高圈多外线费曼积分的解析计算

2024 全国粒子物理学术会议
2024. 7. 14

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Based on

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP 08(2024) 027*

Liu, Matijasic, Miczajka, Peraro, Xu, Xu, YZ, *to appear*

also the package …

“**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals”
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

Outline

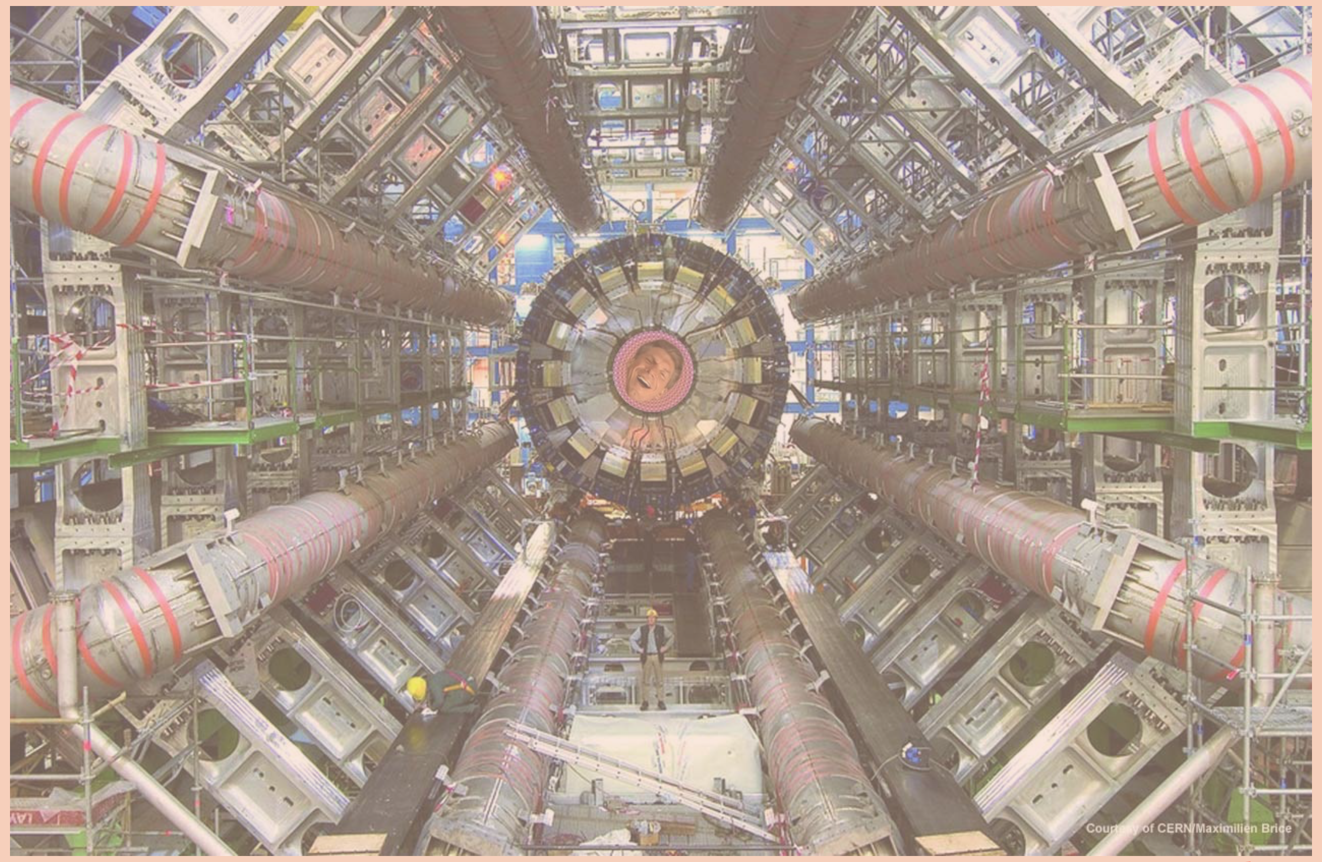
Why Feynman integrals? Why analytic?

Case 1: **2loop 6point Feynman integrals**

Case 2: 3loop 5point Feynman integrals

Summary and Outlook

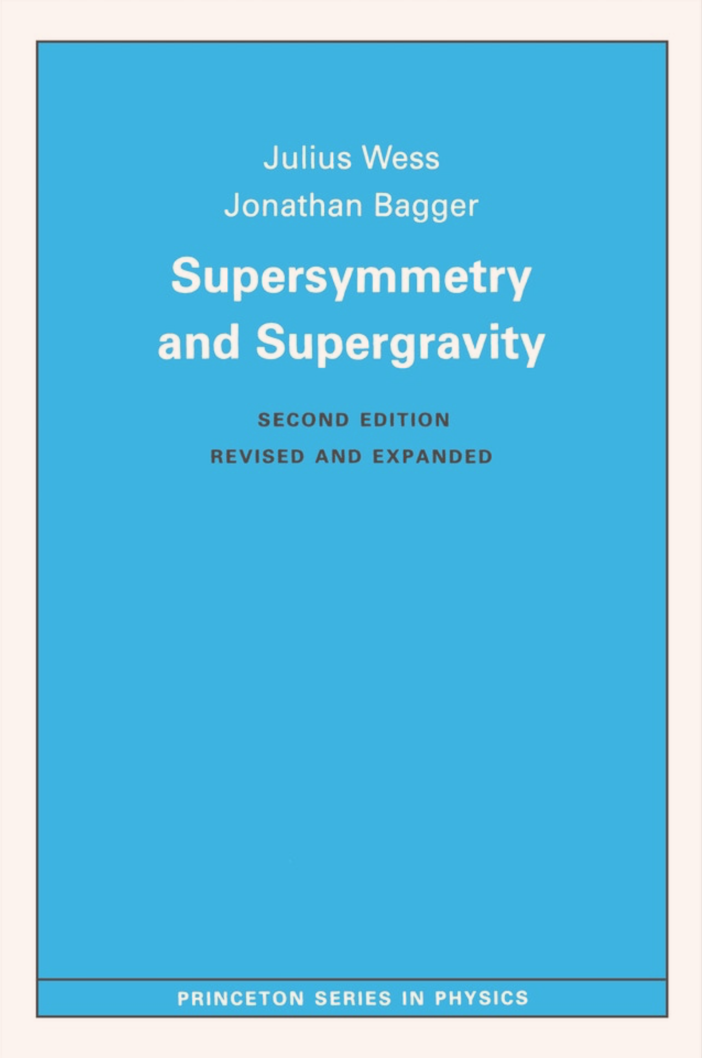
Why Feynman integrals?



Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

Feynman integrals



Formal theory

Gravitational wave template computations

Why *analytic* Feynman integrals?

- Auxiliary Mass Flow and Numeric Monte Carlo methods slow or not available yet

for some multi-loop multi-leg Feynman integrals

for examples: 2loop 6point and 3loop 5point Feynman integrals

- Theoretical aspects of quantum field theory

for examples: 2loop Yang-Mills theory collinear factorization violation

Henn, Ma, Xu, Yan, *YZ*, Zhu, arXiv 2406.14604

- Quantum field theory computation of gravitational wave

for the advance of AMFlow, refer to Yanqing Ma's plenary talk

Our Strategy

Guan, Liu, Ma, Wu, 2024

Integration-by-parts (IBP) reduction — Finite field techniques, Blade, NeatIBP

Uniformly transcendental (UT) basis determination
Canonical differential equation

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon)$$

Alphabet searching

$$A_i = \frac{\partial}{\partial x_i}\tilde{A}, \quad \tilde{A} = \sum_k \tilde{a}_k \log(W_k)$$

Solving differential equation

$$\int d\log(W_{i_1}) \circ \ldots \circ d\log(W_{i_k})$$

$\rightarrow$ polylogarithm functions or one-fold integration

2loop 6point Feynman integrals

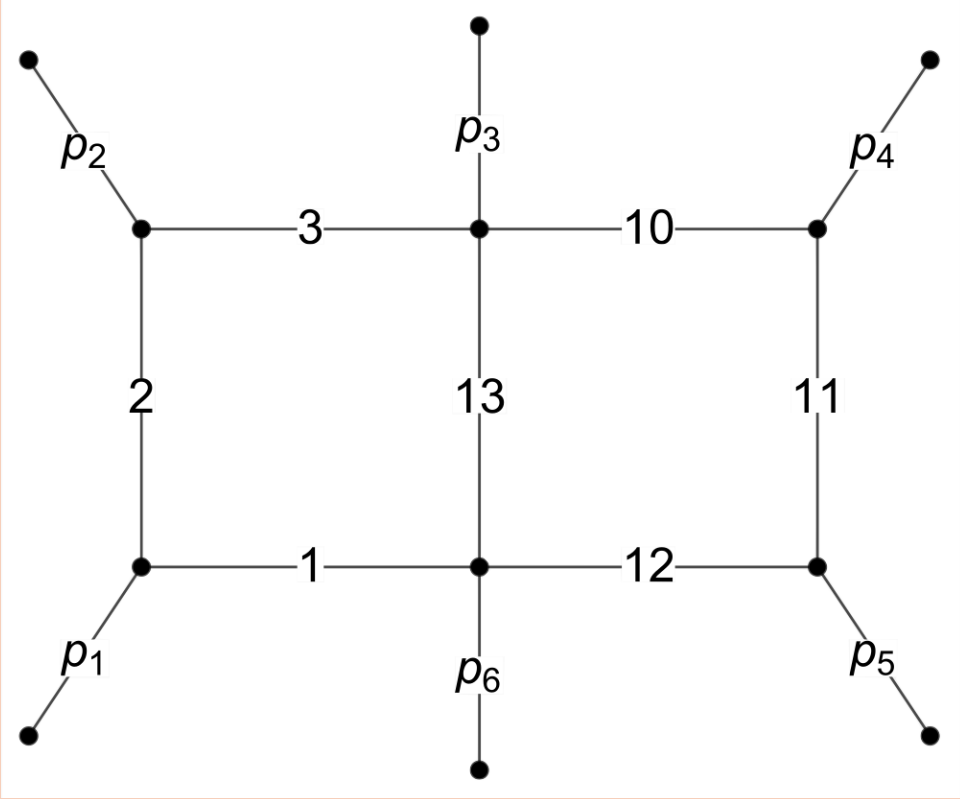
The status of art for analytic computations

