# Recent progress on scattering amplitudes & beyond

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based on works w. N. Arkani-Hamed, 曹趣, 董晋, C. Figueiredo 2312.16282, 2401.00041, 2401.05483, to appear

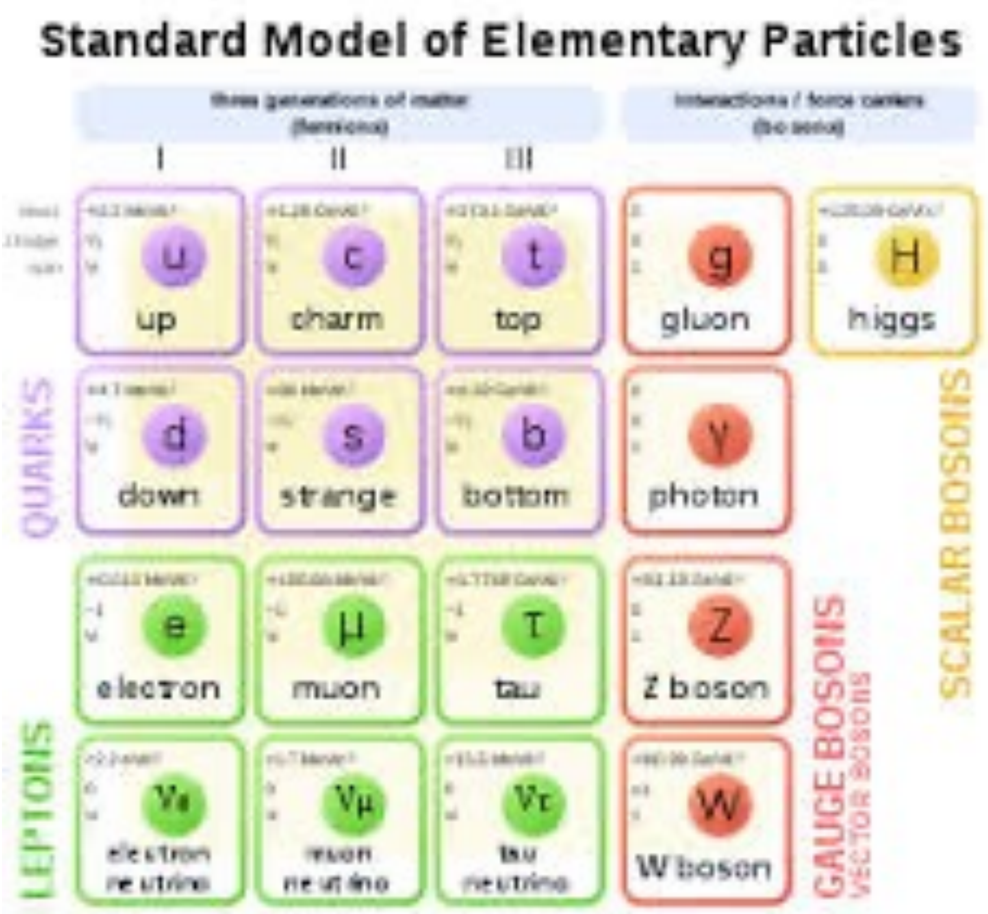
& w. 郭家恺, 黄宇廷 2405.20292; w. 姜旭航、杨清霖、张耀奇 2408.04222, ……

**第十四届全国粒子物理学术会议**

**山东大学（青岛）**
**2024.8.17**

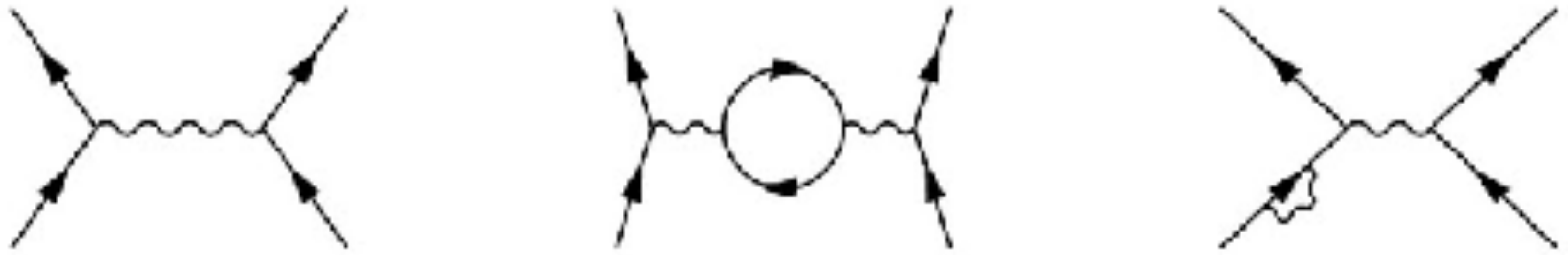
# Quantum Field Theory (QFT)

Most successful theoretical framework to describe Nature: particle physics, condensed matter, cosmology, strings




inevitable & universal: consequence of QM & relativity! fundamental interactions unified @ high energy

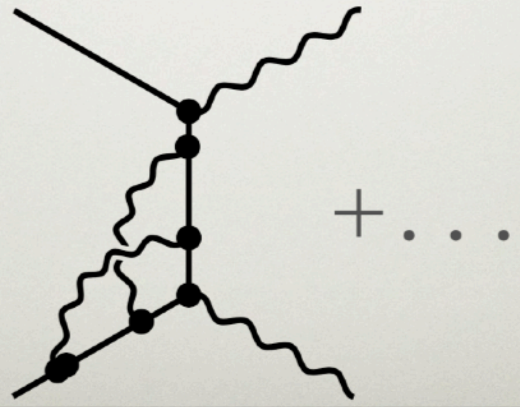
simple picture in perturbation theory: Feynman diagrams



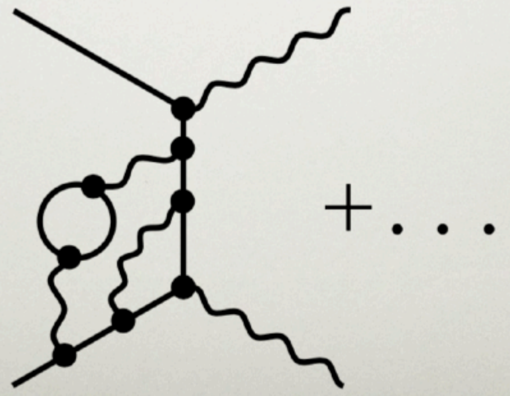
incredible accuracy! e.g. g-factor of electron magnetic dipole moment

$$g_e^{\text{theory}} = 2 + \frac{\alpha}{\pi} + \dots \qquad [1947]$$
$$g_e^{\text{expt}} = 2.0023\dots \qquad [1947]$$

$$g_e^{\text{theory}} = 2.0023193\dots \qquad [1957]$$
$$g_e^{\text{expt}} = 2.00231931\dots \qquad [1972]$$



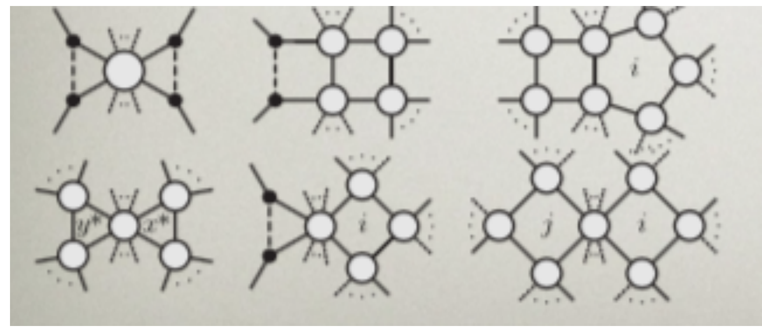
$$g_e^{\text{theory}} = 2.0023193044\dots \qquad [1990]$$
$$g_e^{\text{expt}} = 2.00231931\dots \qquad [1972]$$



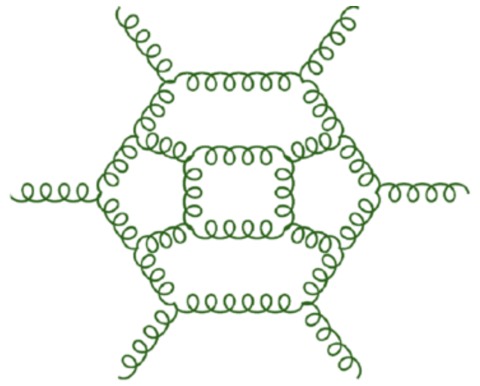
$$g_e^{\text{theory}} = 2.00231930435801\dots \qquad [2012]$$
$$g_e^{\text{expt}} = 2.002319304361\dots \qquad [2011]$$

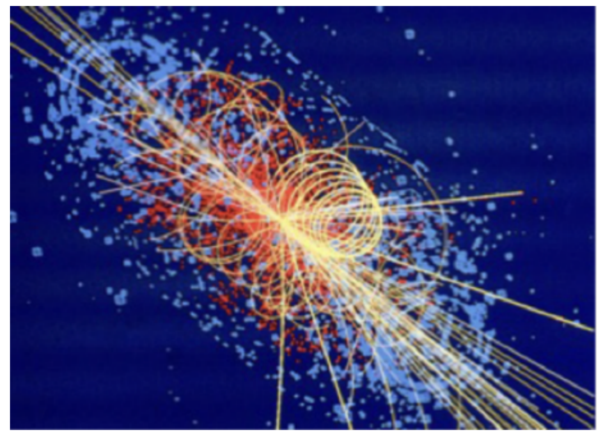
# “Amplitudes”

