Generic EFT for all Masses and Spins

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Educational, Scientific and
Cultural Organization

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INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS ASIA-PACIFIC

Based on:

JHEP **05 (2023) 241,**
JHEP **09 (2023) 101,**
Chin.Phys.C 47 (2023) 023105
Phys.Rev.D **107 (2023) 11, L111901,**
Phys.Rev.D **106 (2022) 11, 116010,**
2409.XXX

Collaborators:

Zi-Yu Dong, Zi-Zheng Zhou, Jing Shu, Ci-hang Li, Qi-ming Qiu
Hong-Kai Liu, Yael Shadmi, Michael Waterbury

Massless Amplitude Basis

Challenge in EFT Basis Construction

- Effective field theory

Wilson coefficients

$$\mathscr{L}_{EFT} = \mathscr{L}_{renormalizable} + \sum \frac{c_i}{\Lambda^{d-4}} \mathcal{O}_i^d$$

Scale

A complete set of EFT bases

$\cup$

All lower energy effects of any UV theory

- Difficulties in constructing a complete set of operator bases

Eliminate redundancy from the constraints:

Equation of motion (EOM)

Integration by part (IBP)

Unsolved problem in traditional field theory!!!

$$\mathcal{O} \longrightarrow \mathcal{O}' + \cdots$$

- EFT Basis number can be counted by Hilbert series technique

Can not construct the bases!!!

How to solve it?

2, 84, 30, 993, 560, 15456, 11962, 261485, ...: Higher dimension operators in the SM EFT

Brian Henning (Yale U.), Xiaochuan Lu (UC, Davis), Tom Melia (UC, Berkeley and LBNL, Berkeley), Hitoshi Murayama (UC, Berkeley and LBNL, Berkeley and Tokyo U., IPMU) (Dec 10, 2015)

Published in: *JHEP* 08 (2017) 016, *JHEP* 09 (2019) 019 (erratum) • e-Print: 1512.03433 [hep-ph]

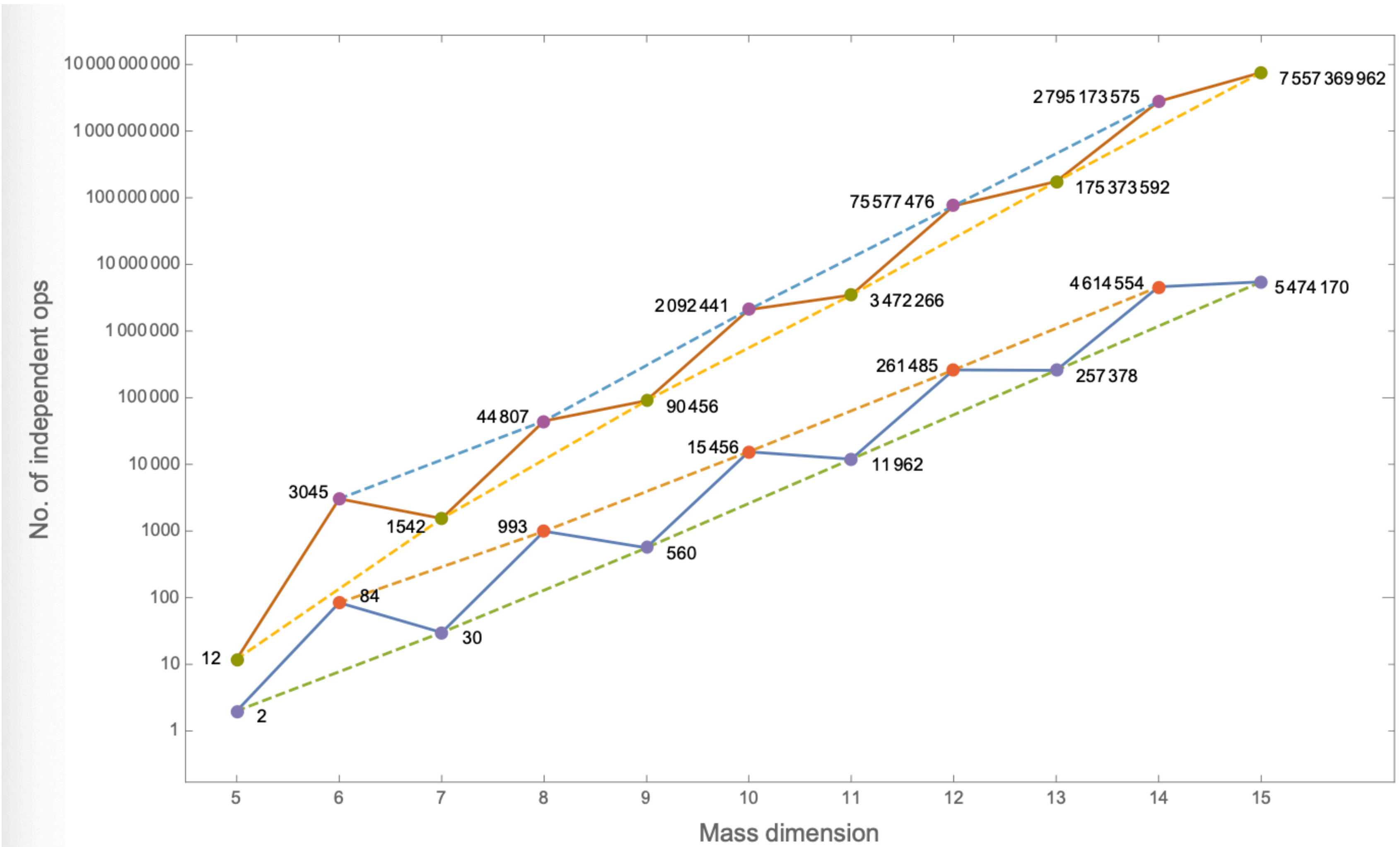
Low-derivative operators of the Standard Model effective field theory via Hilbert series methods

Landon Lehman (Notre Dame U.), Adam Martin (Notre Dame U.) (Oct 1, 2015)

Published in: *JHEP* 02 (2016) 081 • e-Print: 1510.00372 [hep-ph]

What is more on > dim 6

- Hilbert series technique.



The number
IS HUGE
with big
mass dim!

