Black Hole Superradiance and Gravitational Wave Beats

Based on PRD 107 075009 (2023), arXiv:2407.00767, arXiv:2408.xxxxx (in preparation)

Speaker: 郭印达 (山东大学)
Collaborators : 鲍守山, 贾拿云, 张宏, 张鑫

2024/08/16 Qingdao

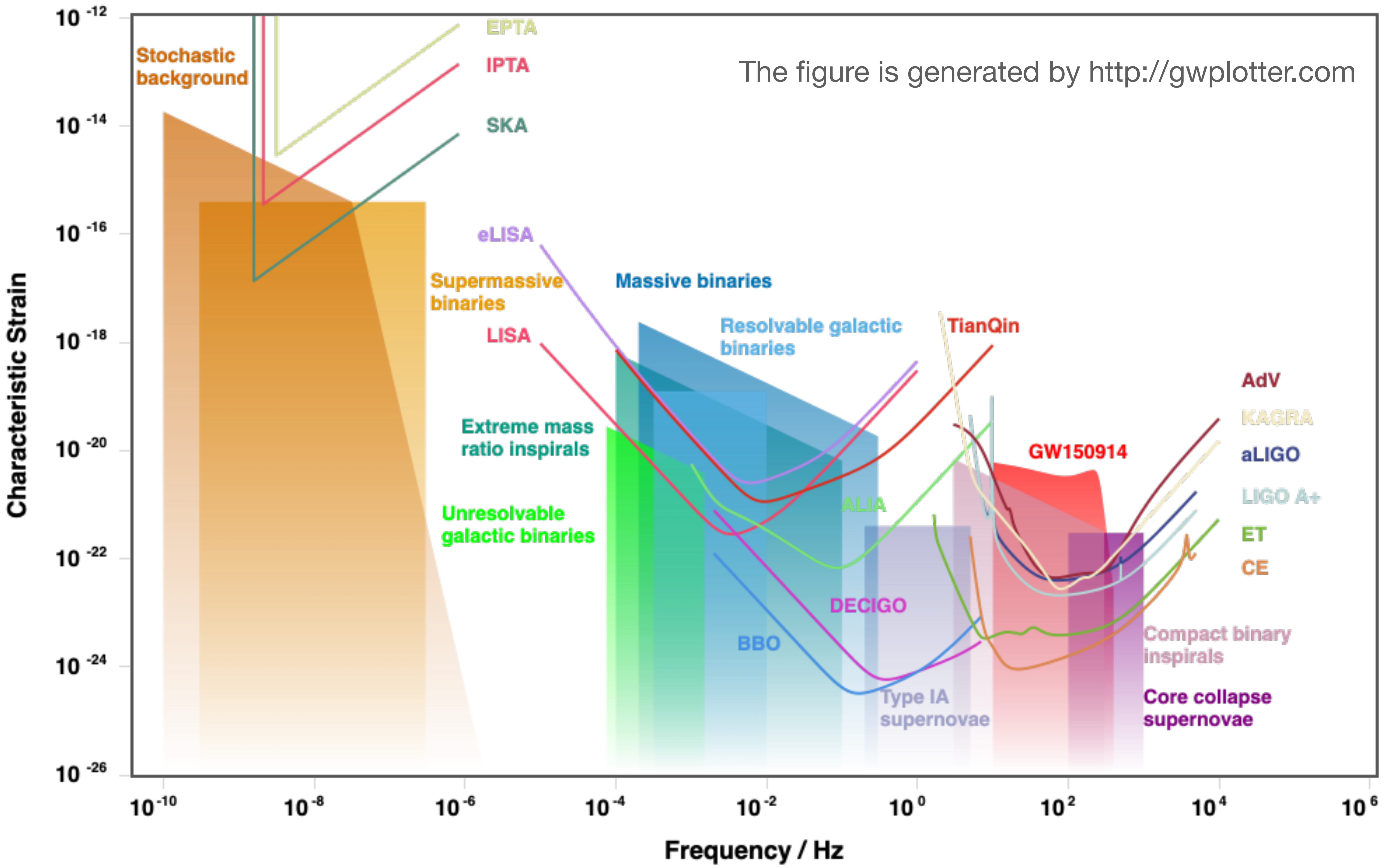
Figure is from https://www.ligo.caltech.edu/

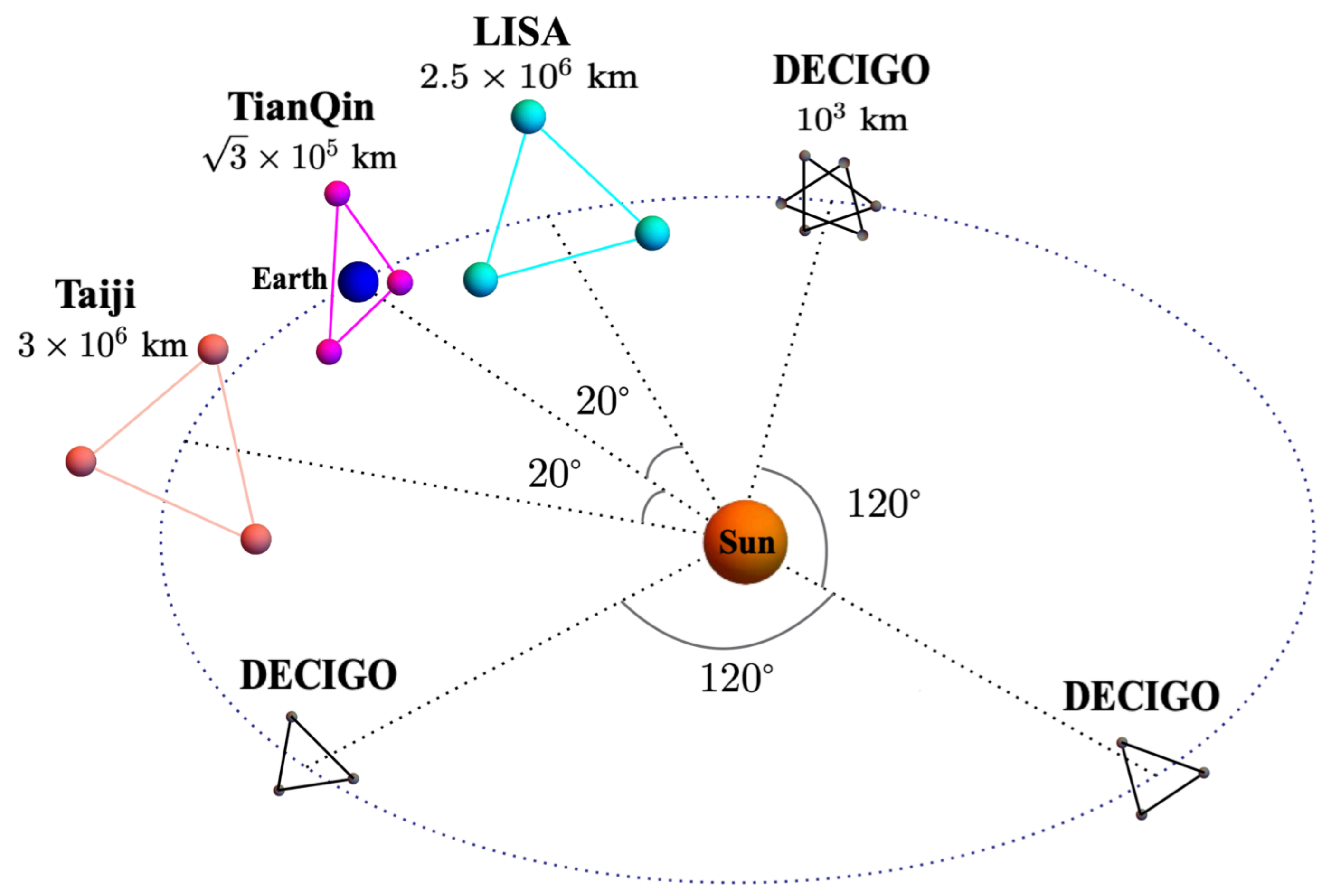
Outline

- **Introduction**
  - Gravitational wave
  - BH–condensate system
- **Evolution**
  - Superradiance rate
  - Time evolution
- **Detection**
  - GW beats
- **Summary**

Introduction

More detectors in the future



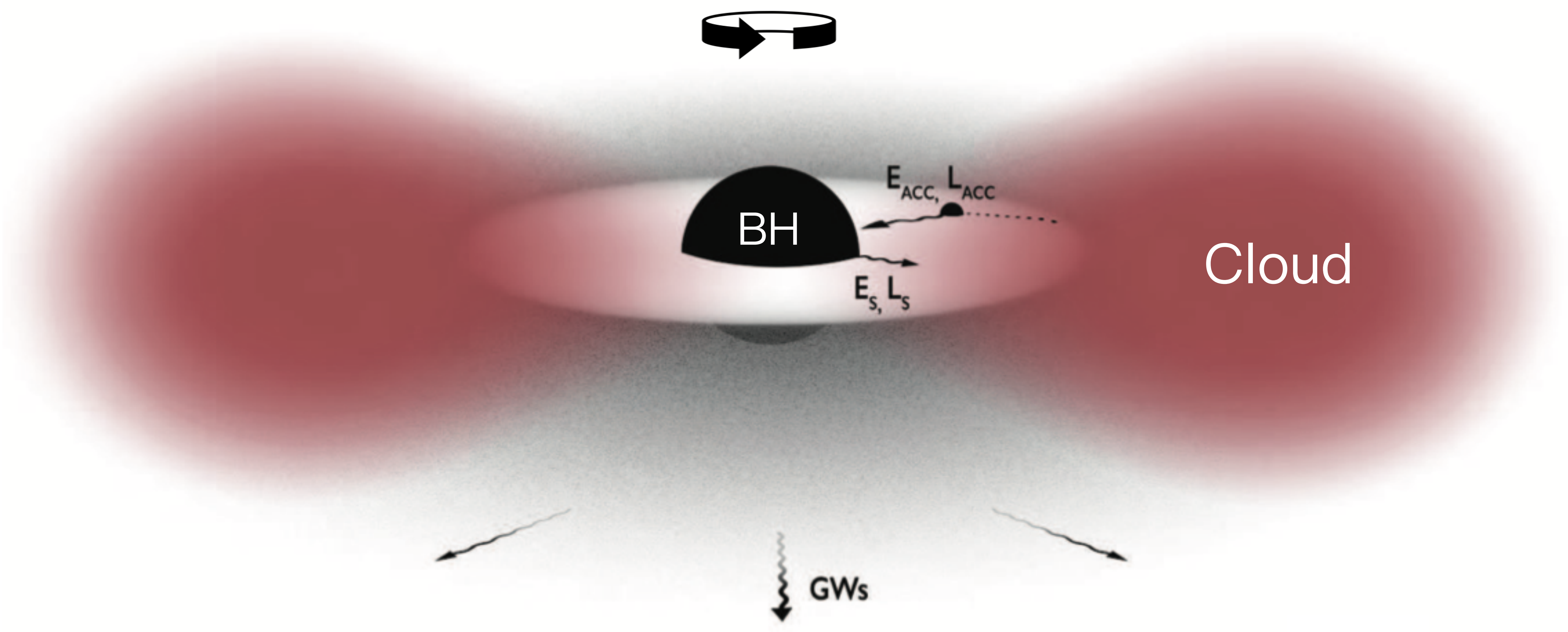


Y. Gong, J. Luo, and B. Wang, Nature Astron **5**, 881 (2021).

We are going to discover weaker GW sources in a broader frequency band

Introduction
Superradiance

BH-condensate system



R. Brito, V. Cardoso, and P. Pani, Class. Quant. Grav. **32**, 134001 (2015).

- Clouds can extract **energy** and **angular momentum.**
- Rotating clouds emit **GWs**.

Klein–Gordon equation

$$\left(g^{ab}\nabla_a\nabla_b + \mu^2\right)\Phi = 0$$

Kerr metric   scalar mass

Eigenfrequency is **complex**

$$\omega_{nlm} = \omega_{nlm}^{(R)} + \boxed{i\omega_{nlm}^{(I)}}$$

similar to hydrogen atom: 3 indexes - (n, l, m)

Superradiance condition:

$$0 < \omega_R < m\Omega_H$$

Ω_H: BH horizon angular velocity

Calculating ω_{nlm} is nontrivial

S. Detweiler, Phys. Rev. D **22**, 2323 (1980).
V. Cardoso and S. Yoshida, J. High Energy Phys. **2005**, 009 (2005).
S. R. Dolan, Phys. Rev. D **76**, 084001 (2007).
S. S. Bao, Q. X. Xu, and H. Zhang, Phys. Rev. D **106**, 064016 (2020).

Outline

- Introduction
  - Gravitational wave
  - BH–condensate system
- Evolution
  - Superradiance rate
  - Time evolution
- Detection
  - GW beats
- Summary

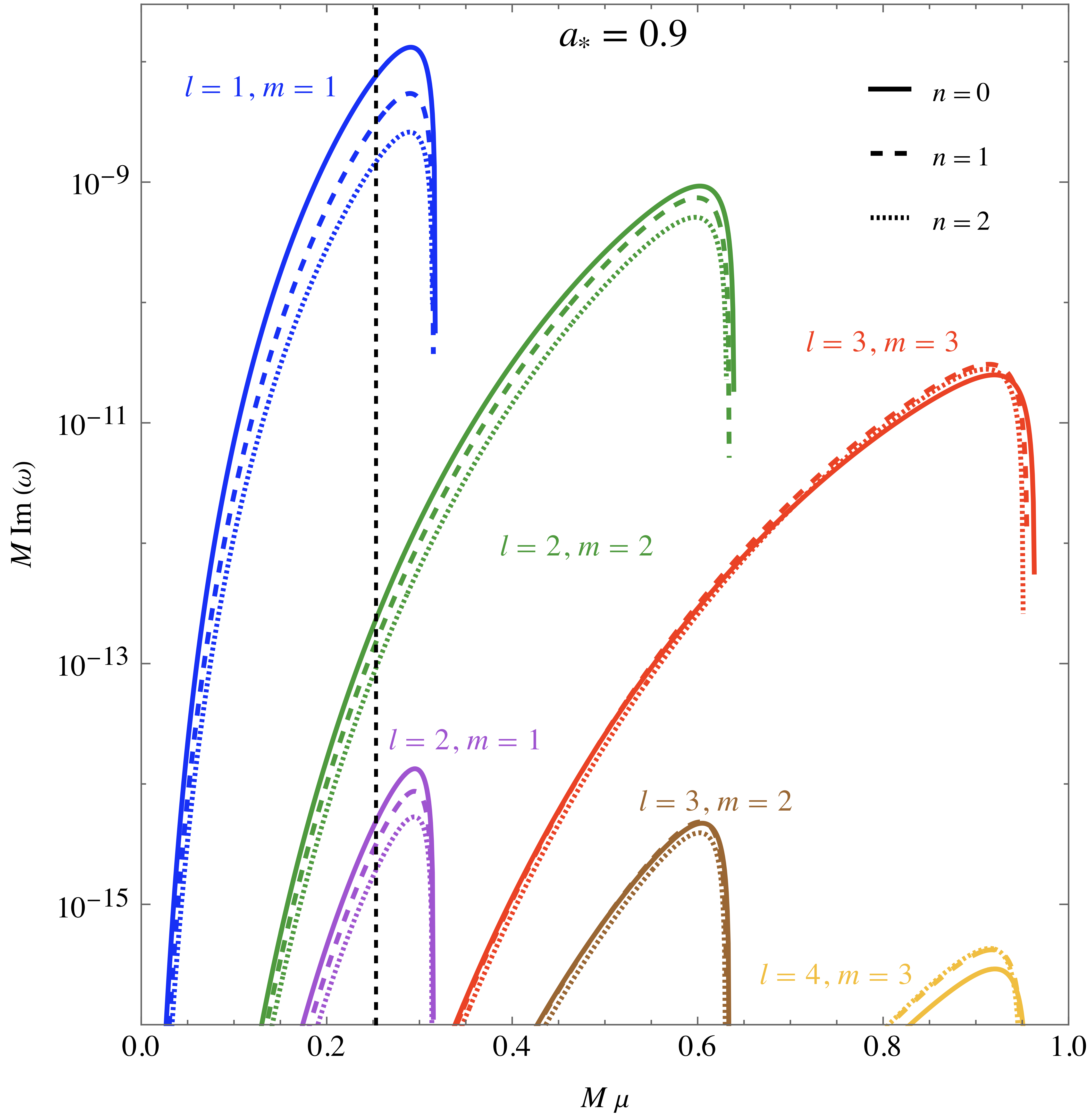
Evolutions

The superradiance rate

$\omega^{(I)}_{nlm}$: 3 indexes – $(n > 0,\ l,\ m)$

$$\omega^{(I)}_{nlm} \rightarrow \dot{M}^{(nlm)}_s = 2M_s\omega^{(I)}_{nlm}$$

- Modes with different m has different superradiant region $\Leftarrow 0 < \omega_R < m\Omega_H$
- Dominant mode: $(0,\ 1,\ 1)$ $(0,\ 2,\ 2)$
- Subdominant mode: $(1,\ 1,\ 1)$ $(1,\ 2,\ 2)$
- Modes with $m < l$ are unimportant.