Scale frontier

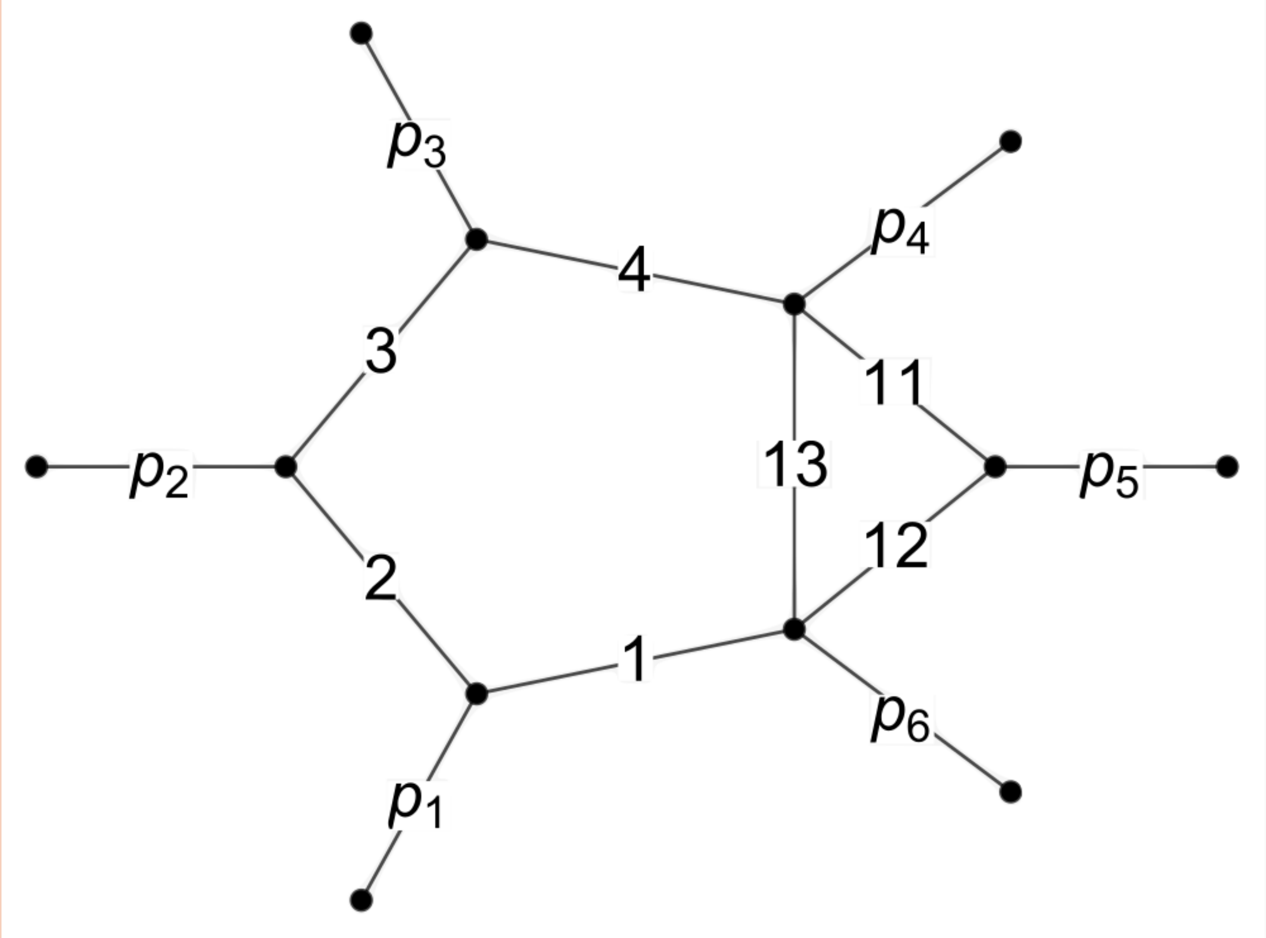
2loop 5point massless	*Gehrmann, Henn, Lo Presti 2015* *Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019*	5 scales
2loop 5point one-mass	*Papadopoulos, Tommasini, Wever 2019* *Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020* *Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2023*	6 scales
2loop 5point two-mass	*Cordero, Figueiredo, Kraus, Page and Reina 2023* for leading-Color pp→ttH amplitudes with a light-quark loop	7 scales
2loop 6point massless	*Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2024* *for NNLO 4 jets production, 2 jets+ 2photons*	8 scales!

