffusion currents

$$\sum_b \tau^{cb}_{\mathscr{J}} \dot{\mathscr{J}}_{b\mu} + \mathscr{J}_{c\mu} = \sum_b \left[\chi_{cb}\nabla_\mu\alpha_b + \overline{\chi}^{cb}\theta\mathscr{J}_{b\mu} + \chi^*_{cb}\theta\nabla_\mu\alpha_b + \widetilde{\chi}^{cb}_3\sigma_{\mu\nu}\nabla^\nu\alpha_b\right] + \chi_{ch}\frac{\beta}{\epsilon+p}\big(-TS^{\alpha\beta}\nabla_\mu\widetilde{\omega}_{\alpha\beta}$$
$$-\nabla_\mu\Pi + \Pi\dot{u}_\mu + \dot{h}_\mu + h^\nu\partial_\nu u_\mu + h_\mu\theta + \Delta_{\mu\sigma}\partial_\nu\pi^{\nu\sigma} + q^\nu\partial_\nu u_\mu - q_\mu\theta - \dot{q}_\mu + \Delta_{\mu\sigma}\partial_\nu\phi^{\nu\sigma}\big)$$
$$+\widetilde{\chi}^c_2\theta\left(Du_\mu + \beta\nabla_\mu T - 4\omega_{\mu\nu}u^\nu\right) + \sum_a \widetilde{\chi}^{ca}_4\nabla^\sigma\alpha_a\left(\Omega_{\mu\sigma} + 2\beta\omega_{\langle\mu\rangle\langle\sigma\rangle}\right)$$
$$+\widetilde{\chi}^c_5\left(Du^\sigma + \beta\nabla^\sigma T - 4\omega^{\sigma\nu}u_\nu\right)\sigma_{\mu\sigma} + \widetilde{\chi}^c_6\left(Du^\sigma + \beta\nabla^\sigma T - 4\omega^{\sigma\nu}u_\nu\right)\left(\Omega_{\mu\sigma} + 2\beta\omega_{\langle\mu\rangle\langle\sigma\rangle}\right)$$
$$+\widetilde{\chi}^c_7\left(\Omega^{\nu\sigma} + 2\beta\omega^{\langle\nu\rangle\langle\sigma\rangle}\right)\Delta_{\{\mu\nu\sigma\}\{\rho\gamma\delta\}}\nabla^\rho\widetilde{\omega}^{\gamma\delta}.$$

$$\overline{\chi}^{cb} = \beta\sum_a \widetilde{\chi}_{ca}\left[\gamma\frac{\partial\left(\chi^{-1}\right)_{ab}}{\partial\beta} - \sum_d \delta_d\frac{\partial\left(\chi^{-1}\right)_{ab}}{\partial\alpha_d} - \delta_{\alpha\beta}\frac{\partial\left(\chi^{-1}\right)_{ab}}{\partial\widetilde{\omega}_{\alpha\beta}}\right],$$

$$\chi^*_{cb} = \widetilde{\chi}^{cb}_1 - \widetilde{\chi}_{ch}\frac{1}{(\epsilon+p)^2}n_b\left[\gamma\left(\epsilon+p\right) + \sum_d \delta_d n_d + \delta_{\alpha\beta}S^{\alpha\beta}\right], \quad \tau^{cb}_{\mathscr{J}} = -\left(\widetilde{\chi}\chi^{-1}\right)_{cb} = -\sum_a \widetilde{\chi}_{ca}\left(\chi^{-1}\right)_{ab},$$

Second-order corrections to the diffusion currents

$$\widetilde{\chi}_1^{ca} = \frac{2}{3}\beta \int d^4x_1 d^4x_2 \left(\hat{\mathscr{J}}_{c\gamma}(x), \hat{\mathscr{J}}_a^{\gamma}(x_1), \hat{p}^*(x_2) \right), \quad \widetilde{\chi}_2^{c} = \frac{2}{3}\beta^2 \int d^4x_1 d^4x_2 \left(\hat{\mathscr{J}}_{c\beta}(x), \hat{q}^{\beta}(x_1), \hat{p}^*(x_2) \right),$$

