An Explicit Expression of Generating Function for One-Loop Tensor Reduction

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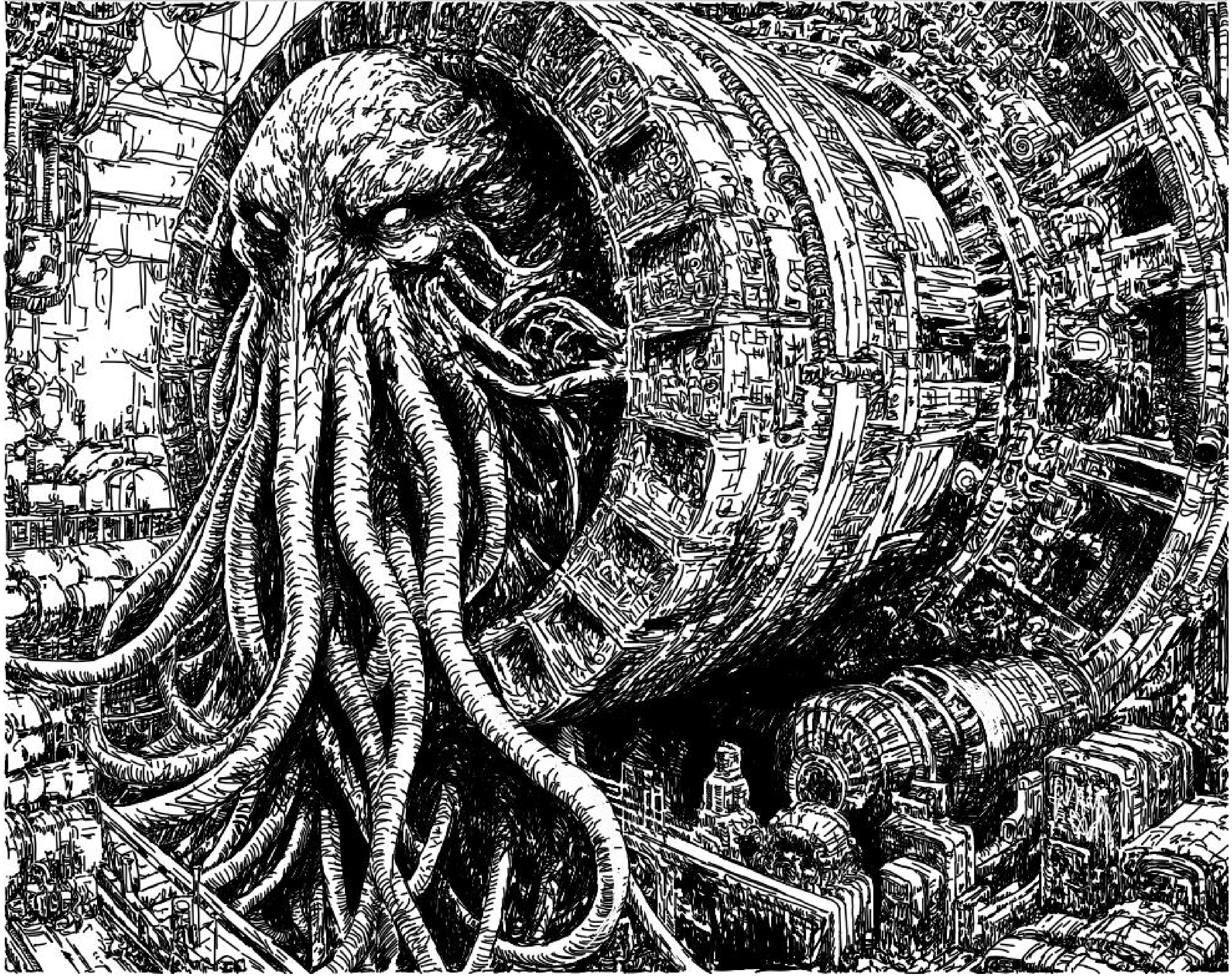

粒子对撞机和克苏鲁的相同点：

都非常巨大，容易让人产生巨物恐惧症

都代表了人类未知的世界

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2024年6月4日于杭州

Background:

In high-energy physics, calculating loop diagrams /Feynman Integral is always an inevitable challenge.

①Studying the analytical structure of Feynman integrals;

② Finding efficient methods for computing loop diagrams.

Background:

For a general one loop tensor Feynman Integral:

$$I_n^{(r)} = \int \frac{d^D l}{i\pi^{D/2}} \frac{(2R \cdot l)^r}{\prod_{j=1}^{n} (l - q_j)^2 - M_j^2}.$$

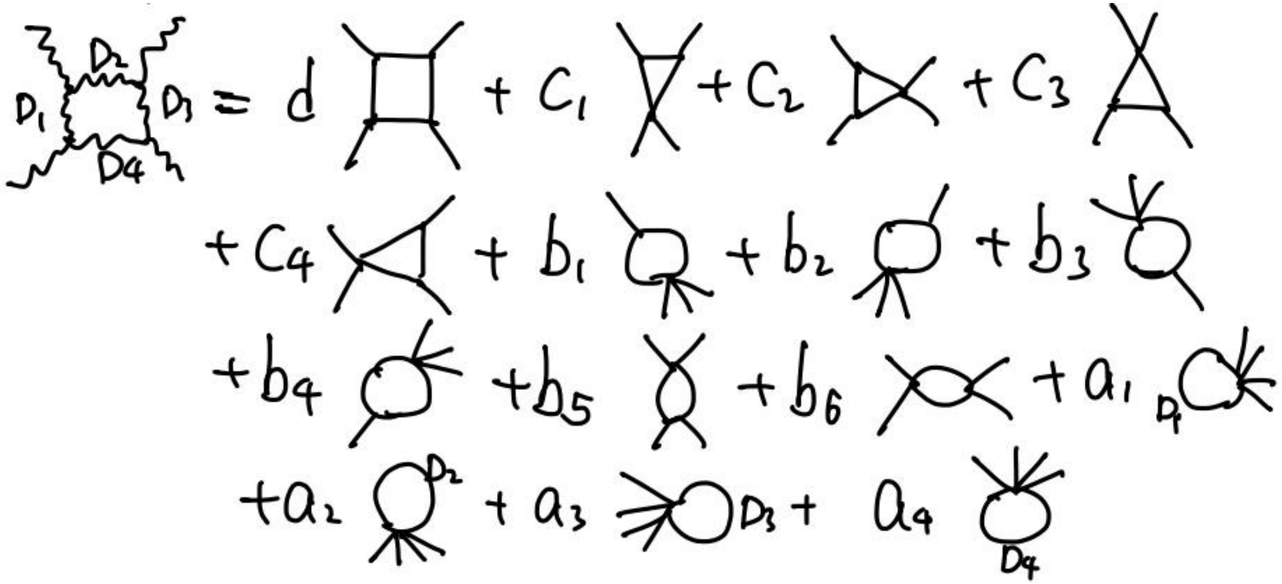
Reduction:

约化系数

$$I_n^{(r)} = \sum_{\mathbf{b} \subseteq \{1,2,\ldots,n\}} C_{n \to n;\widehat{\mathbf{b}}}^{(r)} \cdot I_{n;\widehat{\mathbf{b}}}^{(0)},$$

Master Integrals/Basis

Example:



Background:

Reduction methods:

Passarino-Veltman (PV),

Ossola-Papadopoulos-Pittau (OPP),

Integration by Part (IBP),

Unitarity Cut,

Intersection Number,

Computational Algebraic Geometry...etc.

Package:

Neat IBP;	FIRE;
Kira;	LiteRed;
FIESTA;	Blade;
...etc.	

Generating Function:

$$I_n^{(r)} = \int \frac{d^D l}{i\pi^{D/2}} \frac{(2R\cdot l)^r}{\prod_{j=1}^{n}(l-q_j)^2 - M_j^2}.$$

Ex: Hermite Polynomial Hn(x):

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}\,.$$

Motivation:

$$\psi_1(t) = \sum_{n=0}^{\infty} t^n (2\ell\cdot R)^n = \frac{1}{1-t(2\ell\cdot R)},$$

$$\psi_2(t) = \sum_{n=0}^{\infty} \frac{(2\ell\cdot R)^n t^n}{n!} = e^{t(2\ell\cdot R)}\,.$$

(arxiv:2209.09517)

Generation Function For One-loop Tensor Reduction

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ABSTRACT: For loop integrals, the reduction is the standard method. Having an efficient way to find reduction coefficients is an important topic in scattering amplitudes. In this paper, we present the generation functions of reduction coefficients for general one-loop integrals with arbitrary tensor rank.

KEYWORDS: Generation function, Tensor reduction, One-loop integral.

Feynman parametrization in the projective space:

$$I_n^{(r)} = \frac{2(D+2r-n-2)}{D+r-n-1} \cdot \frac{(\overline{VL})}{(\overline{LL})} \cdot I_n^{(r-1)} + \frac{4(r-1)}{D+r-n-1} \cdot \frac{R^2 - (\overline{VV})}{(\overline{LL})} \cdot I_n^{(r-2)}$$
$$+ \sum_{b=1}^{n} \left(X^{(b)} \cdot I_{n,\widehat{b}}^{(r-1)} + \frac{2(r-1) \cdot Y^{(b)}}{D+r-n-1} \cdot I_{n,\widehat{b}}^{(r-2)} \right),$$

(arxiv:2204.03190)

$$X^{(b)} = \left((\overline{H_b L})(\overline{VL})_b - (\overline{H_b V})(\overline{LL})_b \right) / (\overline{LL}),$$
$$Y^{(b)} = \left((\overline{H_b L}) R^2 + (\overline{H_b V})(\overline{VL})_b - (\overline{H_b L})(\overline{VV})_b \right) / (\overline{LL}).$$

Multiply both sides by t^r and sum over r from $1 \to \infty$. We can have a differential function for the generating function $\mathbf{GF}_{n \to n;\widehat{\mathbf{b}}}(t)$

$$\left((D-n-1) - 2(D-n) \cdot \frac{(\overline{VL})}{(\overline{LL})} t - 4 \cdot \frac{R^2 - (\overline{VV})}{(\overline{LL})} t^2 \right) \mathbf{GF}_{n \to n;\widehat{\mathbf{b}}}(t)$$
$$+ \left(t - 4 \cdot \frac{(\overline{VL})}{(\overline{LL})} t^2 - 4 \cdot \frac{R^2 - (\overline{VV})}{(\overline{LL})} t^3 \right) \mathbf{GF}'_{n \to n;\widehat{\mathbf{b}}}(t)$$
$$= \sum_{b_i \in \mathbf{b}} \Big\{ X^{(b_i)} \left((D-n) t \cdot \mathbf{GF}_{n;\widehat{b_i} \to n;\widehat{\mathbf{b}}}(t) + t^2 \cdot \mathbf{GF}'_{n;\widehat{b_i} \to n;\widehat{\mathbf{b}}}(t) \right)$$
$$+ 2Y^{(b_i)} \left(t^3 \cdot \mathbf{GF}'_{n;\widehat{b_i} \to n;\widehat{\mathbf{b}}}(t) + t^2 \cdot \mathbf{GF}_{n;\widehat{b_i} \to n;\widehat{\mathbf{b}}}(t) \right) \Big\} + (D-n-1)\delta_{\mathbf{b},\emptyset}$$

Notations:

- Some n-dimension vectors:

$$\boldsymbol{L}: \; L_i = 1, \qquad \boldsymbol{V}: \; V_i = R \cdot q_i, \qquad \boldsymbol{H}_b = \{0, \cdots, 0, 1, 0, \cdots, 0\},$$

where 1 is in the b-th position.