on/off-shell, weak/strong coupling, …

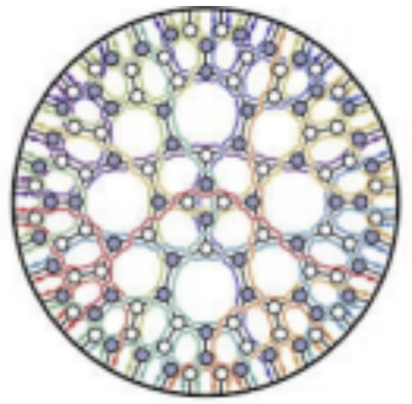


Formal QFT





Geometries, combinatorics, number theory, …

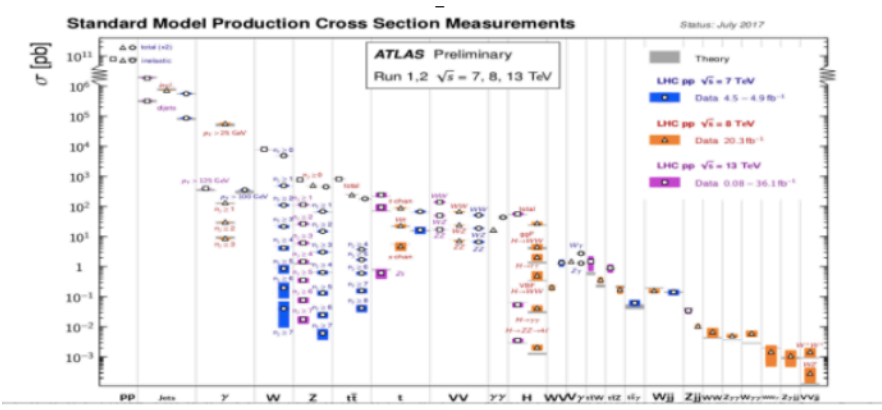


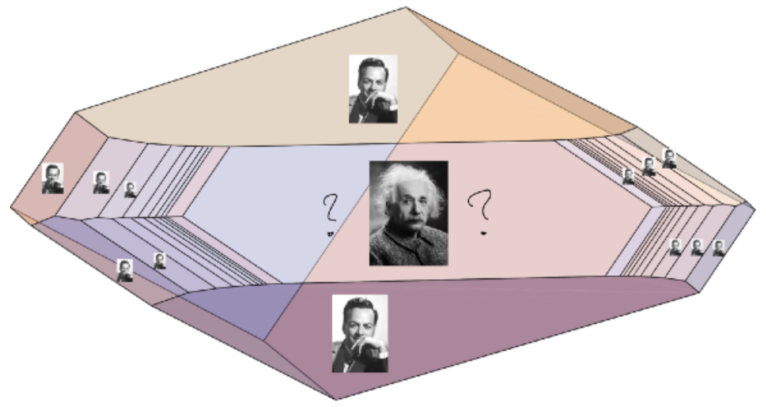
Mathematics

Amplitudes

precision frontier: loop integrands + integrals

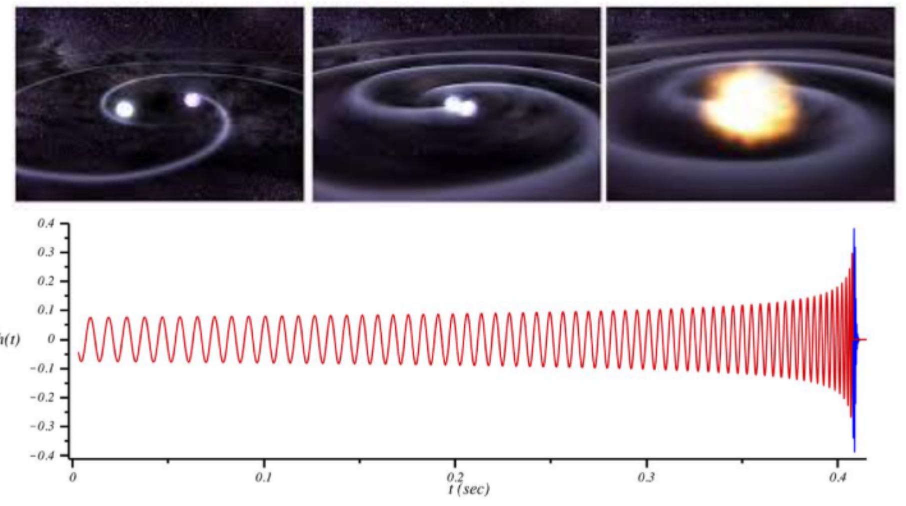
Collider Phenomenology



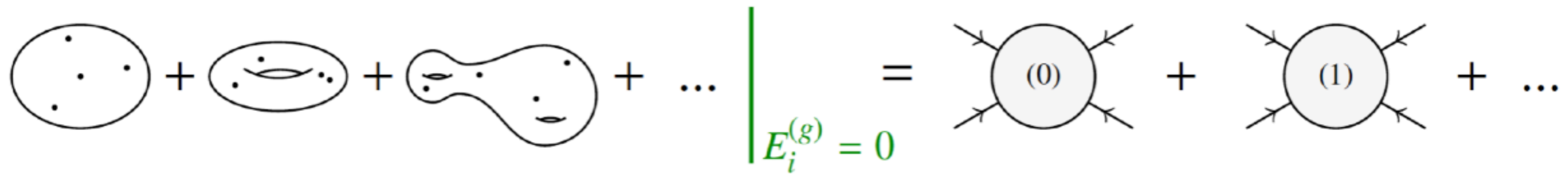



String Theory

Quantum/ Classical Gravity



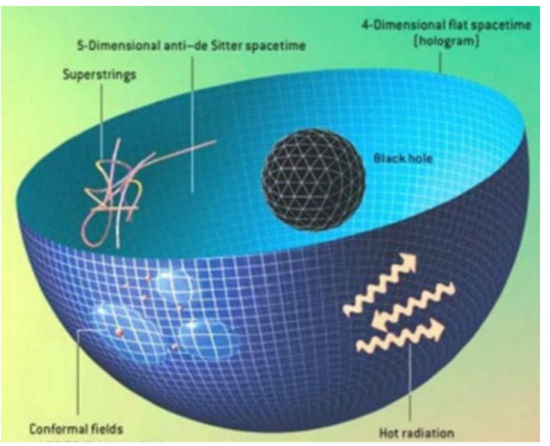
AdS/dS Cosmology

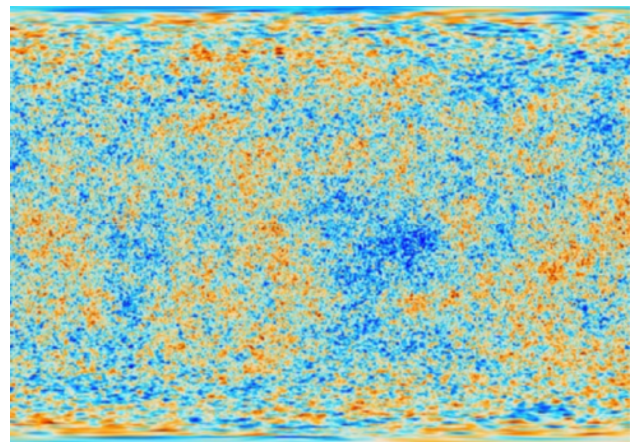


AdS/CFT, curved background, inflation, …

gravity amps, black holes & GW

moduli spaces, string perturbation + CFT…

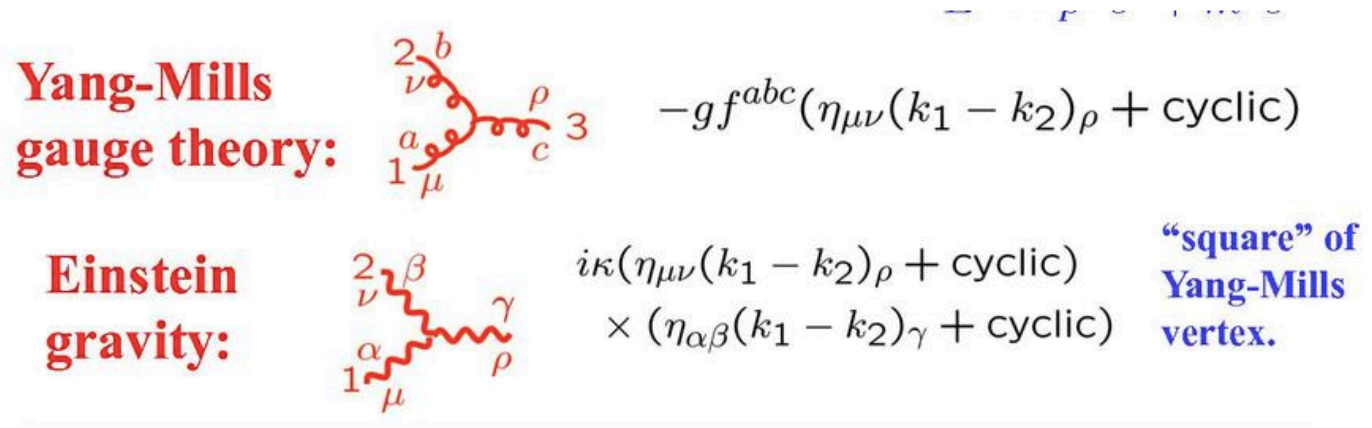


# Gravity=(Gauge Theory)^2

1985: Kawai, Lewellen, Tye (KLT): **"closed string amp=open-string amp^2"**

Field-theory limit:


2008: Bern, Carrasco, Johansson (BCJ): **double-copy construction**

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$$n_s + n_t + n_u = 0$$

$$\mathcal{A}_4^{\text{tree}}\Big|_{c_i \to n_i} \equiv \mathcal{M}_4^{\text{tree}} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

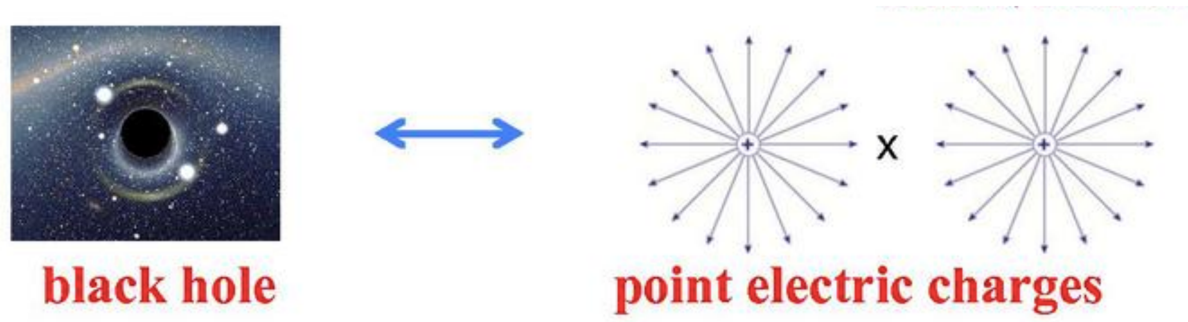
If you have a set of duality satisfying numerators.
To get:

gauge theory ⟶ gravity theory

simply take

color factor ⟶ kinematic numerator

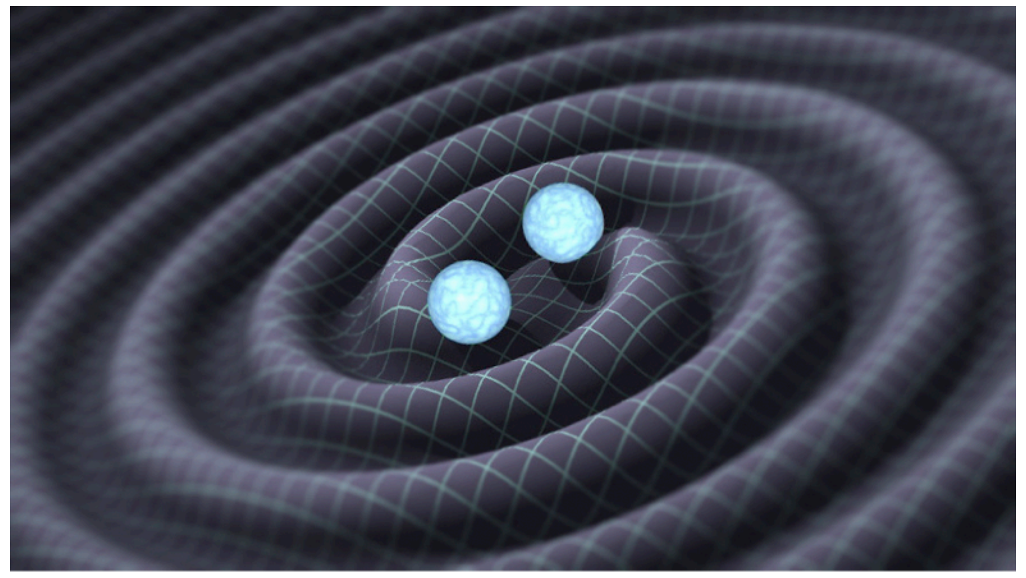
**extended to classical solutions, curved background etc.-> hidden symmetry & structure of classical gravity!**




Schwarzschild ~ (Coulomb)$^2$

# Gravitational waves

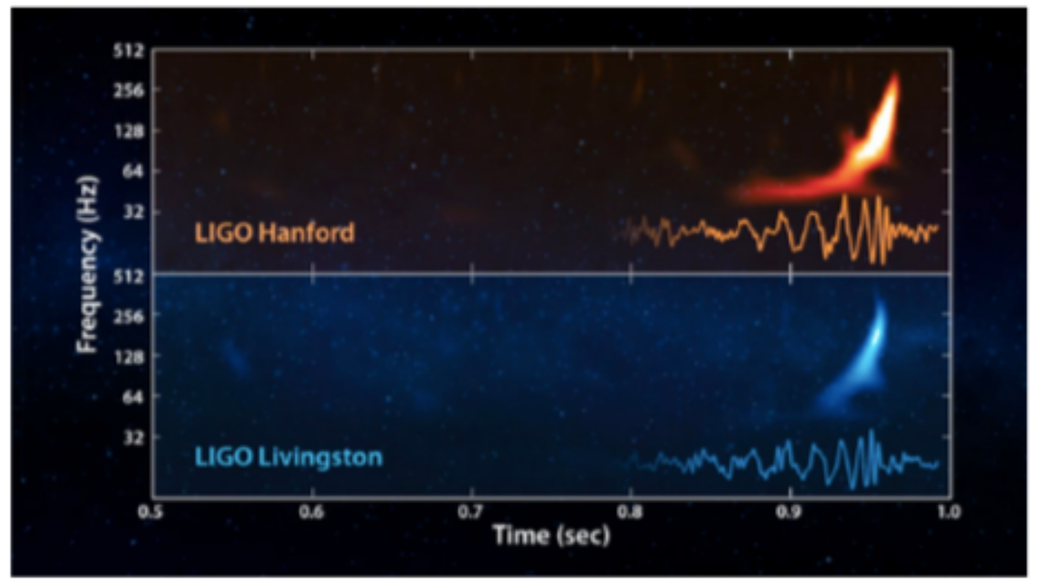
How to help calculations needed for LIGO (inspiral)?