$$\widetilde{\chi}_3^{ca} = -\frac{2}{5}\beta \int d^4x_1 d^4x_2 \left(\hat{\mathscr{J}}_c^{\gamma}(x), \hat{\mathscr{J}}_a^{\delta}(x_1), \hat{\pi}_{\gamma\delta}(x_2) \right), \quad \widetilde{\chi}_4^{ca} = -\frac{2}{3} \int d^4x_1 d^4x_2 \left(\hat{\mathscr{J}}_c^{\gamma}(x), \hat{\mathscr{J}}_a^{\delta}(x_1), \hat{\phi}_{\gamma\delta}(x_2) \right),$$

$$\widetilde{\chi}_5^{c} = -\frac{2}{5}\beta^2 \int d^4x_1 d^4x_2 \left(\hat{\mathscr{J}}_c^{\gamma}(x), \hat{q}^{\delta}(x_1), \hat{\pi}_{\gamma\delta}(x_2) \right), \quad \widetilde{\chi}_6^{c} = -\frac{2}{3}\beta \int d^4x_1 d^4x_2 \left(\hat{\mathscr{J}}_c^{\gamma}(x), \hat{q}^{\delta}(x_1), \hat{\phi}_{\gamma\delta}(x_2) \right),$$

$$\widetilde{\chi}_7^{c} = -2 \int d^4x_1 d^4x_2 \left(\hat{\mathscr{J}}_{c\rho}(x), \hat{\phi}_{\gamma\delta}(x_1), \hat{\varpi}^{\rho\gamma\delta}(x_2) \right), \quad \widetilde{\chi}_{ac} = i\frac{d}{d\omega}\chi_{ac}(\omega)\Big|_{\omega=0} = \frac{T}{6}\frac{d^2}{d\omega^2}\mathrm{Re}\, G^R_{\hat{\mathscr{J}}_a^{\lambda}\hat{\mathscr{J}}_{c\lambda}}(\omega)\Big|_{\omega=0},$$

$$\chi_{ch} = -\frac{1}{3}\int d^4x_1 \left(\hat{\mathscr{J}}_{c\alpha}(x), \hat{h}^{\alpha}(x_1) \right) = \frac{T}{3}\frac{d}{d\omega}\mathrm{Im}\, G^R_{\hat{\mathscr{J}}_c^{\alpha}\hat{h}_{\alpha}}(\omega)\Big|_{\omega=0},$$

$$\widetilde{\chi}_{ch} = i\frac{d}{d\omega}\chi_{ch}(\omega)\Big|_{\omega=0} = \frac{T}{6}\frac{d^2}{d\omega^2}\mathrm{Re}\, G^R_{\hat{\mathscr{J}}_c^{\lambda}\hat{h}_{\lambda}}(\omega)\Big|_{\omega=0},$$

Second-order corrections to the spatial projection of the antisymmetric stress

Relaxation equation for the spatial projection of the antisymmetric stress

$$
\begin{aligned}
\tau_\phi \dot{\phi}_{\mu\nu} + \phi_{\mu\nu} = {} & \gamma\left(\Omega_{\mu\nu} + 2\beta\omega_{\langle\mu\rangle\langle\nu\rangle}\right) + \widetilde{\gamma}\theta\phi_{\mu\nu} + \widetilde{\gamma}_1\theta\left(\Omega_{\mu\nu} + 2\beta\omega_{\langle\mu\rangle\langle\nu\rangle}\right) + \sum_{ab}\widetilde{\gamma}_2^{ab}\nabla_{[\mu}\alpha_a\nabla_{\nu]}\alpha_b \\
& + \sum_a \widetilde{\gamma}_3^a\nabla_{[\mu}\alpha_a\left(Du_{\nu]} + \beta\nabla_{\nu]}T - 4\omega_{\nu]\rho}u^\rho\right) + \sum_a\widetilde{\gamma}_5^a\nabla^\sigma\alpha_a\Delta_{\{\mu\nu\sigma\}\{\rho\gamma\delta\}}\nabla^\rho\widetilde{\omega}^{\gamma\delta} \\
& + \widetilde{\gamma}_4\left(Du_{[\mu} + \beta\nabla_{[\mu}T - 4\omega_{[\mu\rho}u^\rho\right)\left(Du_{\nu]} + \beta\nabla_{\nu]}T - 4\omega_{\nu]\sigma}u^\sigma\right) \\
& + \widetilde{\gamma}_6\left(Du^\sigma + \beta\nabla^\sigma T - 4\omega^{\sigma\epsilon}u_\epsilon\right)\Delta_{\{\mu\nu\sigma\}\{\rho\gamma\delta\}}\nabla^\rho\widetilde{\omega}^{\gamma\delta} + \widetilde{\gamma}_7\sigma_{[\mu\alpha}\sigma^\alpha{}_{\nu]} \\
& + \widetilde{\gamma}_8\sigma_{[\mu\alpha}\left(\Omega^\alpha{}_{\nu]} + 2\beta\omega^{\langle\alpha\rangle}{}_{\langle\nu\rangle]}\right) + \widetilde{\gamma}_9\left(\Omega_{[\mu\alpha} + 2\beta\omega_{[\langle\mu\rangle\langle\alpha\rangle}\right)\left(\Omega^\alpha{}_{\nu]} + 2\beta\omega^{\langle\alpha\rangle}{}_{\langle\nu\rangle]}\right).
\end{aligned}
$$