$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{345}$

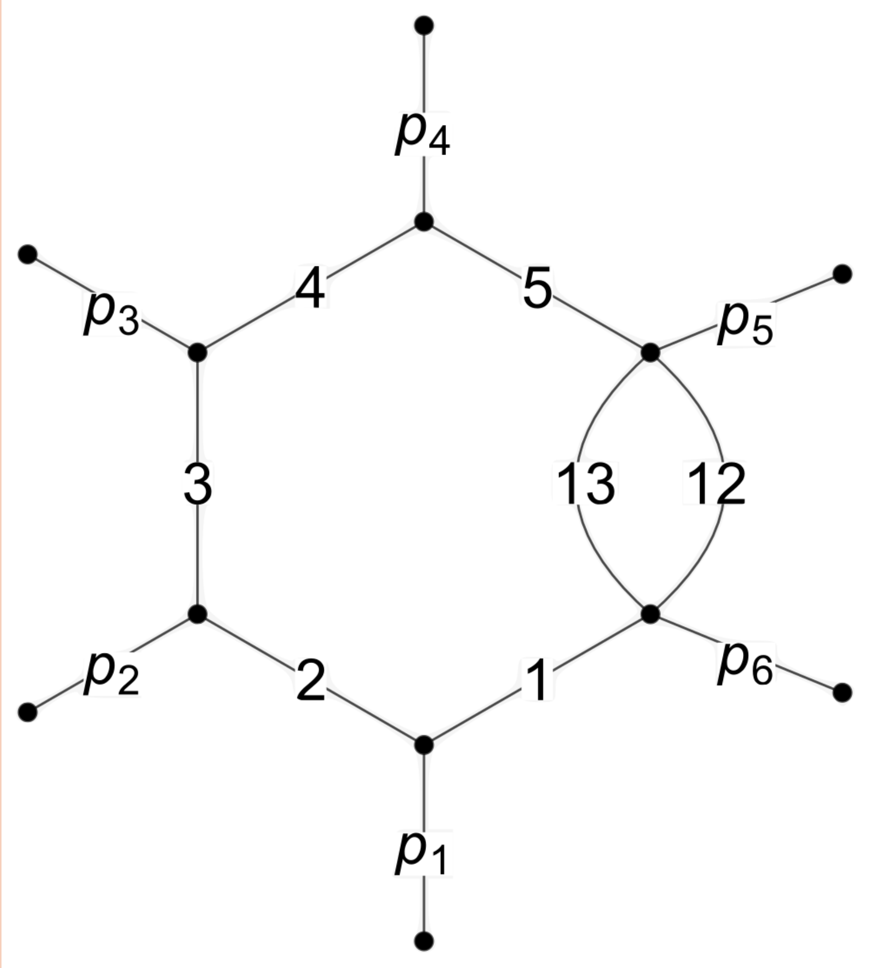
What we achieved



double box



pentagon triangle

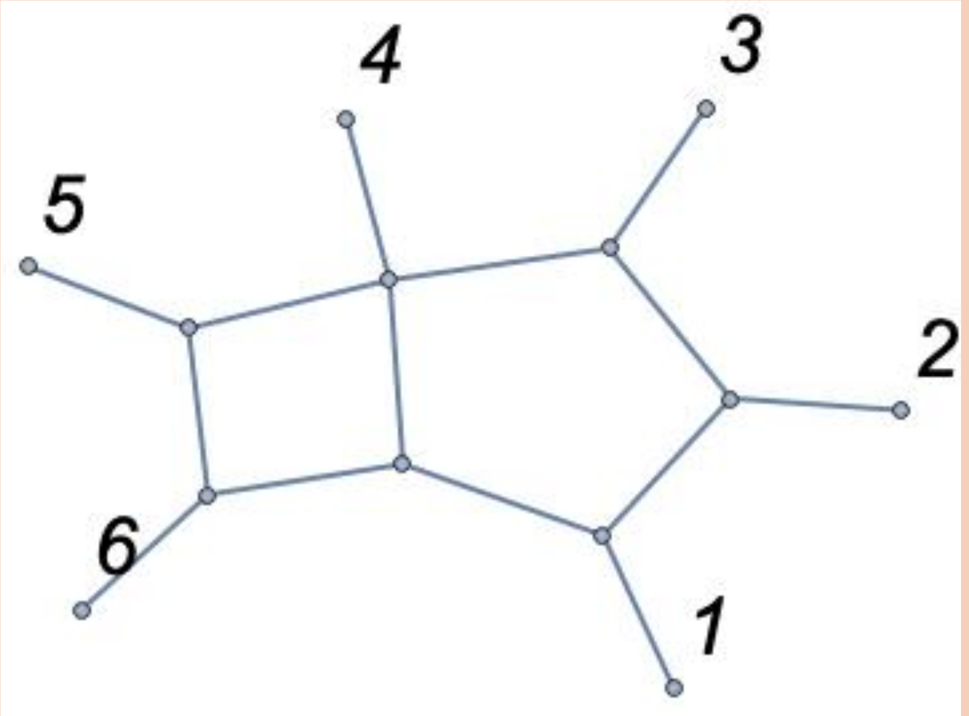


hexagon bubble

✔ UT basis found!

✔ Canonical differential equation solved!

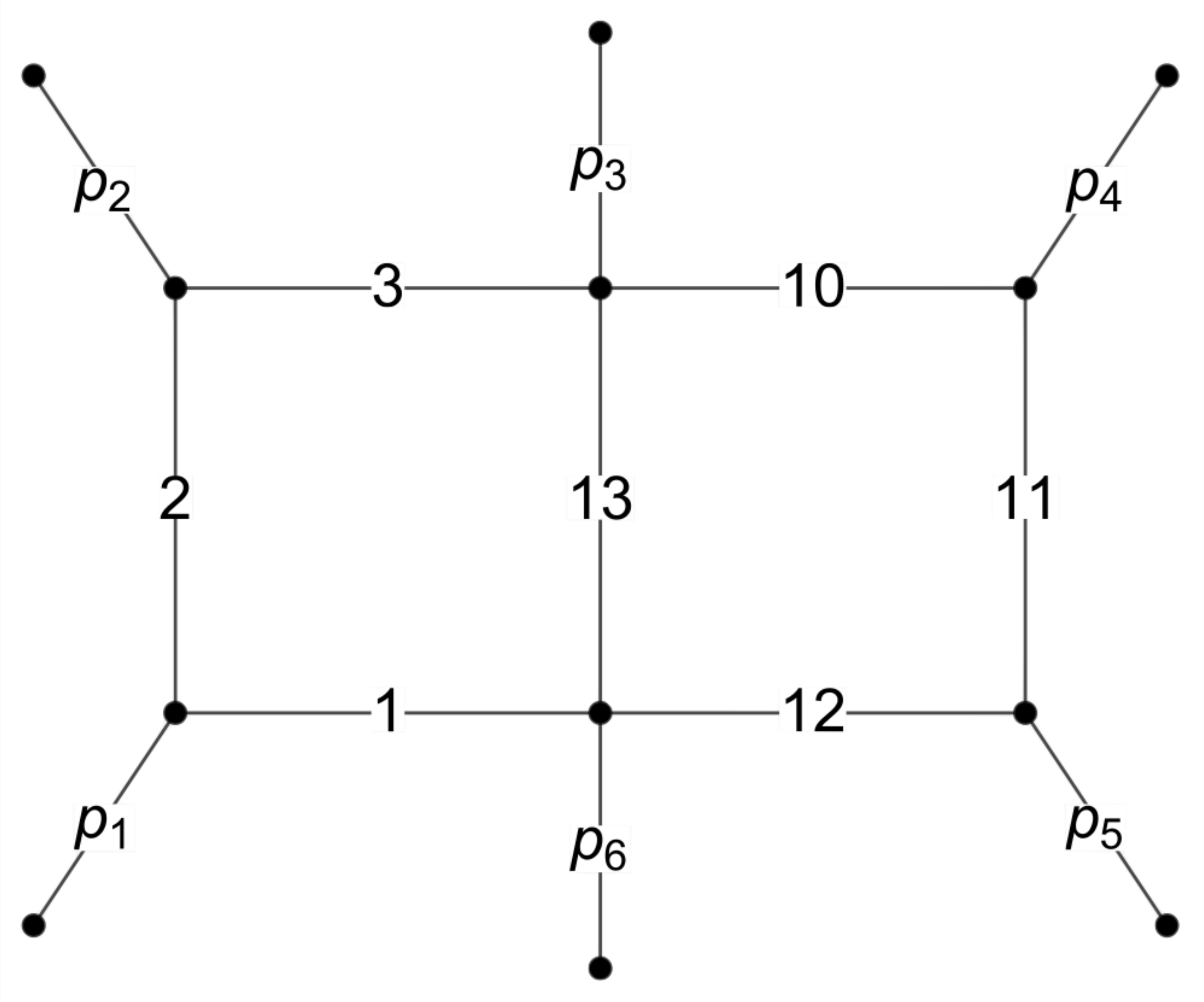
J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, *JHEP08(2024)027*