Classical limits from quantum scattering amplitudes

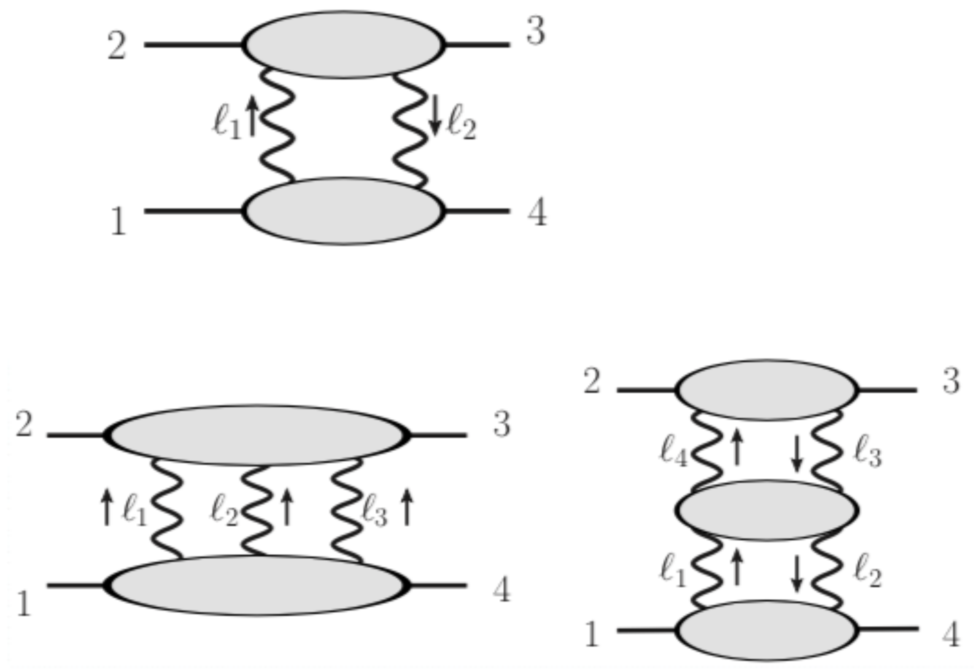
New tools e.g. double-copy simplifies GW calculations

Post-Newtonian/Minkowski from (EFT) amplitudes




[Goldberger, Rothstein, Porto,…]
[Bern, Cheung, Roiban, Shen, Solon, Zeng; …]
[…]

| | | 0PN | 1PN | 2PN | 3PN | 4PN | 5PN | … |
|---|---|---|---|---|---|---|---|---|
| 0PM: | 1 | $v^2$ | $v^4$ | $v^6$ | $v^8$ | $v^{10}$ | $v^{12}$ | … |
| 1PM: | | $1/r$ | $v^2/r$ | $v^4/r$ | $v^6/r$ | $v^8/r$ | $v^{10}/r$ | … |
| 2PM: | | | $1/r^2$ | $v^2/r^2$ | $v^4/r^2$ | $v^6/r^2$ | $v^8/r^2$ | … |
| 3PM: | | | | $1/r^3$ | $v^2/r^3$ | $v^4/r^3$ | $v^6/r^3$ | … |
| 4PM: | | | | | $1/r^4$ | $v^2/r^4$ | $v^4/r^4$ | … |
| … | | | | | | … | … | |



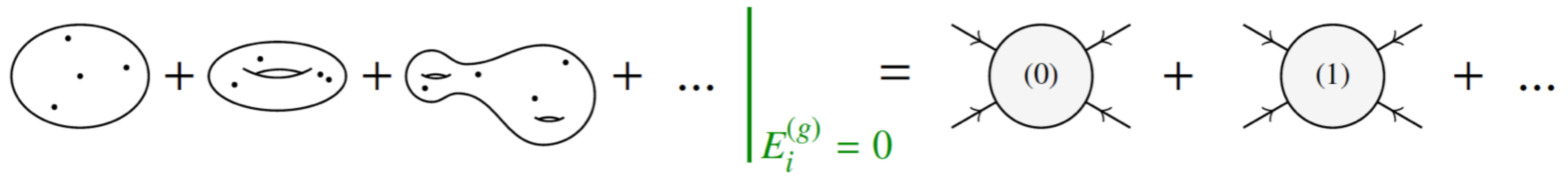
# New formulations of QFT

- Twistor string theory [Witten 2003]: worldsheet model for N=4 SYM tree amps failed at loops, but led to BCFW, CSW & many new developments!

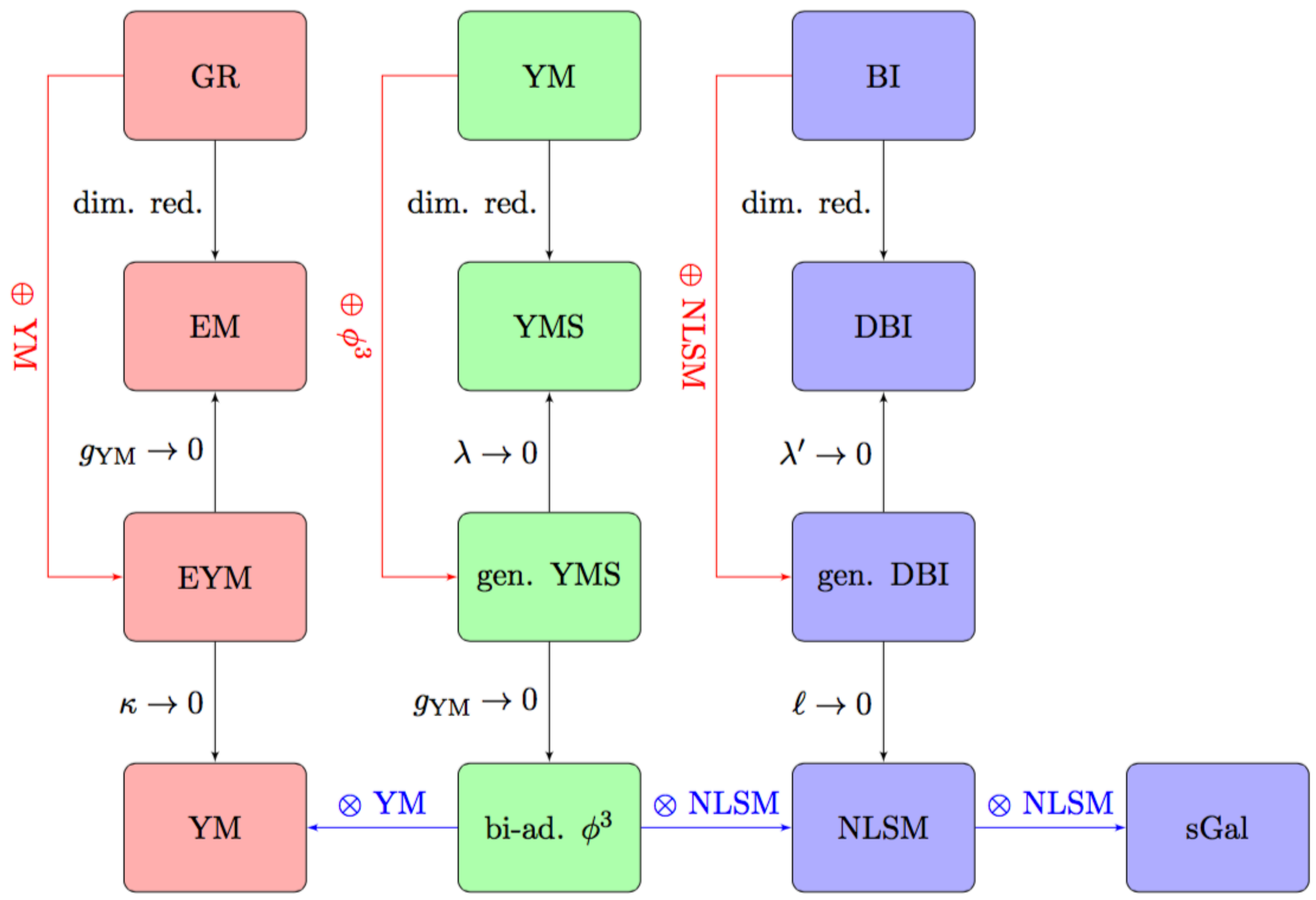
- How universal is Witten's twistor string? no SUSY? any spacetime dim? more general theories: (pure) Yang-Mils, gravity, effective field theories? loop level?

- CHY formulation: scattering of massless particles in any dim [Cachazo, SH, Yuan 2013]

  - *compact formulas* for amps of gluons, gravitons, scalars, (fermions?!) etc.
  - *manifest* gauge (diff) invariance, soft theorems, double-copy & new relations, etc.
  - *worldsheet picture:* ambitwsitor strings etc. [Mason, Skinner; Adamo et al; Berkovits; Siegel…]



# A web of theories & relations




new CHY from old ones by e.g. dim reduction
GR -> Einstein-Maxwell, YM -> YM-scalar

A new operation as direct sum of two particles ->
Einstein-Yang-Mills, Yang-Mills + bi-adjoint scalars

even more interesting relations [CHY 14][Cheung et al]:
pions from special dimension reduction of gluons!

These amplitudes are strongly constrained (even uniquely determined) by symmetries:
gauge invariance & Adler zero; deeply connected to each other!

# Combinatorial Geometries (with ``factorizing bd.") underlying scattering amplitudes & beyond (by 2023)

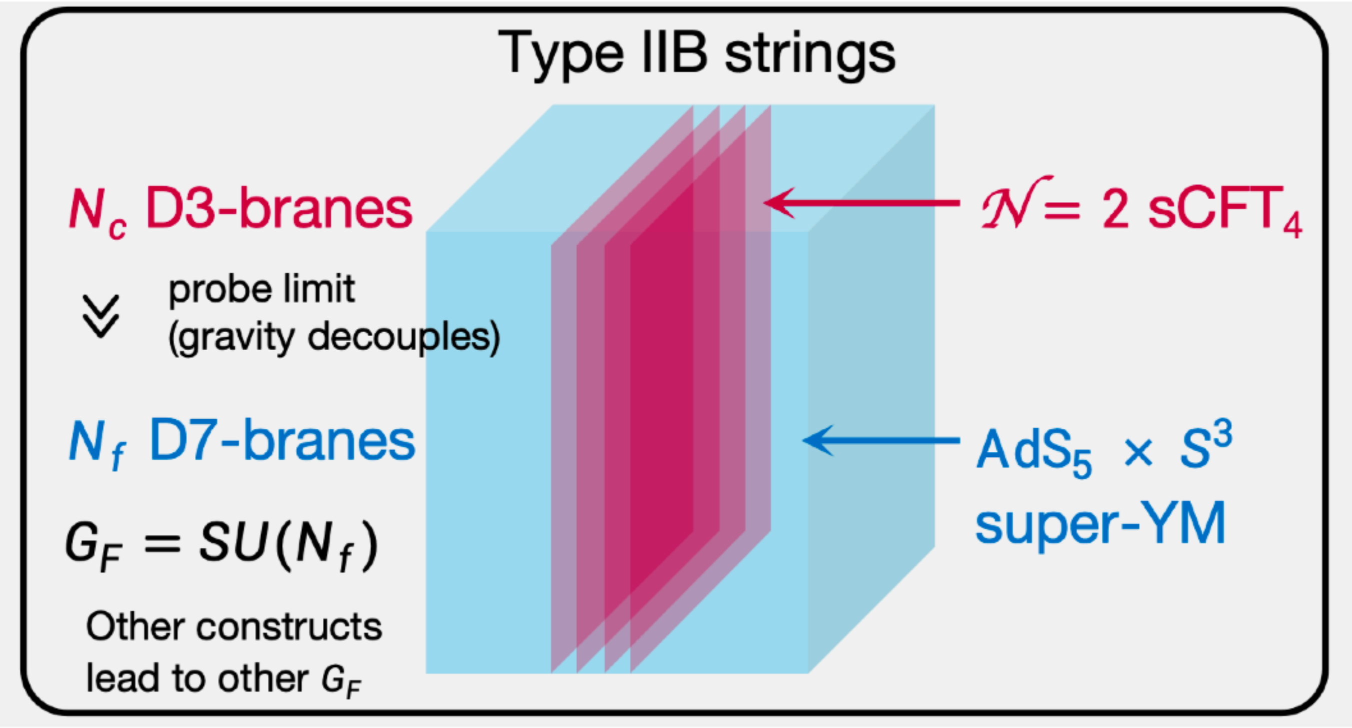
- moduli space $\mathscr{M}_{g,n}$ for conventional & (ambi-)twistor strings [Witten, Berkovits 04'; CHY 13'; Mason, Skinner 14'; …]
- positive Grassmannian $G_+(k,n)$, on-shell diagrams etc. for planar N=4 SYM [Arkani-Hamed 12' et al]

