Non-Abelian Chiral Kinetic Theory

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Xiao-Li Luo, JHG,   JHEP 11 (2021) 115

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中国物理学会高能物理分会第十四届全国粒子物理学术会议 2024.08.13-17 青岛

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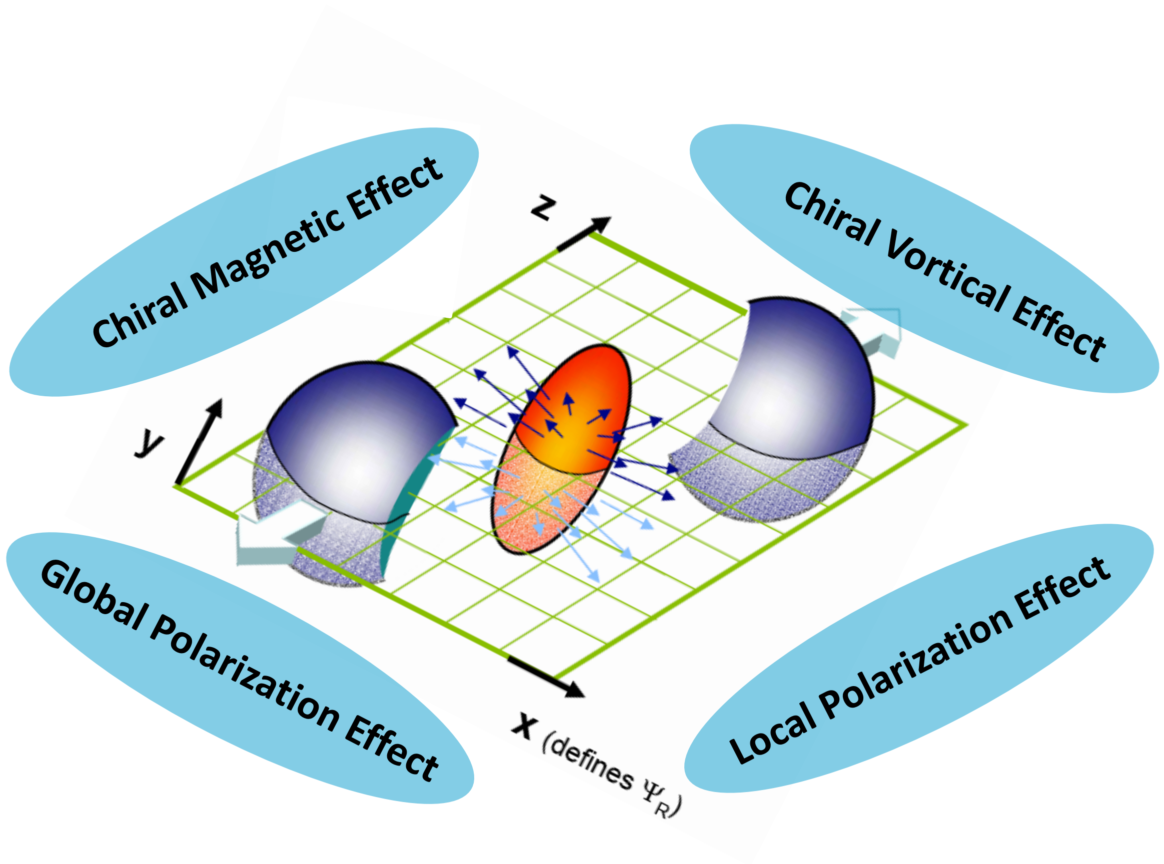
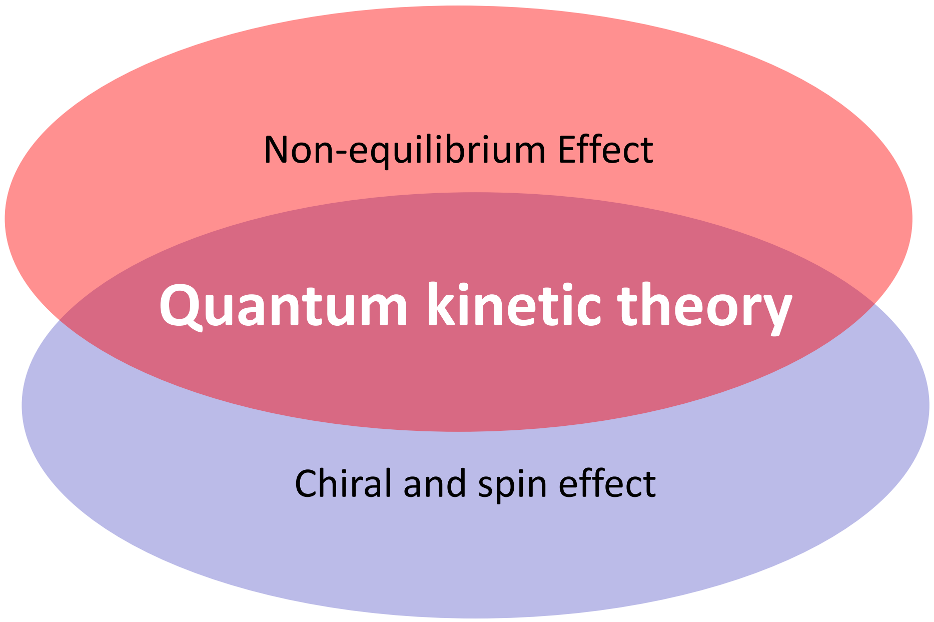
Introduction