2, 84, 30, 993, 560, 15456, 11962, 261485, ...: Higher dimension operators in the SM EFT

Brian Henning (Yale U.), Xiaochuan Lu (UC, Davis), Tom Melia (UC, Berkeley and LBNL, Berkeley), Hitoshi Murayama (UC, Berkeley and LBNL, Berkeley and Tokyo U., IPMU) (Dec 10, 2015)

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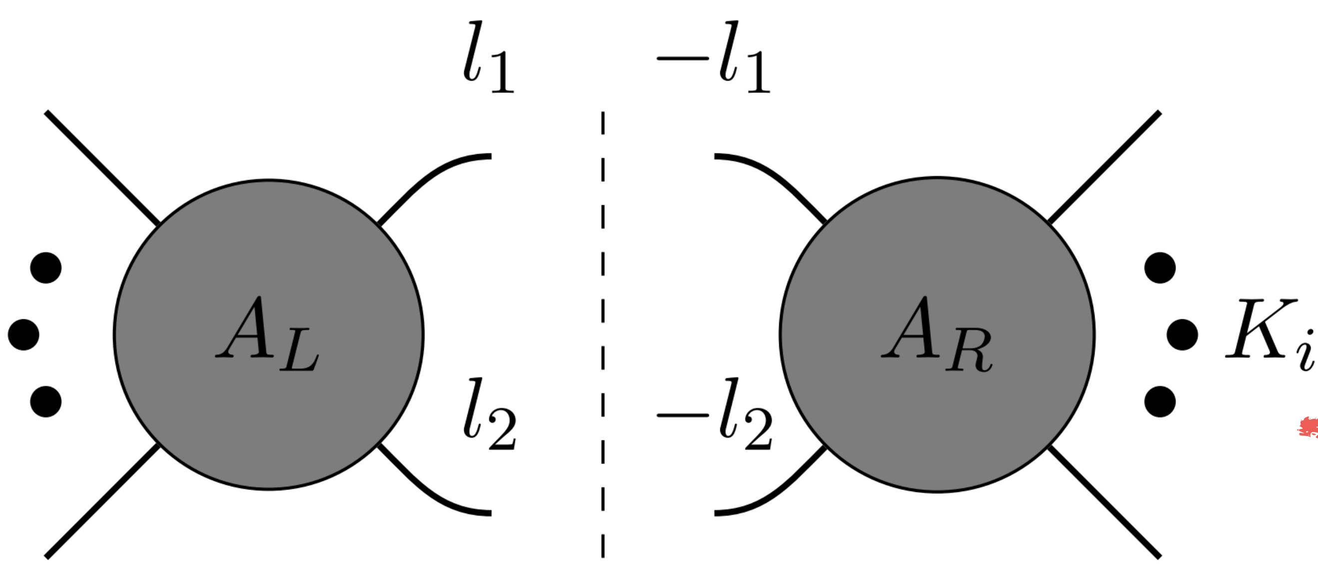
How to systematic generate
the independent basis?

On-shell scattering amplitude

Z. Bern, J. Parra-Martinez and E. Sawyer, JHEP **10**, 211 (2020) doi:10.1007/JHEP10(2020)211 [arXiv:2005.12917 [hep-ph]].
M. Jiang, T. Ma and J. Shu, [arXiv:2005.10261 [hep-ph]].
J. Elias Miró, J. Ingoldby and M. Riembau, JHEP **09**, 163 (2020) doi:10.1007/JHEP09(2020)163 [arXiv:2005.06983 [hep-ph]].
P. Baratella, C. Fernandez and A. Pomarol, Nucl. Phys. B **959**, 115155 (2020) doi:10.1016/j.nuclphysb.2020.115155 [arXiv:2005.07129 [hep-ph]].

- Efficient in massless EFT calculations



RG-running

$$\frac{\mathrm{d}c_i(\mu)}{\mathrm{d}\log\mu} = \sum_j \frac{1}{16\pi^2}\gamma_{ij}c_j$$

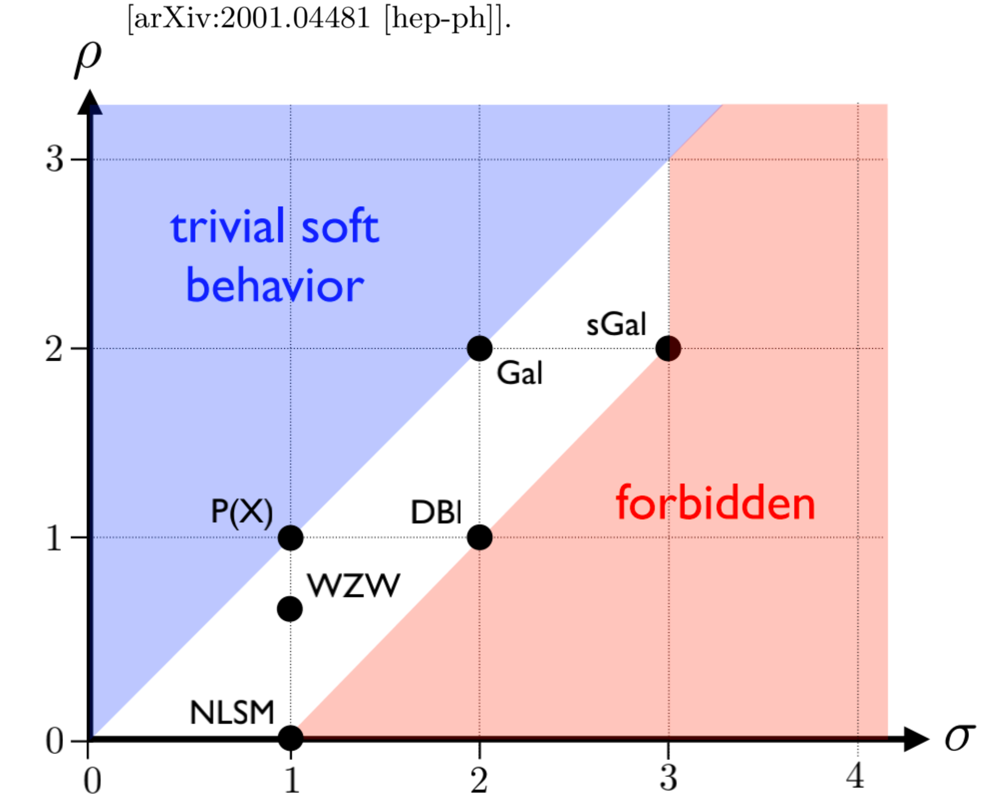
Selection rules

$$\gamma_{ij} = 0$$

C. Cheung and C. H. Shen, Phys. Rev. Lett. **115**, no. 7, 071601 (2015) doi:10.1103/PhysRevLett.115.071601 [arXiv:1505.01844 [hep-ph]].
M. Jiang, J. Shu, M. L. Xiao and Y. H. Zheng, [arXiv:2001.04481 [hep-ph]].