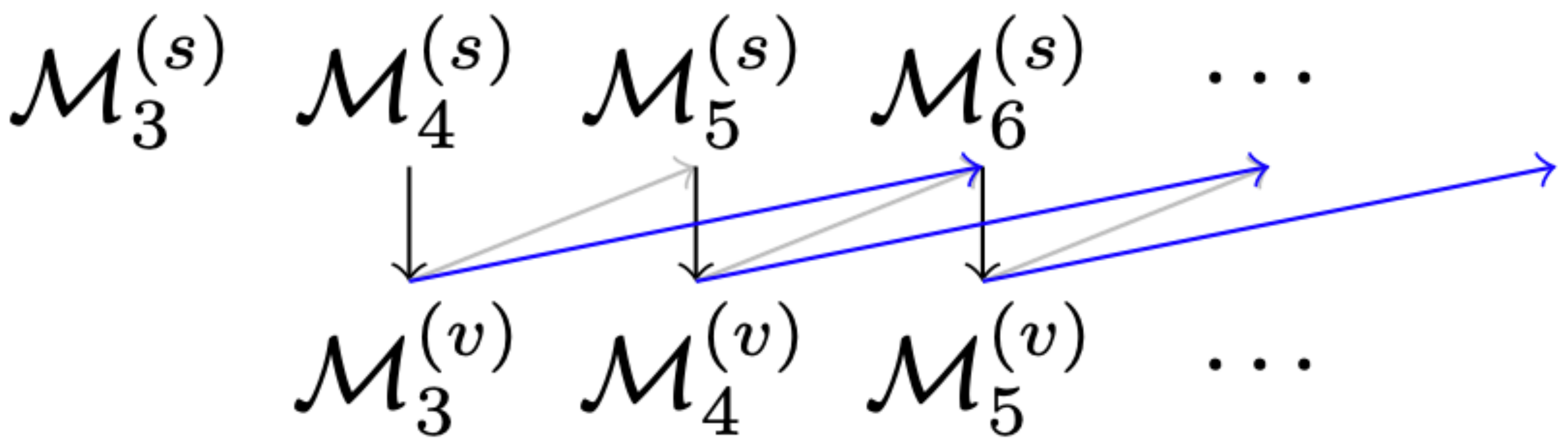
- Amplituhedron: map from $G_+(k,n)$ →all-loop integrand of SYM in momentum twistor space [Arkani-Hamed, Trnka 13' + Thomas;…] => momentum space [SH, Zhang 18'] & momentum amplituhedron [Ferro et al]
- ABJM amplituhedron: reduction to D=3 from SYM amplituhedron —> all-loop integrand of ABJM! [SH, Kuo, Li, Zhang, 22' + Huang 23',…]

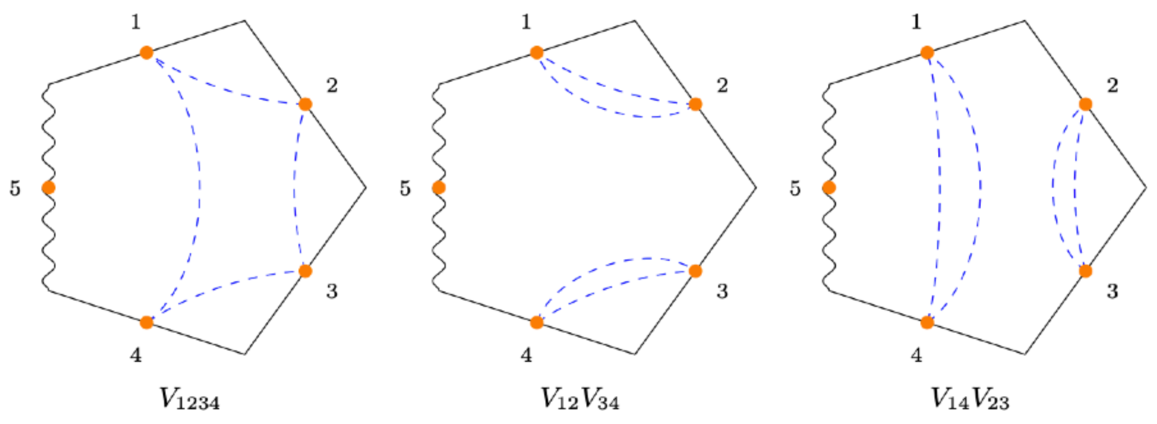
- kinematic associahedron (bi-adjoint $\phi^3$ tree) & worldsheet associahedron [Arkani-Hamed, Bai, SH, Yan, 17']
- cosmological polytopes +"kinematic flow" DE for tree-level wave function of universe [Arkani-Hamed et al 17' , 23',…]
- surfacehedra + binary geometries for surfaces… => "strings & particles without worldsheet"[w. Arkani-Hamed et al 20-]
- more applications of tropical geometry e.g. tropical Grassmannian for "symbology", Feynman integrals, etc…

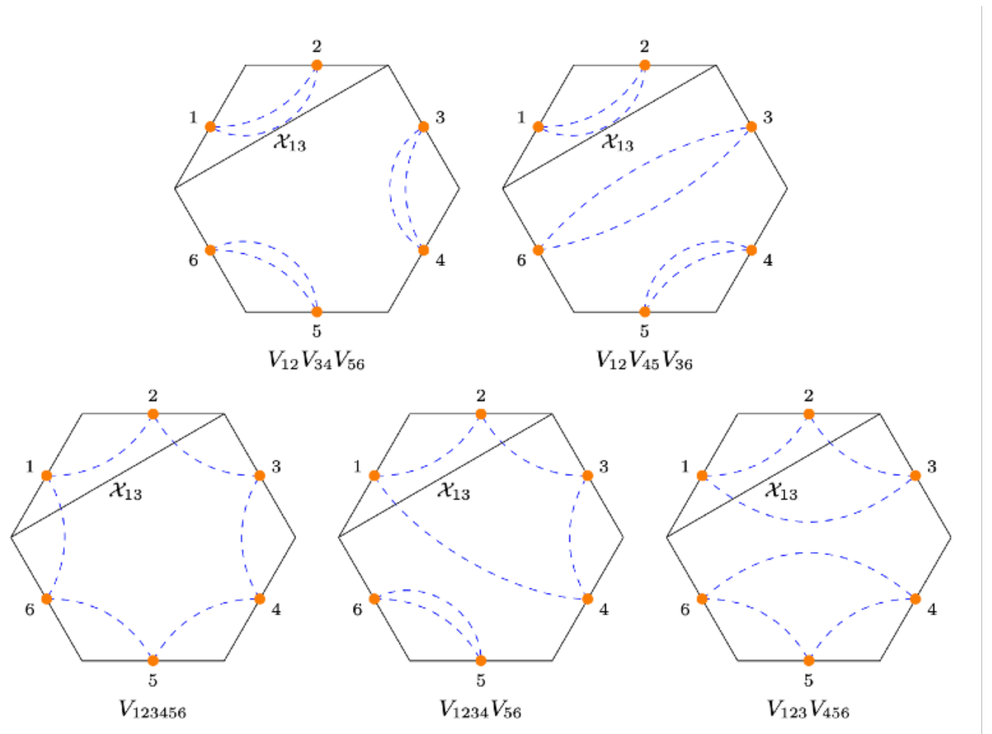
# Holographic correlators to all n [w. 曹趣, 唐一朝, 2312.15484 (PRL), + 李想, 2406.08538]

- Supergluon amplitudes in AdS_5 x S^3 (tree-level): rich data for CFT_4 & “scattering in AdS”; known up to n=6 based on factorizations (OPE) + flat-space limit [Alday, Goncalves, Nocchi, Zhou 2023]
- We find a recursive algorithm for supergluon & spinning amps to all n (“AdS constructibility”)