Introduction

Recent developments in quantum kinetic theory：

J. H. Gao, Z. T. Liang, S. Pu, Q. Wang and X. N. Wang, Phys. Rev. Lett. 109, 232301 (2012)2012
M. A. Stephanov and Y. Yin, Phys. Rev. Lett. 109, 162001 (2012)
J. W. Chen, S. Pu, Q. Wang and X. N. Wang, Phys. Rev. Lett. 110, no. 26, 262301 (2013).
J. Y. Chen, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 115, no. 2, 021601 (2015).
Y. Hidaka, S. Pu and D. L. Yang, Phys. Rev. D 95, no. 9, 091901 (2017).
A. Huang, S. Shi, Y. Jiang, J. Liao and P. Zhuang, Phys. Rev. D 98, no. 3, 036010 (2018).
J. H. Gao, Z. T. Liang, Q. Wang and X. N. Wang, Phys. Rev. D 98, no. 3, 036019 (2018).
Y. C. Liu, L. L. Gao, K. Mameda and X. G. Huang, Phys. Rev. D 99, no.8, 085014 (2019)
J. H. Gao and Z. T. Liang, Phys. Rev. D 100, 056021 (2019).
N. Weickgenannt, X. L. Sheng, E. Speranza, Q. Wang and D. H. Rischke, Phys. Rev. D 100,056018 (2019).
K. Hattori, Y. Hidaka and D. L. Yang, Phys. Rev. D 100, 096011 (2019).
Z. Wang, X. Guo, S. Shi and P. Zhuang, Phys. Rev. D 100, 014015 (2019).
S. Lin and L. Yang, Phys. Rev. D 101, no.3, 034006 (2020)
S. Li and H. U. Yee, Phys. Rev. D 100, no. 5, 056022 (2019)
N. Weickgenannt, E. Speranza, X. l. Sheng, Q. Wang and D. H. Rischke, Phys. Rev. Lett.127, no.5, 052301 (2021)
T. Hayata, Y. Hidaka and K. Mameda, JHEP 05, 023 (2021)
X. L. Luo and J. H. Gao, JHEP 11, 115 (2021)
S. Lin, Phys. Rev. D 105, no.7, 076017 (2022)
S. Fang, S. Pu and D. L. Yang, Phys. Rev. D 106, no.1, 016002 (2022)
……

from massless to massive

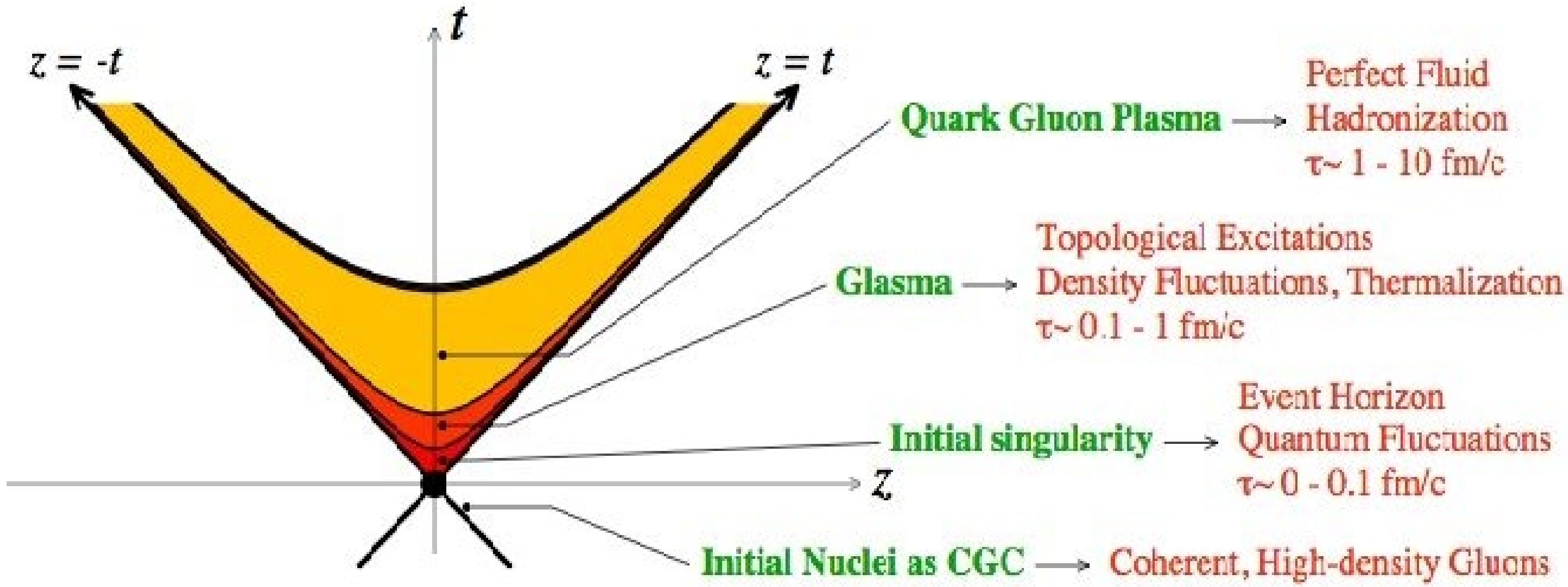
from first to second order

from collisionless to collisional

from flat to curved spacetime

from Abelian to non-Abelian

Introduction



Wigner function formalism: Non-Abelian VS Abelian

Wigner function defined by density operator：
$$W(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \langle \rho(x,y) \rangle$$

Gauge **invariant** density operator in Abelian gauge field：

$$\rho(x,y) = \bar{\psi}\left(x+\frac{y}{2}\right) U\left(x+\frac{y}{2}, x-\frac{y}{2}\right) \psi\left(x-\frac{y}{2}\right) \quad : \quad \rho_{\alpha\beta}(x,y) = \bar{\psi}_\beta\left(x+\frac{y}{2}\right) U\left(x+\frac{y}{2}, x-\frac{y}{2}\right) \psi_\alpha\left(x-\frac{y}{2}\right)$$