✔ canonical differential equation also solved

to appear as the next paper

Uniformal transcendental (UT) basis determination



$$I_{\mathrm{db},i} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}}\,, \qquad i = 1, \ldots, 7$$

$$N_1 = -s_{12}s_{45}s_{156}\,,$$

$$N_2 = -s_{12}s_{45}(l_1+p_5+p_6)^2\,,$$

$$N_3 = \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5+p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix},$$

$$N_4 = \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2-p_6 & p_5 & p_4 & p_1+p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix},$$

$$N_5 = -\frac{1}{4}\frac{\epsilon_{1245}}{G(1,2,5,6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix},$$

$$N_6 = \frac{1}{8} G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2-p_6 & p_4 & p_5 \end{pmatrix} + \frac{D_2 D_{11}(s_{123}+s_{126})}{8}\,,$$

$$N_7 = -\frac{1}{2\epsilon}\frac{\Delta_6}{G(1,2,4,5)D_{13}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}.$$

key step

Chiral numerator
(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)
/ Gram determinant
correspondence

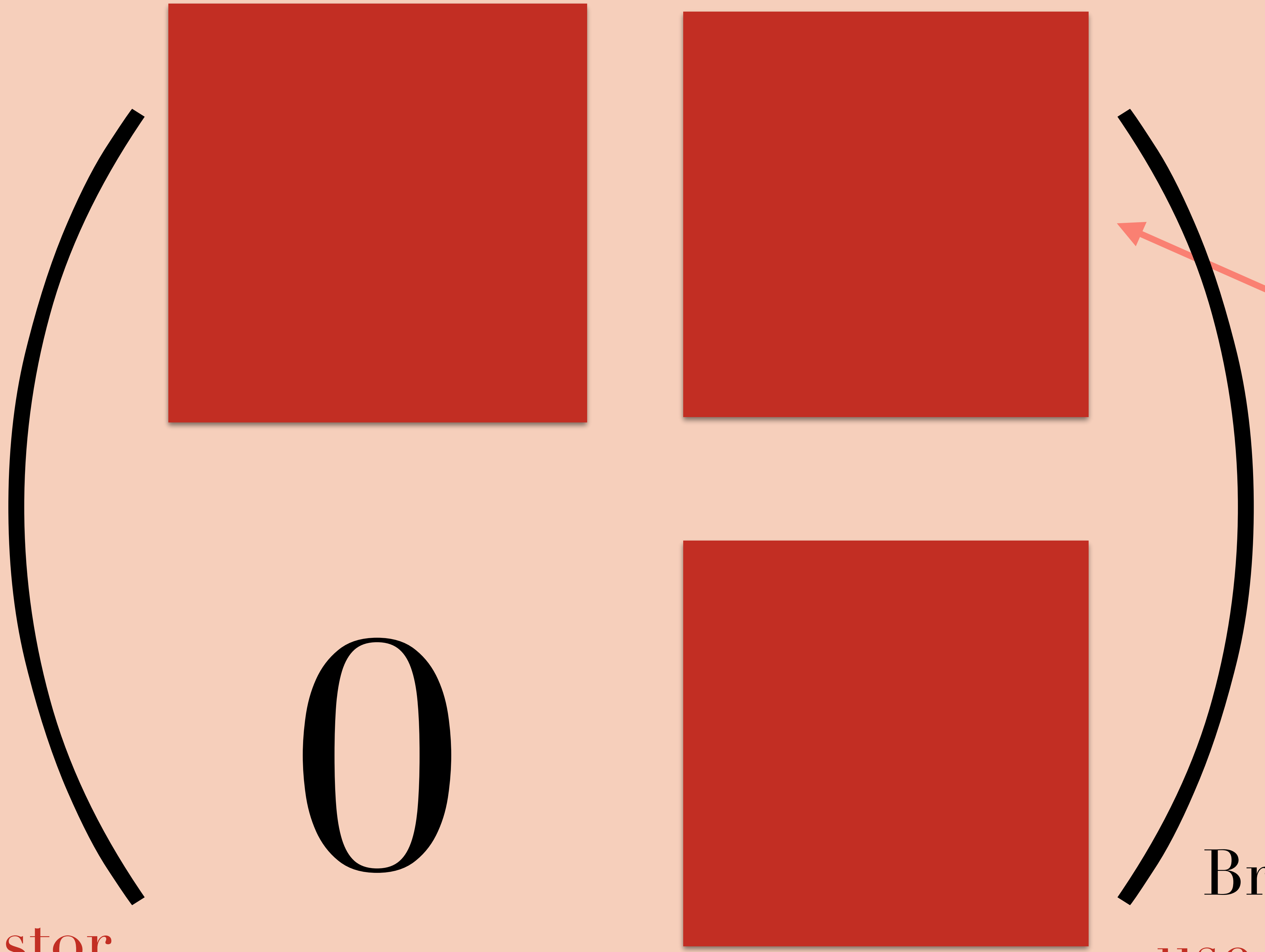
$$\Delta_6 = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - \langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$$