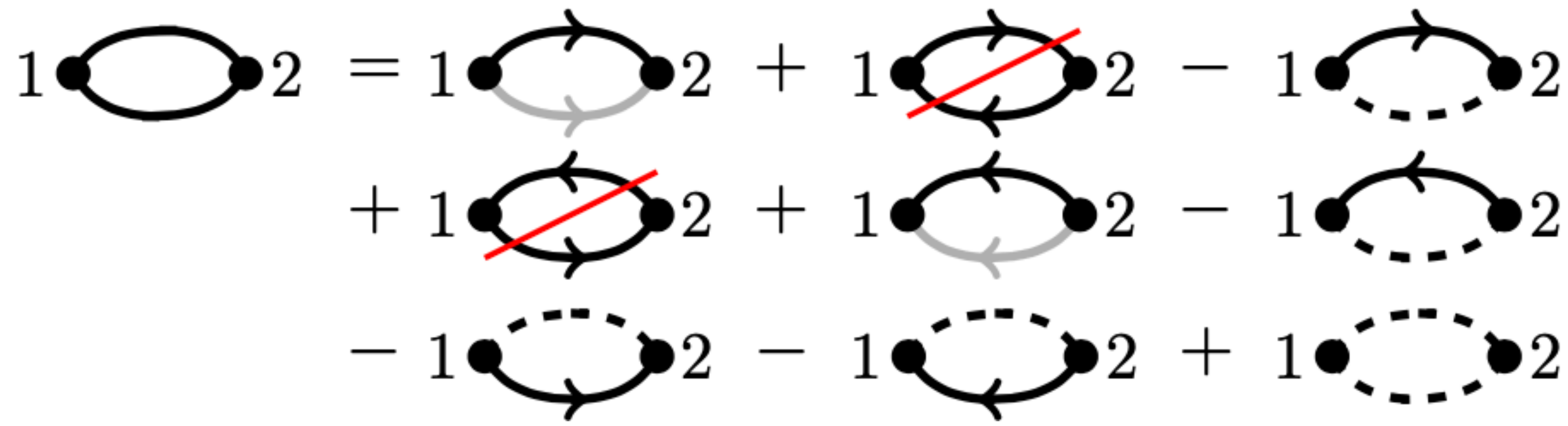



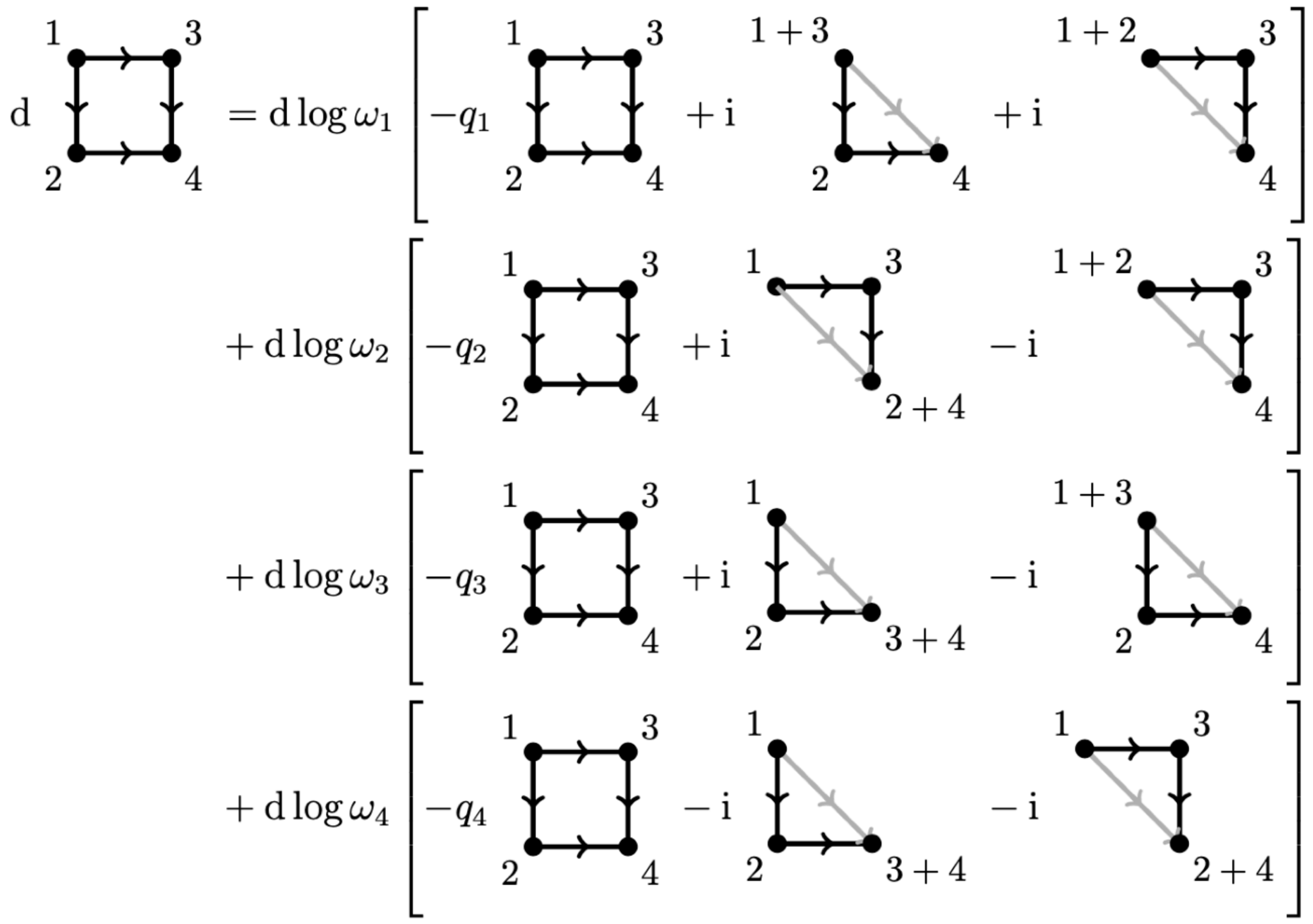




- Explicit, compact results up to n=8 (spinning for n=7), and the simplest R-symmetry case to all n
- New structures: general poles (truncation of descendents), nice Feynman rules, collinear/soft etc.
- They can be viewed as AdS generalizations of “scalar-scaffolded gluons” in flat-space!

# Cosmo. correlators: diff. eqs & recursion [w. 姜旭航, 刘家昊, 杨清霖, 张耀奇, 2407.17715,…]

- wave function coefficients & correlators for (conformal) scalars in FRW universe, e.g. q=0 for de Sitter
- Nested time integrals => naturally decompose into building blocks e.g. family trees, analytically solved in terms of gen. hypergeometric series [Fan, Zhong-zhi, 2024]; in general “loop integrands” —> all directed graphs



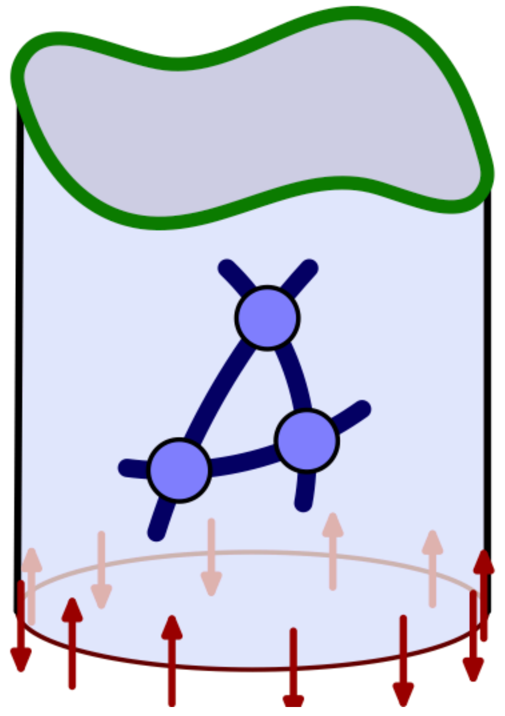


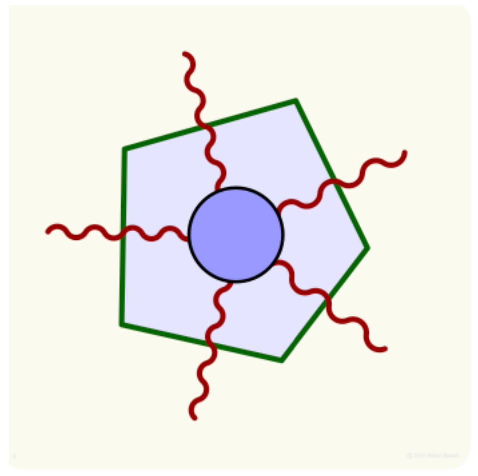

$$
\mathrm{d}\begin{pmatrix} \psi_{\text{2-chain}} \\ -\mathbf{B}_1/q_1 \\ -\mathbf{B}_2/q_2 \\ \mathbf{C}/(q_1q_2) \end{pmatrix} = \begin{pmatrix} -q_1\ell_1 - q_2\ell_2 & q_1\ell_1 - q_1\ell_3 & q_2\ell_2 - q_2\ell_4 & 0 \\ 0 & -q_1\ell_3 - q_2\ell_2 & 0 & q_2\ell_2 - q_2\ell_5 \\ 0 & 0 & -q_1\ell_1 - q_2\ell_4 & q_1\ell_1 - q_1\ell_5 \\ 0 & 0 & 0 & -(q_1+q_2)\ell_5 \end{pmatrix} \begin{pmatrix} \psi_{\text{2-chain}} \\ -\mathbf{B}_1/q_1 \\ -\mathbf{B}_2/q_2 \\ \mathbf{C}/(q_1q_2) \end{pmatrix}
$$

- Simplest DE for cosmo amps of any directed graph: contracting edge one at a time -> recursion relations
- For tree amps: combined to give canonical DE—> “kinematic flow”of Nima et al; same CDE for loops
- byproduct: a compact, recursive formula for de Sitter (q=0) amps (polylogs & symbol structures)

# The simplest QFT

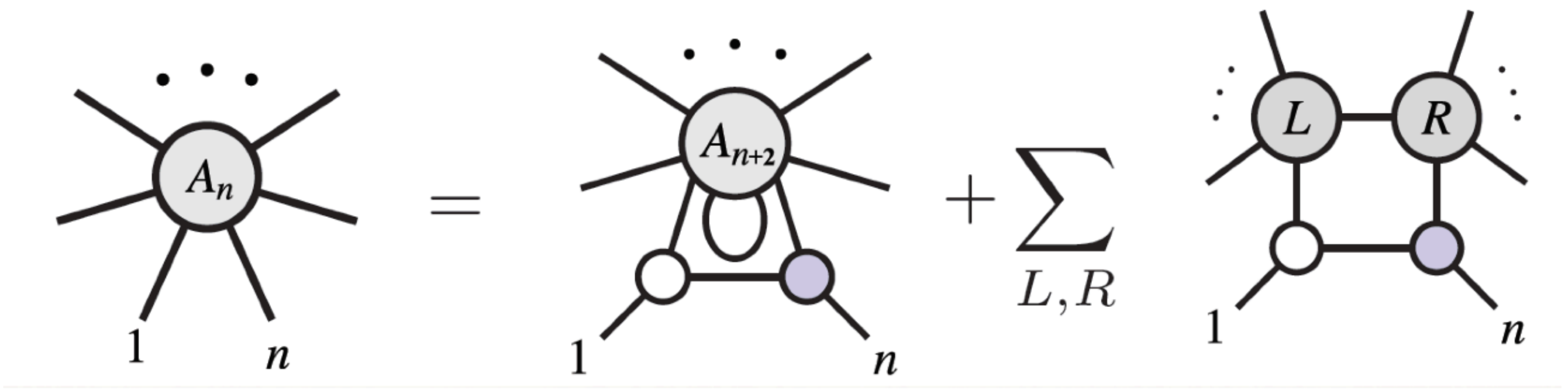
Harmonic oscillator of 21st century: hidden simplicity & structure in $\mathcal{N} = 4$ SYM (planar limit)

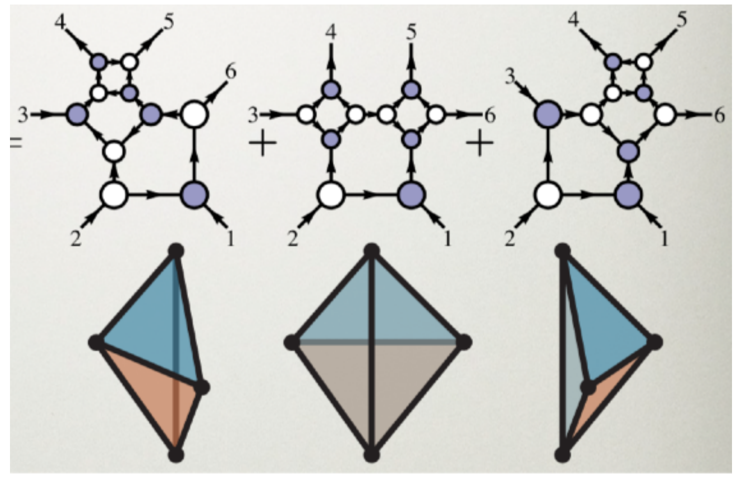




Integrability (planar limit): strong coupling via AdS/CFT, Wilson loops & OPE
Yangian symmetry … Ising model of gauge theories!