- Construct scalar EFT with non-trivial soft limit

$$A_n \sim p^\sigma \quad \text{for} \quad p \to 0$$



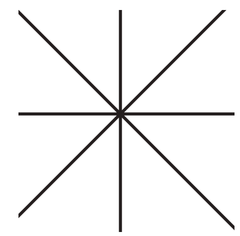
C. Cheung, K. Kampf, J. Novotny and J. Trnka, Phys. Rev. Lett. **114**, no.22, 221602 (2015) doi:10.1103/PhysRevLett.114.221602 [arXiv:1412.4095 [hep-th]].
C. Cheung, K. Kampf, J. Novotny, C. H. Shen and J. Trnka, JHEP **02**, 020 (2017) doi:10.1007/JHEP02(2017)020 [arXiv:1611.03137 [hep-th]].
I. Low, Phys. Rev. D **91**, no.10, 105017 (2015) doi:10.1103/PhysRevD.91.105017 [arXiv:1412.2145 [hep-th]].
I. Low, Phys. Rev. D **91**, no.11, 116005 (2015) doi:10.1103/PhysRevD.91.116005 [arXiv:1412.2146 [hep-ph]].

On-shell amplitude basis

- Efficient in constructing EFT operator bases of massless fields

$\mathcal{O}_{d\geq 4}$

Operator base



Unfactorizable amplitude base

Polynomial of spinor products

T. Ma, J. Shu and M. L. Xiao, [arXiv:1902.06752 [hep-ph]].
H. Elvang, D. Z. Freedman and M. Kiermaier, JHEP **1011**, 016 (2010) doi:10.1007/JHEP11(2010)016 [arXiv:1003.5018 [hep-th]].
Y. Shadmi and Y. Weiss, arXiv:1809.09644 [hep-ph].

- Massless amplitude base is free of EOMs automatically

Null EOM

$$p\,|p] = 0, \quad p\,|p\rangle = 0$$

- IBP redundancy can be systematically removed by $U(N) \supset \otimes_{i=1}^{N} U(1)_i$

Momentum conservation

- The amplitude bases are the basis of some special $U(N)$ representations

B. Henning and T. Melia, Phys. Rev. D **100**, no.1, 016015 (2019) doi:10.1103/PhysRevD.100.016015 [arXiv:1902.06754 [hep-ph]].
B. Henning and T. Melia, [arXiv:1902.06747 [hep-th]].

- It be constructed by the computer programs

(**Field theory can not do it!!!**)

H. L. Li, Z. Ren, J. Shu, M. L. Xiao, J. H. Yu and Y. H. Zheng, [arXiv:2005.00008 [hep-ph]].

Massive Amplitude Basis

Massive Effective Field Operator

- Constructing EFT bases of massive fields is still a problem

- EFT of massive fields has wide application in particle physics

Higgs EFT ⊃ SMEFT ⊃ SM

Dark matter EFT

Lower energy QCD

- Massive fields amplitude base construction is very challenge

Redundancy:

Equation of motion

Integration by part

G. Durieux, T. Kitahara, Y. Shadmi and Y. Weiss, JHEP **01**, 119 (2020) doi:10.1007/JHEP01(2020)119 [arXiv:1909.10551 [hep-ph]].
G. Durieux, T. Kitahara, C. S. Machado, Y. Shadmi and Y. Weiss, JHEP **12**, 175 (2020) doi:10.1007/JHEP12(2020)175 [arXiv:2008.09652 [hep-ph]].
A. Falkowski, G. Isabella and C. S. Machado, [arXiv:2011.05339 [hep-ph]].

Notrivial EOM

$$p\,|p^I] = m\,|p^I\rangle$$

How to solve it?

Massive amplitude basis

- The scattering amplitude can be factorized in two parts:

$$\mathscr{M}^I_{m,n} = \sum_{\{\dot{\alpha}\}} \mathscr{A}^I_{\{\dot{\alpha}\}}\left(\{\epsilon_{s_i}\}\right) G^{\{\dot{\alpha}\}}\left(|j], |j\rangle, p_i\right)$$

Massive LG charged

Massless LG charged

- $\mathscr{A}^I(\{\epsilon_{s_i}\})$ take all the massive LG charges and is required to be the holomorphic function of $|i^I]$s

Linear in massive polarization tensor

$$\epsilon_{s_i} \equiv |i]^{\{I_1}_{\dot{\alpha}_1}, \ldots, |i]^{I_{2s_i}\}}_{\dot{\alpha}_{2s_i}} \in (2s_i + 1, 2s_i + 1) = SU(2)_l \otimes SU(2)_r$$

$\mathscr{A}^I(\{\epsilon_{s_i}\})$ can not be EOM and IBP redundant!

Massive amplitude basis

- The scattering amplitude can be factorized in two parts:

m massive
n massless

$$\mathcal{M}^I_{m,n} = \sum_{\{\dot\alpha\}} \mathscr{A}^I_{\{\dot\alpha\}}\left(\{\epsilon_{s_i}\}\right) G^{\{\dot\alpha\}}\left(|j], |j\rangle, p_i\right)$$

Massive LG charged

Massless LG charged

- $G(|j], |j\rangle, p_i)$ take all the massless LG charges, so is the function of massless spinors $|j], |j\rangle$ and massive momentum p_i

$G(|j], |j\rangle, p_i)$ can be both EOM and IBP redundant!