J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, JHEP08(2024)027

Complete canonical differential equation for 2l6p double box

double box   7 MIs

sub sectors   59 MIs



Differential equation matrix
red blocks are proportional to ε

Brute-force IBP reduction doesn't work
use alphabet to fit the differential equation

Use momentum twistor Variables
(Penrose, Hodges)

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon)$$

$$A_i = \frac{\partial}{\partial x_i}\tilde{A}, \quad \tilde{A} = \sum_k \tilde{a}_k \log(W_k)$$

A new algorithm to search for odd letters

Odd letter $\dfrac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$

$$\boxed{P^2 - Q = c\prod_i W_i^{e_i}, \quad c \in \mathbb{Q}, \quad e_i \in \mathbb{N}}$$

Even letter

An observation (and conjecture) from Heller, von Manteuffel, Schabinger 2020

Algorithm to solve for $c, \quad e_i$ Matijasic, J. Miczajka, to appear soon

package "Effortless"

Even letter, Odd letter and the more complicated ...

114 Even letter $F(s)$ a polynomial in Mandelstam variables and masses

Conjecture: a Feynman integrals' even letters are all from Landau singularity?

94 Odd letter $\dfrac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

square roots are ϵ_{ijkl}, Δ_6, , $\sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

pseudo scalar

leading singularity hexagon

Källin function from massive triangle diagrams

15 More complicated letter $\dfrac{P(s)-\sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s)+\sqrt{Q_1(s)}\sqrt{Q_2(s)}}$

Boundary Values

Numeric boundary values

Liu, Wang, Ma, 2018
Liu, Ma 2022

It is fine to use the package AMFlow to get ~100 digits as the boundary value

Analytic boundary values

It is still possible to *fully analytic* boundary values due to the kinematic symmetry

$$X_0: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \{-1,-1,-1,-1,-1,-1,-1,-1,-1\}$$

UT integrals are not divergent at this point (spurious poles).

Solve the canonical DE on a curve starting with X_0 and require the finite solution
Some known integrals' boundary values

} analytic boundary value

Boundary Values

Analytic boundary values

Boundary values at the initial point, are combination of poly-logarithm of roots of unity

$$
\begin{aligned}
\epsilon^4 I_{\mathrm{db},1}(X_0) &= 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,\\
\epsilon^4 I_{\mathrm{db},2}(X_0) &= 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,\\
I_{\mathrm{db},3}(X_0) &= I_{\mathrm{db},4}(X_0) = I_{\mathrm{db},5}(X_0) = 0,\\
\epsilon^4 I_{\mathrm{db},6}(X_0) &= -\left(\frac{\pi^4}{540} + \frac{4}{3}\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,\\
\epsilon^4 I_{\mathrm{db},7}(X_0) &= 0.
\end{aligned}
$$

from the differential equation
spurious pole asymptotic analysis

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Solution of canonical DE

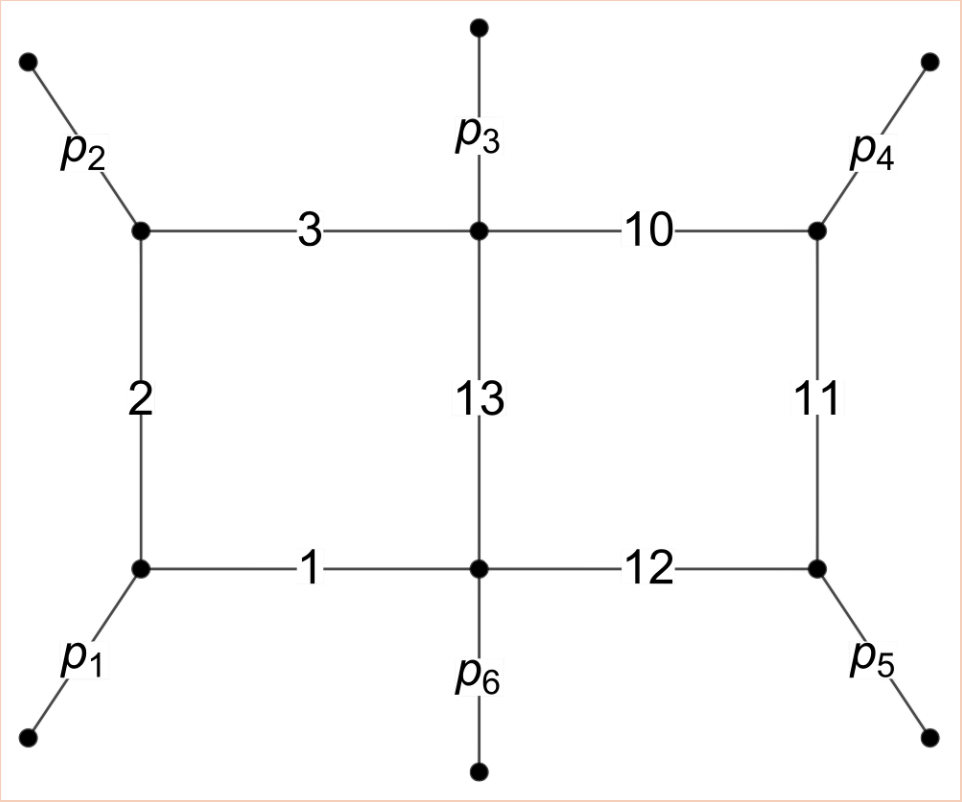
$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-1, weight-2