Second-order corrections to the spatial projection of the antisymmetric stress

$$\widetilde{\gamma} = \tau_\phi \beta \eta^{-1} \left(\gamma \frac{\partial \gamma}{\partial \beta} - \sum_a \delta_a \frac{\partial \gamma}{\partial \alpha_a} - \delta_{\alpha\beta} \frac{\partial \gamma}{\partial \tilde{\omega}_{\alpha\beta}} \right), \quad \widetilde{\gamma}_1 = -\frac{2}{3}\beta \int d^4x_1 d^4x_2 \left(\hat{\phi}_{\gamma\delta}(x), \hat{p}^*(x_1), \hat{\phi}^{\gamma\delta}(x_2) \right),$$

$$\widetilde{\gamma}_2^{ab} = \frac{1}{3} \int d^4x_1 d^4x_2 \left(\hat{\phi}_{\gamma\delta}(x), \hat{\mathscr{J}}_a^{\gamma}(x_1), \hat{\mathscr{J}}_b^{\delta}(x_2) \right), \quad \widetilde{\gamma}_3^a = \frac{2}{3}\beta \int d^4x_1 d^4x_2 \left(\hat{\phi}_{\gamma\delta}(x), \hat{\mathscr{J}}_a^{\gamma}(x_1), \hat{q}^{\delta}(x_2) \right),$$

$$\widetilde{\gamma}_4 = \frac{1}{3}\beta^2 \int d^4x_1 d^4x_2 \left(\hat{\phi}_{\gamma\delta}(x), \hat{q}^{\gamma}(x_1), \hat{q}^{\delta}(x_2) \right), \quad \widetilde{\gamma}_5^a = 2 \int d^4x_1 d^4x_2 \left(\hat{\phi}_{\varepsilon\gamma}(x), \hat{\mathscr{J}}_{a\epsilon}(x_1), \hat{\varpi}^{\varepsilon\gamma\epsilon}(x_2) \right),$$

$$\widetilde{\gamma}_6 = 2\beta \int d^4x_1 d^4x_2 \left(\hat{\phi}_{\varepsilon\gamma}(x), \hat{q}_{\phi}(x_1), \hat{\varpi}^{\varepsilon\gamma\phi}(x_2) \right), \quad \widetilde{\gamma}_7 = -\frac{4}{15}\beta^2 \int d^4x_1 d^4x_2 \left(\hat{\phi}_{\gamma}{}^{\delta}(x), \hat{\pi}_{\delta}{}^{\lambda}(x_1), \hat{\pi}_{\lambda}{}^{\gamma}(x_2) \right),$$

$$\widetilde{\gamma}_8 = -\frac{8}{5}\beta \int d^4x_1 d^4x_2 \left(\hat{\phi}_{\gamma}{}^{\delta}(x), \hat{\pi}_{\delta}{}^{\lambda}(x_1), \hat{\phi}_{\lambda}{}^{\gamma}(x_2) \right), \quad \widetilde{\gamma}_9 = -\frac{4}{3} \int d^4x_1 d^4x_2 \left(\hat{\phi}_{\gamma}{}^{\delta}(x), \hat{\phi}_{\delta}{}^{\lambda}(x_1), \hat{\phi}_{\lambda}{}^{\gamma}(x_2) \right),$$

$$\gamma\tau_\phi = -i \frac{d}{d\omega} \gamma(\omega) \bigg|_{\omega=0} = \frac{T}{6} \frac{d^2}{d\omega^2} \mathrm{Re}\, G^R_{\hat{\phi}_{\mu\nu}\hat{\phi}^{\mu\nu}}(\omega) \bigg|_{\omega=0},$$