- The notations $(\overline{AB})$ and $(\overline{AB})_{\mathbf{b}}$, with a label list $\mathbf{b} = \{b_1, b_2, ..., b_k\}$ for two vectors A and B, are defined as follows:

$$(\overline{AB}) = A \cdot Q^{-1} \cdot B, \qquad (\overline{AB})_{\mathbf{b}} = A_{\widehat{\mathbf{b}}} \cdot (Q_{\widehat{\mathbf{b}}\widehat{\mathbf{b}}})^{-1} \cdot B_{\widehat{\mathbf{b}}}.$$

For example, with $n = 4$ and $\mathbf{b} = \{2, 3\}$

$$(\overline{VL})_{2,3} = \left(R \cdot q_1, \; R \cdot q_4\right) \begin{pmatrix} M_1^2 & \frac{M_1^2 + M_4^2 + (q_1 - q_4)^2}{2} \\ \frac{M_4^2 + M_1^2 + (q_4 - q_1)^2}{2} & M_4^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- $I^{(r)}_{n,\widehat{\mathbf{b}}}$ represents a one-loop integral where all propagators of $I^{(r)}_n$ with index $b_i \in \mathbf{b}$ are removed

$$I^{(r)}_{n,\widehat{\mathbf{b}}} = \int \frac{d^D l}{i\pi^{D/2}} \frac{(2R \cdot l)^r}{\prod_{j=1, j \notin \mathbf{b}}^{n} (l - q_j)^2 - M_j^2}.$$

- Q is an $n \times n$ matrix defined as

$$Q_{ij} = \frac{M_i^2 + M_j^2 - (q_i - q_j)^2}{2}.$$

- If Ω is an analytic expression composed of $(\overline{AB})$ or $(\overline{AB})_{\mathbf{b}}$, then $[\Omega]_{\mathbf{a}}$ with a label set $\mathbf{a}$ outside the square brackets represents the analytic expression obtained by appending subscript $\mathbf{a}$ on each term of the form $(\overline{AB})$ or $(\overline{AB})_{\mathbf{b}}$ present in Ω. For example

$$[(\overline{VL})]_{1,2} = (\overline{VL})_{1,2}, \quad [(\overline{LL})_2]_3 = (\overline{LL})_{2,3}.$$

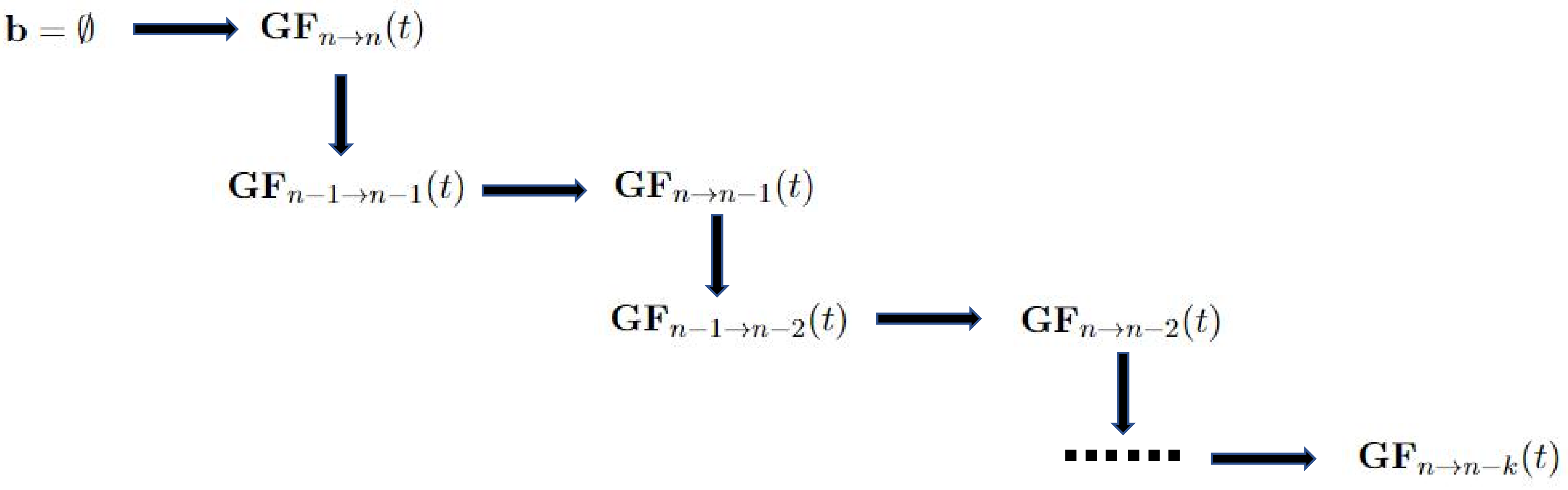
If

$$P = \frac{(\overline{H_1 L})(\overline{VV})_3 + (\overline{LL})_2 R^2}{(\overline{VL})_2 (H_3 V)_1},$$

then

$$[P]_{4,5} = \frac{(\overline{H_1 L})_{4,5}(\overline{VV})_{3,4,5} + (\overline{LL})_{2,4,5} R^2}{(\overline{VL})_{2,4,5}(H_3 V)_{1,4,5}}.$$

$$
\begin{aligned}
&\left((D-n-1)-2(D-n)\cdot\frac{(\overline{VL})}{(\overline{LL})}t-4\cdot\frac{R^2-(\overline{VV})}{(\overline{LL})}t^2\right)\mathbf{GF}_{n\to n;\widehat{\mathbf{b}}}(t)\\
&+\left(t-4\cdot\frac{(\overline{VL})}{(\overline{LL})}t^2-4\cdot\frac{R^2-(\overline{VV})}{(\overline{LL})}t^3\right)\mathbf{GF}'_{n\to n;\widehat{\mathbf{b}}}(t)\\
=&\sum_{b_i\in\mathbf{b}}\left\{X^{(b_i)}\left((D-n)t\cdot\mathbf{GF}_{n;\widehat{b_i}\to n;\widehat{\mathbf{b}}}(t)+t^2\cdot\mathbf{GF}'_{n;\widehat{b_i}\to n;\widehat{\mathbf{b}}}(t)\right)\right.\\
&\left.+2Y^{(b_i)}\left(t^3\cdot\mathbf{GF}'_{n;\widehat{b_i}\to n;\widehat{\mathbf{b}}}(t)+t^2\cdot\mathbf{GF}_{n;\widehat{b_i}\to n;\widehat{\mathbf{b}}}(t)\right)\right\}+(D-n-1)\delta_{\mathbf{b},\emptyset}
\end{aligned}
$$