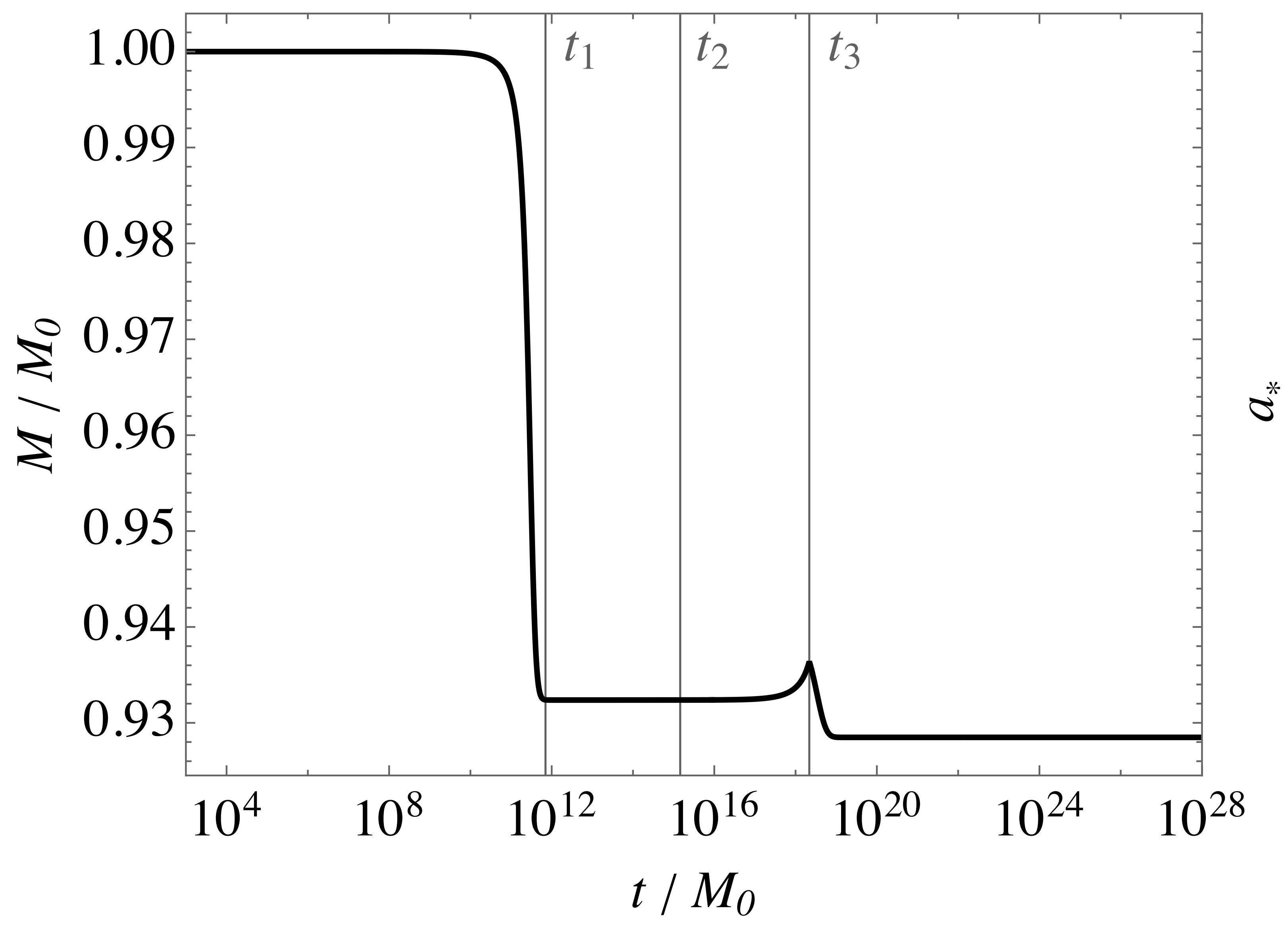
YDG, S. S. Bao and H. Zhang, Phys. Rev. D **107**, 075009 (2023).

Evolutions

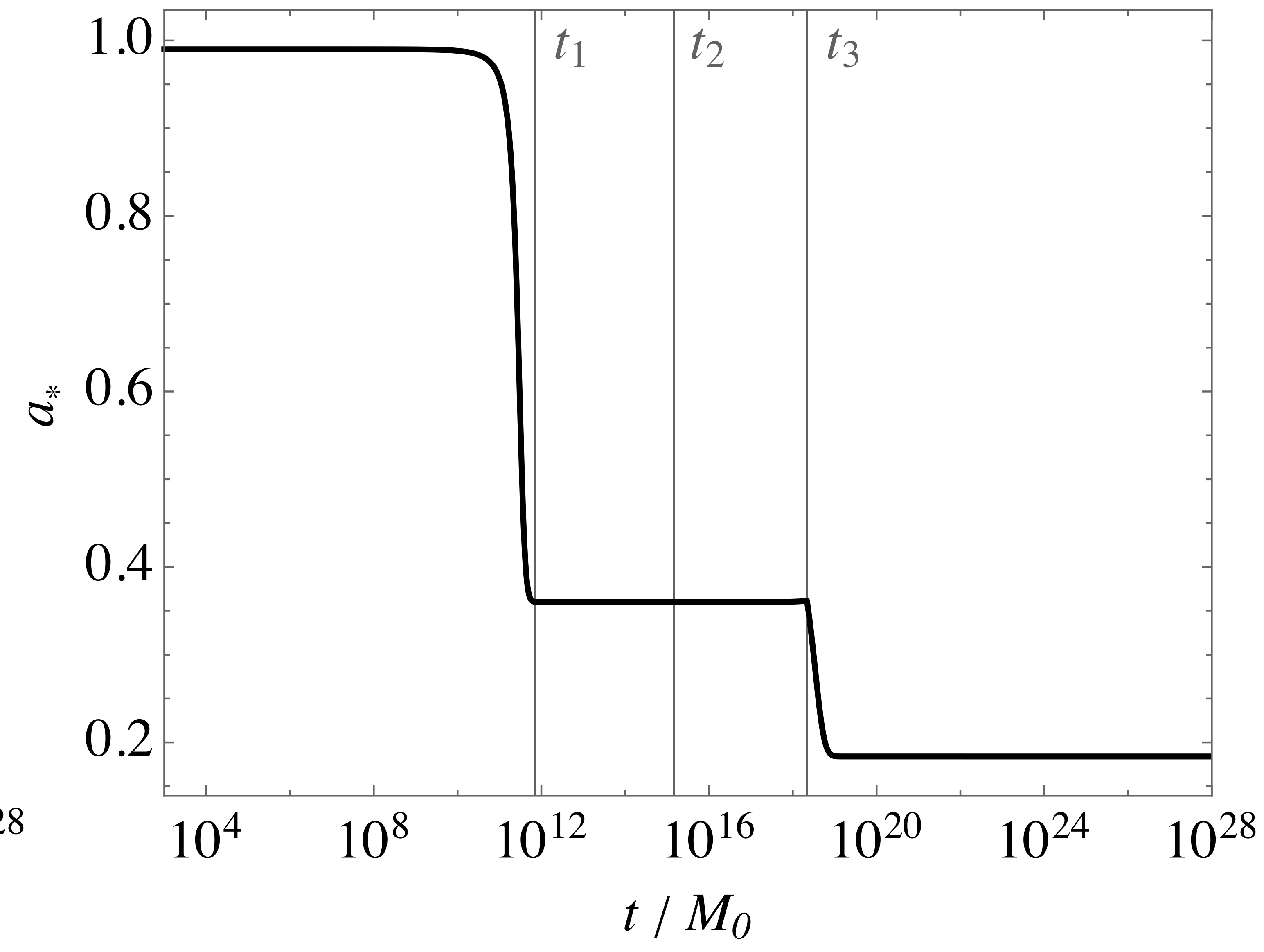
Evolutions with GW emission

$$M_0 = 1.56 \times 10^{-13}\text{yr}\left(\frac{M_0}{M_\odot}\right)$$

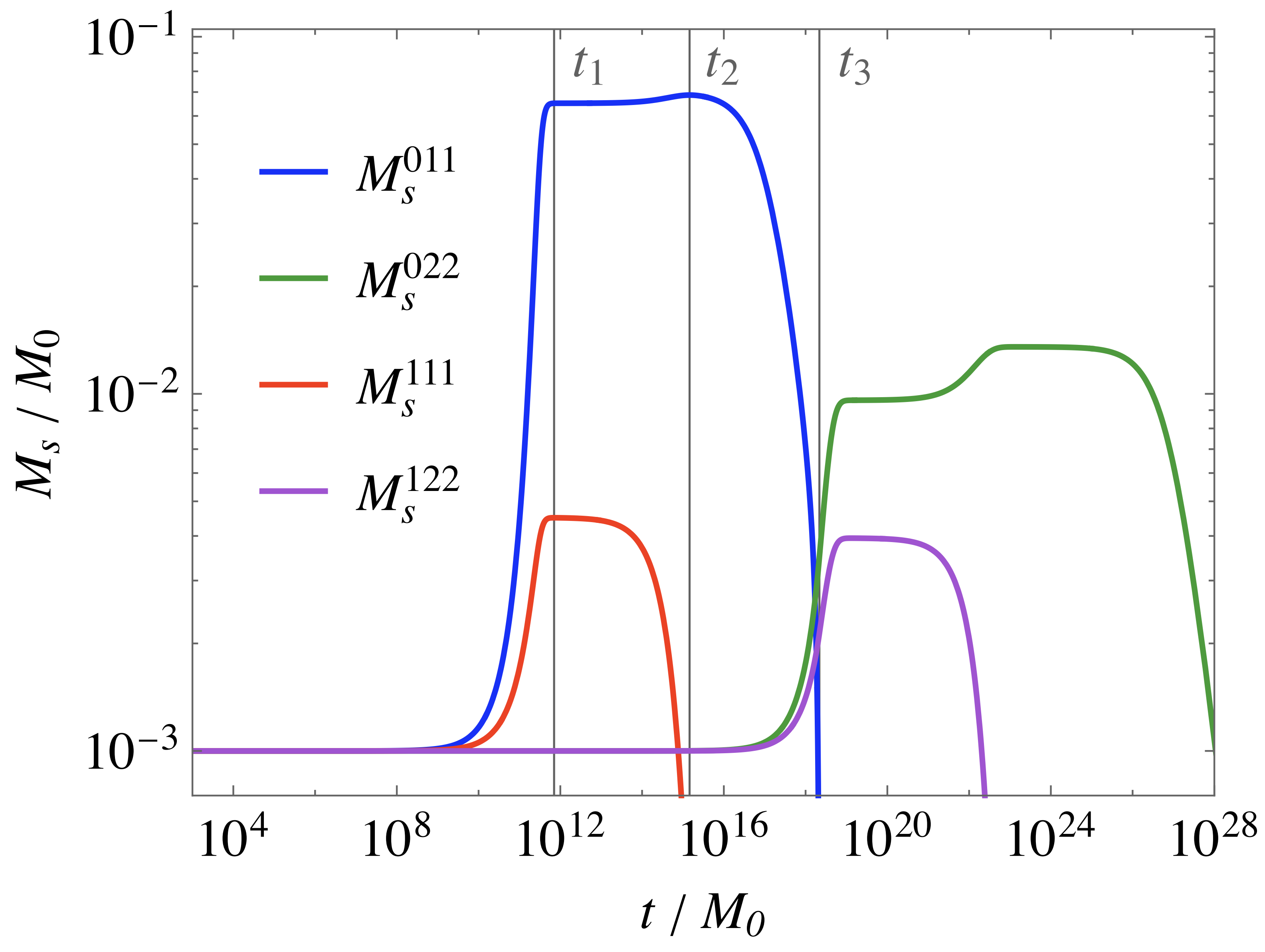
BH mass



BH spin



Scalar cloud mass



1. With scalar condensate, BH mass and spin are "**discrete**". ⇒ constrain the saclar mass
2. Modes with different m are important **at different stages**.
3. (011) mode mass is the largest ~ **10% BH mass.**
4. Only (011) mode ⇒ **Monochromatic** GWs.

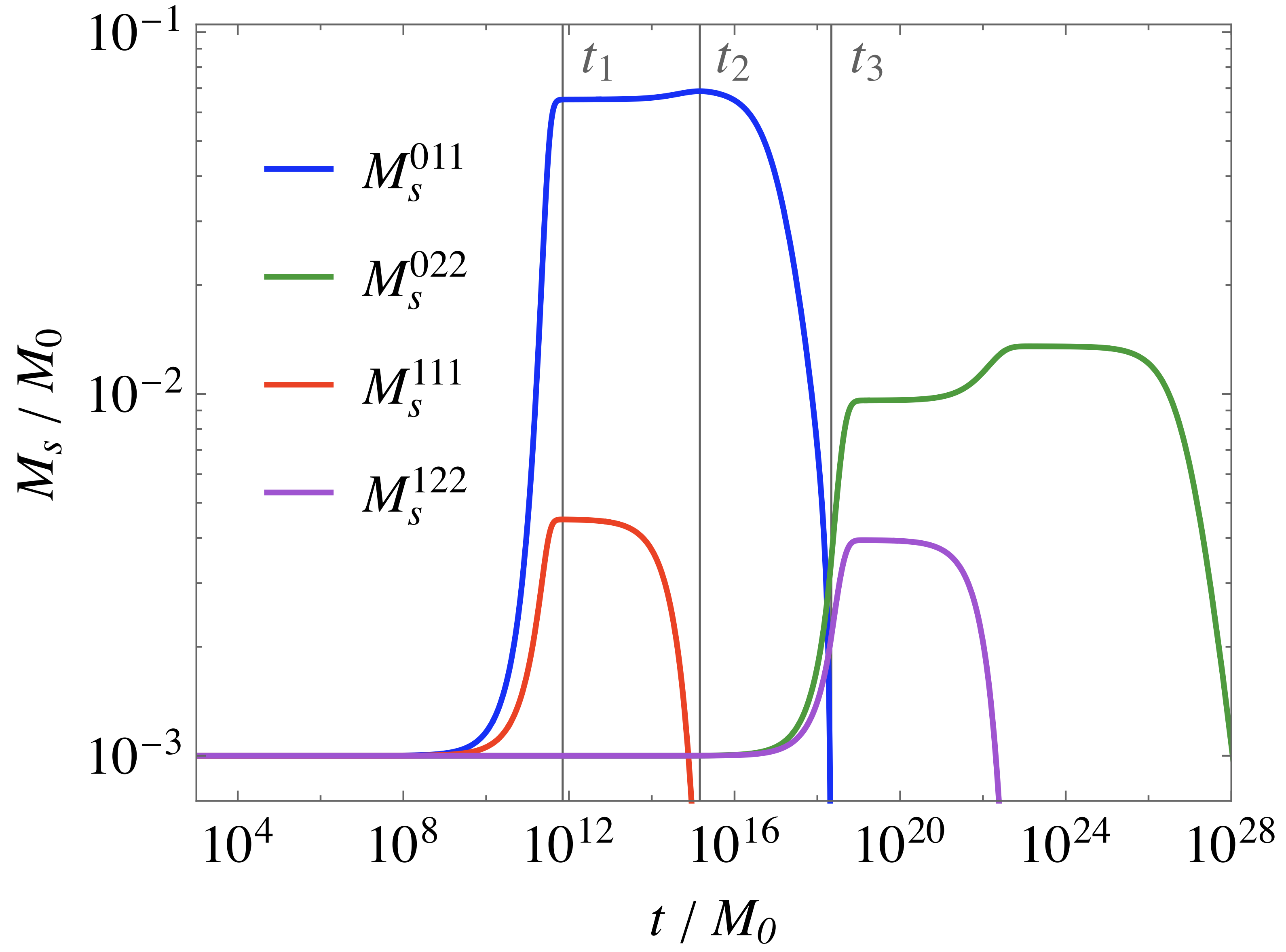
? Difficult to be distinguished from other monochromatic GW sources. e.g. neutron stars

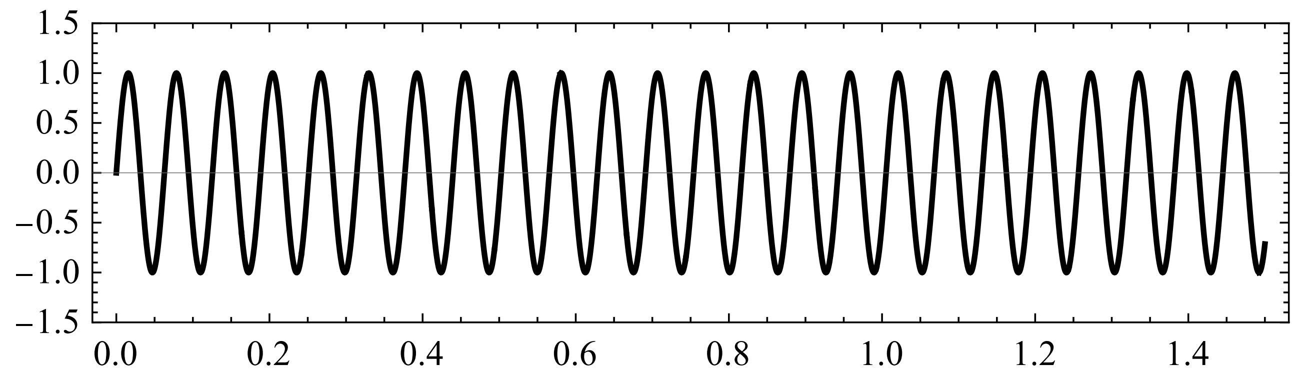
YDG, S. S. Bao and H. Zhang, Phys. Rev. D **107**, 075009 (2023).