Gauge **covariant** density operator in Non-Abelian gauge field：

$$\rho(x,y) = \bar{\psi}\left(x+\frac{y}{2}\right) U\left(x+\frac{y}{2}, x\right) \otimes U\left(x, x-\frac{y}{2}\right) \psi\left(x-\frac{y}{2}\right) \quad : \quad \rho^{ij}_{\alpha\beta}\left(x+\frac{y}{2}, x-\frac{y}{2}\right) = \bar{\psi}^{j'}_\beta\left(x+\frac{y}{2}\right) U^{j'j}\left(x+\frac{y}{2}, x\right) U^{ii'}\left(x, x-\frac{y}{2}\right) \psi^{i'}_\alpha\left(x-\frac{y}{2}\right)$$

Wigner function formalism: Non-Abelian VS Abelian

Only spinor decomposition involved in **Abelian** gauge field：

$$W(x,p) = \frac{1}{4}\left[\mathscr{F} + i\gamma^5\mathscr{P} + \gamma^\mu\mathscr{V}_\mu + \gamma^5\gamma^\mu\mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right]$$

scalar ($\mathscr{F}$), pseudo ($\mathscr{P}$), vector ($\mathscr{V}_\mu$), axial ($\mathscr{A}_\mu$), tensor ($\mathscr{S}_{\mu\nu}$)

Additional color decomposition involved in **non-Abelian** gauge field:

$$W(x,p) \;=\; W^I \mathbf{1} + W^a t^a$$

Singlet ($W^I \mathbf{1}$), multiplet ($W^a t^a$)

$$[t^a, t^b] = i f^{abc} t^c$$

$$A_\mu \;\rightarrow\; A^a_\mu t^a \qquad F_{\mu\nu} \;\rightarrow\; F^a_{\mu\nu} t^a$$

Chiral limit $m = 0$: $\mathscr{V}_\mu$, $\mathscr{A}_\mu$ decouple with $\mathscr{F}$, $\mathscr{P}$, $\mathscr{S}_{\mu\nu}$

Chiral Wigner functions: $\mathscr{J}^\mu_s \equiv \frac{1}{2}(\mathscr{V}_\mu + s\mathscr{A}_\mu)$ right-handed/left-handed ($s = \pm 1$)

Wigner function formalism: Non-Abelian VS Abelian

Wigner equations in background **Abelian** field:

$$\epsilon_{\mu\nu\rho\sigma} G^{\rho} \mathscr{J}^{\sigma} = -2s\left(\Pi_{\mu}\mathscr{J}_{\nu} - \Pi_{\nu}\mathscr{J}_{\mu}\right), \qquad \Pi_{\mu}\mathscr{J}^{\mu} = 0, \qquad G_{\mu}\mathscr{J}^{\mu} = 0$$

$$\Pi^{\mu} = p_{\mu} + \frac{g}{2}\int_0^1 ds\, e^{-\frac{1}{2}is\Delta} F_{\mu\nu} is\partial_p^{\nu} \qquad G_{\mu} = \partial_{\mu}^{x} + \frac{g}{2}\int_0^1 ds\, e^{-\frac{1}{2}is\Delta} F_{\mu\nu}\partial_p^{\nu} \qquad \Delta \equiv \partial^{p}\cdot\partial_{x}$$

Wigner equations in background **non-Abelian** field:

$$\epsilon_{\mu\nu\rho\sigma}[G^{\rho}, \mathscr{J}^{\sigma}] = -s\left(\{\Pi_{\mu}, \mathscr{J}_{\nu}\} - \{\Pi_{\nu}, \mathscr{J}_{\mu}\}\right), \quad \{\Pi_{\mu}, \mathscr{J}^{\mu}\} = 0, \quad [G_{\mu}, \mathscr{J}^{\mu}] = 0$$

$$\Pi^{\mu} = p_{\mu} + \frac{g}{2}\int_0^1 ds\, e^{-\frac{1}{2}is\Delta} F_{\mu\nu} is\partial_p^{\nu} \qquad G_{\mu} = D_{\mu} + \frac{g}{2}\int_0^1 ds\, e^{-\frac{1}{2}is\Delta} F_{\mu\nu}\partial_p^{\nu} \qquad \Delta \equiv \partial^{p}\cdot\mathscr{D}$$

Color decomposition of Wigner function:

$$\mathscr{J}_{\mu}(x,p) \;=\; \mathscr{J}_{\mu}^{I}(x,p)1 + \mathscr{J}_{\mu}^{a}(x,p)t^{a},$$

Non-Abelian CKT in conventional semiclassical expansion

Semiclassical expansion for Wigner equations:

$$\hbar\epsilon_{\mu\nu\rho\sigma}[G^\rho, \mathscr{J}^\sigma] = -s(\{\Pi_\mu, \mathscr{J}_\nu\} - \{\Pi_\nu, \mathscr{J}_\mu\}), \quad \{\Pi_\mu, \mathscr{J}^\mu\} = 0, \quad [G_\mu, \mathscr{J}^\mu] = 0$$

Triangle or Δ expansion for operators:

$$G_\mu = D_\mu + \frac{g}{2}\int_0^1 ds\, e^{-\frac{1}{2}is\hbar\Delta} F_{\mu\nu}\partial_p^\nu \qquad \Pi^\mu = p_\mu + \frac{g}{2}\hbar\int_0^1 ds\, e^{-\frac{1}{2}is\hbar\Delta} F_{\mu\nu} is\partial_p^\nu$$

$$e^{-\frac{is\hbar\Delta}{2}} F_{\nu\mu}(x) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{is}{2}\right)^k [\hbar\partial_p \cdot \mathscr{D}(x)]^k F_{\nu\mu}(x)$$