All-loop integrands <-> positive Grassmannian + amplituhedron [Arkani-Hamed, Trnka]



(Integrated) amplitudes + Feynman Integrals: extremely rich laboratory for perturbative QFT!
iterated integrals (polylogs & beyond), symbology, cluster algebras, differential eqs, bootstrap + Qbar,…

# All-loop geometry for (4pt) correlator! [w. Y. Huang, C. Kuo 2405.20292]

**canonical form** ⟷ **physical observables**

Correlation function:

half BPS operator k=2

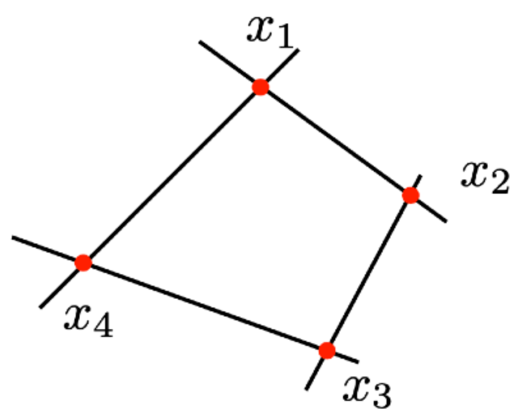
$\mathcal{O}_k(x,y)$

**4-pt (loop) correlation function**

$$\mathcal{G}^{(\ell)}_{2222} = (2x^2_{12}x^2_{13}x^2_{14}x^2_{23}x^2_{24}x^2_{34})\underline{R_{1234}} \times \mathcal{H}^{(\ell)}_{2222}(x_i), \quad \ell \geq 1$$

factor out $y$-dependent

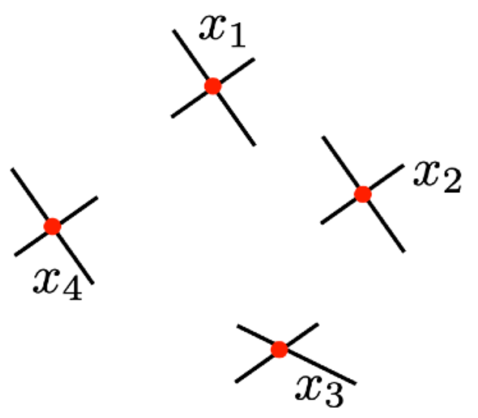
**Amplitudes**




$p_i^2 = 0$

ordering (cyclic)

**Correlators**




$x^2_{i,j} \neq 0$

permutation

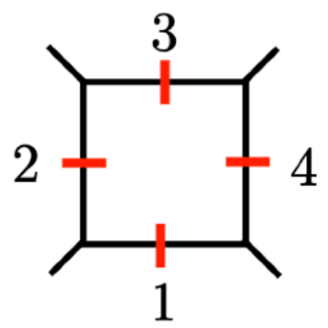
**Conjecture**

$\mathcal{G}^{(\ell)}_{2222}$ for $\ell \geq 1$ related to the canonical form defined in Correlahedron.

n=4 L-loop geom:

| | |
|---|---|
| **Kinematic** | $Y \in \mathrm{Gr}(4,8),\ \mathrm{X}_i \in \mathrm{Gr}(2,8)$ <br> $\langle Y \mathrm{X}_i \mathrm{X}_j \rangle > 0$ for $i,j = 1,2,3,4.$ |
| **Loop/ AB space** | $\dfrac{\langle Y(\mathrm{AB})_a \mathrm{X}_i\rangle}{\langle Y(\mathrm{AB})_a \mathrm{X}_1\rangle} > 0, \quad \dfrac{\langle Y(\mathrm{AB})_a(\mathrm{AB})_b\rangle}{\langle Y(\mathrm{AB})_a \mathrm{X}_1\rangle\langle Y(\mathrm{AB})_b \mathrm{X}_1\rangle} > 0$ |

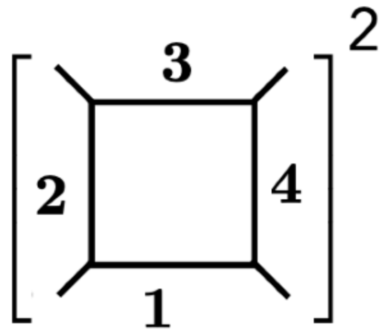
$\Delta^2 > 0:$ 4-mass box integrand



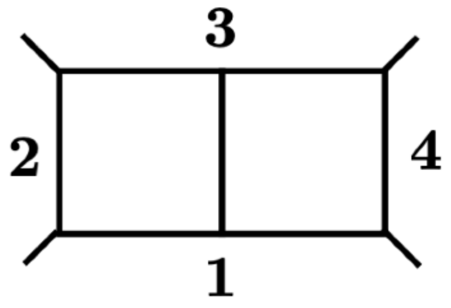

$$\frac{\pm\Delta}{\langle (AB)X_1\rangle\langle (AB)X_2\rangle\langle (AB)X_3\rangle\langle (AB)X_4\rangle} d\mu_{AB}$$

$$\Delta \equiv \langle X_1X_3\rangle\langle X_2X_4\rangle\sqrt{(1-v-w)^2-4vw}$$

$$\Delta^2 > 0: \quad I^{\ell=2}_{\pm} = \left(\frac{\Delta^2}{2} g^2_{1234} \pm \Delta\big(h_{12,34} + 5\,\text{perms}\big)\right)$$



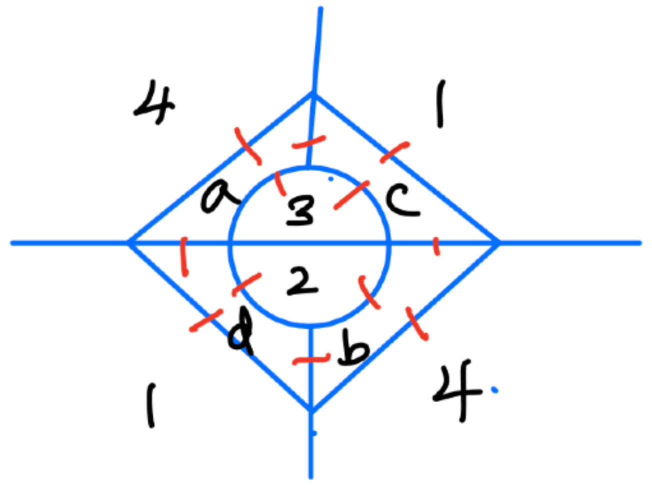

$g^2_{1234}$



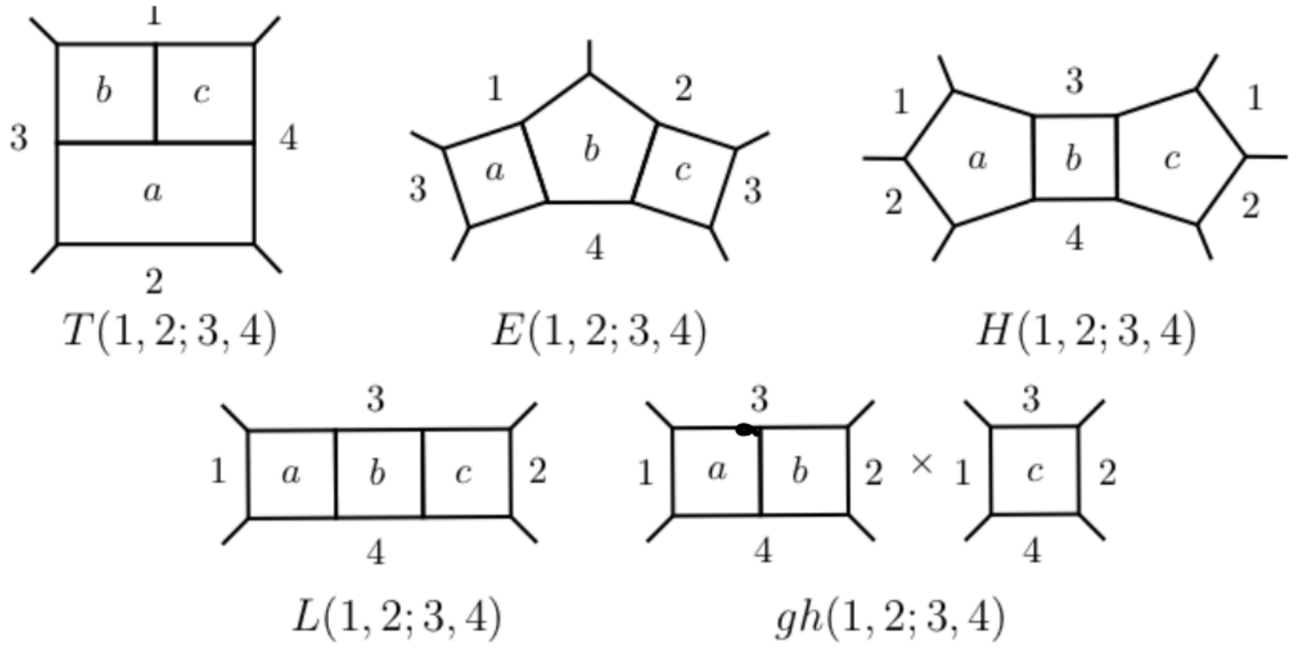

$h_{12,34}$

# Loops as fibration over trees [w. Y. Huang, C. Kuo 2405.20292 + to appear]

- In general, non-renormalizable theorem (prefactor for all loops)follows from tree geometry
- Starting L=3: distinct loop forms for different tree/kinematic regions! (new leading singularities etc)
- Computed loop forms for L=3,4 (sum over all 6 chambers/different LS)! 4-loop elliptic cut?

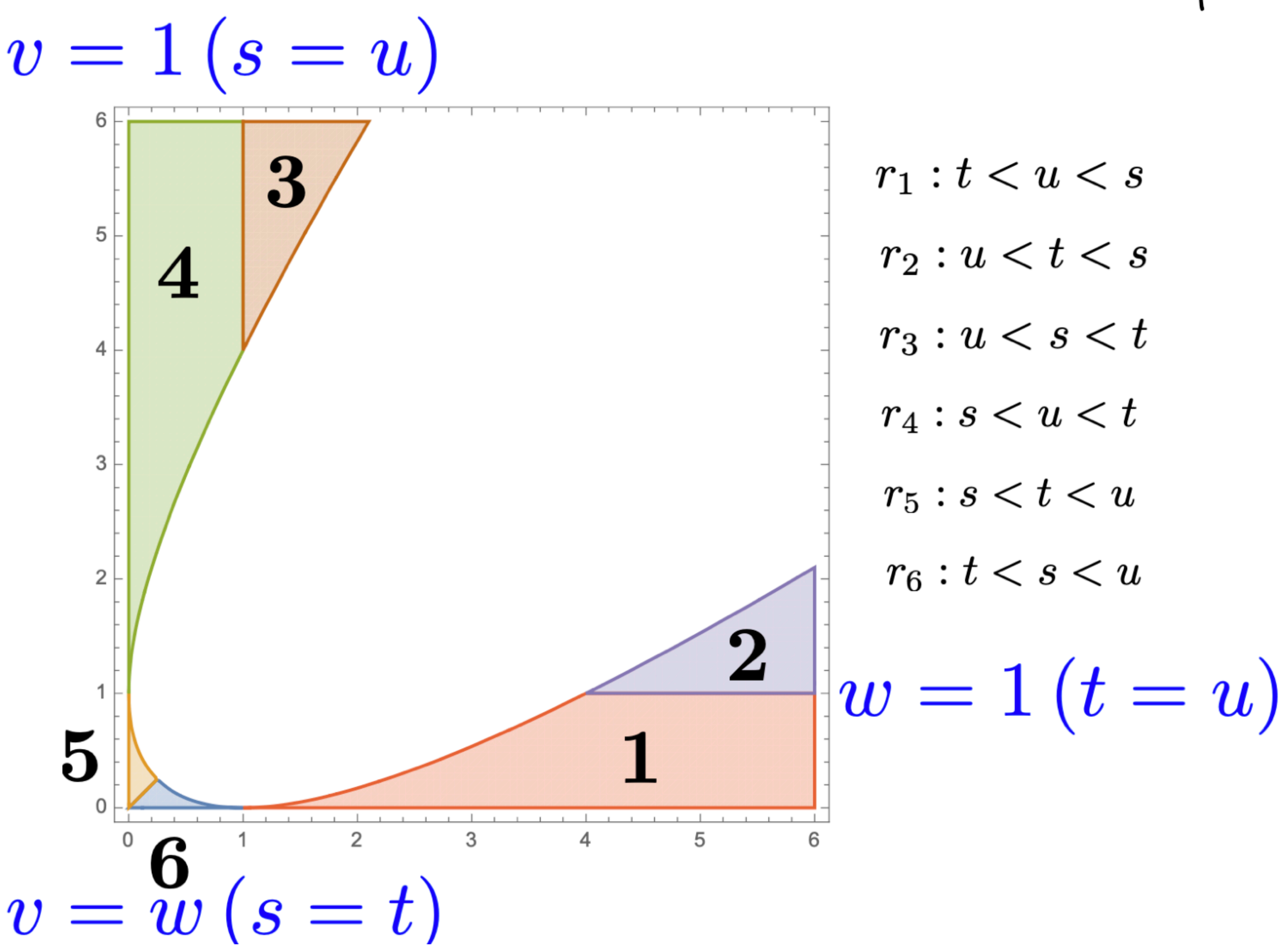


$\ell = 3$:




$$\Omega^{(3)\pm}_{r_i} = \Delta^2 A_{\sigma_3} \pm \Delta\big(B - \sigma_1(C_{\sigma_2} + C_{\sigma_3}) - \sigma_2 C_{\sigma_1}\big),$$

$$\begin{aligned} A_s :=& \big[H(1,4;2,3) - E'(1,4;2,3) + (1,4)\leftrightarrow(2,3)\big] + (3\leftrightarrow 4) \\ &+ gh(1,2;3,4) + gh(3,4;1,2), \\ B :=& T(1,2;3,4) + E(1,2;3,4) + 11\,\text{perms.} \\ &+ L(1,2;3,4) + (t+u)E'(1,2;3,4) + 5\,\text{perms.}, \\ C_s :=& 4(E'(1,2;3,4) + E'(3,4;1,2)) \end{aligned}$$