Generating function of n-gon to n-gon:

$$t^{1-D+n}(1-x_+\cdot t)^{\frac{D-n-2}{2}}(1-x_-\cdot t)^{\frac{D-n-2}{2}}\cdot C_1+\frac{1}{1-x_+\cdot t}\cdot {}_2F_1\left(\begin{matrix}1,\frac{D-n}{2}\\ D-n\end{matrix}\middle|\frac{(x_--x_+)\cdot t}{1-x_+\cdot t}\right),$$

Not a Taylor series of *t*

initial condition $\mathbf{GF}_{n\to n}(0)=C^{(0)}_{n\to n}=1.$

where

$$x_\pm=\frac{2\left((\overline{VL})\pm\sqrt{(\overline{LL})R^2+(\overline{VL})^2-(\overline{LL})(\overline{VV})}\right)}{(\overline{LL})}.$$

The Generalized Hypergeometric Function is defined as:

$${}_pF_q\left(\begin{matrix}a_1,a_2,\cdots,a_p\\ b_1,b_2,\cdots,b_q\end{matrix}\middle|z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{z^n}{n!},$$

where we have used the **Pochhammer symbol**

$$(x)_n\equiv\frac{\Gamma(x+n)}{\Gamma(x)}=\prod_{i=1}^{n}(x+(i-1)).$$

The final result:

$$\mathbf{GF}_{n\to n}(t)=\frac{1}{1-x_+\cdot t}\cdot {}_2F_1\left(\begin{matrix}1,\frac{D-n}{2}\\ D-n\end{matrix}\middle|\frac{(x_--x_+)t}{1-x_+\cdot t}\right).$$

Derivative formula:

$$\frac{d}{dz}\,{}_pF_q\left(\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix}\middle|z\right)=\frac{\prod_{i=1}^{p}a_i}{\prod_{j=1}^{q}b_j}\cdot\ {}_pF_q\left(\begin{matrix}1+a_1,...,1+a_p\\ 1+b_1,...,1+b_q\end{matrix}\middle|z\right).$$

$$\frac{d}{dz}\,{}_pF_q\left(\begin{matrix}a_1,a_2,\cdots,a_{p-1},\alpha\\ b_1,b_2,\cdots,b_{q-1},\alpha+1\end{matrix}\middle|z\right)$$
$$=\frac{\alpha}{z}\cdot\left({}_{(p-1)}F_{(q-1)}\left(\begin{matrix}a_1,a_2,\cdots,a_{p-1}\\ b_1,b_2,\cdots,b_{q-1}\end{matrix}\middle|z\right)-\ {}_pF_q\left(\begin{matrix}a_1,a_2,\cdots,a_{p-1},\alpha\\ b_1,b_2,\cdots,b_{q-1},\alpha+1\end{matrix}\middle|z\right)\right).$$

Integral formula:

$$\int dz\ {}_pF_q\left(\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix}\middle|k\cdot z\right)\cdot z^m=\frac{z^{m+1}}{m+1}\cdot\ {}_{(p+1)}F_{(q+1)}\left(\begin{matrix}a_1,...,a_p,1+m\\ b_1,...,b_q,2+m\end{matrix}\middle|k\cdot z\right)+Const.$$

$$\mathbf{GF}_{n\to n}(t)=\frac{1}{1-x_+\cdot t}\cdot\ {}_2F_1\left(\begin{matrix}1,\frac{D-n}{2}\\ D-n\end{matrix}\middle|\frac{(x_--x_+)t}{1-x_+\cdot t}\right).$$

$$\left((D-n-1)-2(D-n)\cdot\frac{(\overline{VL})}{(\overline{LL})}t-4\cdot\frac{R^2-(\overline{VV})}{(\overline{LL})}t^2\right)\mathbf{GF}_{n\to n;\widehat{\mathbf{b}}}(t)$$
$$+\left(t-4\cdot\frac{(\overline{VL})}{(\overline{LL})}t^2-4\cdot\frac{R^2-(\overline{VV})}{(\overline{LL})}t^3\right)\mathbf{GF}'_{n\to n;\widehat{\mathbf{b}}}(t)$$
$$=\sum_{b_i\in\mathbf{b}}\left\{X^{(b_i)}\left((D-n)t\cdot\mathbf{GF}_{n;\widehat{b_i}\to n;\widehat{\mathbf{b}}}(t)+t^2\cdot\mathbf{GF}'_{n;\widehat{b_i}\to n;\widehat{\mathbf{b}}}(t)\right)\right.$$
$$\left.+2Y^{(b_i)}\left(t^3\cdot\mathbf{GF}'_{n;\widehat{b_i}\to n;\widehat{\mathbf{b}}}(t)+t^2\cdot\mathbf{GF}_{n;\widehat{b_i}\to n;\widehat{\mathbf{b}}}(t)\right)\right\}+(D-n-1)\delta_{\mathbf{b},\emptyset}$$