Evolutions

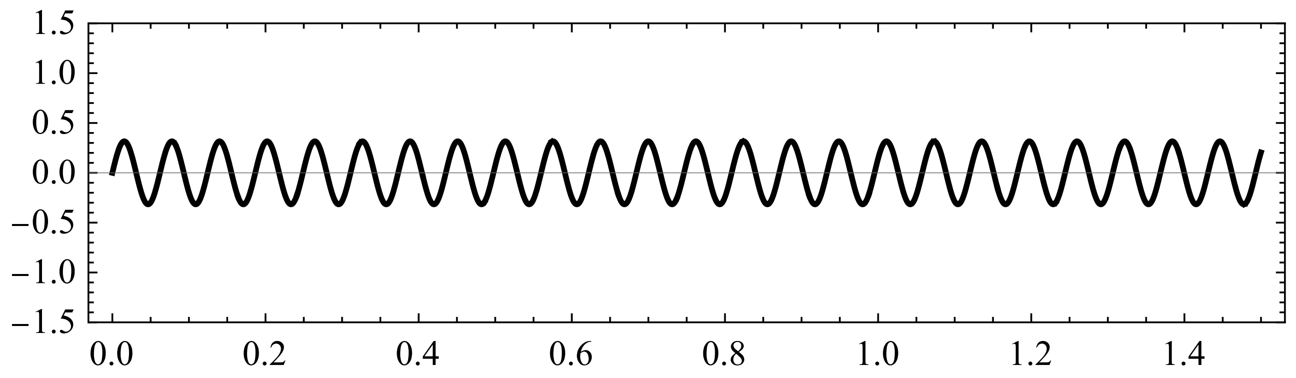
GW interference

- $n > 0$ and $n = 0$ modes **coexist** for a long time.
- $\mathbf{N_{111}/N_{011} \sim 0.1}$ in coexistence period.
- **Coexistence ⇒ GW beat**
  - Waveform

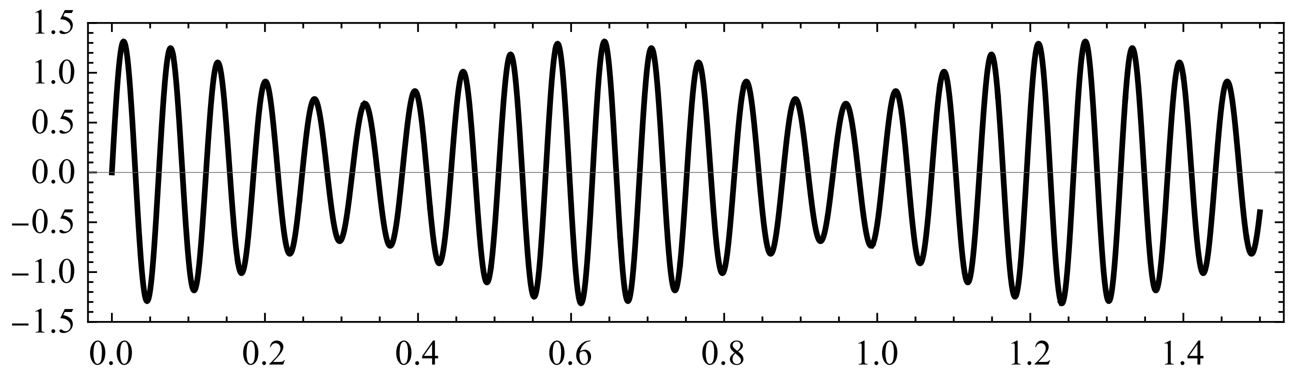




+



⇒



  - Estimate beat strength

$$2 \times (011) \to \text{graviton} \qquad \text{Amp} \propto N_{011}$$

$$(011) + (111) \to \text{graviton} \qquad \text{Amp} \propto \sqrt{N_{011} N_{111}}$$

The interference term is mildly suppressed by $\mathbf{\sqrt{N_{111}/N_{011}} \sim 30\,\%}$ **.**

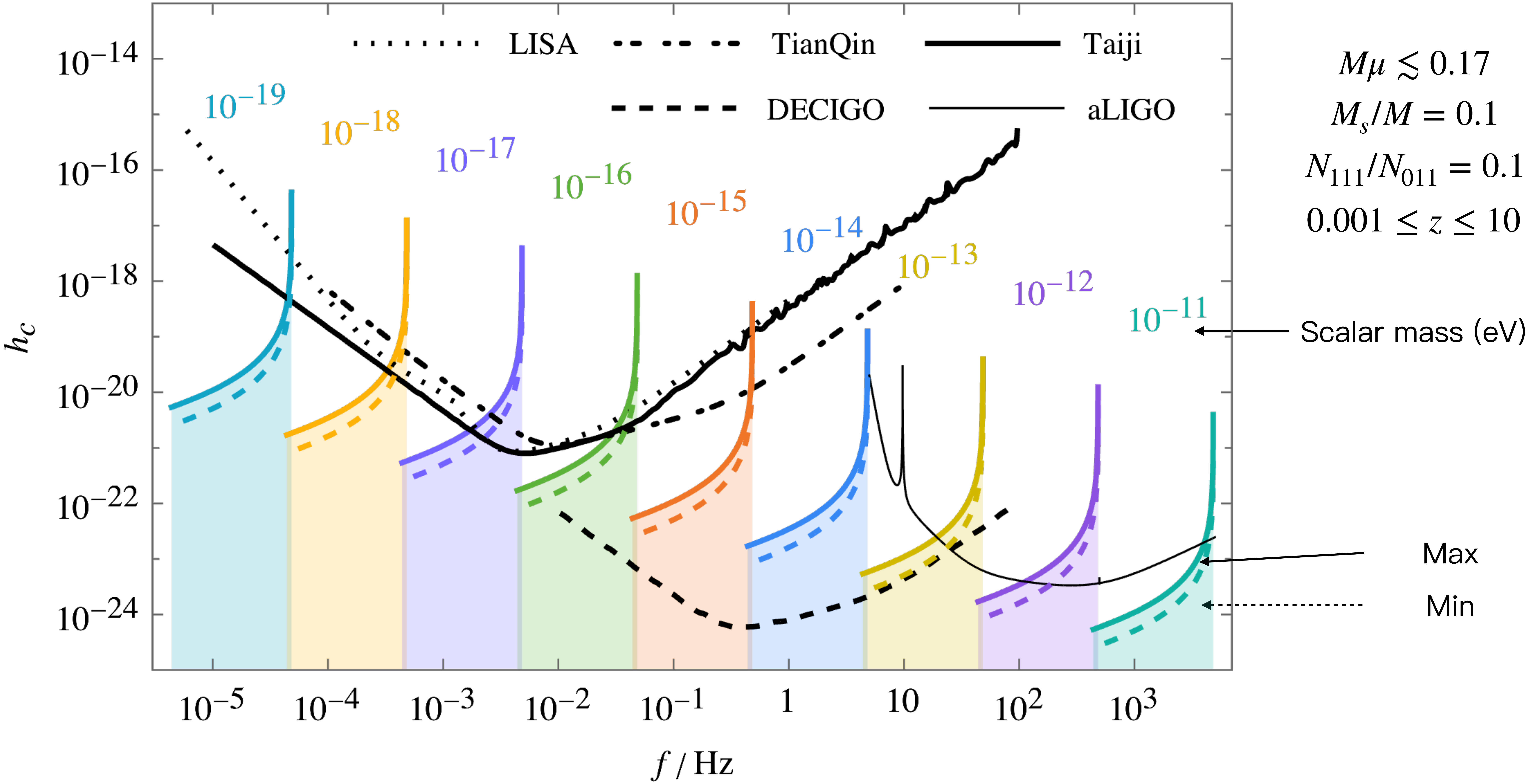
YDG, S. S. Bao and H. Zhang, Phys. Rev. D **107**, 075009 (2023).

Outline

- Introduction
  - Gravitational wave
  - BH–condensate system
- Evolution
  - Superradiance rate
  - Time evolution
- Detection
  - GW beats
- Summary

Detection

GW beats observation from scalar

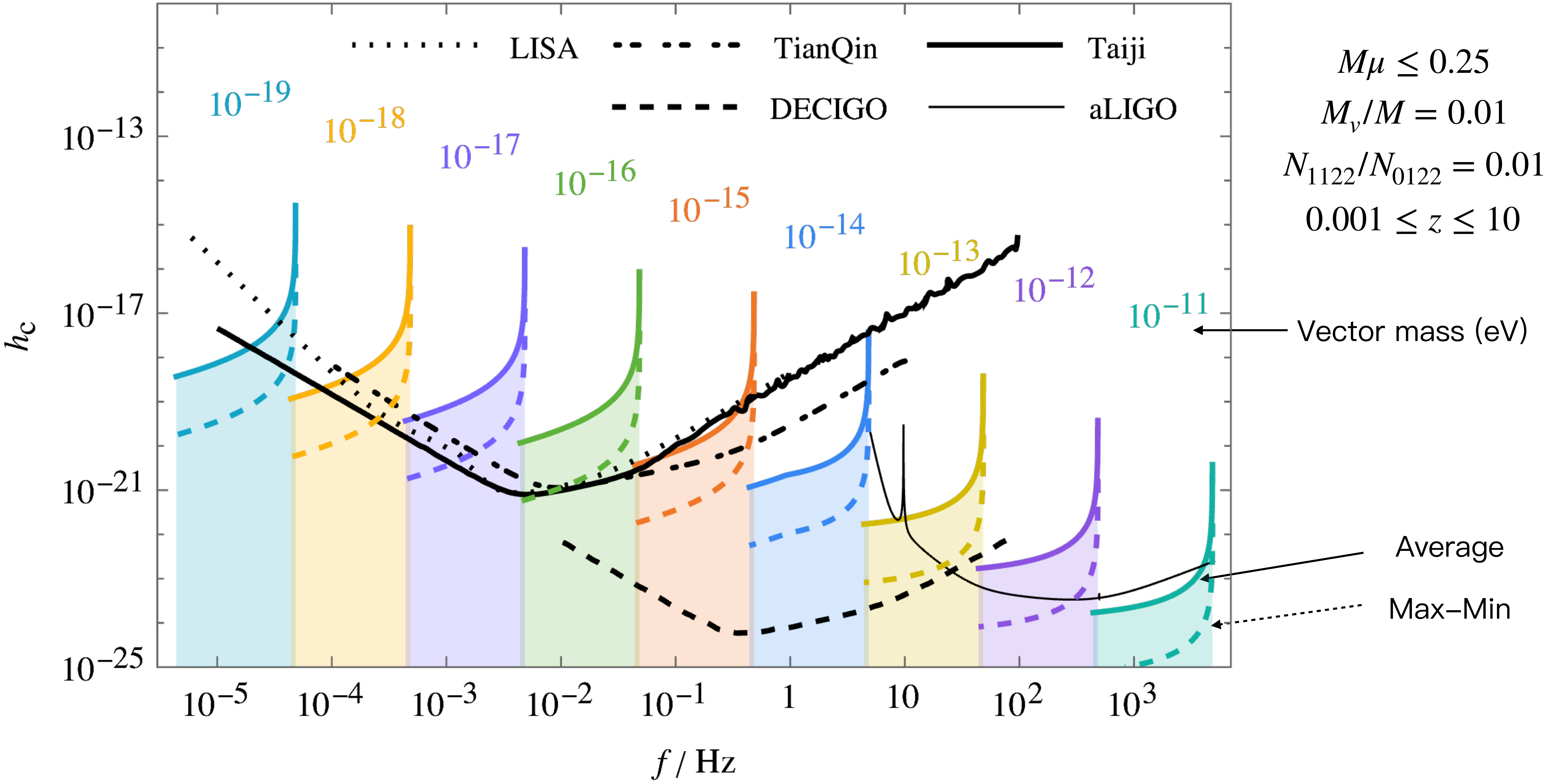


GW beats could be detected by TianQin, Taiji, LISA and so on.

YDG, S. S. Bao and H. Zhang, Phys. Rev. D **107**, 075009 (2023).

Detection

GW beats observation from vector



Stronger GW signals with weaker GW beats

YDG, N. Jia, S. S. Bao, H. Zhang and X. Zhang, [arXiv:2407.00767 [gr-qc]]

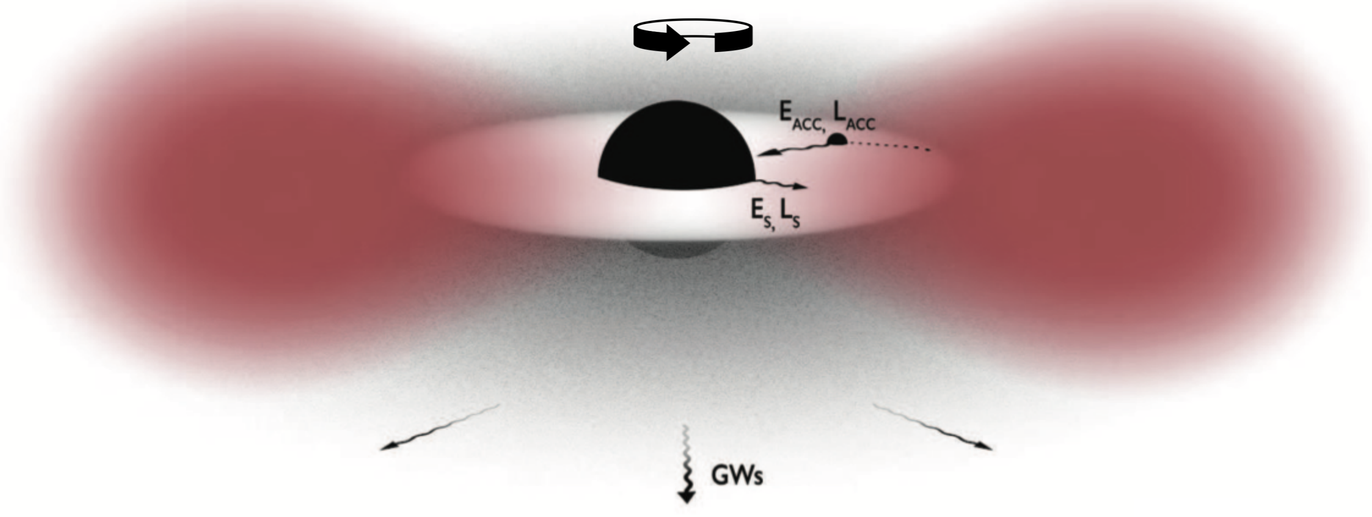
Summary

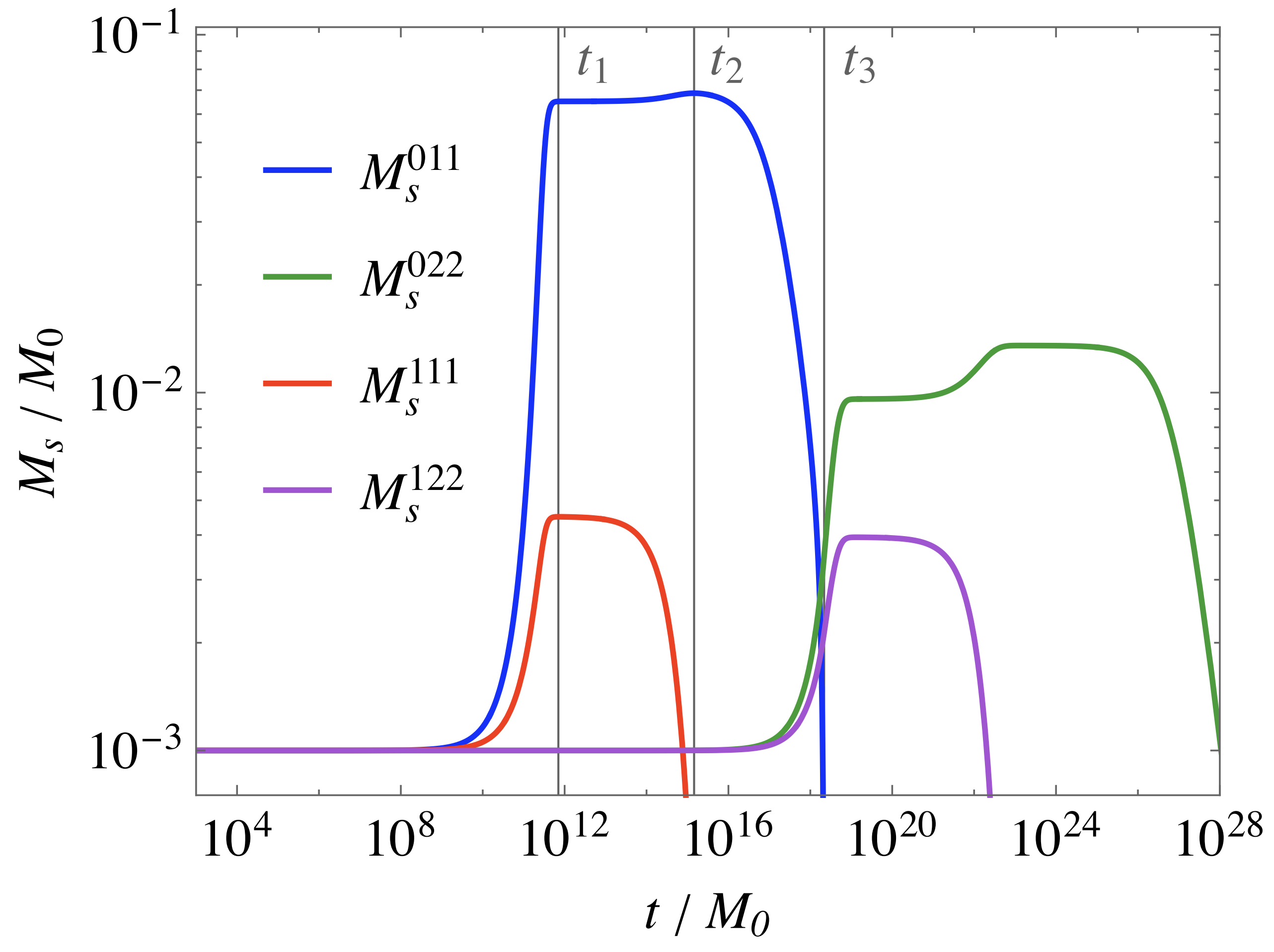
✓ The bosonic cloud increases exponentially by **superradiance** and emits **gravitational waves**.