- Example: 3-pt local amplitude of $W - \psi - \psi$

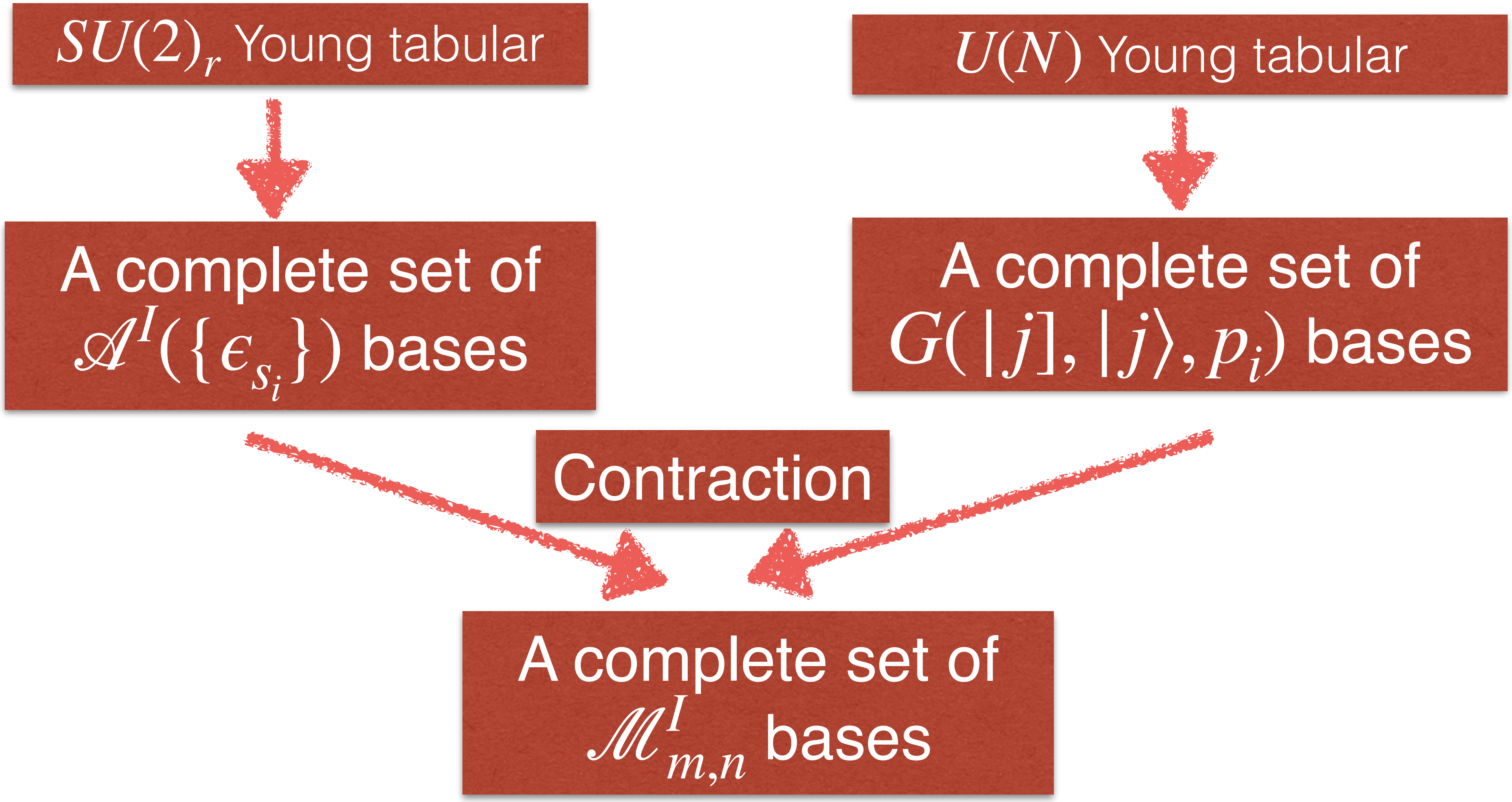
$$[\mathbf{1}^{\{\mathbf{I}}2]\langle\mathbf{1}^{\mathbf{J}\}}3\rangle = \frac{[\mathbf{1}^{\mathbf{I}}2][\mathbf{1}^{\mathbf{J}\}}\mathbf{p}_1 3\rangle}{m_W} \to \underbrace{([\mathbf{1}^{\{\mathbf{I}}|[\mathbf{1}^{\mathbf{J}\}}|)}_{\mathscr{A}} \cdot \underbrace{(|2]\,\mathbf{p}_1|3\rangle)}_{G}$$

Massive amplitude basis

$$\mathcal{M}^I_{m,n} = \sum_{\{\dot\alpha\}} \mathcal{A}^I_{\{\dot\alpha\}}\left(\{\epsilon_{s_i}\}\right) G^{\{\dot\alpha\}}\left(|j], |j\rangle, p_i\right)$$

Zi-Yu Dong, **T.Ma**, Jing Shu, Phys.Rev.D 107 (2023) 11, L111901

- The general framework to construct a complete set of massive amplitude bases



Dimension Reduction of Massive Amplitude Basis

- The $\{\mathscr{A} . G\}$ bases can not be directly mapped into operator bases due to their dimension mismatch

Trivial mass factors : $$O_d = m_i^2 O_{d-2} + m_i^4 O_{d-4} + \cdots$$

- Should find a complete set of lowest dimensional amplitude bases that can directly map into physical operators

- Lowest dimensional amplitude bases means that their dimension can not be reduced further by EOM $p_i | i^I] / m_i = | i^I \rangle$

Dimension Reduction of Massive Amplitude Basis

How ?

Zi-Yu Dong, **T.Ma**, Jing Shu, Yu-Hui Zheng, Phys.Rev.D 106 (2022) 11, 116010

- Step one: construct a redundant and complete set of amplitude bases $\{\mathscr{C}\,.\,F\}$ that can always contain a complete set of lowest dimensional amplitude bases

- Step two: decompose this redundant $\{\mathscr{C}\,.\,F\}$ bases into independent $\{\mathscr{A}\,.\,G\}$ bases from low to high dimension and remove the linear correlation bases

Computer Programs

- Massive EFT operators construction can be automatically done by computer programs: https://github.com/hamiguazzz/Massive

Zi-Yu Dong, **T.Ma**, Jing Shu, Zi-Zheng Zhou, JHEP 09 (2023) 101

A.91 Type: $ZZZZ$

A.91.1 Dimension = 4, $\mathcal{O}_4^1$

Type: $ZZZZ$ $\quad d=4 \quad \mathcal{O}_4^1$
$Z_\mu Z_\nu Z_\rho Z_\sigma \, \mathrm{Tr}\,(\sigma^\mu \bar\sigma^\rho)\, \mathrm{Tr}\,(\sigma^\nu \bar\sigma^\sigma)$