Generating function of n-gon to (n-k)-gon:

$$\mathbf{GF}_{n\to n;\widehat{\mathbf{I}_k}}(t) = (1-x_+\cdot t)^{\frac{-2+D-n}{2}}\cdot(1-x_-\cdot t)^{\frac{-2+D-n}{2}}$$
$$\times\sum_{m_1,\ldots,m_k=0}^{\infty}\left\{\frac{1}{\sum_{i=1}^k m_i+D-n+k-1}\cdot\frac{t^{\sum_{i=1}^k m_i+k}}{(1-[x_+]_{\mathbf{I}_k}\cdot t)^{\sum_{i=1}^k m_i+D-n+k-1}}\right.$$
$$\times\sum_{\{a'_1,\ldots,a'_k\}\in\sigma(\mathbf{I}_k),}\ \sum_{b_1,b_2,\ldots,b_{k-1}=0}^{1}\mathbf{C}^{(a'_1,\ldots,a'_k)}_{\{b_1,\ldots,b_{k-1}\}}(n)$$
$$\times\Big\{[\mathbf{C}^{(1)}_{a'_1}(n-k+1;m_1)]_{a'_2,\ldots,a'_k}\cdot\mathbf{HG}_1(n,k;\{b_1,b_2,\ldots,b_{k-1},1\};\mathscr{W}_{\mathbf{I}_k}(t))$$
$$\left.+[\mathbf{C}^{(2)}_{a'_1}(n-k+1;m_1)]_{a'_2,\ldots,a'_k}\cdot\mathbf{HG}_2(n,k;\{b_1,b_2,\ldots,b_{k-1},1\};\mathscr{W}_{\mathbf{I}_k}(t))\Big\}\right\},$$

where

$$\mathscr{W}_{\mathbf{I}_k}(t)=\frac{[x_--x_+]_{\mathbf{I}_k}\cdot t}{1-[x_+]_{\mathbf{I}_k}\cdot t}.$$

$[x_\pm]_{\mathbf{b}}\ \ [X^{(i)}]_{\mathbf{b}}\ \ [Y^{(i)}]_{\mathbf{b}}.$

相当于对动量q，质量m等参数进行层层“打包”。这些打包后的参数在数值计算的时候可以并行计算出，从而加快计算效率。

Examples:

$$C^{(2)}_{3\to 3;\widehat{2,3}} = X^{(2)}\cdot[X^{(3)}]_2 + X^{(3)}\cdot[X^{(2)}]_3,$$

$$C^{(3)}_{3\to 3;\widehat{2,3}} = \frac{1}{2(D-1)}\Big\{ D\left(X^{(2)}\cdot[X^{(3)}]_2\cdot[x_+ + x_-]_2 + X^{(3)}\cdot[X^{(2)}]_3\cdot[x_+ + x_-]_3\right)$$
$$+\left(X^{(2)}\cdot[X^{(3)}]_2 + X^{(3)}\cdot[X^{(2)}]_3\right)\left((D-1)\cdot[x_+ + x_-]_{2,3} + (D+1)\cdot(x_+ + x_-)\right)$$
$$+8\left(Y^{(2)}\cdot[X^{(3)}]_2 + Y^{(3)}\cdot[X^{(2)}]_3\right) + 4\left(X^{(2)}\cdot[Y^{(3)}]_2 + X^{(3)}\cdot[Y^{(2)}]_3\right)\Big\}.$$

k	0	1	2	3	4
number	2	6	10	14	18
Building Blocks	$(x_+ + x_-)$ $(x_+ - x_-)$	$X^{(1)}$ $Y^{(1)}$ $(x_+ + x_-)$, $[x_+ + x_-]_1$ $[x_+ - x_-]_1$ $(x_+ \cdot x_-)$	$X^{(2)}$, $[X^{(1)}]_2$ $Y^{(2)}$, $[Y^{(1)}]_2$ $[x_+ + x_-]_{12}$, $[x_+ + x_-]_2$ $(x_+ + x_-)$ $[x_+ - x_-]_{12}$ $[x_+ \cdot x_-]_2$, $(x_+ \cdot x_-)$	$X^{(3)}$, $[X^{(2)}]_3$, $[X^{(1)}]_{23}$ $Y^{(3)}$, $[Y^{(2)}]_3$, $[Y^{(1)}]_{23}$ $[x_+ + x_-]_{123}$, $[x_+ + x_-]_{23}$, $[x_+ + x_-]_3$, $(x_+ + x_-)$ $[x_+ - x_-]_{123}$ $[x_+ \cdot x_-]_{23}$, $[x_+ \cdot x_-]_3$, $(x_+ \cdot x_-)$	$\cdots$

由Building blocks生成的“基”

$$C^{(2)}_{3\to3,2,3} = X^{(2)}\cdot\left[X^{(3)}\right]_2 + X^{(3)}\cdot\left[X^{(2)}\right]_3,$$

1 in total

$$\begin{aligned}C^{(3)}_{3\to3,2,3} =& \frac{1}{2(D-1)}\Big\{D\left(X^{(2)}\cdot\left[X^{(3)}\right]_2\cdot[x_+ + x_-]_2 + X^{(3)}\cdot\left[X^{(2)}\right]_3\cdot[x_+ + x_-]_3\right)\\ &+(D-1)\left(X^{(2)}\cdot\left[X^{(3)}\right]_2 + X^{(3)}\cdot\left[X^{(2)}\right]_3\right)\cdot[x_+ + x_-]_{2,3}\\ &+(D+1)\left(X^{(2)}\cdot\left[X^{(3)}\right]_2 + X^{(3)}\cdot\left[X^{(2)}\right]_3\right)\cdot(x_+ + x_-)\\ &+8\left(Y^{(2)}\cdot\left[X^{(3)}\right]_2 + Y^{(3)}\cdot\left[X^{(2)}\right]_3\right)\\ &+4\left(X^{(2)}\cdot\left[Y^{(3)}\right]_2 + X^{(3)}\cdot\left[Y^{(2)}\right]_3\right)\Big\}.\end{aligned}$$