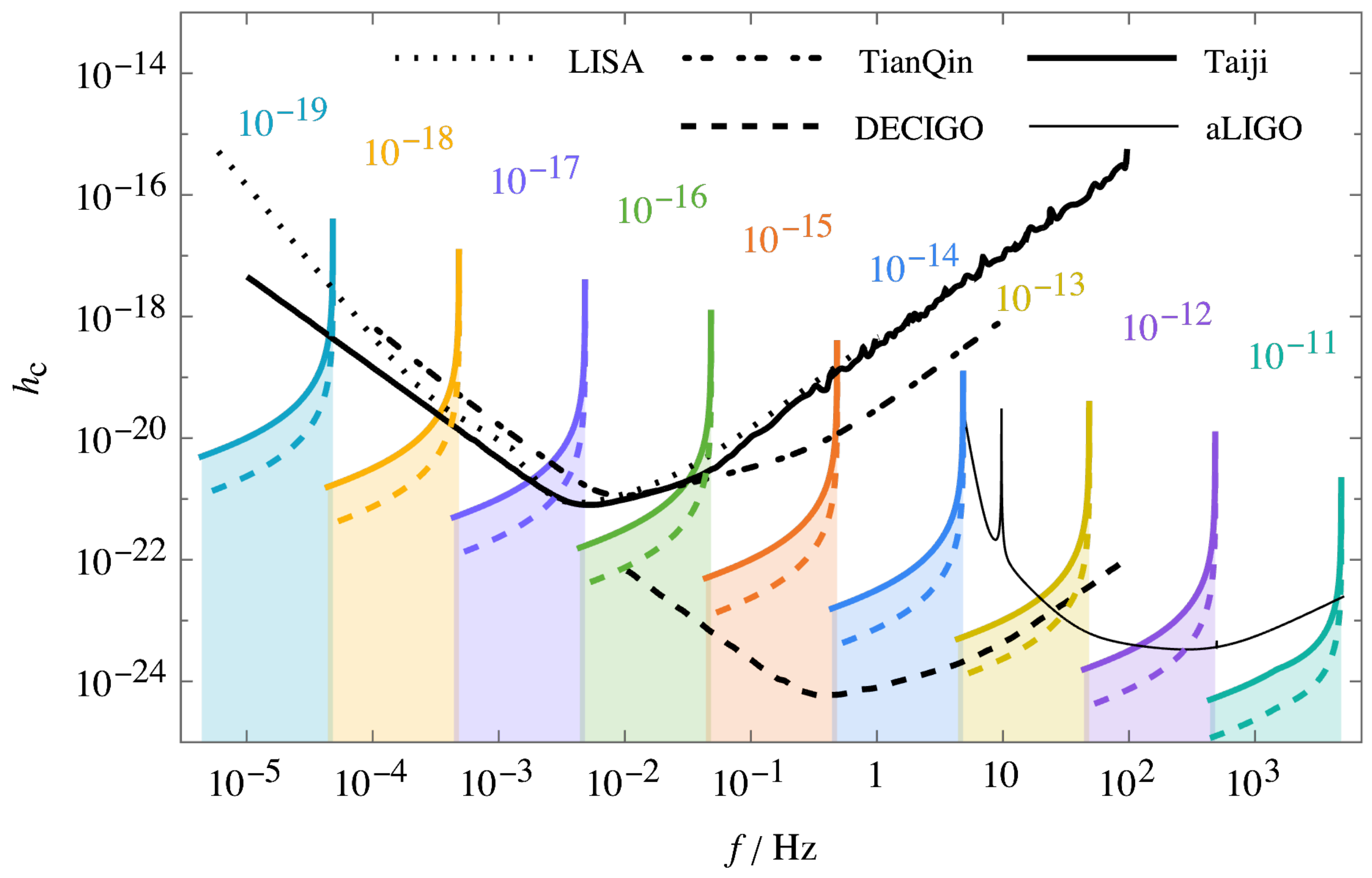
✓ We study the evolution of BH–condensate system with $n > 0$ **modes**, which have important contributions.

✓ Unique GW signal: **GW beats** can be detected by Taiji, TianQin, LISA, etc.

Thank You!



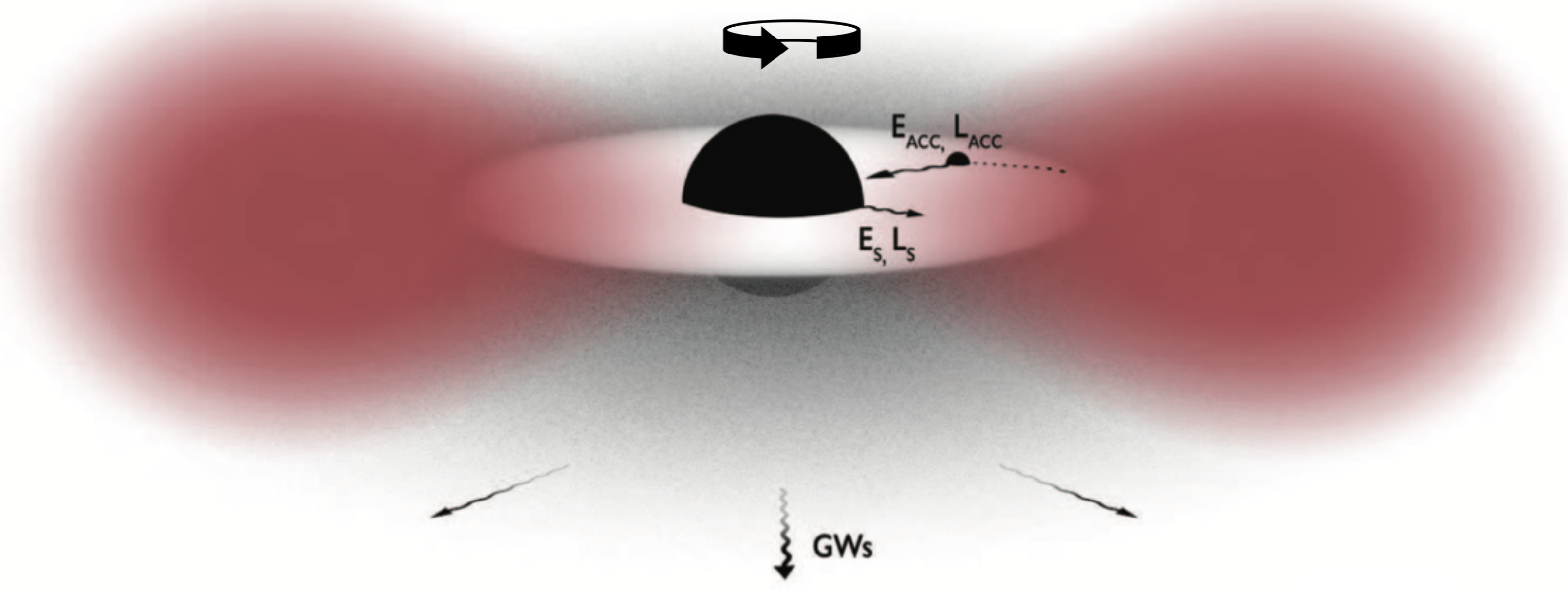




Back Up

Evolutions

GW interference



- **Single mode**: Monochromatic GWs

$$2\times(011)\to\text{graviton}\qquad \omega\sim\mu \qquad \tilde{\omega}\sim 2\mu \qquad T_{\rm GW}\sim\pi/\mu\approx 21\,\sec\left(\frac{10^{-16}\text{eV}}{\mu}\right)$$

Difficult to be distinguished from other monochromatic GW sources   e.g. neutron stars

- **Multiple modes:** {0,1,1} and {1,1,1}
    - Modulation period

$$T_{\rm mod}=\frac{2\pi}{\omega^{(111)}-\omega^{(011)}}=2880\left(\frac{0.1}{M\mu}\right)^2 T_{\rm GW}\approx 6.0\times 10^4\,\sec\left(\frac{10^{-16}\text{eV}}{\mu}\right)\left(\frac{0.1}{M\mu}\right)^2$$

    - Particle picture

$$2\times(011)\to\text{graviton}\qquad \text{Amp}\propto N_{011}$$

$$(011)+(111)\to\text{graviton}\qquad \text{Amp}\propto\sqrt{N_{011}N_{111}}$$

The interference term is mildly suppressed by $\sqrt{N_{111}/N_{011}}$.

Detection

GW emission energy flux

$N_{111}/N_{011} \ll 1$

LO

$\tilde{\omega}_1 \equiv 2\omega^{(011)},\ \tilde{\omega}_2 \equiv 2\omega^{(111)},$
$\tilde{\omega}_3 \equiv \omega^{(011)} + \omega^{(111)},\ \tilde{\omega}_4 \equiv \omega^{(111)} - \omega^{(011)}$

$$\frac{dE_{\rm GW}}{dt} = \frac{1}{8\pi}\sum_{\tilde{l}}\Bigg\{\frac{N_{011}^2}{\omega^{(011)2}}\frac{\left|U_{l2}^{(\tilde{\omega}_1)}\right|^2}{\tilde{\omega}_1^2} + \frac{N_{111}^2}{\omega^{(111)2}}\frac{\left|U_{l2}^{(\tilde{\omega}_2)}\right|^2}{\tilde{\omega}_2^2} + 4\frac{N_{011}N_{111}}{\omega^{(011)}\omega^{(111)}}\frac{\left|U_{l2}^{\tilde{\omega}_3}\right|^2}{\tilde{\omega}_3^2}$$
$$+ 4\sqrt{\frac{N_{011}^3 N_{111}}{\omega^{(011)3}\omega^{(111)}}}\frac{\left|U_{\tilde{l}2}^{(\tilde{\omega}_1)}\right|\left|U_{\tilde{l}2}^{(\tilde{\omega}_3)}\right|}{\tilde{\omega}_1\tilde{\omega}_3}\cdot\cos\left[\tilde{\omega}_4(t-r_*) - \phi_{\tilde{l}2}^{(\tilde{\omega}_3)} + \phi_{\tilde{l}2}^{(\tilde{\omega}_1)}\right]$$
$$+ 2\frac{N_{011}N_{111}}{\omega^{(011)}\omega^{(111)}}\frac{\left|U_{\tilde{l}2}^{(\tilde{\omega}_1)}\right|\left|U_{\tilde{l}2}^{(\tilde{\omega}_2)}\right|}{\tilde{\omega}_1\tilde{\omega}_2}\cdot\cos\left[2\tilde{\omega}_4(t-r_*) - \phi_{\tilde{l}2}^{(\tilde{\omega}_2)} + \phi_{\tilde{l}2}^{(\tilde{\omega}_1)}\right]$$
$$+ 4\sqrt{\frac{N_{011}N_{111}^3}{\omega^{(011)}\omega^{(111)3}}}\frac{\left|U_{\tilde{l}2}^{(\tilde{\omega}_2)}\right|\left|U_{\tilde{l}2}^{(\tilde{\omega}_3)}\right|}{\tilde{\omega}_2\tilde{\omega}_3}\cdot\cos\left[\tilde{\omega}_4(t-r_*) - \phi_{\tilde{l}2}^{(\tilde{\omega}_2)} + \phi_{\tilde{l}2}^{(\tilde{\omega}_3)}\right]\Bigg\}.$$