A.91.2 Dimension = 6, $\mathcal{O}_6^{1\sim 4}$

Type: $ZZZZ$ $\quad d=6 \quad \mathcal{O}_6^{1\sim 4}$
$Z_\mu Z_\nu Z^+_{\rho\sigma} Z^+_{\xi\tau}\, \mathrm{Tr}\left(\sigma^\mu \sigma^\nu \bar\sigma^{\xi\tau} \bar\sigma^{\rho\sigma}\right)$
$Z^-_{\mu\nu} Z_\rho Z_\sigma Z^+_{\xi\tau}\, \mathrm{Tr}\left(\sigma^\rho \sigma^{\mu\nu} \sigma^\sigma \bar\sigma^{\xi\tau}\right)$
$Z^-_{\mu\nu} Z^-_{\rho\sigma} Z_\xi Z_\tau\, \mathrm{Tr}\left(\sigma^\xi \sigma^{\mu\nu} \sigma^{\rho\sigma} \bar\sigma^\tau\right)$
$(D_\mu Z_\rho)\, Z_\sigma\, (D_\nu Z_\xi)\, Z_\tau\, \mathrm{Tr}\left(\sigma^\sigma \bar\sigma^\mu \sigma^\xi \bar\sigma^\tau\right) \mathrm{Tr}\,(\bar\sigma^\nu \sigma^\rho)$

A.86 Type: $ZZ\gamma^+\gamma^-$

A.86.1 Dimension = 6, $\mathcal{O}_6^1$

Type: $ZZ\gamma^+\gamma^-$ $\quad d=6 \quad \mathcal{O}_6^1$
$Z_\mu Z_\nu \gamma^+_{\rho\sigma} \gamma^-_{\xi\tau}\, \mathrm{Tr}\left(\sigma^\mu \sigma^{\xi\tau} \sigma^\nu \bar\sigma^{\rho\sigma}\right)$

A.86.2 Dimension = 8, $\mathcal{O}_8^{1\sim 4}$

Type: $ZZ\gamma^+\gamma^-$ $\quad d=8 \quad \mathcal{O}_8^{1\sim 4}$
$Z^-_{\mu\nu} Z^+_{\rho\sigma} \gamma^+_{\xi\tau} \gamma^-_{\zeta\eta}\, \mathrm{Tr}\left(\bar\sigma^{\rho\sigma} \bar\sigma^{\xi\tau}\right) \mathrm{Tr}\left(\sigma^{\mu\nu} \sigma^{\zeta\eta}\right)$
$Z^-_{\nu\rho} Z_\sigma \gamma^+_{\xi\tau} \left(D_\mu \gamma^-_{\zeta\eta}\right) \mathrm{Tr}\left(\sigma^\sigma \sigma^{\zeta\eta} \sigma^{\nu\rho} \bar\sigma^\mu \bar\sigma^{\xi\tau}\right)$
$Z_\nu Z^+_{\rho\sigma} \left(D_\mu \gamma^+_{\xi\tau}\right) \gamma^-_{\zeta\eta}\, \mathrm{Tr}\left(\bar\sigma^\mu \sigma^{\zeta\eta} \sigma^\nu\right) \mathrm{Tr}\left(\bar\sigma^{\rho\sigma} \bar\sigma^{\xi\tau}\right)$
$Z_\rho\, (D_\nu Z_\sigma)\, \gamma^+_{\xi\tau} \left(D_\mu \gamma^-_{\zeta\eta}\right) \mathrm{Tr}\left(\sigma^\rho \sigma^\mu \bar\sigma^\nu \sigma^{\zeta\eta} \sigma^\sigma \bar\sigma^{\xi\tau}\right)$

Massless EFT VS Massive EFT

- Map SMEFT bases to HEFT bases

SMEFT bases $$\mathcal{A}(H^i H_k^\dagger H^j H_l^\dagger) \supset c^+_{HHHH}\frac{s_{13}}{\Lambda^2}T^{+\,ij}_{\;\;kl} + c^-_{HHHH}\frac{s_{12}-s_{14}}{\Lambda^2}T^{-\,ij}_{\;\;kl}$$

HEFT bases $$\boldsymbol{\mathcal{M}}(W^+W^-hh) = C^{00}_{6,WWhh}\frac{[\mathbf{12}]\langle\mathbf{12}\rangle}{\Lambda^2}$$

Just bold the spinors

$$\mathcal{A}(G^+G^-hh) = \frac{1}{2}\left(\mathcal{A}(H^1H_1^\dagger H^2 H_2^\dagger) + \mathcal{A}(H^1H_1^\dagger H_2^\dagger H^2)\right) = -\frac{c^+_{(H^\dagger H)^2} - 3c^-_{(H^\dagger H)^2}}{2}\frac{s_{12}}{2\Lambda^2}$$