All in logarithm and classical poly-logarithm



$$\begin{aligned}
I^{(2)}_{\mathrm{db},1} = & -\log(-v_1)\log(-v_2) - \log(-v_1)\log(-v_3) + \log(-v_1)\log(-v_4) - \log(-v_1)\log(-v_5) - \\
& \log(-v_1)\log(-v_6) + 4\log(-v_1)\log(-v_8) + \frac{1}{2}\log^2(-v_1) + \log(-v_2)\log(-v_3) - \\
& \log(-v_2)\log(-v_4) - \mathrm{Li}_2\left(1 - \frac{v_2 v_5}{v_7 v_8}\right) + \log(-v_2)\log(-v_6) + \log(-v_2)\log(-v_7) - \\
& 2\mathrm{Li}_2\left(1 - \frac{v_2}{v_8}\right) - \log(-v_2)\log(-v_8) - \log^2(-v_2) - \log(-v_3)\log(-v_4) + \log(-v_3)\log(-v_5) - \\
& \mathrm{Li}_2\left(1 - \frac{v_3 v_6}{v_8 v_9}\right) - 2\mathrm{Li}_2\left(1 - \frac{v_3}{v_8}\right) - \log(-v_3)\log(-v_8) + \log(-v_3)\log(-v_9) - \\
& \log^2(-v_3) - \log(-v_4)\log(-v_5) - \log(-v_4)\log(-v_6) + 4\log(-v_4)\log(-v_8) + \\
& \frac{1}{2}\log^2(-v_4) + \log(-v_5)\log(-v_6) + \log(-v_5)\log(-v_7) - 2\mathrm{Li}_2\left(1 - \frac{v_5}{v_8}\right) - \log(-v_5)\log(-v_8) - \\
& \log^2(-v_5) - 2\mathrm{Li}_2\left(1 - \frac{v_6}{v_8}\right) - \log(-v_6)\log(-v_8) + \log(-v_6)\log(-v_9) - \log^2(-v_6) - \\
& \log(-v_7)\log(-v_8) - \frac{1}{2}\log^2(-v_7) - \log(-v_8)\log(-v_9) + 3\log^2(-v_8) - \frac{1}{2}\log^2(-v_9) + \\
& \frac{\pi^2}{6}
\end{aligned}$$

Solution of canonical DE

$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-3, weight-4

$$\begin{aligned}\vec{I}^{(4)} &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{I}^{(3)}(\vec{x}_0) + \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_1} \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_2} \vec{f}^{(2)}(t_2) \\ &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t)\right) \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{f}^{(2)}(t) \right).\end{aligned}$$