NLO

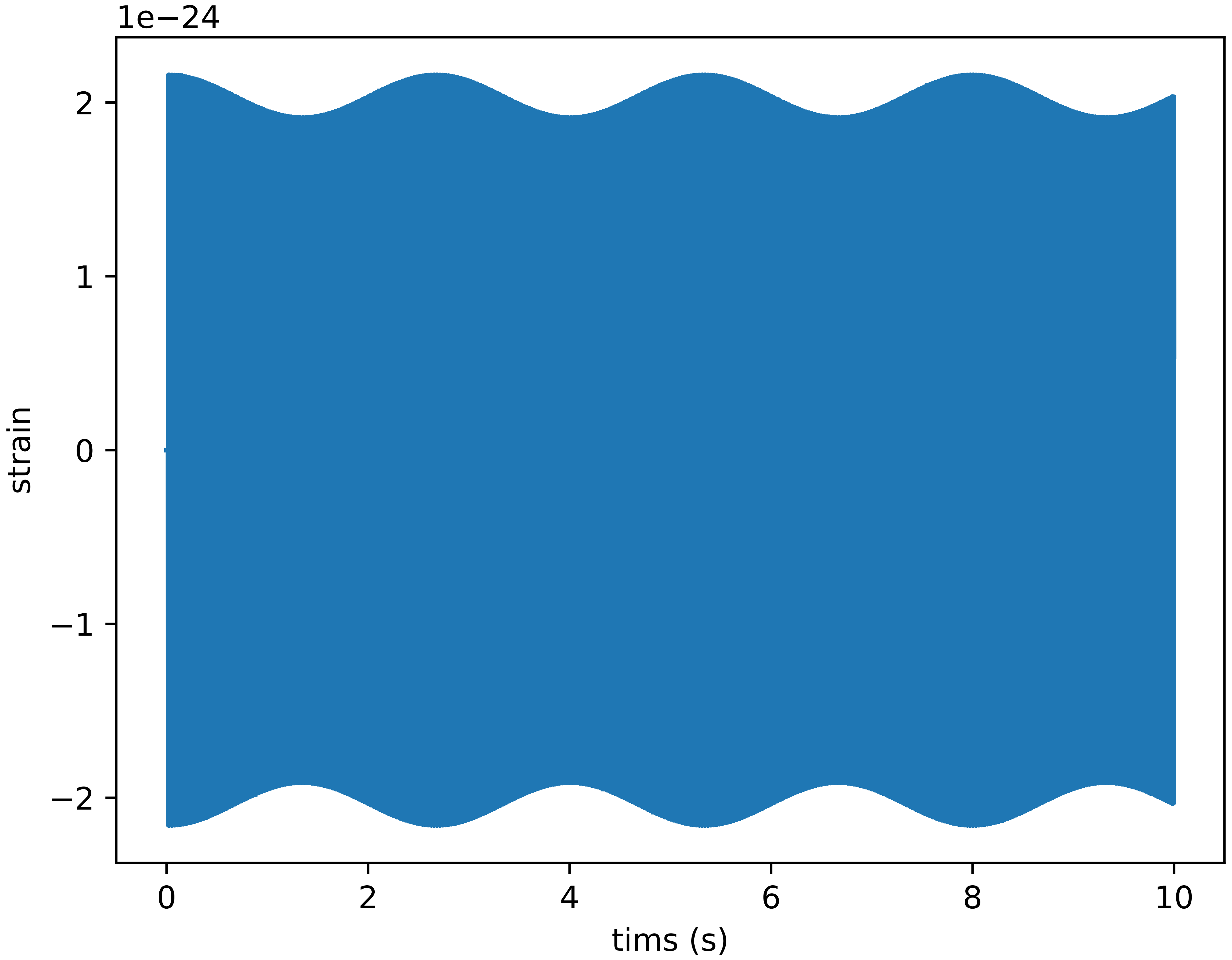
$\sqrt{N_{111}/N_{011}}$ suppressed

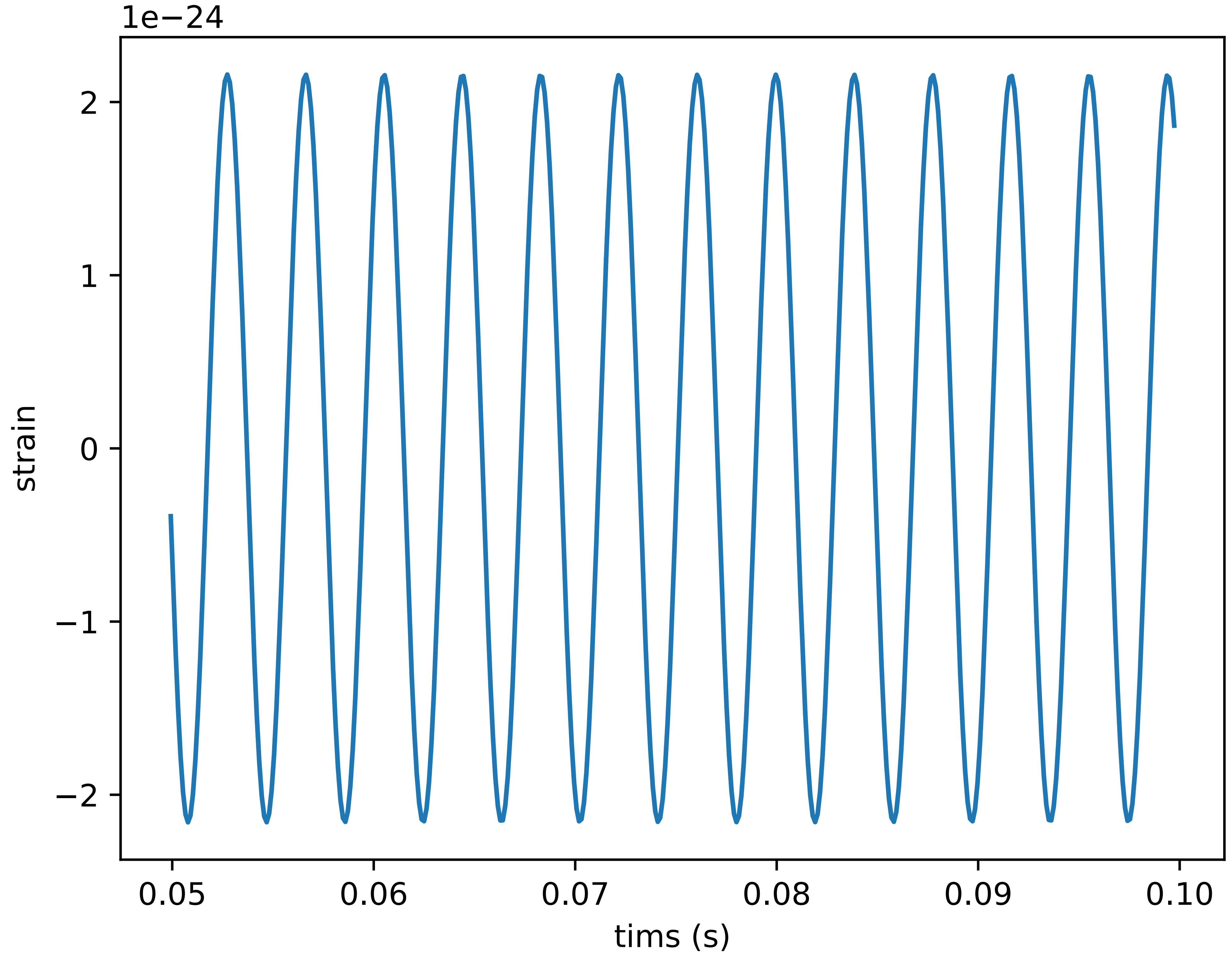
$N_{111}/N_{011} = 0.01 \quad \Rightarrow \quad$ modulation ~ 10%

YDG, S. S. Bao and H. Zhang, Phys. Rev. D **107**, 075009 (2023).

Detection

GW beats observation





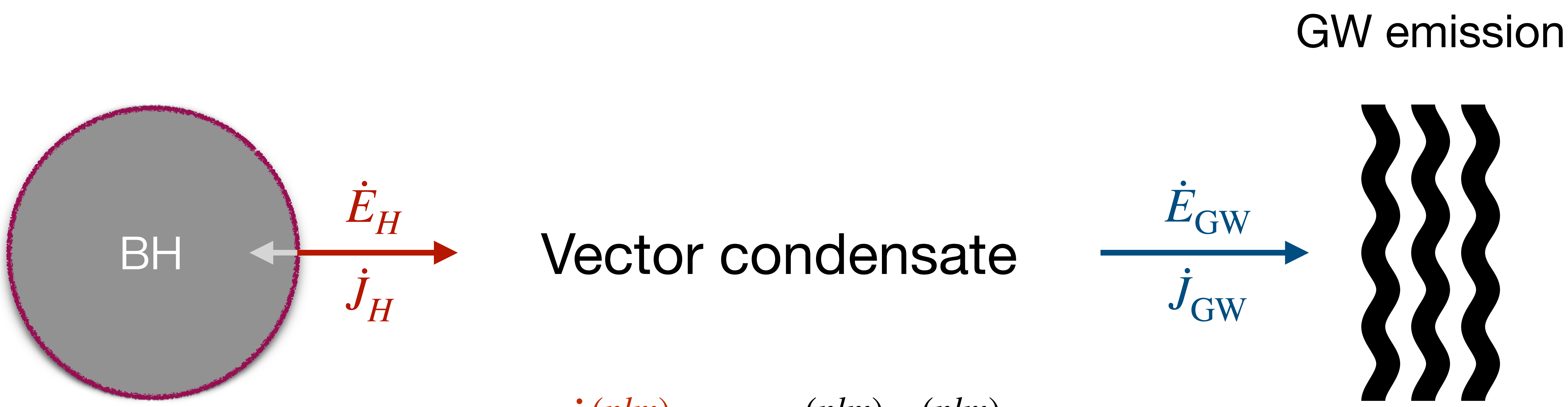
The GW frequency in the detector frame is 258.4 Hz.

The GW beat frequency in the detector frame is 0.3587 Hz.

Redshift: $2.50 \times 10^{-4} \sim 1\,$Mpc. Mass: 5.35×10^{-13} eV SNR: 1.29

Evolutions

Evolution equations



$$\dot{E}_H^{(nlm)} = 2M_s^{(nlm)}\omega_I^{(nlm)}$$

$$\dot{M} = -\sum_{nlm} \dot{E}_H^{(nlm)}, \qquad \dot{J} = -\sum_{nlm} m\dot{E}_H^{(nlm)}/\omega_R^{(nlm)}$$

$$\dot{M}_s^{(nlm)} = \dot{E}_H^{(nlm)} - \dot{E}_{\rm GW}^{(nlm)}, \quad \dot{J}_s = m\left(\dot{E}_H^{(nlm)} - \dot{E}_{\rm GW}^{(nlm)}\right)/\omega_R^{(nlm)}$$

Evolutions

Different initial parameters

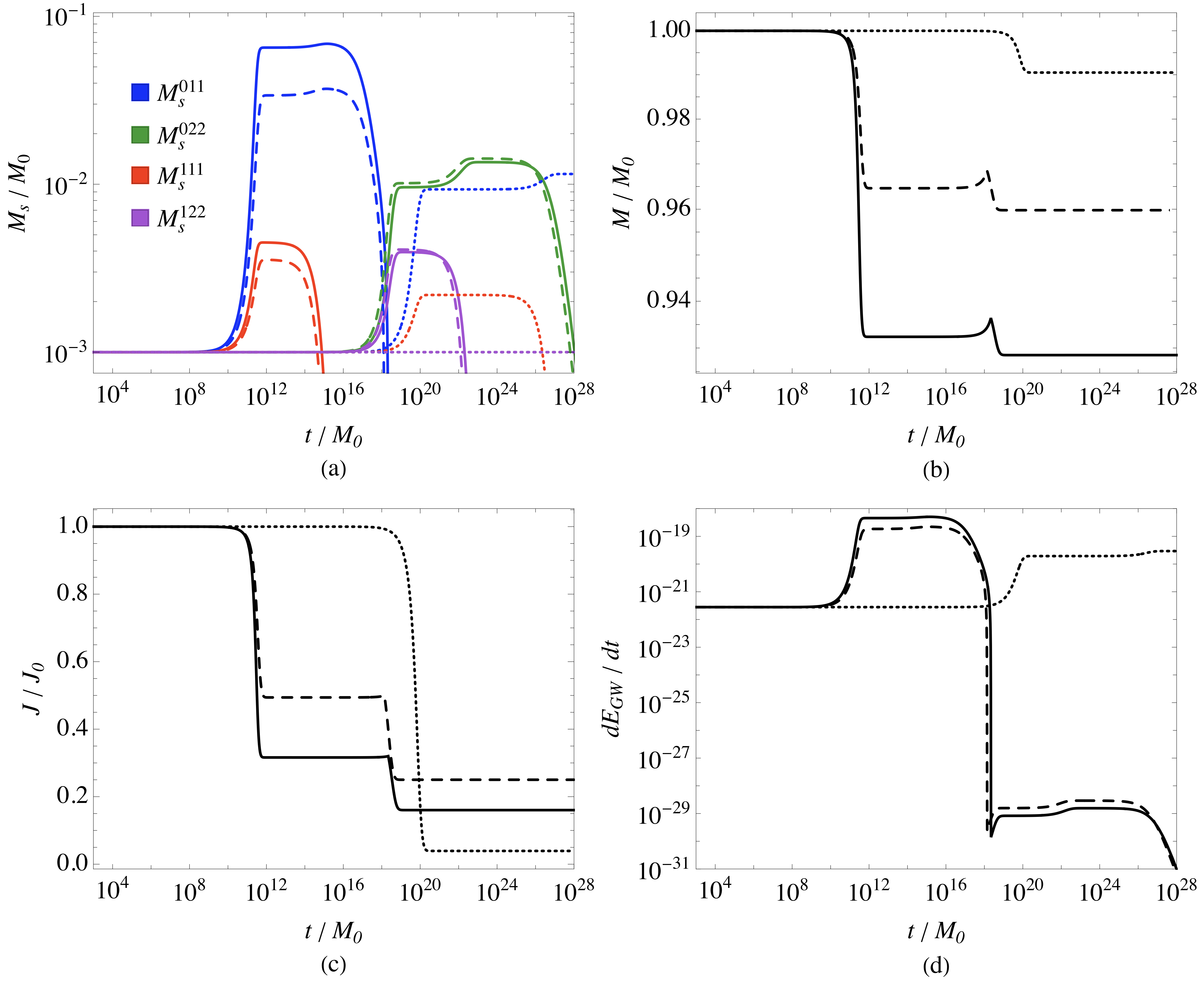


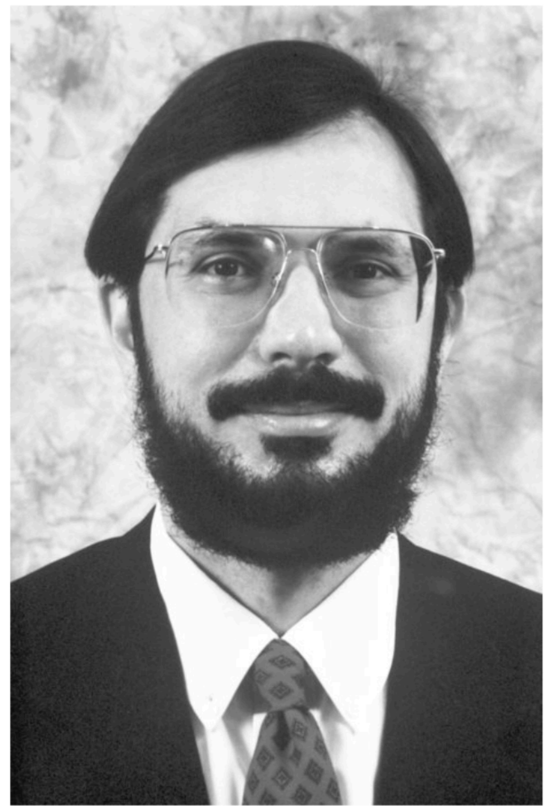
FIG. 7. The evolution of the scalar cloud masses, BH mass, BH spin, and the GW emission luminosity as a function of time. The solid curves are the baseline case with initial parameters $M_0\mu = 0.1$ and $a_{*0} = 0.99$, same as in Fig. 4. The dashed curves are with initial parameters $M_0\mu = 0.1$ and $a_{*0} = 0.7$. The dotted curves are with initial parameters $M_0\mu = 0.01$ and $a_{*0} = 0.9$. The initial mass of each mode is set as $10^{-3} M_0$ for all cases.

Introduction

GW observation

- Indirect observation
  - Hulse—Taylor binary (1974)
- Direct observation
  - GW150914 (2015)
  - GW170817 (2017)
  - 90 confirmed events (by 2024.07.21)

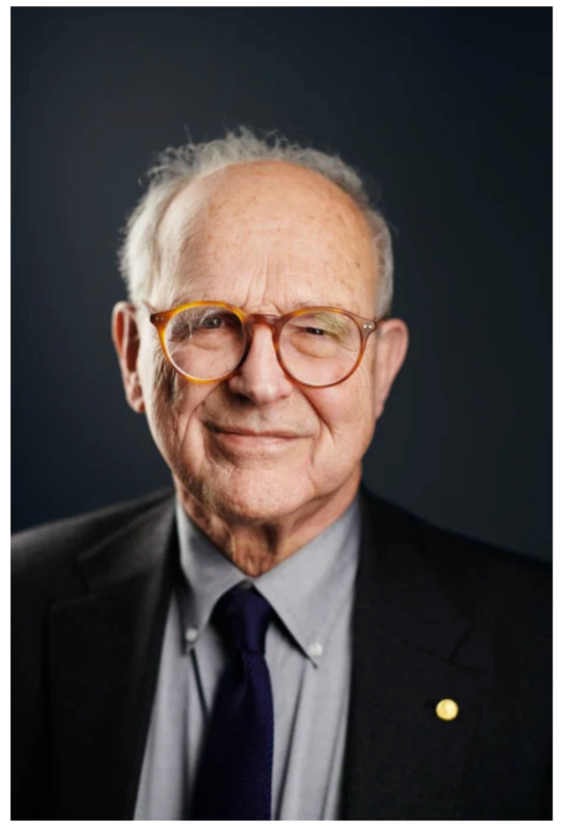
The O4 LIGO-Virgo-KAGRA observing
15:00 UTC, 24 May 2023



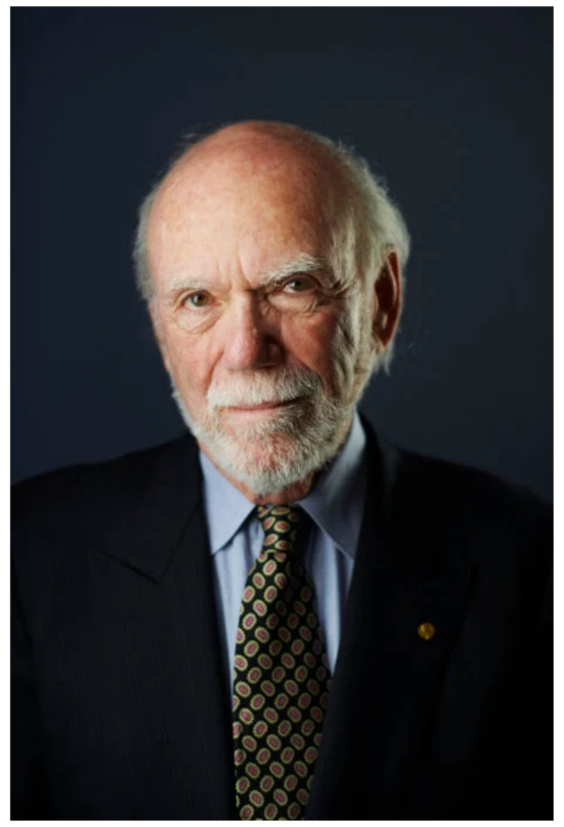
Russell A. Hulse

Joseph H. Taylor Jr.