Semiclassical expansion for Wigner functions:

$$\mathscr{J}_\mu = \mathscr{J}_\mu^{(0)} + \hbar\,\mathscr{J}_\mu^{(1)} + \hbar^2\,\mathscr{J}_\mu^{(2)} + \cdots$$

Xiao-Li Luo, JHG, JHEP 11 (2021) 115； S. Ochs and U.W. Heinz, Annals Phys. 266 (1998) 351

Non-Abelian CKT in conventional semiclassical expansion

Time- and space-like decomposition: $\mathscr{J}_\mu = \mathscr{J}_n n_\mu + \bar{\mathscr{J}}_\mu, \quad n^2=1, \quad \mathscr{J}_n = n\cdot\mathscr{J}, \quad n\cdot\bar{\mathscr{J}}=0$

$$\hbar\epsilon_{\mu\nu\rho\sigma}[G^\rho,\mathscr{J}^\sigma] = -s\left(\{\Pi_\mu,\mathscr{J}_\nu\}-\{\Pi_\nu,\mathscr{J}_\mu\}\right)$$

$\bar{\mu}\, n$ ⟹

$$\hbar\epsilon_{\mu\nu\rho\sigma}n^\nu[G^\rho,\mathscr{J}^\sigma] = -s\left(\{\bar{\Pi}_\mu,\mathscr{J}_n\}-\{\Pi_n,\bar{\mathscr{J}}_\mu\}\right)$$

⇓

$$\bar{\mathscr{J}}^{(0)}_\mu = \frac{\bar{p}_\mu}{p_n}\mathscr{J}^{(0)}_n,$$
$$\bar{\mathscr{J}}^{(1)}_\mu = \frac{\bar{p}_\mu}{p_n}\mathscr{J}^{(1)}_n - \frac{s}{2p_n}\epsilon^{\mu\nu\rho\sigma}n_\nu G^{(0)}_\sigma\left(\frac{p_\rho}{p_n}\mathscr{J}^{(0)}_n\right) + \frac{1}{2p_n}\left(\left\{\bar{\Pi}^{(1)\mu},\mathscr{J}^{(0)}_n\right\} - \left\{\Pi^{(1)}_n,\frac{\bar{p}^\mu}{p_n}\mathscr{J}^{(0)}_n\right\}\right),$$
$$\cdots \qquad \cdots\cdots$$

$$\{\Pi_\mu,\mathscr{J}^\mu\}=0, \quad + \quad [G_\mu,\mathscr{J}^\mu]=0$$

✗ ⟸

$\bar{\mu}\,\bar{\nu}$ ⇓

$$\hbar\epsilon_{\mu\nu\rho\sigma}n^\rho\left([G_n,\mathscr{J}^\sigma]-[G^\sigma,\mathscr{J}_n]\right) = -s\left(\left\{\bar{\Pi}_\mu,\bar{\mathscr{J}}_\nu\right\}-\left\{\bar{\Pi}_\nu,\bar{\mathscr{J}}_\mu\right\}\right)$$

⇓

$$\left[\tilde{F}^{\alpha\beta},\mathscr{J}^{(0)}_\alpha\right]=0,$$

$$\left[\tilde{F}^{\alpha\beta},\mathscr{J}^{(1)}_\alpha\right] = -\frac{3}{32}\left[\left[\tilde{F}^{\nu\alpha}\partial^\beta_p - \tilde{F}^{\nu\beta}\partial^\alpha_p, F_{\nu\kappa}\partial^\kappa_p\right],\mathscr{J}^{(0)}_\alpha\right]$$

In conventional semiclassical expansion, some equations are satisfied automatically for Abelian field, but they are not for non-Abelian case.

Some constraint conditions emerge!

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Non-Abelian CKT in conventional semiclassical expansion

Chiral anomaly:

$$\partial^x_\mu j_5^{(1)I\mu} = -\frac{g^2}{2N} F^a_{\mu\lambda}\tilde{F}^{a,\mu\nu}\int d^4p\,\partial_p^\lambda\left[p_\nu f_{\rm V}^{(0)}\delta'(p^2)\right] = \frac{g^2}{8\pi^2 N}E^a\cdot B^a,$$

$$\mathscr{D}^{ac}_\mu j_5^{(1)c\mu} = -\frac{g^2}{2}d^{bca}F^b_{\mu\lambda}\tilde{F}^{c,\mu\nu}\int d^4p\,\partial_p^\lambda\left[p_\nu f_{\rm V}^{(0)}\delta'(p^2)\right] = \frac{g^2}{8\pi^2}d^{bca}E^b\cdot B^c$$

Vacuum contribution： $f_{\rm V}^{(0)}$

Vector currents in global equilibrium:

$$j^{(1)I\mu} = \xi^{\rm s}\omega^\mu + \xi^a_B B^{a\mu},$$
$$j^{(1)a\mu} = \xi^a\omega^\mu + \xi^{ab}_B B^{b\mu}$$

$$\xi^I = \frac{1}{2\pi^2 N}\sum_i \mu^i\mu_5^i,$$
$$\xi^a_B = -\frac{g}{2\pi^2 N}\sum_i t^a_{ii}\mu_5^i,$$
$$\xi^a = \frac{4}{\pi^2\hbar^2}\sum_i t^a_{ii}\mu^i\mu_5^i,$$
$$\xi^{ab}_B = -\frac{g}{2\pi^2}\left(\frac{\delta^{ab}}{N}\sum_i\mu_5^i + \frac{d^{bca}}{\hbar}\sum_i t^c_{ii}\mu_5^i\right)$$