5 in total

PV的基:

$$\begin{aligned}C^{(2)}_{3\to3,2,3}:&\Big\{g^{\mu_1\mu_2},\ k_1^{\mu_1}k_1^{\mu_2},\ k_1^{\mu_1}(k_2^{\mu_2}+k_3^{\mu_2})+k_1^{\mu_2}(k_2^{\mu_1}+k_3^{\mu_1}),\\ &k_2^{\mu_1}k_2^{\mu_2}+k_3^{\mu_1}k_3^{\mu_2},\ k_2^{\mu_1}k_3^{\mu_2}+k_2^{\mu_2}k_3^{\mu_1}\Big\}\end{aligned}$$

5 in total

$$\begin{aligned}C^{(3)}_{3\to3,2,3}:&\Big\{g^{\mu_1\mu_2}k_1^{\mu_3}+\sigma(\mu_1,\mu_2,\mu_3),\ g^{\mu_1\mu_2}(k_1^{\mu_3})+\sigma(\mu_1,\mu_2,\mu_3),\ k_1^{\mu_1}k_1^{\mu_2}k_1^{\mu_3}+\sigma(\mu_1,\mu_2,\mu_3),\\ &k_1^{\mu_1}k_1^{\mu_2}(k_2^{\mu_3}+k_3^{\mu_3})+\sigma(\mu_1,\mu_2,\mu_3),\ k_1^{\mu_1}(k_2^{\mu_2}k_2^{\mu_3}+k_3^{\mu_2}k_3^{\mu_2})+\sigma(\mu_1,\mu_2,\mu_3),\\ &k_2^{\mu_1}k_2^{\mu_2}k_3^{\mu_3}+k_3^{\mu_1}k_3^{\mu_2}k_3^{\mu_3}+\sigma(\mu_1,\mu_2,\mu_3),\ k_2^{\mu_1}k_2^{\mu_2}k_2^{\mu_3}+k_3^{\mu_1}k_3^{\mu_2}k_3^{\mu_3}+\sigma(\mu_1,\mu_2,\mu_3),\\ &k_1^{\mu_1}k_2^{\mu_2}k_3^{\mu_3}+\sigma(\mu_1,\mu_2,\mu_3),\Big\}\end{aligned}$$

8 in total

与传统PV约化基个数的对比：（Box约化）

$$\int d^D l(1) \frac{l(1)^{a1}\, l(1)^{a2}\, l(1)^{a3}\, l(1)^{a4}}{(l(1)^2 - m1^2)((l(1)-p2)^2 - m2^2)((l(1)-p3)^2 - m3^2)((l(1)-p4)^2 - m4^2)}$$

r \ k	box	triangle	bubble	tadpole
1	1	1	0	0
2	2	3	1	0
3	2	7	5	1
4	3	13	16	7
5	3	22	40	29
6	4	34	86	91
7	4	50	166	239
8	5	70	296	533

r \ k	box	triangle	bubble	tadpole
1	1	2	0	0
2	3	5	6	0
3	4	9	10	9
4	8	16	20	16
5	10	25	30	25
6	17	39	50	39
7	21	56	70	56
8	32	80	105	80

计算时间测试

$$\int d^D l(1) \frac{l(1)^{a1}\, l(1)^{a2}\, l(1)^{a3}\, l(1)^{a4}}{(l(1)^2 - m1^2)\,((l(1)-p2)^2 - m2^2)\,((l(1)-p3)^2 - m3^2)\,((l(1)-p4)^2 - m4^2)}$$

Kira:

```
In[8]:= tmp1 = TI[{l[1]}, {{l[1], m1}, {l[1] - p2, m2}, {l[1] - p3, m3}, {l[1] - p4, m4}}, FV[l[1], a1] × FV[l[1], a2] × FV[l[1], a3] × FV[l[1], a4]]
```

Out[8]= $-\frac{i\,\pi^{-D/2}}{\Gamma(\epsilon+1)} \int d^D l(1) \frac{l(1)^{a1}\, l(1)^{a2}\, l(1)^{a3}\, l(1)^{a4}}{(l(1)^2 - m1^2)\,((l(1)-p2)^2 - m2^2)\,((l(1)-p3)^2 - m3^2)\,((l(1)-p4)^2 - m4^2)}$

```
In[9]:= DoKira[AlphaParametrize[tmp1, Method → 1], "UserDefinedSystem" → True, Method → 1] // AbsoluteTiming
```

Out[9]= $\{3849.57,\ i\,(\ldots 1 \ldots)\Big(\frac{p4^{a1}\, p4^{a2}\, p4^{a3}\, p4^{a4}\,((p2{\cdot}p3)^8 - 4\, p2^2\, p3^2\, (p2{\cdot}p3)^6 + 6\, p2^4\, p3^4\, (p2{\cdot}p3)^4 - 4\, p2^6\, p3^6\, (p2{\cdot}p3)^2 + p2^8\, p3^8)}{16\,(p4^2\,(p2{\cdot}p3)^2 - 2\, p2{\cdot}p4\, p3{\cdot}p4\, p2{\cdot}p3 + p2^2\,(p3{\cdot}p4)^2 + (p2{\cdot}p4)^2\, p3^2 - p2^2\, p3^2\, p4^2)^4} + \frac{p4^{a3}\, p4^{a4}\,(\ldots 1 \ldots)\,(g^{a1\,a2} - p4^{a1}\,(\ldots 1 \ldots) - p3^{a1}\,(\ldots 1 \ldots) - p2^{a1}\,(\ldots 1 \ldots))}{16\,(D-3)\,(\ldots 1 \ldots)^3} + \ldots 134 \ldots + \frac{p2^{a1}\,(\ldots 3 \ldots)\,(\ldots 1 \ldots)}{16\,(\ldots 1 \ldots)^4} + \frac{p2^{a1}\, p2^{a2}\, p2^{a3}\, p2^{a4}\,((p2{\cdot}p4)^4\, p3^8 - 4\, p2{\cdot}p3\,(p2{\cdot}p4)^3\, p3{\cdot}p4\, p3^6 + 6\,(p2{\cdot}p3)^2\,(p2{\cdot}p4)^2\,(p3{\cdot}p4)^2\, p3^4 - 4\,(p2{\cdot}p3)^3\, p2{\cdot}p4\,(p3{\cdot}p4)^3\, p3^2 + (p2{\cdot}p3)^4\,(p3{\cdot}p4)^4)}{16\,(p4^2\,(p2{\cdot}p3)^2 - 2\, p2{\cdot}p4\, p3{\cdot}p4\, p2{\cdot}p3 + (\ldots 1 \ldots) + (\ldots 1 \ldots)^2\,(\ldots 1 \ldots) - p2^2\, p3^2\, p4^2)^4}\Big) + \ldots 67 \ldots + \ldots 1 \ldots\}$