Second-order corrections to the temporal projection of the antisymmetric stress

Relaxation equation for the temporal projection of the antisymmetric stress

$$\begin{aligned}\tau_q \dot{q}_\mu + q_\mu =& \lambda\left(Du_\mu + \beta\nabla_\mu T - 4\omega_{\mu\nu}u^\nu\right) + \lambda^*\theta\left(Du_\mu + \beta\nabla_\mu T - 4\omega_{\mu\nu}u^\nu\right) + \widetilde{\lambda}\theta q_\mu + \sum_b \widetilde{\lambda}_1^b \theta\nabla_\mu\alpha_b \\ &+ \sum_a \widetilde{\lambda}_3^a \sigma_{\mu\sigma}\nabla^\sigma\alpha_a + \sum_a \widetilde{\lambda}_4^a\left(\Omega_{\mu\sigma} + 2\beta\omega_{\langle\mu\rangle\langle\sigma\rangle}\right)\nabla^\sigma\alpha_a + \widetilde{\lambda}_5\sigma_{\mu\sigma}\left(Du^\sigma + \beta\nabla^\sigma T - 4\omega^{\sigma\nu}u_\nu\right) \\ &+ \widetilde{\lambda}_6\left(Du^\sigma + \beta\nabla^\sigma T - 4\omega^{\sigma\nu}u_\nu\right)\left(\Omega_{\mu\sigma} + 2\beta\omega_{\langle\mu\rangle\langle\sigma\rangle}\right) + \widetilde{\lambda}_7\left(\Omega^{\alpha\beta} + 2\beta\omega^{\langle\alpha\rangle\langle\beta\rangle}\right)\Delta_{\{\mu\alpha\beta\}\{\gamma\delta\varepsilon\}}\nabla^\gamma\widetilde{\omega}^{\delta\varepsilon}.\end{aligned}$$

$$\widetilde{\lambda}_q = \tau_q\lambda^{-1}\beta\left(\frac{\partial\lambda}{\partial\beta}\gamma - \sum_d \frac{\partial\lambda}{\partial\alpha_d}\delta_d - \frac{\partial\lambda}{\partial\widetilde{\omega}_{\alpha\beta}}\delta_{\alpha\beta}\right), \quad \lambda^* = \widetilde{\lambda}_2 + \gamma\widetilde{\lambda}, \quad \tau_q = -\widetilde{\lambda}\lambda^{-1},$$

$$\widetilde{\lambda}_1^b = \frac{2}{3}\beta\int d^4x_1 d^4x_2\left(\hat{q}_\beta(x), \hat{p}^*(x_1), \hat{\mathscr{J}}_b^\beta\right), \quad \widetilde{\lambda}_2 = \frac{2}{3}\beta^2\int d^4x_1 d^4x_2\left(\hat{q}_\beta(x), \hat{p}^*(x_1), \hat{q}^\beta(x_2)\right),$$

$$\widetilde{\lambda}_3^a = -\frac{2}{5}\beta\int d^4x_1 d^4x_2\left(\hat{q}^\gamma(x), \hat{\mathscr{J}}_a^\delta(x_1), \hat{\pi}_{\gamma\delta}(x_2)\right), \quad \widetilde{\lambda}_4^a = -\frac{2}{3}\int d^4x_1 d^4x_2\left(\hat{q}^\gamma(x), \hat{\mathscr{J}}_a^\delta(x_1), \hat{\phi}_{\gamma\delta}(x_2)\right),$$

$$\widetilde{\lambda}_5 = -\frac{2}{5}\beta^2\int d^4x_1 d^4x_2\left(\hat{q}^\gamma(x), \hat{q}^\delta(x_1), \hat{\pi}_{\gamma\delta}(x_2)\right), \quad \widetilde{\lambda}_6 = -\frac{2}{3}\beta\int d^4x_1 d^4x_2\left(\hat{q}^\gamma(x), \hat{q}^\delta(x_1), \hat{\phi}_{\gamma\delta}(x_2)\right),$$

$$\widetilde{\lambda}_7 = -2\int d^4x_1 d^4x_2\left(\hat{q}_\gamma(x), \hat{\phi}_{\delta\epsilon}(x_1), \hat{\varpi}^{\gamma\delta\epsilon}(x_2)\right), \quad \widetilde{\lambda} = i\frac{d}{d\omega}\lambda(\omega)\Big|_{\omega=0} = \frac{1}{6}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{q}^\lambda\hat{q}_\lambda}(\omega)\Big|_{\omega=0}.$$