Axial currents in global equilibrium:

$$j_5^{(1)I\mu} = \xi_5^I\omega^\mu + \xi^a_{B5}B^{a\mu},$$
$$j_5^{(1)a\mu} = \xi_5^a\omega^\mu + \xi^{ab}_{B5}B^{b\mu}$$

$$\xi_5^I = -\frac{T^2}{6} - \frac{1}{2\pi^2 N}\sum_i(\mu^{i\,2} + \mu_5^{i\,2}),$$
$$\xi^a_{B5} = \frac{g}{2\pi^2 N}\sum_i t^a_{ii}\mu^i,$$
$$\xi_5^a = -\frac{2}{\pi^2\hbar^2}\sum_i t^a_{ii}(\mu^{i\,2} + \mu_5^{i\,2}),$$
$$\xi^{ab}_{B5} = \frac{g}{2\pi^2}\left(\frac{\delta^{ab}}{N}\sum_i\mu^i + \frac{d^{bca}}{\hbar}\sum_i t^c_{ii}\mu^i\right)$$

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Non-Abelian CKT in conventional semiclassical expansion

For non-Abelian gauge field:

Triangle expansion $\neq$ Semiclassical $\hbar$ expansion

Covariant derivative: $$D_\mu(x) = \partial_\mu - \frac{ig}{\hbar} A_\mu(x)$$

Iterative process could lead to order mixing: $$[D_\mu, D_\nu] = -\frac{ig}{\hbar} F_{\mu\nu}(x)$$

Such issue never appears for Abelian field because only ordinary derivative $\boldsymbol{\partial_\mu}$ is involved!

Non-Abelian CKT in self-consistent semiclassical expansion

Redefine the non-Abelian gauge field: $\frac{A_\mu(x)}{\hbar} \rightarrow A_\mu(x)$

Covariant derivative: $D_\mu(x) = \partial_\mu - \frac{ig}{\hbar}A_\mu(x) \rightarrow D_\mu(x) = \partial_\mu - igA_\mu(x)$

Commutator: $[D_\mu, D_\nu] = -\frac{ig}{\hbar}F_{\mu\nu}(x) \rightarrow [D_\mu, D_\nu] = -igF_{\mu\nu}(x)$

Field strength tensor: $F_{\mu\nu}(x) \rightarrow \hbar F_{\mu\nu}(x)$

Operators:

$$G_\mu = D_\mu + \frac{g}{2}\int_0^1 ds\, e^{-\frac{1}{2}is\hbar\Delta}F_{\mu\nu}\partial_p^\nu \rightarrow G_\mu = D_\mu + \frac{g}{2}\hbar\int_0^1 ds\, e^{-\frac{1}{2}is\hbar\Delta}F_{\mu\nu}\partial_p^\nu$$

$$\Pi^\mu = p_\mu + \frac{g}{2}\hbar\int_0^1 ds\, e^{-\frac{1}{2}is\hbar\Delta}F_{\mu\nu}is\partial_p^\nu \rightarrow \Pi^\mu = p_\mu + \frac{g}{2}\hbar^2\int_0^1 ds\, e^{-\frac{1}{2}is\hbar\Delta}F_{\mu\nu}is\partial_p^\nu$$

Non-Abelian CKT in self-consistent semiclassical expansion

Zeroth-order Wigner functions and equations in **8d** phase space:

$$\mathscr{J}_n^{(0)} = p_n \mathcal{J}_n^{(0)} \delta\left(p^2\right) \qquad \mathscr{J}_\mu^{(0)} = p_\mu \mathcal{J}_n^{(0)} \delta\left(p^2\right) \qquad 0 = \mathscr{D}_\mu \left(p^\mu \frac{\mathscr{J}_n^{(0)}}{p_n}\right)$$

First-order Wigner functions and equations in **8d** phase space:

$$\mathscr{J}_n^{(1)} = p_n \mathcal{J}_n^{(1)} \delta\left(p^2 - m^2\right) \qquad \mathscr{J}_\mu^{(1)} = p_\mu \mathcal{J}_n^{(1)} \delta\left(p^2\right) + \frac{s}{2p_n^2} \bar{\epsilon}_{\mu\alpha\beta} \bar{p}^\beta \mathscr{D}^\alpha \mathscr{J}_n^{(0)}$$

Contribute for both Abelian and non-Abelian

$$0 = \mathscr{D}_\mu \left(p^\mu \frac{\mathscr{J}_n^{(1)}}{p_n}\right) + \frac{s}{2p_n} \bar{\epsilon}^{\mu\alpha\beta} \mathscr{D}_\mu \mathscr{D}_\alpha \mathscr{J}_\beta^{(0)} + \frac{1}{2} \partial_p^\nu \left\{ p^\mu \frac{\mathscr{J}_n^{(0)}}{p_n}, F_{\mu\nu} \right\}$$

Contribute only for non-Abelian

All other Wigner equations are satisfied automatically and no constraint conditions arise!

Non-Abelian CKT in self-consistent semiclassical expansion

Second-order Wigner functions and equations in **8d** phase space:

$$\mathscr{J}_n^{(2)} = p_n \mathcal{J}_n^{(2)}\delta\left(p^2\right) - \frac{s}{4}\bar{\epsilon}_{\mu\alpha\beta}\bar{p}^{\beta}\left\{\mathcal{J}_n^{(0)}, F^{\alpha\mu}\right\}\delta'\left(p^2\right) - \frac{1}{4p_n}\left(\bar{p}^{\mu}\bar{p}^{\nu} - \bar{p}^2\Delta^{\mu\nu}\right)\bar{\mathscr{D}}_{\mu}\bar{\mathscr{D}}_{\nu}\mathcal{J}_n^{(0)}\delta'\left(p^2\right) - \frac{i}{4}p_n\partial_p^{\nu}\left[p^{\mu}\mathcal{J}_n^{(0)}, F_{\mu\nu}\right]\delta'\left(p^2\right)$$