1993



Rainer Weiss



Barry C. Barish

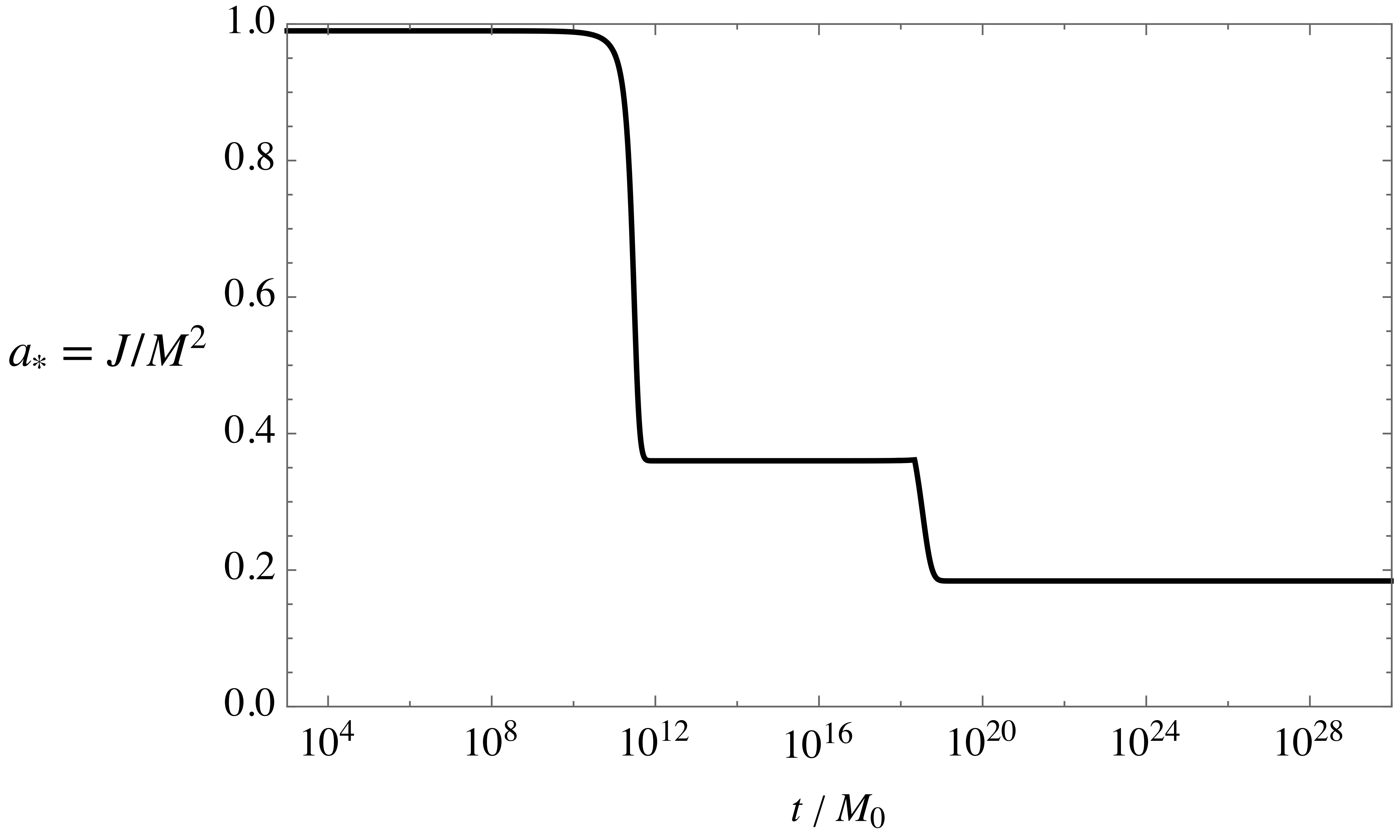
Kip S. Thorne

2017

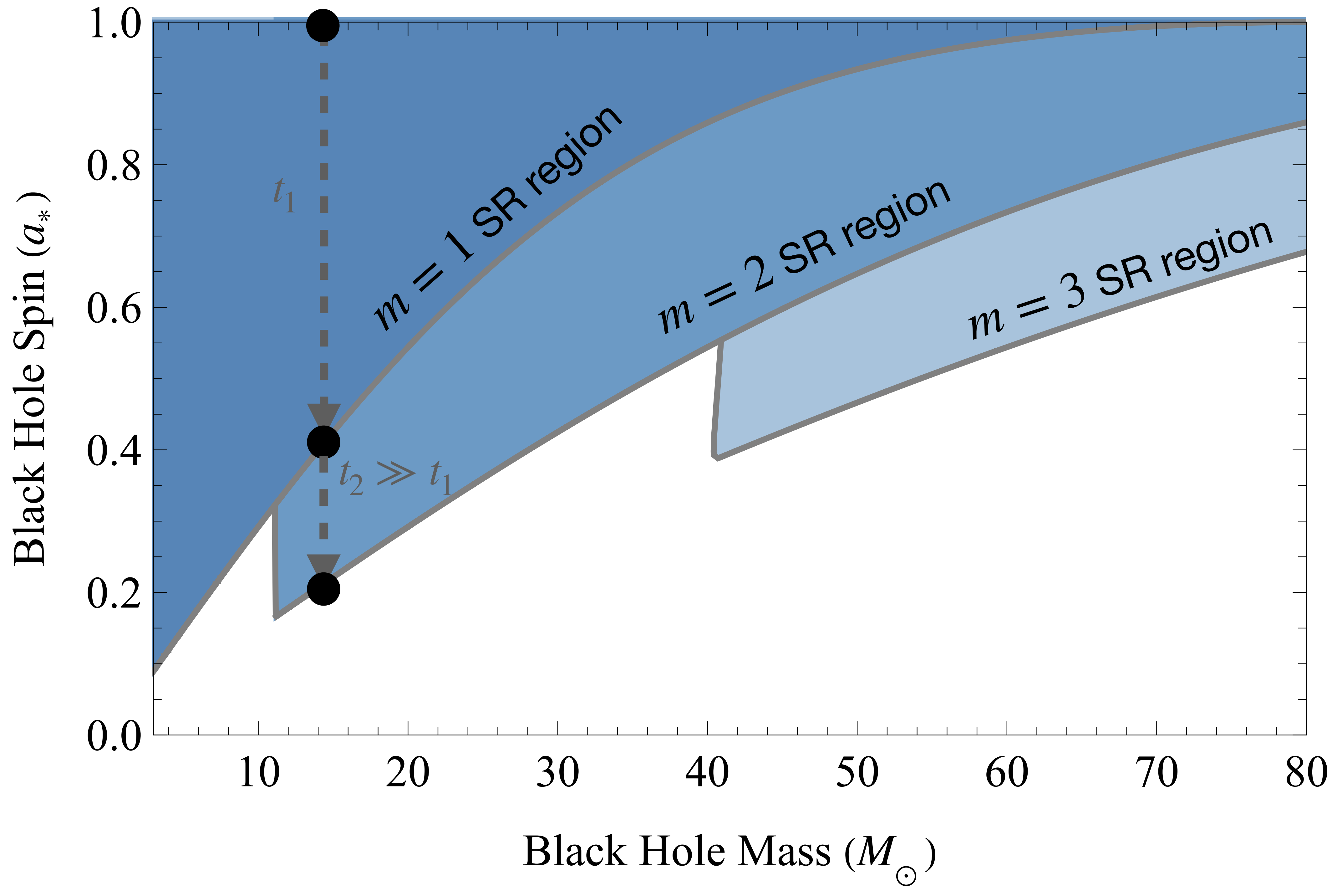
https://www.nobelprize.org/prizes/physics/1993/summary/
https://www.nobelprize.org/prizes/physics/2017/summary/

Introduction

With superradiance, BH spin is discrete



BH spin against time



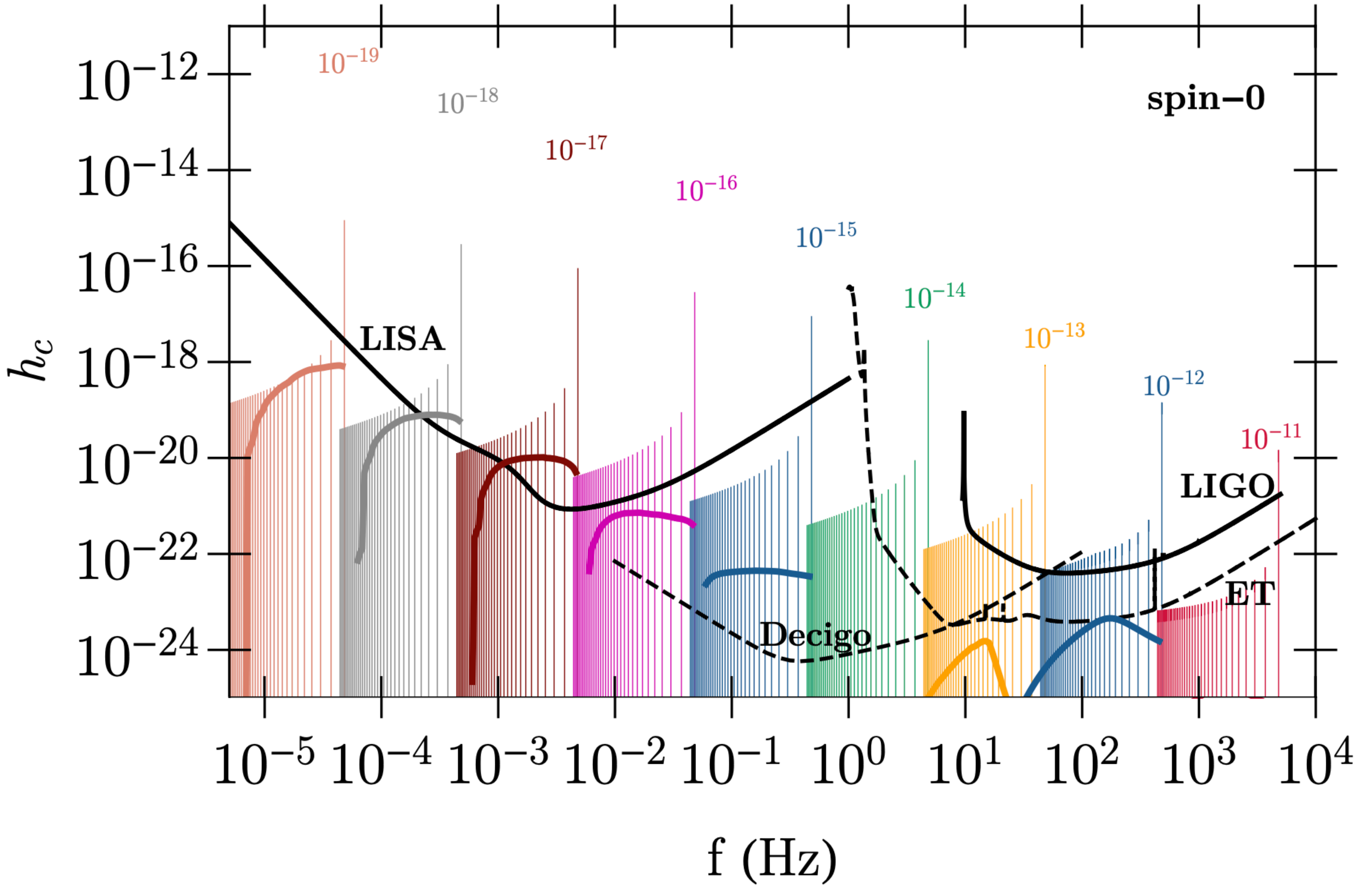
Regge plane

Superradiance affects the BH spin distribution

Introduction

Monochromatic GWs

GW emission



- With only the fastest scalar mode: $n = 0$ and $l = m = 1$
- Monochromatic GWs
- Difficult to identify

The period of monochromatic GWs

$$2 \times (011) \rightarrow \text{graviton} \qquad \omega \sim \mu \qquad \tilde{\omega} \sim 2\mu$$

$$T_{\rm GW} \sim \pi/\mu \approx 21\,\text{sec}\left(\frac{10^{-16}\text{eV}}{\mu}\right)$$

R. Brito, V. Cardoso, and P. Pani, arXiv:1501.06570

GW emission

Linearized Einstein field equations

$$g_{\mu\nu} \to g^{\mathrm{B}}_{\mu\nu} + h_{\mu\nu}$$

$$\Box^{\mathrm{B}} h_{\mu\nu} = -16\pi T_{\mu\nu} + 2R^{\mathrm{B}}_{\rho\mu\nu\sigma} h^{\rho\sigma}$$

- **Two observed polarizations:** $h_+ \; h_\times$ — transverse-traceless gauge
- **Energy flux**

$$\frac{d^2E^{(\mathrm{out})}}{dt d\Omega} = \lim_{r\to\infty} \frac{r^2}{16\pi}\left(\dot{h}_+^2 + \dot{h}_\times^2\right)$$

- **Kerr case**

$$f_1 = a^2 - 6r^2 + a^2\cos(2\theta)$$

$$f_2 = 3a^2 - 2r^2 + 3a^2\cos(2\theta)$$

$$\Box^{\mathrm{B}} h_{tr} = -16\pi G T_{tr} + \frac{3a^3 f_1 h_{\varphi\theta}\sin(2\theta)}{2\Delta\Sigma^4} + \frac{3a^2 f_1\left(a^2+r^2\right)h_{t\theta}\sin(2\theta)}{2\Delta\Sigma^4} - \frac{3af_2 r h_{r\varphi}}{\Sigma^4} + \frac{rh_{tr}\left(3a^4\cos(4\theta) - 27a^4 - 4a^2\left(6a^2+7r^2\right)\cos(2\theta) - 4a^2r^2 + 16r^4\right)}{4\Sigma^4}$$

$$\Box^{\mathrm{B}} h_{t\theta} = -16\pi G T_{t\theta} + \frac{f_2 r h_{t\theta}\left(-a^2\cos(2\theta) + 2a^2 + r^2\right)}{\Sigma^4} + \frac{3a^2\Delta f_1 h_{tr}\sin(2\theta)}{2\Sigma^4} + \frac{3af_2 r h_{\varphi\theta}}{\Sigma^4} + \frac{3a\Delta f_1 h_{r\varphi}\cot(\theta)}{\Sigma^4}$$

… (other 8 equations)

Considerable algebraic complexity!

GW emission

Newman-Penrose formalism

- Null tetrads: $\{l^\mu, n^\mu, m^\mu, m^{*\mu}\}$

$$l^\mu = \left[\left(r^2 + a^2\right)/\Delta, 1, 0, a/\Delta\right], \quad n^\mu = \left[r^2 + a^2, -\Delta, 0, a\right] \Big/ (2\Sigma),$$
$$m^\mu = \left[ia \sin\theta, 0, 1, i/\sin\theta\right] \Big/ \left[\sqrt{2}(r + ia\cos\theta)\right].$$

- Orthogonality relations

$$l^\mu n_\mu = 1 \qquad m^\mu m^*_\mu = -1$$
others = 0

- Projection of tensors

$$T_{\mu\nu} \to T_{ab} = T_{\mu\nu} a^\mu b^\nu, \quad a, b = \{l, n, m, m^*\}$$
e.g.: $T_{12} = T_{\mu\nu} l^\mu n^\nu$

$$R_{\mu\nu\sigma\rho} \to R_{abcd} = R_{\mu\nu\sigma\rho} a^\mu b^\nu c^\sigma d^\rho, \quad a, b, c, d = \{l, n, m, m^*\}$$
$$R_{2424} = R_{\mu\nu\sigma\rho} n^\mu m^{*\nu} n^\sigma m^{*\rho}$$

S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Clarendon Press, 1998.