large output | show less | show more | show all | set size limit...

```
(*Box到tadpole单个约化系数计算时间测试*)
{timeTaken, rules} = AbsoluteTiming[GenerateRules[red4to1]];
                     绝对时间

Print["Time taken to generate rules: ", timeTaken, " seconds"];
打印
Print["Memory size of rules: ", ByteCount[rules], " bytes"];
打印                             字节数
Export["result4to1.m", red4to1 /. rules] // AbsoluteTiming
导出                                       绝对时间

Total number of unique items found: 214
Time taken to generate rules: 1.66137 seconds
Memory size of rules: 252071840 bytes
{81.3598, result4to1.m}
```

```
(*Box到bubble单个约化系数计算时间测试*)
{timeTaken, rules} = AbsoluteTiming[GenerateRules[red4to2]];
                     绝对时间

Print["Time taken to generate rules: ", timeTaken, " seconds"];
打印
Print["Memory size of rules: ", ByteCount[rules], " bytes"];
打印                             字节数
Export["result4to2.m", red4to2 /. rules] // AbsoluteTiming
导出                                       绝对时间

Total number of unique items found: 84
Time taken to generate rules: 1.51656 seconds
Memory size of rules: 201920336 bytes
Out[65]= {52.4218, result4to2.m}
```

```
(*Box到Triangle单个约化系数计算时间测试*)
{timeTaken, rules} = AbsoluteTiming[GenerateRules[red4to3]];
                     绝对时间

Print["Time taken to generate rules: ", timeTaken, " seconds"];
打印
Print["Memory size of rules: ", ByteCount[rules], " bytes"];
打印                             字节数
Export["result4to3.m", red4to3 /. rules] // AbsoluteTiming
导出                                       绝对时间

Total number of unique items found: 30
Time taken to generate rules: 1.02067 seconds
Memory size of rules: 118637376 bytes
{16.345, result4to3.m}
```

```
(*Box到Box单个约化系数计算时间测试*)
{timeTaken, rules} = AbsoluteTiming[GenerateRules[red4to4]];
                     绝对时间

Print["Time taken to generate rules: ", timeTaken, " seconds"];
打印
Print["Memory size of rules: ", ByteCount[rules], " bytes"];
打印                             字节数
Export["result4to4.m", red4to4 /. rules] // AbsoluteTiming
导出                                       绝对时间

Total number of unique items found: 10
Time taken to generate rules: 0.0004083 seconds
Memory size of rules: 23770536 bytes
{1.04486, result4to4.m}
```

主积分	主积分个数	计算单个系数时间（秒）	总时间（秒）
Tadpole	4	83	332
Bubble	6	54	324
Triangle	4	18	72
Box	1	1	1
总时间			729
Kira			3859

Generating function for higher poles:

arxiv: 2403.16040

$$F(a_1, a_2, ..., a_n) = \int \frac{d^D l}{D_1^{a_1} \cdot D_2^{a_2} \cdot \cdots \cdot D_n^{a_n}} = \int \frac{d^D l}{\prod_{k=1}^{n} \left((l - q_k)^2 - m_k^2 \right)^{a_k}}$$

$$\frac{1}{D_1 - t_1} = \frac{1}{D_1} \cdot \frac{1}{1 - \frac{t_1}{D_1}} = \frac{1}{D_1} \sum_{a_1=0}^{\infty} \frac{t_1^{a_1}}{D_1^{a_1}} = \sum_{a_1=1}^{\infty} \frac{t_1^{a_1-1}}{D_1^{a_1}}$$

$$\begin{aligned} G_n(t_1, t_2, \cdots, t_n) &= \frac{1}{(D_1 - t_1)(D_2 - t_2) \cdots (D_n - t_n)} \\ &= \sum_{a_1, \cdots, a_n = 1}^{\infty} \frac{t_1^{a_1-1} \cdot t_2^{a_2-1} \cdot \cdots \cdot t_n^{a_n-1}}{D_1^{a_1} \cdot D_2^{a_2} \cdot \cdots \cdot D_n^{a_n}} \end{aligned}$$

IBP relations:

$$0 = \int d^D l \cdot \frac{\partial}{\partial (l-q_i)^\mu} \cdot \frac{l^\mu - q_i^\mu}{\prod_{k=1}^n \left((l-q_k)^2 - m_k^2\right)^{a_k}}$$

Define the following operators i^+, i^-:

$$i^+ F(a_1, \cdots, a_n) = F(a_1, \cdots, a_i + 1, \cdots, a_n)$$
$$i^- F(a_1, \cdots, a_n) = F(a_1, \cdots, a_i - 1, \cdots, a_n).$$

$$Q'_{ij} = \frac{m_i^2 + m_j^2 - (q_i - q_j)^2}{2}.$$

$$2Q' \begin{bmatrix} a_1 1^+ \\ a_2 2^+ \\ \vdots \\ a_n n^+ \end{bmatrix} F(a_1, \cdots, a_n) = \begin{bmatrix} D - \sum_{k=1}^n a_k - \left(\sum_{k=1}^n a_k \cdot k^+\right) 1^- \\ D - \sum a_k - \left(\left(\sum_{k=1}^n a_k \cdot k^+\right) 2^- \\ \vdots \\ D - \sum a_k - \left(\left(\sum_{k=1}^n a_k \cdot k^+\right) n^- \end{bmatrix} F(a_1, \cdots, a_n).$$