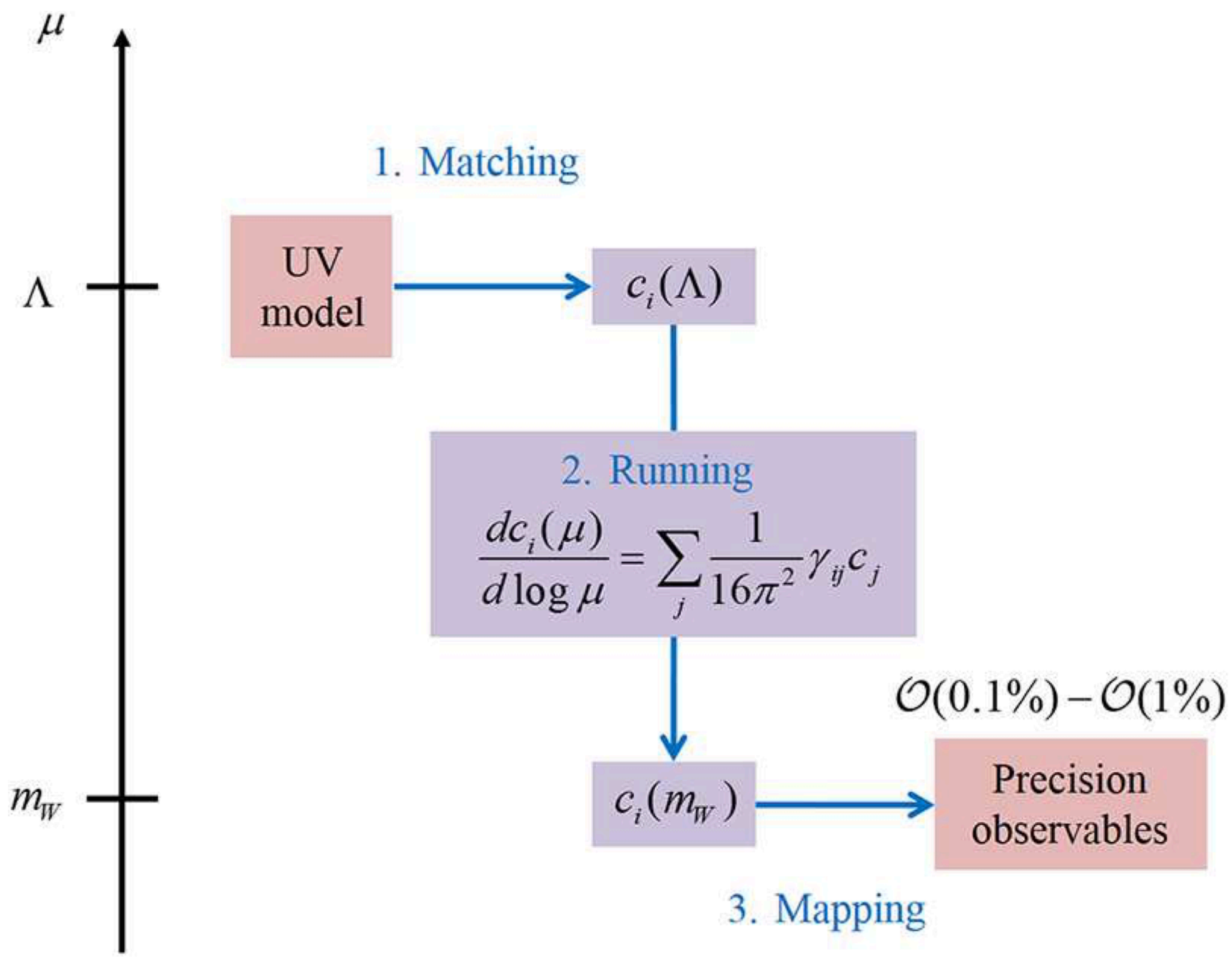
$$-\frac{c^+_{(H^\dagger H)^2} - 3c^-_{(H^\dagger H)^2}}{2}\frac{s_{12}}{2\Lambda^2} \longrightarrow \frac{c^+_{(H^\dagger H)^2} - 3c^-_{(H^\dagger H)^2}}{2}\frac{[\mathbf{12}]\langle\mathbf{12}\rangle}{\Lambda^2}$$

Hongkai Liu, **T.Ma**, Yael Shadmi, Michael Waterbury, JHEP 05 (2023) 241

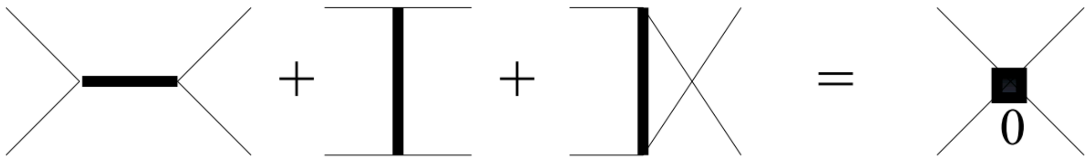
EFT Amplitude Basis Matching

On-shell EFT matching

- Matching between IR EFT and UV for precision measurements

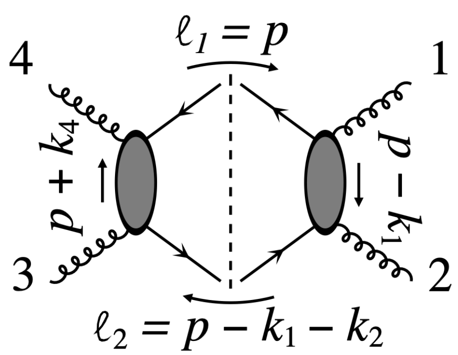


- Matching between on-shell EFT and on-shell UV is efficient



On-shell EFT matching

- Amplitudes with poles and branch cuts contribute to EFT matching



$$i\mathcal{M}_4^{\text{1-loop}}\Big|_{s\text{-cut}} = \int \frac{d^{4-2\epsilon}L}{(2\pi)^{4-2\epsilon}} \frac{1}{L_1^2 - m^2} \frac{1}{L_2^2 - m^2} \sum_{\text{states}} \mathcal{M}_4^{\text{tree}}(1,2,L_1,L_2)\mathcal{M}_4^{\text{tree}}(-L_1,-L_2,3,4)\Big|_{s\text{-cut}}$$

Efficiently constructed by unitary cuts

- The amplitude parts with no singularity only contribute to RG running

$$A_4 = \text{Cut-constructed part} + \tilde{d}_1\, I_1 + \tilde{d}_2\, I_2(0)$$

Just ignore them!

On-shell EFT matching

- The on-shell EFT matching becomes a simple algebra problem without loop calculation

Unitary cuts and sewing method

↓

The loop amplitude integrand

↓ Passarino and Veltman decomposition

$$\mathcal{A}_{\text{loop}} = \sum_a C_1^{(a)} I_1^{(a)} + \sum_b C_2^{(b)} I_2^{(b)} + \sum_c C_3^{(c)} I_3^{(c)} + \sum_d C_4^{(d)} I_4^{(d)} + R$$

↓ Big mass expansion

On-shell EFT amplitude bases

Zi-Yu Dong, **T.Ma**, Zi-Zheng Zhou, Ci-hang Li, Qi-ming Qiu, arXiv 2409.XXXX

Summary

- Massive EFT amplitude bases can be efficiently constructed by $SU(2)_r \times U(N)$ representations

- Based on our theory, EFT operators of any massive fields can be automatically constructed by computer programs

- Using on-shell method, SMEFT can be easily mapped into HEFT

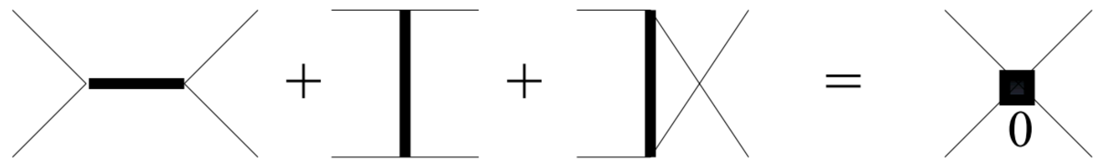
- On-shell method can be efficient in EFT matching

Thanks !!