GW emission

Teukolsky equation

- Einstein equation: $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T^{\mu\nu}$$
- Ricci identity: $$R^{\mu}{}_{\nu\rho\sigma} Z_{\mu} = Z_{\nu;\rho;\sigma} - Z_{\nu;\sigma;\rho}$$
- Bianchi identity: $$R_{\mu\nu[\rho\sigma;\lambda]} = \frac{1}{3}\left(R_{\mu\nu\rho\sigma;\lambda} + R_{\mu\nu\sigma\lambda;\rho} + R_{\mu\nu\lambda\rho;\sigma}\right) = 0$$

Projecting onto $\{l^{\mu}, n^{\mu}, m^{\mu}, m^{*\mu}\}$

$$\left[\frac{\left(r^2+a^2\right)^2}{\Delta} - a^2 \sin^2\theta\right] \frac{\partial^2 \psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t \partial \varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2} - \Delta^2 \frac{\partial}{\partial r}\left(\Delta^{-1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$

Teukolsky equation

$$+4\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi} + 4\left[\frac{M\left(r^2-a^2\right)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + \left(4\cot^2\theta + 2\right)\psi = 4\pi\Sigma T$$

S. A. Teukolsky, ApJ **185**, 635 (1973).
S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Clarendon Press, 1998.

GW emission

Weyl scalar $\psi_4 = \psi \rho^4$

$$\rho = -1/(r - ia\cos\theta)$$

$$\psi_4 = -C_{2424} = -C_{\mu\nu\rho\sigma} n^\mu m^{*\nu} n^\rho m^{*\sigma}.$$

- Weyl tensor

S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Clarendon Press, 1998.

$$C_{abcd} = R_{abcd} - \frac{1}{2}\left(\eta_{ac}R_{bd} - \eta_{bc}R_{ad} - \eta_{ad}R_{bc} + \eta_{bd}R_{ac}\right) + \frac{1}{6}\left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\right) R$$

$$C_{2424} = R_{2424}$$

- At infinity

$$l^\mu \to \left(\hat{t}^\mu + \hat{r}^\mu\right) \qquad n^\mu \to \left(\hat{t}^\mu - \hat{r}^\mu\right)/2 \qquad m^\mu \to \left(\hat{\theta}^\mu + i\hat{\varphi}^\mu\right)/\sqrt{2}$$

$$\frac{d^2E^{(\text{out})}}{dtd\Omega} = \lim_{r\to\infty} \frac{r^2}{16\pi}\left(\dot{h}_+^2 + \dot{h}_\times^2\right)$$

Recall

$$\psi_4 \to \frac{1}{2}\left(\ddot{h}_+ - i\ddot{h}_\times\right)$$

$$h_+ \equiv h_{\hat{\theta}\hat{\theta}},\ h_\times \equiv h_{\hat{\theta}\hat{\varphi}}$$

Teukolsky equation $\Rightarrow$ $\psi = \psi_4 \rho^{-4}$ $\Rightarrow$ GW emission flux $dE^{(\text{out})}/dt$

GW emission

GW solution

- Separation of variables

S. A. Teukolsky, ApJ **185**, 635 (1973).

$$4\pi\Sigma T = \int d\tilde{\omega} \sum_{\tilde{l},\tilde{m}} G_{\tilde{l}\tilde{m}}(r)\,{}_sS_{\tilde{l}\tilde{m}}(\theta)e^{i\tilde{m}\phi}e^{-i\tilde{\omega}t},$$

$$\psi = \int d\tilde{\omega} \sum_{\tilde{l},\tilde{m}} R_{\tilde{l}\tilde{m}}(r)\,{}_sS_{\tilde{l}\tilde{m}}(\theta)e^{i\tilde{m}\phi}e^{-i\tilde{\omega}t}.$$

- Angular equation

spin-weighted spheroidal harmonics

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\,{}_sS_{\tilde{l}\tilde{m}}}{d\theta}\right) + \left(a^2\tilde{\omega}^2\cos^2\theta - \frac{\tilde{m}^2}{\sin^2\theta} - 2a\tilde{\omega}s\cos\theta - \frac{2\tilde{m}s\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A\right){}_sS_{\tilde{l}\tilde{m}} = 0$$

$a = a_* M$

- $a = 0$ and $s = 0 \Rightarrow$ Associated Legendre polynomials $P_{\tilde{l}\tilde{m}}$
- Exact results $\Leftarrow$ Black Hole Perturbation Toolkit
- $s = -2$ for outgoing GW

https://bhptoolkit.org/toolkit.html

GW emission

GW solution

- Radial equation $s = -2$

$$\Delta^2 \frac{\partial}{\partial r}\left(\frac{1}{\Delta}\frac{\partial R_{\tilde{l}\tilde{m}}(r)}{\partial r}\right) + \left[\frac{\tilde{K}^2 + 4i(r-M)\tilde{K}}{\Delta} - 8i\tilde{\omega}r - {}_{-2}\lambda_{\tilde{l}\tilde{m}}\right]R_{\tilde{l}\tilde{m}}(r) = -G_{\tilde{l}\tilde{m}}(r)$$

<u>**Green's function method**</u>

<u>Tortoise coordinate</u>

$dr_*/dr = (r^2 + a^2)/\Delta$

$r \to r_+,\ r_* \to -\infty$

Two Green's functions

$$g^{\infty}_{\tilde{l}\tilde{m}} \to \begin{cases} A^{\text{out}}_{\tilde{l}\tilde{m}} e^{i\tilde{k}r_*} + \Delta^2 A^{\text{in}}_{\tilde{l}\tilde{m}} e^{-i\tilde{k}r_*} & \text{at } r \to r_+, \\ r^3 e^{i\tilde{\omega}r_*} & \text{at } r \to +\infty, \end{cases}$$

$$g_{\tilde{l}\tilde{m}} \to \begin{cases} \Delta^2 e^{-i\tilde{k}r_*} & \text{at } r \to r_+ \\ r^3 B^{\text{out}}_{\tilde{l}\tilde{m}} e^{i\tilde{\omega}r_*} + r^{-1} B^{\text{in}}_{\tilde{l}\tilde{m}} e^{-i\tilde{\omega}r_*} & \text{at } r \to +\infty \end{cases}$$

Solution

$$R_{\tilde{l}\tilde{m}} = \frac{(-1)}{W_{\tilde{l}\tilde{m}}}\left\{ g^{\infty}_{\tilde{l}\tilde{m}} \int_{r_+}^{r} dr' \frac{g_{\tilde{l}\tilde{m}} G_{\tilde{l}\tilde{m}}}{\Delta^2} + g_{\tilde{l}\tilde{m}} \int_{r}^{\infty} dr' \frac{g^{\infty}_{\tilde{l}\tilde{m}} G_{\tilde{l}\tilde{m}}}{\Delta^2} \right\}$$

$$W_{\tilde{l}\tilde{m}} = \frac{g_{\tilde{l}\tilde{m}}}{\Delta}\frac{dg^{\infty}_{\tilde{l}\tilde{m}}}{dr} - \frac{g^{\infty}_{\tilde{l}\tilde{m}}}{\Delta}\frac{dg_{\tilde{l}\tilde{m}}}{dr}$$

$$\lim_{r\to\infty} W_{\tilde{l}\tilde{m}} = 2i\tilde{\omega}B^{\text{in}}_{\tilde{l}\tilde{m}}$$

Difficult to obtain $B^{\text{in}}_{\tilde{l}\tilde{m}}$

R. Brito, *et al*. Phys. Rev. D **96**, 064050 (2017).

GW emission

GW solution

Solving $B^{\mathrm{in}}_{\tilde{l}\tilde{m}}$

Homogeneous

$$R''(r) - AR'(r) - BR(r) = 0,$$

Asymptotic solutions

At infinity

$$\varphi_1 = r^3 e^{i\tilde{\omega}r_*}\left[1 + \mathcal{O}(1/r)\right],$$

$$\varphi_2 = \frac{1}{r}e^{-i\tilde{\omega}r_*}[1 + \mathcal{O}(1/r)]\,.$$

Different behaviors!

Auxiliary function

$$\varphi_1 - S_1 = \mathcal{O}(r^0)\,.$$

$$\chi_i \equiv \frac{d}{dr}\left(\frac{\varphi_i}{S_1}\right) \quad \text{with } i = 1,2$$

$$\chi_1 = \mathcal{O}(1/r^4),$$

$$\chi_2 = -2i\tilde{\omega}\frac{dr_*}{dr}e^{-2i\tilde{\omega}r_*}\mathcal{O}(1/r^4).$$

W. H. Press and S. A. Teukolsky, ApJ **185**, 649-674 (1973).

- GW emission energy flux

$$\frac{d^2E^{(\mathrm{out})}}{dtd\Omega} = \lim_{r\to\infty}\sum_i \frac{r^2\tilde{\omega}_i^2}{16\pi}\left[h_+^2(\tilde{\omega}_i) + h_\times^2(\tilde{\omega}_i)\right] + \text{interference terms}$$

Superradiance

- In 1954, Dicke introduced the concept of superradiance, standing for a collective phenomena whereby radiation is amplified by coherence of emitters.

For want of a better term, a gas which is radiating strongly because of coherence will be called 'super-radiant'.

- In 1971 Zel'dovich showed that scattering of radiation off rotating absorbing surfaces results, under certain conditions, in waves with a larger amplitude.

- This phenomenon is now widely known also as (rotational) superradiance

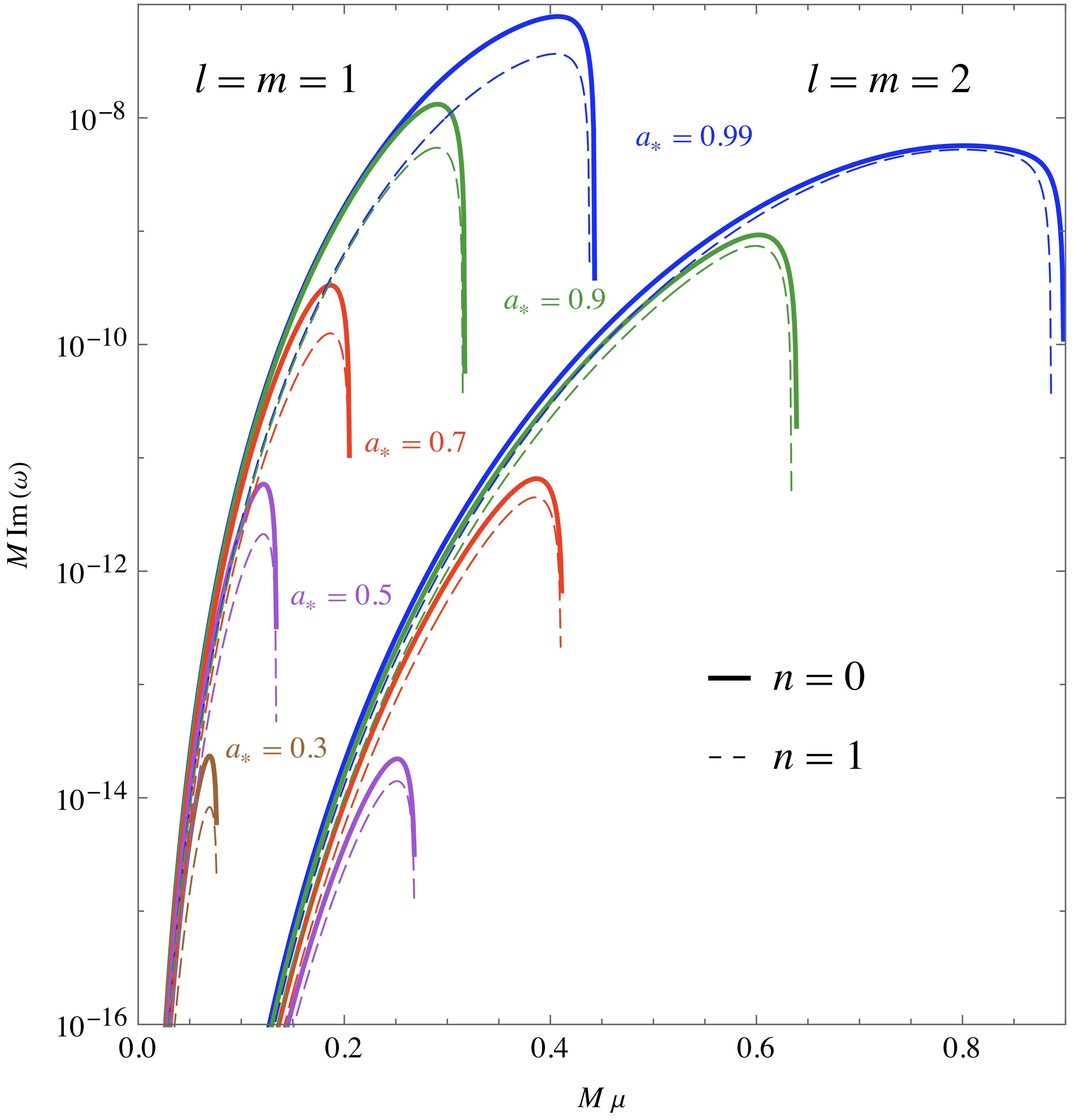
R. Brito, V. Cardoso, and P. Pani, Superradiance -- the 2020 Edition (2020).

Characteristic strain

- The characteristic strain his designed to include the effect of integrating an inspiralling signal.
- The correct identification of characteristic strain for a monochromatic source is the amplitude of the wave times the square root of the number of periods observed.

C. J. Moore, R. H. Cole, and C. P. L. Berry, Class. Quantum Grav. 32, 015014 (2015).

为什么111态会先被吸收



为什么111态被吸收时黑洞参数基本不变, 但对于011不成立

$$\dot{M} = -\sum_{nlm} \dot{E}_s^{(nlm)}, \quad \dot{J} = -\sum_{nlm} m\dot{E}_s^{(nlm)}/\omega_R^{(nlm)}$$

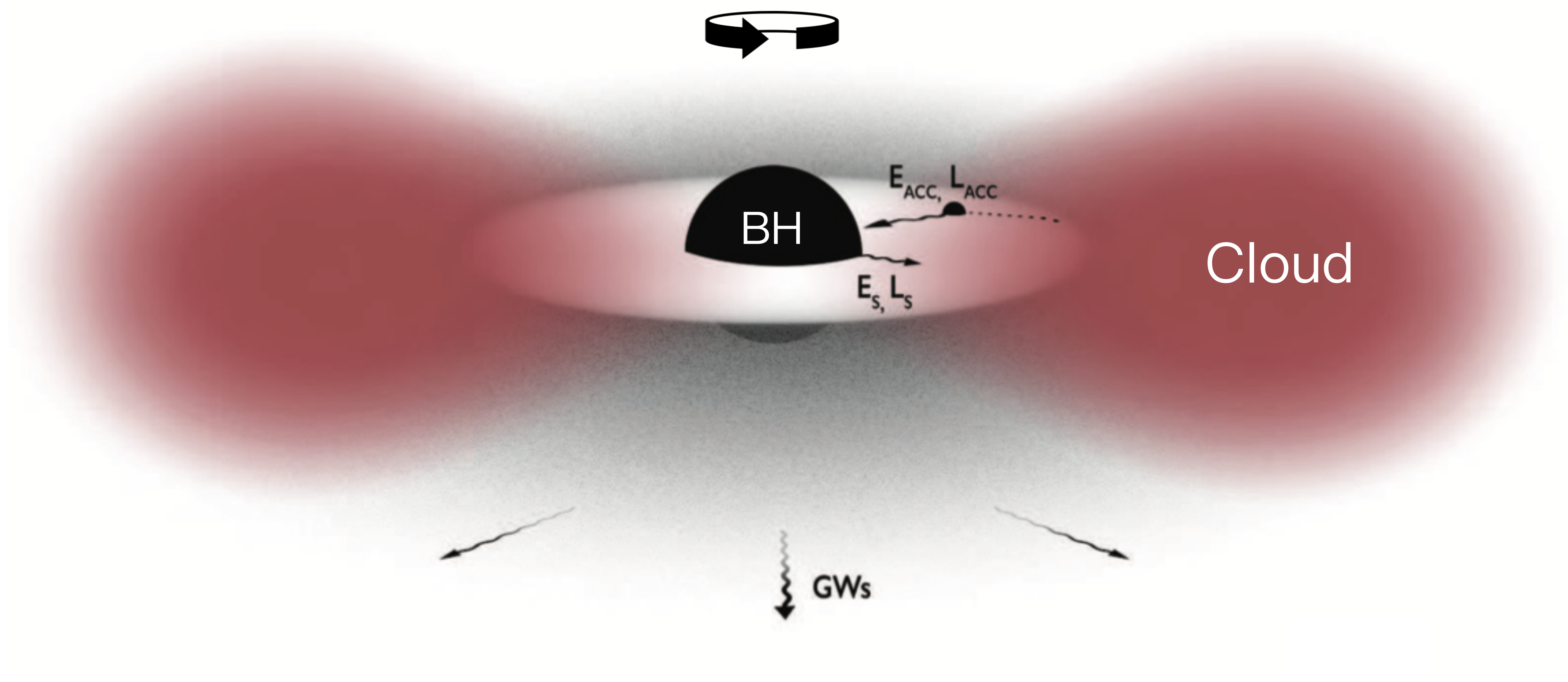
$$\dot{M}_s^{(nlm)} = \dot{E}_s^{(nlm)} - \dot{E}_{\mathrm{GW}}^{(nlm)}, \quad \dot{E}_s^{(nlm)} = 2M_s^{(nlm)}\omega_I^{(nlm)},$$

What's transverse-traceless gauge

- 横向条件：度规扰动的空间分量是横向的，即满足$\nabla_i h_{ij} = 0$，其中∇_i表示空间导数，h_{ij}表示度规扰动。
- 无迹条件：度规扰动的迹为零，即$h_{ii} = 0$。
- 这些条件有效地消除了度规扰动中的纵向和标量模式，只留下与引力波相关的横向张量模式。
- 简单的来说，对于零质量的矢量粒子，m_z 不再是一个好量子数，此时的一个好量子数是helicity，其本征值为± 1

Introduction

BH-condensate system



- Scalar clouds can extract energy and angular momentum by **superradiance**.

Strongest when

bosonic Compton wavelength ~ radius of BH

- Scalar clouds can emits **GWs**.