Multiply both sides by $t_1^{a_1-1} t_2^{a_2-1} \cdots t_n^{a_n-1}$, and sum over $a_1, a_2, \cdots, a_n$ from 1 to ∞.

$$2Q\left(t_1, t_2, \cdots, t_n\right) \begin{bmatrix} \partial_{t_1} G_n \\ \partial_{t_2} G_n \\ \vdots \\ \partial_{t_n} G_n \end{bmatrix} = \left(D - (n+1)\right) \begin{bmatrix} G_n \\ G_n \\ \vdots \\ G_n \end{bmatrix} - \begin{bmatrix} \partial \hat{t}_1 G_{n:\hat{1}} \\ \partial \hat{t}_2 G_{n:\hat{2}} \\ \vdots \\ \partial \hat{t}_n G_{n:\hat{n}} \end{bmatrix}.$$

$$Q_{ij}(t_1, t_2, \cdots, t_n) = \frac{m_i^2 + t_i + m_j^2 + t_j - (q_i - q_j)^2}{2}$$

$$G_{n:\hat{i}}\left(t_1, \cdots, \not{t}_i, \cdots, t_n\right) = \frac{1}{\prod_{k \neq i} \left((l - q_k)^2 - m_k^2 - t_k\right)}$$

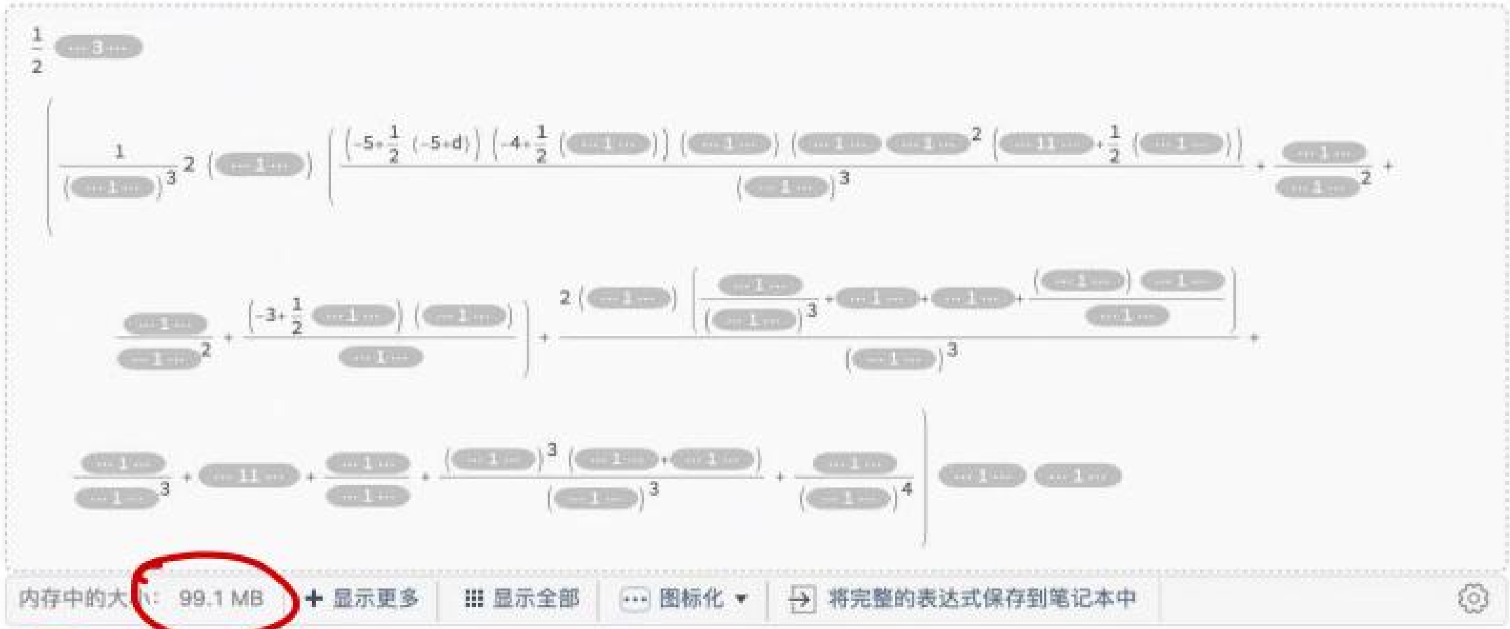
$$GF_{n\to n}(t_1,\cdots,t_n)=\left(\frac{|\,Q\,(t_1,\cdots,t_n)\,|}{|Q(0,\cdots 0)|}\right)^{\frac{D-(n+1)}{2}}$$

$$Q_{ij}(t_1,t_2,\cdots,t_n)=\frac{m_i^2+t_i+m_j^2+t_j-(q_i-q_j)^2}{2}$$

```
In[12]:= D[(Det[Qt]/Det[Q0])^((d-5)/2), {t[1], 3}, {t[2], 2}, {t[3], 2}, {t[4], 2}] /. {t[_] → 0}

Print[AbsoluteTiming[%][[1]]]
```

Out[12]=




```
0.007895
```

Advantage of Generating Function:

①**More concise expression;**

②**Understand the analytical structure;**

③**Accelerate the computation.**

$$x_{\pm} = \frac{2\left((\overline{VL}) \pm \sqrt{(\overline{LL})R^2 + (\overline{VL})^2 - (\overline{LL})(\overline{VV})}\right)}{(\overline{LL})}$$

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