$r_1 : t < u < s$

$r_2 : u < t < s$

$r_3 : u < s < t$

$r_4 : s < u < t$

$r_5 : s < t < u$

$r_6 : t < s < u$

# Amplitude^2 —> Energy Correlators [w. 姜旭航、杨清霖、张耀奇, 2408]

- Correlation of energy flux: Infrared finite object measurable at experiments, lots of studies in QCD & N=4 SYM!
- Mostly EEC (N=2), recent works on N=3,4 @ leading order -> phase space integral of "EC integrand"
- In N-pt collinear limit, nice energy-integral of important "splitting function" = amplitude^2 (or form factor^2)

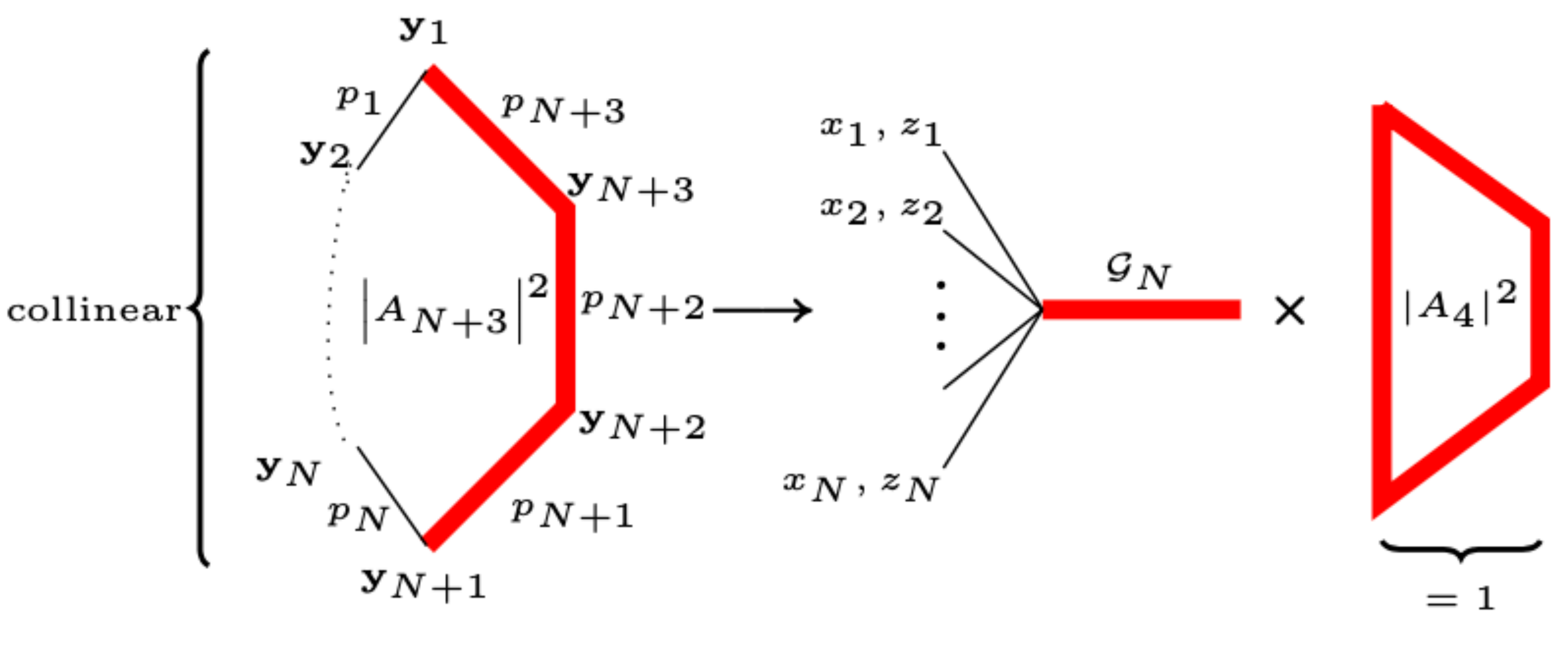



FIG. 1. The $1 \to N$ splitting function from collinear limit of squared amplitudes with $n = N{+}3$ legs.

$$\mathbf{EC}^{(N)}(\{z_i\}) = \frac{I_N(z_1,\cdots,z_N)}{|z_{1,2}\cdots z_{N-1,N}|^2} + \text{perm}(1,2,\cdots,N)\,,$$

$$I_N := \int_0^\infty \frac{d^N x}{\text{GL}(1)} x_{12\cdots N}^{-N} \mathcal{G}_N(x_1,\cdots,x_N; z_1,\cdots,z_N)$$

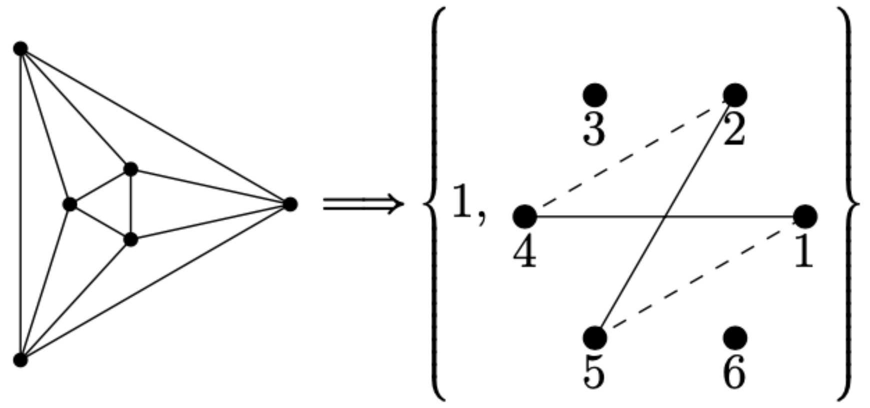
$$\mathcal{G}_N := \lim_{1\|2\cdots\|N} \frac{|A_n|^2}{|A_{n,\text{MHV}}|^2} = \lim_{1\|2\cdots\|N} \underbrace{\frac{1}{2}\sum_{k=0}^{n-4} \frac{A_{n,k} * A_{n,n-4-k}}{A_{n,0} * A_{n,n-4}}}_{r_n},$$

a beautiful (dual) conformal-invariant function, much simpler than amplitudes!

# EC integrand from geometry [w. 姜旭航、杨清霖、张耀奇, 2408]

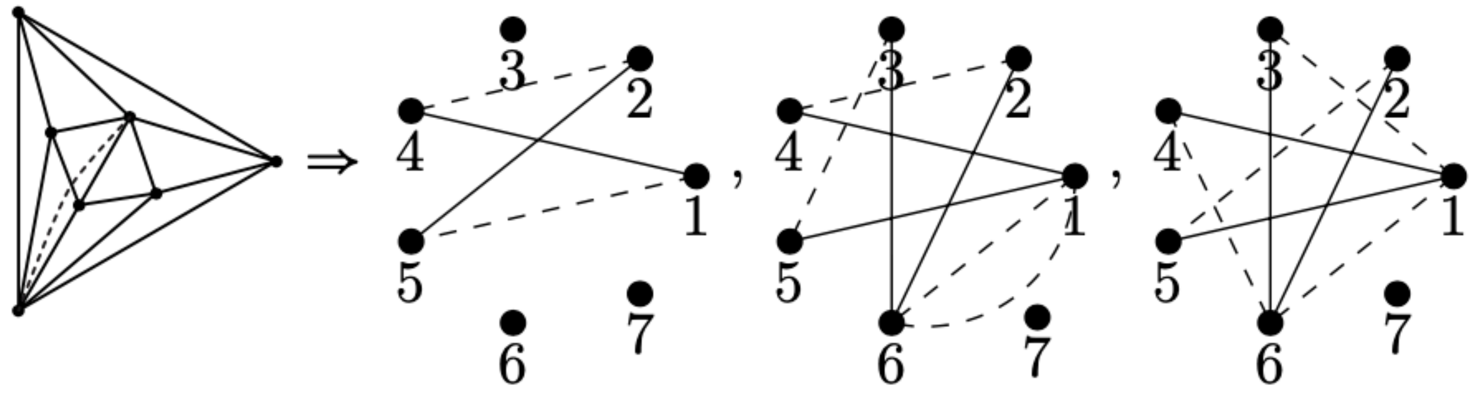
- An extremely simple geometry for $|A_{n=N+3}|^2$ ("amplituhedron squared") —> all-n result!
- Practically extract from "f graphs" => 4pt @ n-4 loops, 5pt @ n-5 loops … known up to n=14 (n>14 in progress)
- Purely graphical rules to obtain (tree) squared amps —> (collinear) EC integrands (all-N pole structure etc.)

| n | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|
| terms | 4 | 21 | 181 | 2085 | 29016 | 464640 | 9105364 | 209639703 |
| seeds | 2 | 3 | 22 | 134 | 1574 | 21423 | 377307 | 7811985 |
| $f$ graphs | 1 | 1 | 3 | 7 | 26 | 127 | 1060 | 10525 |



$$r_6 = 1 + \frac{(2,4)(1,5)}{(1,4)(2,5)} + \frac{(2,6)(3,5)}{(3,6)(2,5)} + \frac{(4,6)(1,3)}{(1,4)(3,6)},$$

$$\mathcal{G}_3 = 1 + \frac{x_1 x_3}{x_{1,2} x_{2,3}} + \frac{s_{1,2} x_{1,2,3}}{s_{1,2,3} x_{1,2}} + \frac{s_{2,3} x_{1,2,3}}{s_{1,2,3} x_{2,3}},$$



$$\frac{(2,4)(1,5)}{(1,4)(2,5)}, \frac{(1,6)^2(2,4)(3,5)}{(1,4)(1,5)(2,6)(3,6)}, \frac{(1,3)(1,6)(2,5)(4,6)}{(1,4)(1,5)(2,6)(3,6)},$$

Universal behavior: soft & multi-collinear limits!

$$\lim_{\mathbf{y}_i \to \mathbf{y}_{i-1}} r_n = 2\, r_{n-1}(1, \cdots, i-1, i+1, \cdots, n),$$

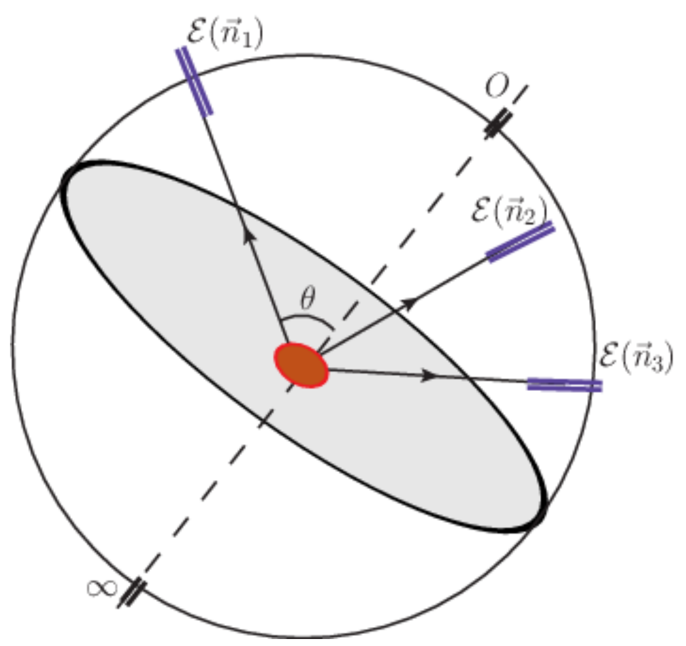
$$\Longrightarrow \quad \mathcal{G}_N(x_N \to 0) \to 2\mathcal{G}_{N-1}$$