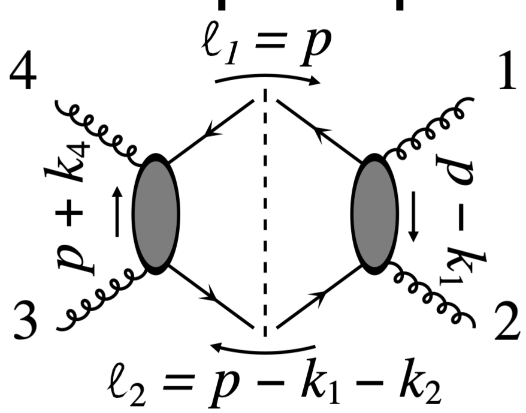
Back up

On-shell EFT matching

- Matching between on-shell EFT and on-shell UV is efficient

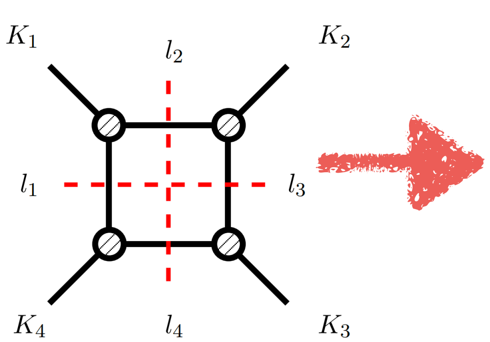


- Loop amplitude can be efficiently constructed by unitary cuts

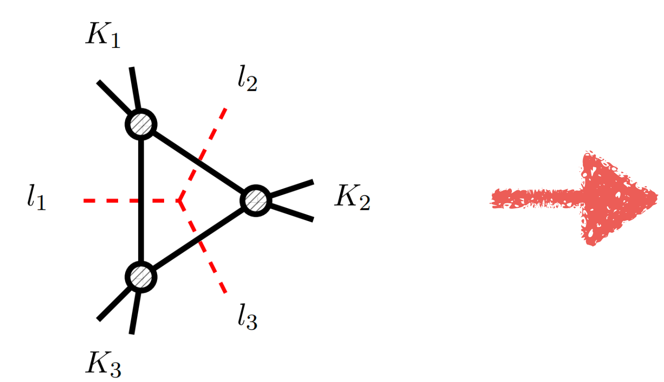


$$\mathcal{A}_{\text{loop}} = \sum_a C_1^{(a)} I_1^{(a)} + \sum_b C_2^{(b)} I_2^{(b)} + \sum_c C_3^{(c)} I_3^{(c)} + \sum_d C_4^{(d)} I_4^{(d)} + R$$

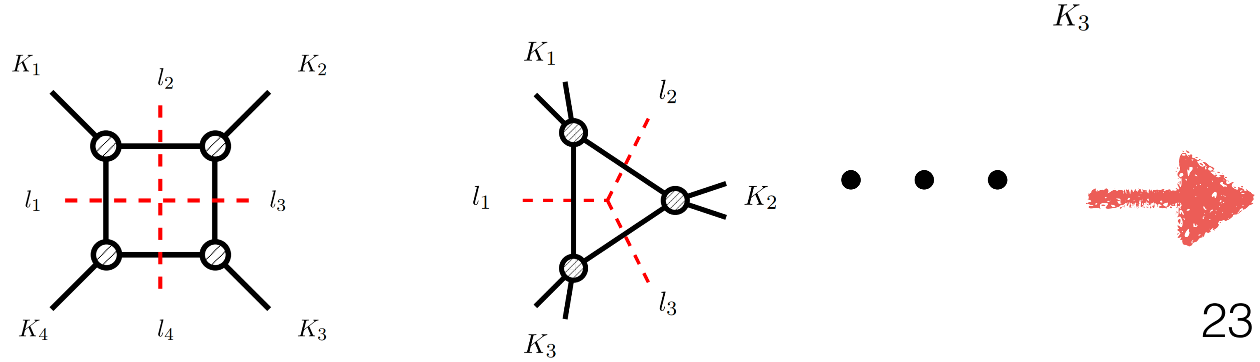
- The full amplitude calculation is not systematically （branch cuts and rational terms seperately calculated）



C_4



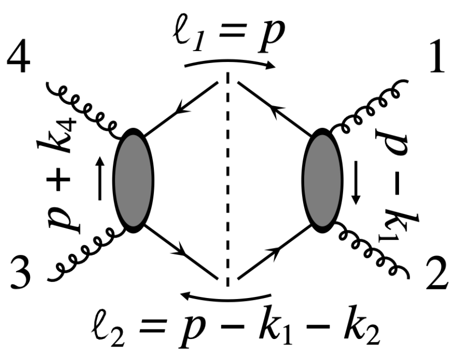
C_3 . . .



R

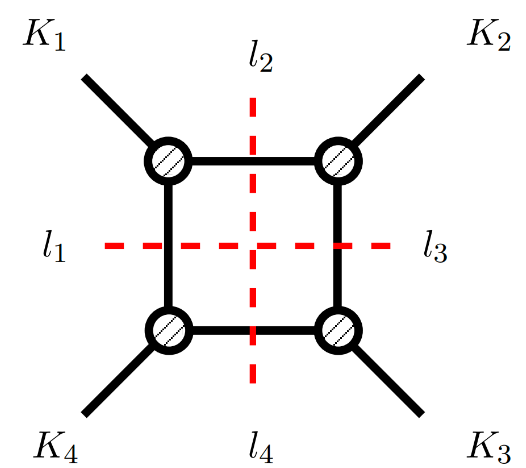
On-shell EFT matching

- Propose a new sewing method to construct on-shell amplitudes



$$iM_4^{\text{1-loop}}\Big|_{s\text{-cut}} = \int \frac{d^{4-2\epsilon}L}{(2\pi)^{4-2\epsilon}} \frac{1}{L_1^2 - m^2} \frac{1}{L_2^2 - m^2} \sum_{\text{states}} \mathcal{M}_4^{\text{tree}}(1, 2, L_1, L_2) \mathcal{M}_4^{\text{tree}}(-L_1, -L_2, 3, 4)\Big|_{s\text{-cut}}$$