R. Brito, V. Cardoso, and P. Pani, arXiv:1501.06570

Massive scalar in Kerr spacetime

$$\left(\nabla^\nu \nabla_\nu + \mu^2\right) \Phi = 0$$

Eigenfrequency is complex

$$\omega_{n\ell m} = \omega^{(R)}_{n\ell m} + \boxed{i\omega^{(I)}_{n\ell m}}$$

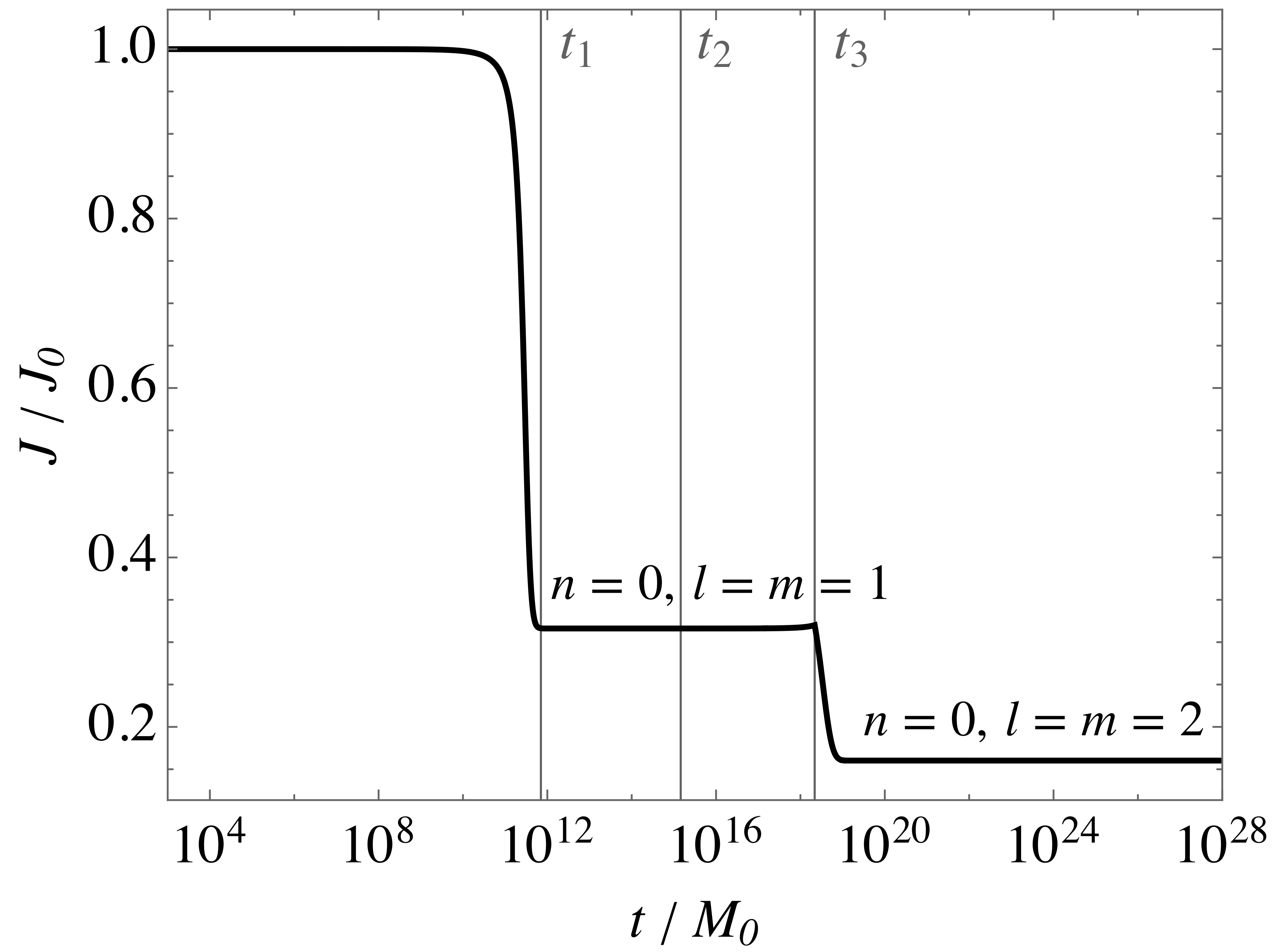
3 indexes - $(n,\ \ell,\ m)$

$$\Phi = e^{\omega_I t} e^{-i\omega_R t} \phi \overset{\omega_I > 0}{\Longrightarrow} \text{Increase}$$

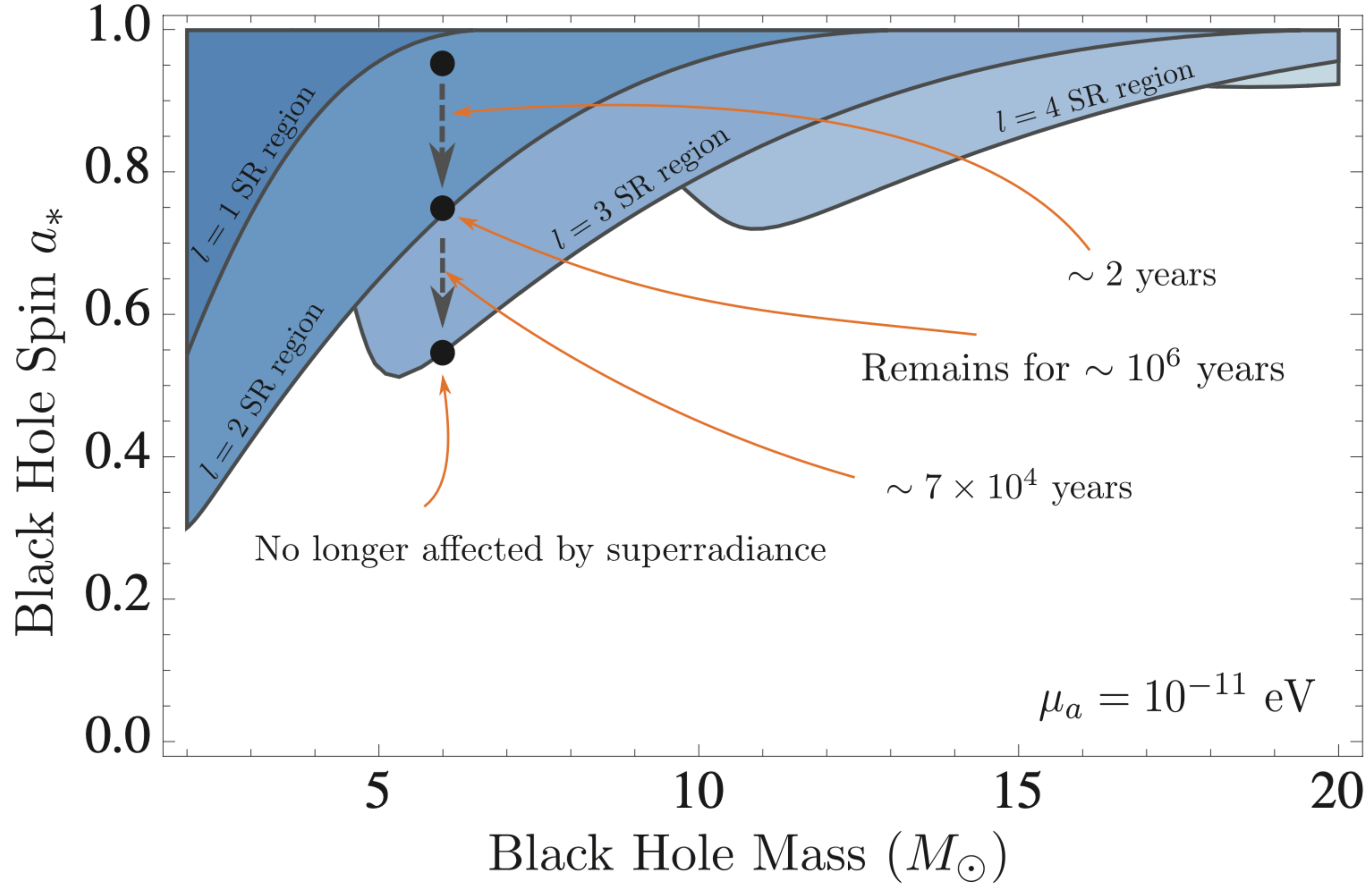
Getting the solution is difficult !

S. Detweiler, Phys. Rev. D **22**, 2323 (1980).
S. S. Bao, Q. X. Xu, and H. Zhang, Phys. Rev. D **106**, 064016 (2020).
V. Cardoso and S. Yoshida, J. High Energy Phys. **2005**, 009 (2005).
S. R. Dolan, Phys. Rev. D **76**, 084001 (2007).

With dominant modes, BH spin is discrete



BH spin against time

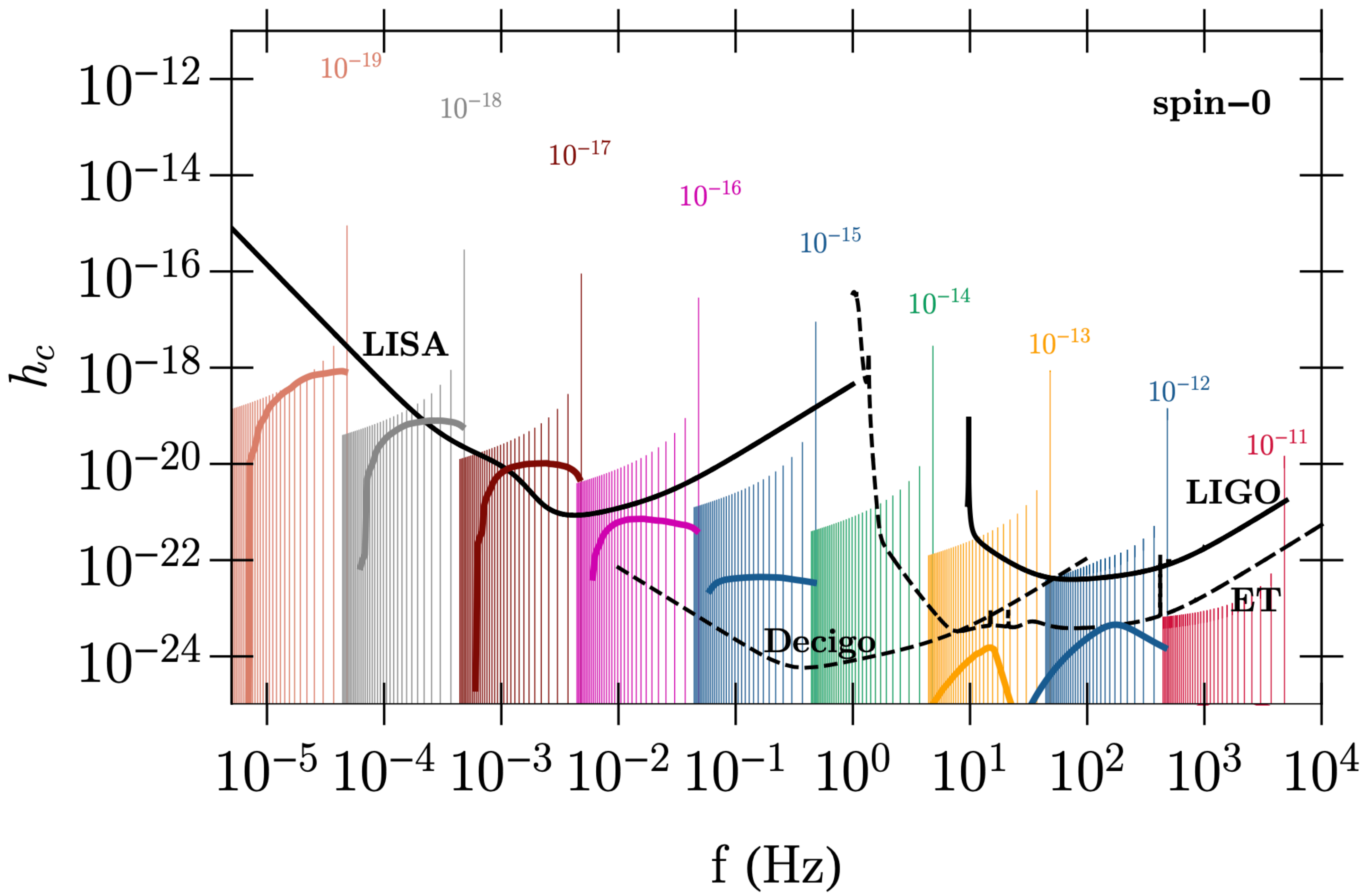


BH evolution

R. Brito, V. Cardoso, and P. Pani, arXiv:1501.06570

Introduction
Monochromatic GWs

GW emission



- Only dominant mode: $n = 0$ and $l = m = 1$
- Monochromatic GWs
- Difficult to distinguish

The period of monochromatic GWs

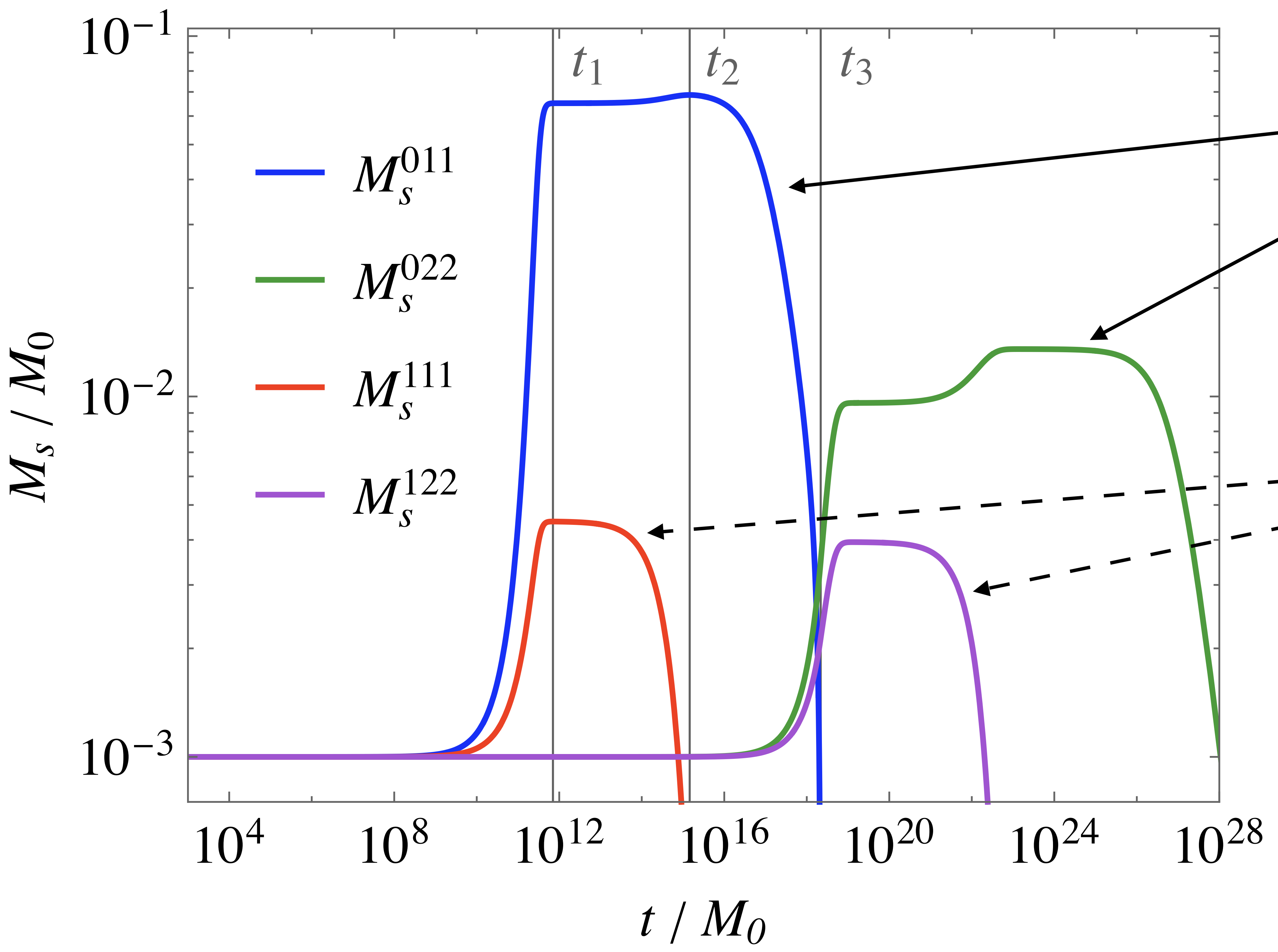
$$2 \times (011) \to \text{graviton}$$
$$\omega \sim \mu \qquad \tilde{\omega} \sim 2\mu$$

$$T_{\rm GW} \sim \pi/\mu \approx 21\,\text{sec}\left(\frac{10^{-16}\text{eV}}{\mu}\right)$$

R. Brito, V. Cardoso, and P. Pani, arXiv:1501.06570

Introduction

Dominant and subdominant modes



dominant modes: n=0

subdominant modes: n>0

Coexistence $\sim \frac{M_0}{M_\odot} 10^2$ yr

- Coexistence $\longrightarrow$ GW beats $\longrightarrow$ to distinguish
- n>0 modes $\longrightarrow$ BH evolution trajectory changes?

YDG, S. S. Bao and H. Zhang, Phys. Rev. D **107**, 075009 (2023).