Contribute from both Abelian and non-Abelian — Contribute only from non-Abelian

$$\mathscr{J}^{(2)\mu} = \frac{p_{\mu}}{p_n}\mathscr{J}_n^{(2)} + \frac{s}{2p_n}\bar{\epsilon}_{\mu\alpha\beta}\mathscr{D}^{\alpha}\mathscr{J}^{(1)\beta} + \frac{s}{4p_n}\bar{\epsilon}_{\mu\alpha\beta}\partial_p^{\lambda}\left\{\mathscr{J}^{(0)\beta}, F^{\alpha}{}_{\lambda}\right\} + \frac{i}{8p_n}\partial_p^{\lambda}\left(\left[\mathscr{J}_{\mu}^{(0)}, F_{n\lambda}\right] - \left[\mathscr{J}_n^{(0)}, F_{\mu\lambda}\right]\right)$$

$$0 = \mathscr{D}^{\mu}\mathscr{J}_{\mu}^{(2)} + \frac{1}{2}\partial_p^{\nu}\left\{\mathscr{J}^{(1)\mu}, F_{\mu\nu}\right\} + \frac{i}{8}\partial_p^{\nu}\partial_p^{\lambda}\left[\mathscr{J}^{(0)\mu}, (\mathscr{D}_{\lambda}F_{\mu\nu})\right]$$

All other Wigner equations are satisfied automatically and no constraint conditions arise!

Such expansion is consistent with the fact: **No macroscopic classical non-Abelian field strength!**

Non-Abelian CKT in self-consistent semiclassical expansion

How to organize the Wigner functions in **7d** phase space:

$$j_\mu = \int d^3p\, \Omega\, \dot{x}_\mu\, f$$

$$= \int d^3p \left(\Omega^{(0)} \dot{x}_\mu^{(0)} f^{(0)} + \Omega^{0} \dot{x}_\mu^{(0)} f^{(1)} + \Omega^{(0)} \dot{x}_\mu^{(1)} f^{(0)} + \Omega^{(1)} \dot{x}_\mu^{(0)} f^{(0)} + \cdots\cdots \right)$$

Distribution function: f Effective velocity: $\dot{x}_\mu$ Correction to invariant phase space: Ω

Non-Abelian CKT in self-consistent semiclassical expansion

Zeroth-order Wigner functions and equations in **7d** phase space:

$$\int dp_n \mathscr{J}^{(0)I}_{s\mu} = \frac{1}{2}\sum_\lambda \dot{x}^{(0)}_\mu \mathcal{J}^{(0)I}_{sn},$$
$$\int dp_n \mathscr{J}^{(0)a}_{s\mu} = \frac{1}{2}\sum_\lambda \dot{x}^{(0)}_\mu \mathcal{J}^{(0)a}_{sn}$$

$$0 = \dot{x}^{(0)\mu}\partial^x_\mu \mathcal{J}^{(0)I}_{sn},$$
$$0 = \dot{x}^{(0)\mu}\mathscr{D}^{ac}_\mu \mathcal{J}^{(0)c}_{sn}$$

$\lambda =+/-$: positive/negative energy

Zeroth-order velocity: $\dot{x}^{(0)}_\mu = \lambda n_\mu + \hat{\bar{p}}_\mu,$

First-order Wigner functions and equations in **7d** phase space:

$$\mathscr{D}^{ac}_\mu = \delta^{ca}\partial^x_\mu + g f^{bca} A^b_\mu$$

$$\int dp_n \mathscr{J}^{(1)I}_{s\mu} = \frac{1}{2}\sum_\lambda \left(\dot{x}^{(0)}_\mu \mathcal{J}^{(1)I}_{sn} + \dot{x}^{(1)I}_\mu \mathcal{J}^{(0)I}_{sn}\right),$$
$$\int dp_n \mathscr{J}^{(1)a}_{s\mu} = \frac{1}{2}\sum_\lambda \left(\dot{x}^{(0)}_\mu \mathcal{J}^{(1)a}_{sn} + \dot{x}^{(1)ac}_\mu \mathcal{J}^{(0)c}_{sn}\right)$$

$$0 = \dot{x}^{(0)\mu}\partial^x_\mu \mathcal{J}^{(1)I}_{sn} + \dot{x}^{(1)\mu,I}\partial^x_\mu \mathcal{J}^{(0)I}_{sn} + \dot{x}^{(0)\mu}\nabla^{(1)Ia}_\mu \mathcal{J}^{(0)a}_{sn}$$
$$0 = \dot{x}^{(0)\mu}\mathscr{D}^{ac}_\mu \mathcal{J}^{(1)c}_{sn} + \dot{x}^{(1)\mu,ab}\mathscr{D}^{bc}_\mu \mathcal{J}^{(0)c}_{sn} + \dot{x}^{(0)\mu}\nabla^{(1)a}_\mu \mathcal{J}^{(0)I}_{sn} + \dot{x}^{(0)\mu}\nabla^{(1)ac}_\mu \mathcal{J}^{(0)c}_{sn} + \left(\mathscr{D}^{ab}_\mu \dot{x}^{(1)\mu,bc} - \dot{x}^{(1)\mu,ab}\mathscr{D}^{bc}_\mu\right)\mathcal{J}^{(0)c}_{sn}$$

First-order velocity $\dot{x}^{(1)I}_\mu = \frac{\lambda s}{2|\bar{p}|^2}\bar{\epsilon}_{\mu\alpha\beta}\bar{p}^\beta\partial^\alpha_x, \quad \dot{x}^{(1)ac}_\mu = \frac{\lambda s}{2|\bar{p}|^2}\bar{\epsilon}_{\mu\alpha\beta}\bar{p}^\beta\mathscr{D}^{ac,\alpha},$

First-order gradient operator: $\nabla^{(1)Ia}_\mu = \frac{1}{2N}F^a_{\mu\nu}\bar{\partial}^\nu_p, \quad \nabla^{(1)a}_\mu = F^a_{\mu\nu}\bar{\partial}^\nu_p, \quad \nabla^{(1)ac}_\mu = \frac{1}{2}d^{cba}F^b_{\mu\nu}\bar{\partial}^\nu_p$