Second-order corrections to the spin-related dissipative current

Relaxation equation for the spin-related dissipative current

$$
\begin{aligned}
\tau_\varphi \dot{\varpi}^{\lambda\mu\nu} + \varpi^{\lambda\mu\nu} =& \varphi \Delta^{\{\lambda\mu\nu\}}_{\{\rho\sigma\delta\}} \nabla^\rho \widetilde{\omega}^{\sigma\delta} + \widetilde{\varphi}\theta\varpi^{\lambda\mu\nu} + \widetilde{\varphi}_1 \theta \Delta^{\{\lambda\mu\nu\}}_{\{abc\}} \nabla^a \widetilde{\omega}^{bc} \\
&+ \sum_a \widetilde{\varphi}_2^a \Delta^{\{\lambda\mu\nu\}\{\rho\sigma\delta\}} \nabla_\rho \alpha_a \left(\Omega_{\sigma\delta} + 2\beta\omega_{\langle\sigma\rangle\langle\delta\rangle}\right) \\
&+ \widetilde{\varphi}_3 \Delta^{\{\lambda\mu\nu\}\{\rho\sigma\delta\}} \left(Du_\rho + \beta\nabla_\rho T - 4\omega_{\rho\alpha}u^\alpha\right)\left(\Omega_{\sigma\delta} + 2\beta\omega_{\langle\sigma\rangle\langle\delta\rangle}\right).
\end{aligned}
$$

$$
\widetilde{\varphi} = \varphi^{-1}\beta\tau_\varphi \left(\frac{\partial\varphi}{\partial\beta}\gamma - \sum_a \frac{\partial\varphi}{\partial\alpha_a}\delta_a - \frac{\partial\varphi}{\partial\widetilde{\omega}_{\alpha\beta}}\delta_{\alpha\beta}\right), \quad \widetilde{\varphi}_1 = 2\beta \int d^4x_1 d^4x_2 \left(\hat{\varpi}^{\gamma\varepsilon\zeta}(x), \hat{p}^*(x_1), \hat{\varpi}_{\gamma\varepsilon\zeta}(x_2)\right),
$$

$$
\widetilde{\varphi}_2^a = -2 \int d^4x_1 d^4x_2 \left(\hat{\varpi}^{\gamma\varepsilon\zeta}(x), \hat{\mathscr{J}}_{a\gamma}(x_1), \hat{\phi}_{\varepsilon\zeta}(x_2)\right), \quad \widetilde{\varphi}_3 = -2\beta \int d^4x_1 d^4x_2 \left(\hat{\varpi}^{\gamma\varepsilon\zeta}(x), \hat{q}_\gamma(x_1), \hat{\phi}_{\varepsilon\zeta}(x_2)\right),
$$

$$
\varphi\tau_\varphi = -i\frac{d}{d\omega}\varphi(\omega)\bigg|_{\omega=0} = -\frac{T}{2}\frac{d^2}{d\omega^2}\mathrm{Re}\,G^R_{\hat{\varpi}_{lmn}\hat{\varpi}^{lmn}}(\omega)\bigg|_{\omega=0},
$$

$$
\varphi(x) = -\int d^4x_1 \left(\hat{\varpi}^{\varepsilon\zeta\eta}(x), \hat{\varpi}_{\varepsilon\zeta\eta}(x_1)\right) = T\frac{d}{d\omega}\mathrm{Im}\,G^R_{\hat{\varpi}^{\varepsilon\zeta\eta}\hat{\varpi}_{\varepsilon\zeta\eta}}(\omega)\bigg|_{\omega=0}.
$$

Summary and Outlook

Summary: We present a new derivation of relativistic second-order spin hydrodynamics for quantum systems based on Zubarev's non-equilibrium statistical operator formalism. Novel transport coefficients characterizing the relaxation dynamics of second-order dissipative processes are identified and expressed via two-point retarded Green's functions.

Outlook: The three-point correlation function will be recast in terms of the retarded three-point Green's function.

Thank you for your attention!