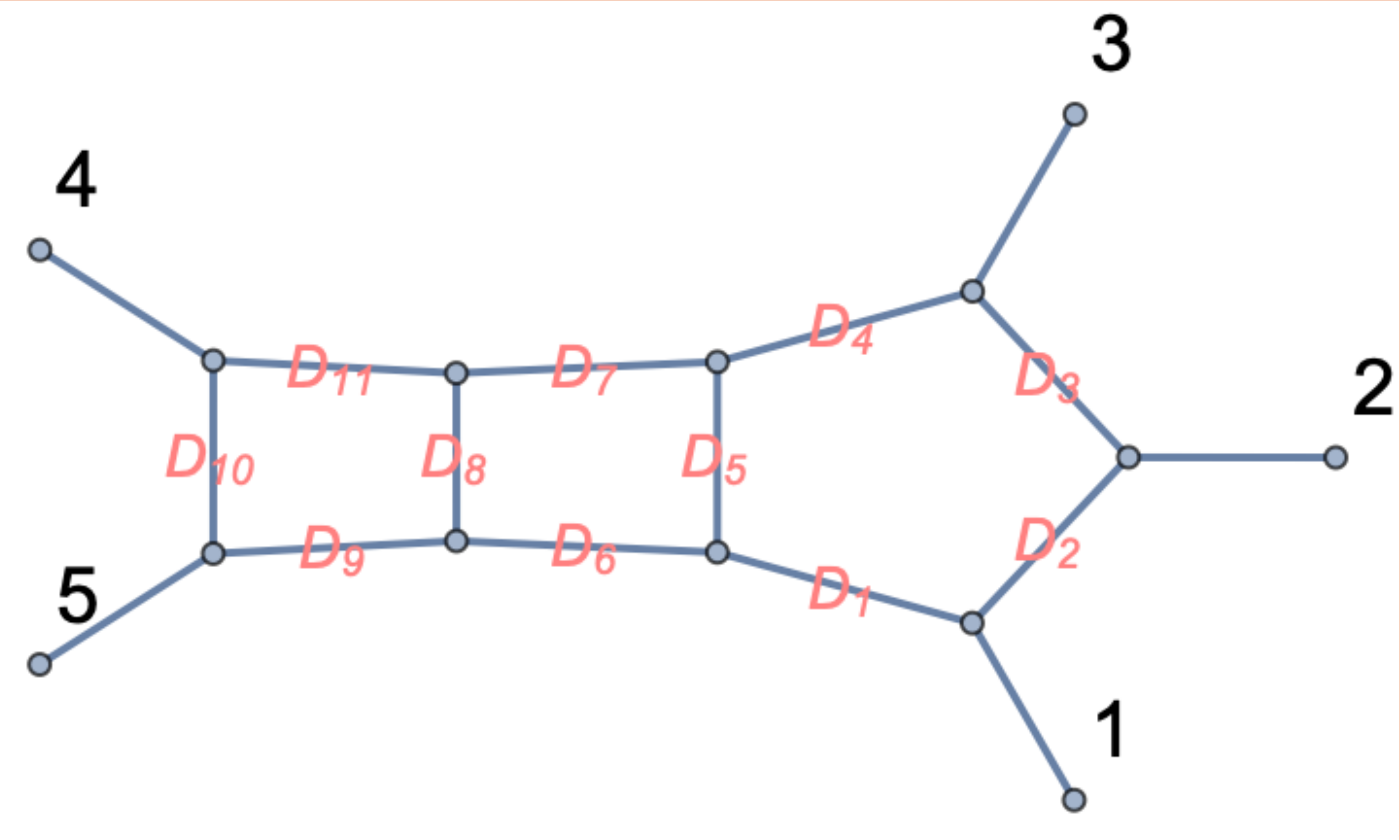
one-fold integration

It takes minutes on a laptop to get 20 digits from our analytic solution

3loop 5point Feynman integrals

from the request of John Ellis …

What we achieved

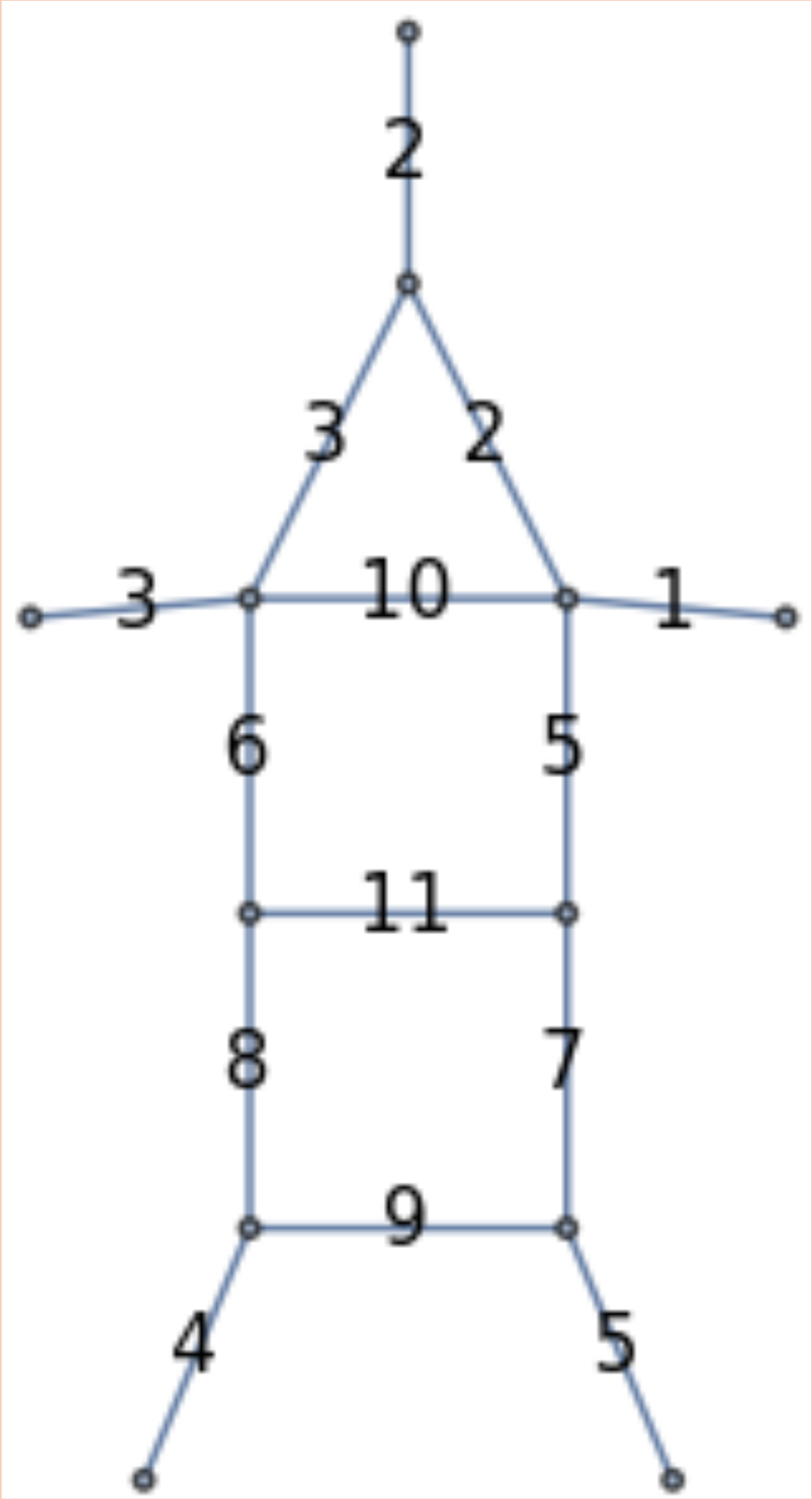


5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

✔ UT basis found!

✔ Canonical differential derived from NeatIBP



For this “rocket” subfamily, the analytic boundary values are obtained

Liu, Matijasic, Miczajka, Peraro, Xu, Xu, YZ, *to appear*

Summary

With the latest progress on IBP reduction, UT basis determination, alphabet searching, we analytically computed 2loop 6point massless Feynman integrals.

This is the analytic computation of 2loop 8-scale Feynman integrals in DR.

and the work on 3loop 5point will appear soon.

The analytic computation of many more multi-loop multi-leg multi-scale Feynman integrals would be possible.

Thank you