Non-Abelian CKT in self-consistent semiclassical expansion

Second-order Wigner functions and equations in **7d** phase space:

$$\int dp_n \mathscr{J}^{(2)I}_{s\mu} = \frac{1}{2}\sum_\lambda \left(\dot{x}^{(0)}_\mu \mathcal{J}^{(2)I}_{sn} + \dot{x}^{(1)I}_\mu \mathcal{J}^{(1)I}_{sn} + \Omega^{(2)I}\dot{x}^{(0)}_\mu \mathcal{J}^{(0)I}_{sn} + \Omega^{(2)Ia}\dot{x}^{(0)}_\mu \mathcal{J}^{(0)a}_{sn} + \dot{x}^{(2)I}_\mu \mathcal{J}^{(0)I}_{sn} + \dot{x}^{(2)Ia}_\mu \mathcal{J}^{(0)a}_{sn} \right),$$

$$\int dp_n \mathscr{J}^{(2)a}_{s\mu} = \frac{1}{2}\sum_\lambda \left(\dot{x}^{(0)}_\mu \mathcal{J}^{(2)a}_{sn} + \dot{x}^{(1)ab}_\mu \mathcal{J}^{(1)b}_{sn} + \Omega^{(2)a}\dot{x}^{(0)}_\mu \mathcal{J}^{(0)I}_{sn} + \Omega^{(2)ab}\dot{x}^{(0)}_\mu \mathcal{J}^{(0)b}_{sn} + \dot{x}^{(2)a}_\mu \mathcal{J}^{(0)I}_{sn} + \dot{x}^{(2)ab}_\mu \mathcal{J}^{(0)b}_{sn} \right)$$

$$\begin{aligned} 0 = {} & \dot{x}^{(0)\mu}\partial^x_\mu \mathcal{J}^{(2)I}_{sn} + \dot{x}^{(1)I,\mu}\partial^x_\mu \mathcal{J}^{(1)I}_{sn} + \dot{x}^{(2)I,\mu}\partial^x_\mu \mathcal{J}^{(0)I}_{sn} + \dot{x}^{(0)\mu}\nabla^{(1)Ic}_\mu \mathcal{J}^{(1)c}_{sn} + \dot{x}^{(1)I,\mu}\nabla^{(1)Ic}_\mu \mathcal{J}^{(0)c}_{sn} + \dot{x}^{(2)Ia,\mu}\mathscr{D}^{ac}_\mu \mathcal{J}^{(0)c}_{sn} \\ & + \left(\nabla^{(1)Ib}_\mu \dot{x}^{(1)bc,\mu} - \dot{x}^{(1)I,\mu}\nabla^{(1)Ic}_\mu\right)\mathcal{J}^{(0)c}_{sn} + \left(\mathscr{D}^{ca}_\mu \dot{x}^{(2)Ia,\mu}\right)\mathcal{J}^{(0)c}_{sn} + \left(\mathscr{D}^{ca}_\mu \Omega^{(2)Ia}\right)\dot{x}^{(0)\mu}\mathcal{J}^{(0)c}_{sn} \end{aligned}$$

$$\begin{aligned} 0 = {} & \dot{x}^{(0)\mu}\mathscr{D}^{ac}_\mu \mathcal{J}^{(2)c}_{sn} + \dot{x}^{(1)ab,\mu}\mathscr{D}^{bc}_\mu \mathcal{J}^{(1)c}_{sn} + \dot{x}^{(2)ab,\mu}\mathscr{D}^{bc}_\mu \mathcal{J}^{(0)c}_{sn} + \dot{x}^{(0)\mu}\nabla^{(1)ac}_\mu \mathcal{J}^{(1)c}_{sn} + \dot{x}^{(1)ab,\mu}\nabla^{(1)bc}_\mu \mathcal{J}^{(0)c}_{sn} + \dot{x}^{(0)\mu}\nabla^{(2)ac}_\mu \mathcal{J}^{(0)c}_{sn} \\ & + \left(\mathscr{D}^{ab}_\mu \dot{x}^{(1)bc,\mu} - \dot{x}^{(1)ab,\mu}\mathscr{D}^{bc}_\mu\right)\mathcal{J}^{(1)c}_{sn} + \left(\nabla^{(1)ab}_\mu \dot{x}^{(1)bc,\mu} - \dot{x}^{(1)ab,\mu}\nabla^{(1)bc}_\mu\right)\mathcal{J}^{(0)c}_{sn} \\ & + \left(\nabla^{(2)ac}_\mu \dot{x}^{(0)\mu} - \dot{x}^{(0)\mu}\nabla^{(2)ac}_\mu\right)\mathcal{J}^{(0)c}_{sn} + \left(\mathscr{D}^{ab,\mu}\dot{x}^{(2)bc}_\mu - \dot{x}^{(2)ab,\mu}\mathscr{D}^{bc}_\mu\right)\mathcal{J}^{(0)c}_{sn} \\ & + \left(\mathscr{D}^{ab,\mu}\Omega^{(2)bc}\dot{x}^{(0)}_\mu - \Omega^{(2)ab}\dot{x}^{(0)\mu}\mathscr{D}^{bc}_\mu\right)\mathcal{J}^{(0)c}_{sn} + \dot{x}^{(0)\mu}\nabla^{(1)a}_\mu \mathcal{J}^{(1)I}_{sn} + \dot{x}^{(1)ab,\mu}\nabla^{(1)b}_\mu \mathcal{J}^{(0)I}_{sn} + \dot{x}^{(2)a,\mu}\partial^x_\mu \mathcal{J}^{(0)I}_{sn} \\ & + \left(\nabla^{(1)a}_\mu \dot{x}^{(1)I,\mu} - \dot{x}^{(1)ab,\mu}\nabla^{(1)b}_\mu\right)\mathcal{J}^{(0)I}_{sn} + \left(\mathscr{D}^{ab,\mu}\dot{x}^{(2)b}_\mu\right)\mathcal{J}^{(0)I}_{sn} + \left(\mathscr{D}^{ab,\mu}\Omega^{(2)b}\dot{x}^{(0)}_\mu\right)\mathcal{J}^{(0)I}_{sn} \end{aligned}$$