Notations

$$\hbar = k_B = c = 1, \quad g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)\,, \quad \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu, \quad \epsilon^{0123} = -\epsilon_{0123} = 1,$$

$$X^{(\mu\nu)} = \frac{1}{2}\left(X^{\mu\nu} + X^{\nu\mu}\right), \quad X^{[\mu\nu]} = \frac{1}{2}\left(X^{\mu\nu} - X^{\nu\mu}\right), \quad X^{\langle\mu\rangle} = \Delta^{\mu\nu} X_\nu,$$

$$X^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} X^{\alpha\beta} \equiv \frac{1}{2}\left(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\mu{}_\beta \Delta^\nu{}_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right) X^{\alpha\beta},$$

$$X^{\langle[\mu\nu]\rangle} = \Delta^{[\mu\nu]}_{[\alpha\beta]} X^{\alpha\beta} \equiv \frac{1}{2}\left(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta - \Delta^\mu{}_\beta \Delta^\nu{}_\alpha\right) X^{\alpha\beta},$$

$$\begin{aligned} X^{\{\lambda\mu\nu\}} = \Delta^{\{\lambda\mu\nu\}}_{\{\alpha\beta\gamma\}} X^{\alpha\beta\gamma} = \frac{1}{6}\Big(&\Delta^\lambda{}_\alpha \Delta^\mu{}_\beta \Delta^\nu{}_\gamma + \Delta^\mu{}_\alpha \Delta^\nu{}_\beta \Delta^\lambda{}_\gamma + \Delta^\nu{}_\alpha \Delta^\lambda{}_\beta \Delta^\mu{}_\gamma \\ &- \Delta^\mu{}_\alpha \Delta^\lambda{}_\beta \Delta^\nu{}_\gamma - \Delta^\nu{}_\alpha \Delta^\mu{}_\beta \Delta^\lambda{}_\gamma - \Delta^\lambda{}_\alpha \Delta^\nu{}_\beta \Delta^\mu{}_\gamma\Big) X^{\alpha\beta\gamma}, \end{aligned}$$

$$\partial_\mu = u_\mu D + \nabla_\mu, \quad D = u^\mu \partial_\mu, \quad \nabla_\mu = \Delta_\mu{}^\alpha \partial_\alpha, \quad \theta = \partial_\mu u^\mu, \quad \sigma_{\mu\nu} = \Delta^{\alpha\beta}_{\mu\nu} \partial_\alpha u_\beta,$$

$$\partial_\mu u_\nu = u_\mu D u_\nu + \frac{1}{3}\theta \Delta_{\mu\nu} + \sigma_{\mu\nu} + \nabla_{[\mu} u_{\nu]}, \quad \Omega^{\mu\nu} = \beta \nabla^{[\mu} u^{\nu]} = \Delta^\mu_\alpha \Delta^\nu_\beta \partial^{[\alpha} \beta^{\beta]},$$

Notations

In the derivation of the first-order transport coefficients and the second-order equations of motion for the dissipative currents we encounter integrals of the type

$$\beta \int d^4x_1 \left(\hat{X}(x), \hat{Y}(x_1)\right) = -\frac{d}{d\omega} \mathrm{Im}\, G^R_{\hat{X}\hat{Y}}(\omega)\Big|_{\omega=0},$$

$$\beta \int d^4x_1 \left(\hat{X}(x), \hat{Y}(x_1)\right)\left(x_1 - x\right)^{\tau} = K\left[\hat{X}, \hat{Y}\right] u^{\tau},$$

$$K\left[\hat{X}, \hat{Y}\right] = -\frac{1}{2}\frac{d^2}{d\omega^2} G^R_{\hat{X}\hat{Y}}(\omega)\Big|_{\omega=0} = -\frac{1}{2}\frac{d^2}{d\omega^2} \mathrm{Re}\, G^R_{\hat{X}\hat{Y}}(\omega)\Big|_{\omega=0},$$

where

$$G^R_{\hat{X}\hat{Y}}(\omega) = -i \int_0^{\infty} dt e^{i\omega t} \int d^3x \left\langle \left[\hat{X}(\boldsymbol{x}, t), \hat{Y}(\boldsymbol{0}, 0)\right] \right\rangle_l$$

is the Fourier transform of the retarded two-point correlator taken in the zero-wavenumber limit and the square brackets denote the commutator.