- But sew the different channels will result in redundancy



- Redundancy can be systematically removed by momentum flow algebra

$$A^1 = \sum_{s_i \in \{s\}} C(s_i)\mathcal{P}\left(\sum_{\text{monomials}} \prod_{\text{poles}} S^{(\dots)}(\dots)\right)$$
$$+ \frac{1}{2}\sum_{s_i}\sum_{s_j} C(s_i)C(s_j)\mathcal{P}\left(\sum_{\text{monomials}} \prod_{\text{poles}} S^{(\dots)}(\dots)\right)$$
$$+ \frac{1}{3}\sum_{s_i}\sum_{s_j}\sum_{s_k} C(s_i)C(s_j)C(s_k)\dots$$

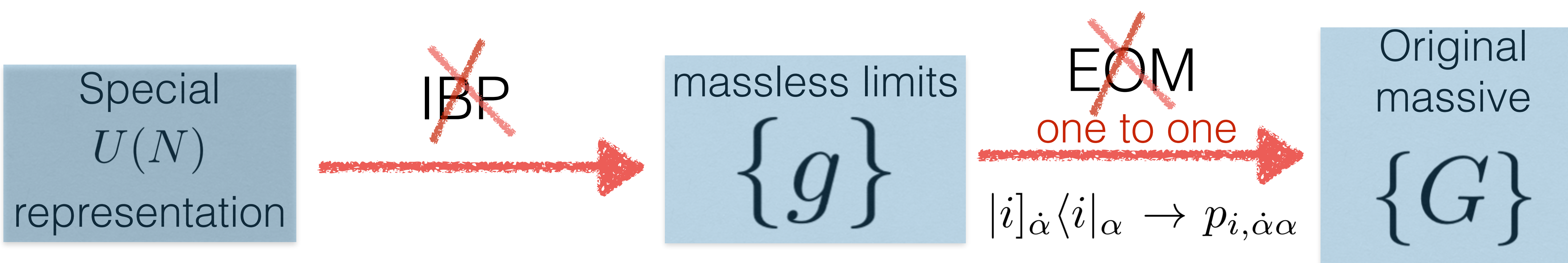
Massive amplitude basis

- $G(|j], |j\rangle, p_i)$ bases suffer from EOM and IBP redundancy

- EOM redundancy can be removed by first constructing $G(|j], |j\rangle, p_i)$ massless limit bases

Construct massless limits $p_{i,\dot{\alpha}\alpha} \to |i]_{\dot{\alpha}}\langle i|_{\alpha}$: $G(|j], |j\rangle, p_i) \to g \equiv G(|j], |j\rangle, |i]\langle i|)$

- IBP redundancy can be removed by $U(N)$ symmetry



Lowest dimensional massive amplitude basis

Massive amplitude basis

- MLGTS basis $\mathscr{A}^I(\{\epsilon_{s_i}\})$ is the linear function of polarization tensor ϵ_{s_i}

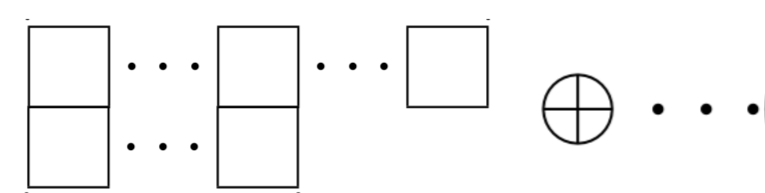
$$\epsilon_{s_i} \equiv |i]^{\{I_1}_{\dot\alpha_1}, \ldots, |i]^{I_{2s_i}\}}_{\dot\alpha_{2s_i}} \sim \underbrace{\boxed{i}\cdots\boxed{i}_{\,r}}_{(2s_i)}$$

$(2s_i+1)\ SU(2)_r$ Rep of ϵ_{s_i}

- Any $\mathscr{A}^I(\{\epsilon_{s_i}\})$ basis must belong to the out product of all ϵ_{s_i}'s $SU(2)_r$ Reps

$SU(2)_r$ Rep of ϵ_{s_i}

$$\mathscr{A}^I(\{\epsilon_{s_i}\}) \subset \bigotimes_{i=1}^{m} \underbrace{\boxed{i}\cdots\boxed{i}_{\,r}}_{(2s_i)} =$$



- A complete set of $\{\mathscr{A}^I_{\dot\alpha}\}$ bases can be constructed by finding all the $SU(2)_r$ irreducible representations from the out product of all ϵ_{s_i}'s $SU(2)_r$ Reps

Dimension Reduction of Massive Amplitude Basis

How to systematically construct redundant $\{\mathscr{C}\,.\,F\}$ bases that contain complete lowest dimensional amplitude bases???

- Amplitude bases can be classified by massive polarization tensor configurations $\{\ldots, \epsilon_{s_i}^{l_i}, \epsilon_{s_{i+1}}^{l_{i+1}}, \ldots\}$

Polarization tensor of massive particle-i with spin-s_i

$$\epsilon_{s_i}^{l_i} \equiv |\left(i^I\rangle\right)^{l_i}\left(|i^I]\right)^{2s_i-l_i}, \; 0 \le l_i \le 2s_i$$

Different value of l_i represent different polarization tensor configuration

- A complete set of bases $\{\mathscr{C}\,.\,F\}^{\{\ldots, l_i, l_{i+1}, \ldots\}}$ with one kind of polarization tensor configuration $\{\ldots, \epsilon_{s_i}^{l_i}, \epsilon_{s_{i+1}}^{l_{i+1}}, \ldots\}$ can be constructed by $SU(2)_r \times U(N)$ SSYTs

- $\{\mathscr{C}\,.\,F\}$ bases consist of all $\{\mathscr{C}\,.\,F\}^{\{\ldots, l_i, l_{i+1}, \ldots\}}$,

$$\mathscr{C}\,.\,F = \sum_{\{\ldots, l_i, l_j, \ldots\}}^{\ldots, 0\le l_i \le 2s_i, \ldots} \left\{ \{\mathscr{C}\,.\,F\}^{\{\ldots, l_i, l_j, \ldots\}} \right\}$$