Non-Abelian CKT in self-consistent semiclassical expansion

Second-order velocity:

$$\dot{x}_{\mu}^{(2)I} = -\frac{1}{8|\bar{p}|^5}\left[\bar{p}_{\alpha}\bar{p}_{\beta}\bar{p}_{\mu} + \bar{p}^2\left(\Delta_{\alpha\beta}\bar{p}_{\mu} - \Delta_{\mu\beta}\bar{p}_{\alpha} - \Delta_{\mu\alpha}\bar{p}_{\beta}\right)\right]\partial_x^{\alpha}\partial_x^{\beta}$$

$$\dot{x}_{\mu}^{(2)Ia} = \frac{\dot{x}_{\mu}^{(2)a}}{2N} = -\frac{\lambda s \hat{\bar{p}}_{\mu}\bar{\epsilon}_{\gamma\alpha\beta}\bar{p}^{\beta}}{8N|\bar{p}|^3}F^{a,\alpha\gamma} + \frac{s\bar{\epsilon}_{\mu\alpha\beta}\bar{p}^{\beta}}{4N|\bar{p}|^3}F^{a,\alpha n} + \bar{\partial}_{\beta}^{p}\left(\frac{\lambda s\bar{\epsilon}_{\mu\alpha\nu}\bar{p}^{\nu}}{4N|\bar{p}|^2}F^{a,\alpha\beta}\right),$$

$$\dot{x}_{\mu}^{(2)ab} = -\frac{1}{8|\bar{p}|^5}\left[\bar{p}_{\alpha}\bar{p}_{\beta}\bar{p}_{\mu} + \bar{p}^2\left(\Delta_{\alpha\beta}\bar{p}_{\mu} - \Delta_{\mu\beta}\bar{p}_{\alpha} - \Delta_{\mu\alpha}\bar{p}_{\beta}\right)\right]\mathscr{D}^{ac,\alpha}\mathscr{D}^{cb,\beta} - \frac{\lambda s\bar{\epsilon}_{\gamma\alpha\beta}\bar{p}^{\beta}}{8|\bar{p}|^3}d^{bca}F^{c,\alpha\gamma}\hat{\bar{p}}_{\mu} + \frac{s\bar{\epsilon}_{\mu\alpha\beta}\bar{p}^{\beta}}{4|\bar{p}|^3}d^{bca}F^{c,\alpha n} + \frac{\lambda\bar{p}^{\alpha}\left(\hat{\bar{p}}_{\mu}n^{\nu} - \lambda\Delta_{\mu}^{\nu}\right)}{8|\bar{p}|^3}f^{bca}F_{\alpha\nu}^{c} + \bar{\partial}_{\beta}^{p}\left(\frac{\lambda s\bar{\epsilon}_{\mu\alpha\nu}\bar{p}^{\nu}}{4|\bar{p}|^2}d^{bca}F^{c,\alpha\beta} + \frac{\bar{p}_{\mu}\bar{p}_{\alpha} - \bar{p}^2\Delta_{\alpha\mu}}{8|\bar{p}|^3}f^{bca}F^{c,\alpha\beta}\right)$$

Second-order gradient operator: $$\nabla_{\mu}^{(2)ac} = -\frac{1}{8}f^{cba}\bar{\partial}_p^{\nu}\bar{\partial}_p^{\lambda}\left(\mathscr{D}_{\lambda}^{be}F_{\mu\nu}^{e}\right)$$

Correction to invariant phase space $$\Omega^{(2)I} = -\frac{1}{4|\bar{p}|^4}\left(\bar{p}^{\alpha}\bar{p}^{\beta} - \bar{p}^2\Delta^{\alpha\beta}\right)\partial_{\alpha}^{x}\partial_{\beta}^{x},$$ $$\Omega^{(2)Ia} = \frac{\Omega^{(2)a}}{2N} = -\frac{\lambda s}{8N|\bar{p}|^3}\bar{\epsilon}^{\nu\alpha\beta}\bar{p}_{\beta}F_{\alpha\nu}^{a};$$

$$\Omega^{(2)ab} = -\frac{\lambda s}{8|\bar{p}|^3}\bar{\epsilon}_{\nu\alpha\beta}\bar{p}^{\beta}d^{bca}F_{\alpha\nu}^{c} - \frac{1}{4|\bar{p}|^4}\left(\bar{p}^{\alpha}\bar{p}^{\beta} - \bar{p}^2\Delta^{\alpha\beta}\right)\bar{\mathscr{D}}_{\alpha}^{ac}\bar{\mathscr{D}}_{\beta}^{cb} - \frac{\lambda}{8|\bar{p}|^3}f^{bca}F_{\mu n}^{c}\bar{p}^{\mu} - \bar{\partial}_p^{\nu}\left(\frac{\lambda}{8|\bar{p}|}f^{bca}F_{n\nu}^{c}\right)$$

Summary and outlook

- Triangle expansion leads to constraint conditions in non-Abelian CKT!
- The anomalous currents can be induced by non-Abelian field.
- A self-consistent expansion is proposed and leads to no constraint conditions.
- Within self-consistent expansion, the non-Abelian CKT has been derived.
- From non-Abelian CKT with $m=0$ to non-Abelian GCKT with $m\neq0$

Thanks for your attention !