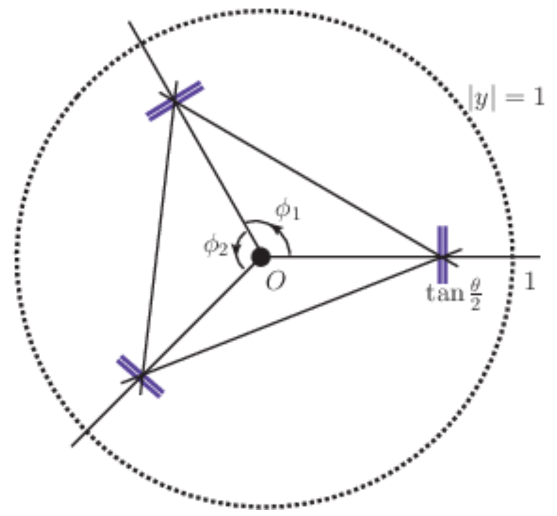
$$\lim_{z_1, \cdots, z_m \sim \epsilon} \mathcal{G}_N = 2\, \mathcal{G}_m(x_1, \cdots, x_m; z_1, \cdots, z_m) \times \mathcal{G}_{N-m+1}(x_{1\cdots m}, x_{m+1}, \cdots, x_n; z_m, z_{m+1}, \cdots, z_N) + \mathcal{O}(\epsilon).$$

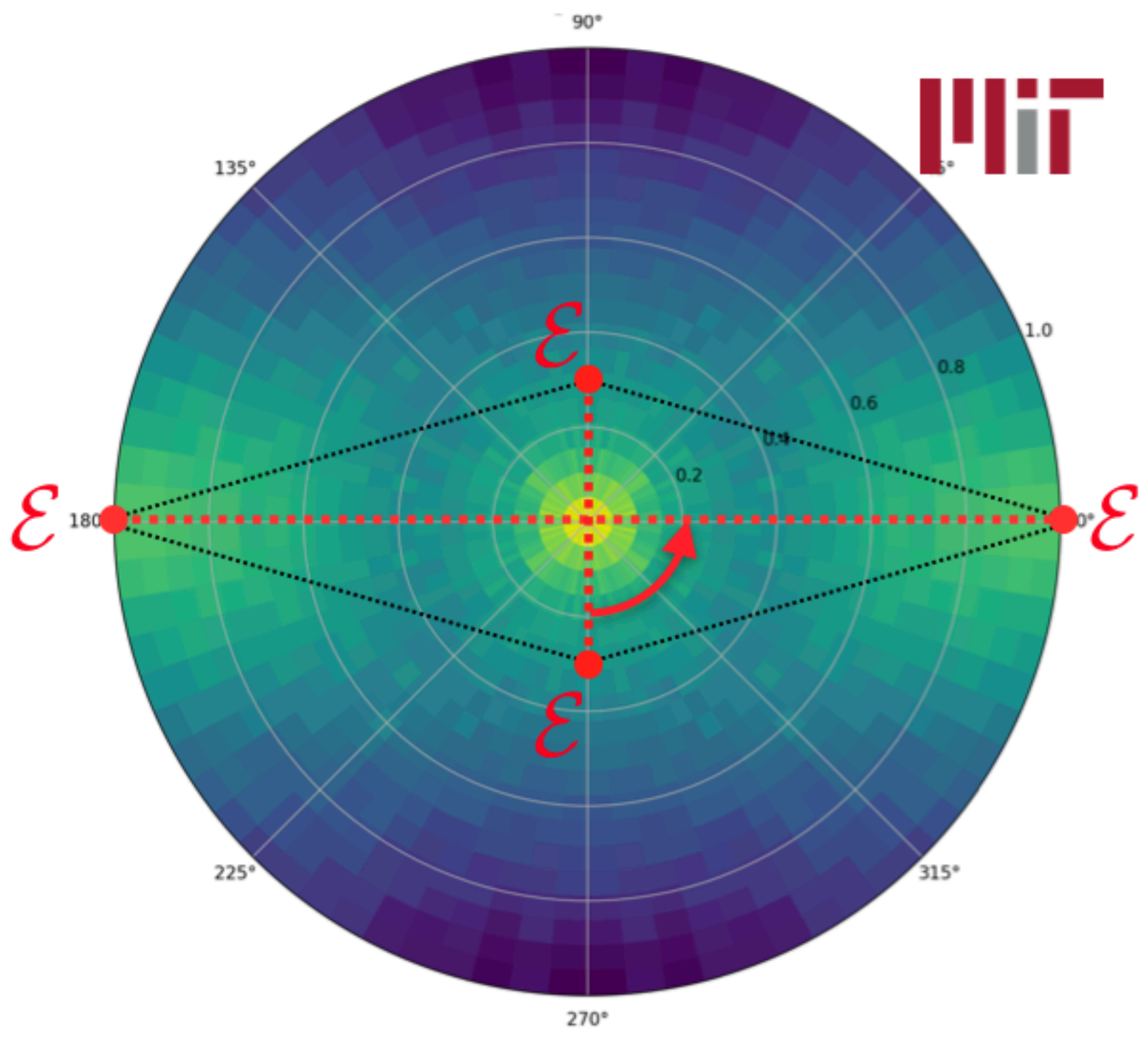
e.g. post-integration: => $\lim_{z_1, \cdots, z_{N-1} \sim \epsilon} \mathbf{EC}^{(N)} = \frac{4}{(N-1)|z_N|^2} \mathbf{EC}^{(N-1)} + \mathcal{O}(\epsilon).$

# Integrations: elliptic curves & beyond

- EC integrands => accurate numeric result for N=5,6,… (ideal playground for high-efficiency numeric tools)
- Analytic integrations arbitrarily complicated: new tools for direct integrations, IBP, symbology + bootstrap, etc.
- Bootstrap (more challenging than amps): e.g. N=3,4 "symbol letters" + prefactors by computing residues




$$\int \frac{\mathrm{d}^5 x_i}{GL(1)} \frac{s_{12}^{-a_4} s_{23}^{-a_5} s_{34}^{-a_6} s_{45}^{-a_7}}{x_{12345}^{a_1} s_{123}^{a_2} s_{345}^{a_3}}$$

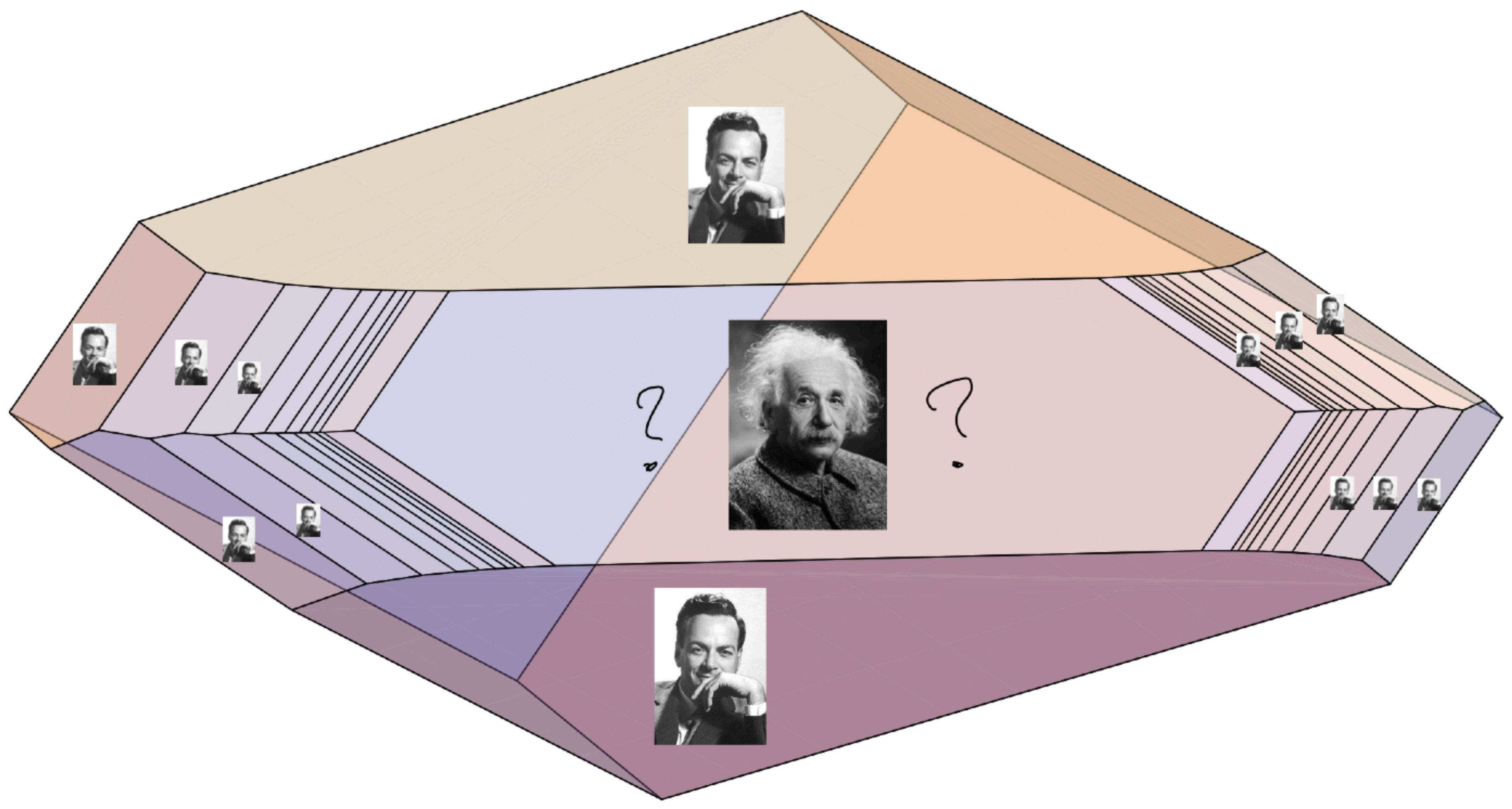
$$A_1 = \int \frac{\mathrm{d}^5 x_i}{GL(1)} \frac{1}{x_{12345} s_{123} s_{345}}, \; A_2 = \int \frac{\mathrm{d}^5 x_i}{GL(1)} \frac{x_1}{x_{12345}^2 s_{123} s_{345}},$$

$$A_3 = \int \frac{\mathrm{d}^5 x_i}{GL(1)} \frac{1}{x_{12345} s_{123}^2}, \; A_4 = \int \frac{\mathrm{d}^5 x_i}{GL(1)} \frac{1}{x_{12345} s_{345}^2}$$

$$A_1 = \int_0^\infty \frac{\mathrm{d}x_4}{\sqrt{P(x_4)}} \mathcal{I}_1^{(3)}(x_4)$$

- Rich structures: N=4 polylogs (up to cubic roots), N=5 elliptic MPLs (numerous curves) + polylogs up to 6th roots
- Even richer for N>5: hyper-elliptic curves, varieties up to dim N-4 (generally non-Calabi-Yau), what functions???
- Beyond collinear limit (|form factor|^2), higher orders (EEC from 4pt correlator), from SYM to QCD???

# Toy Models —> Real World




Combinatorial/geometries: e.g. SYM/ABJM, or Tr $\phi^3$ (simplest colored scalars)
Amps uniquely determined by long-distance sing. or "denominators" !

(see ABHY 18, Arkani-Hamed, Salvatori, Frost, Plamondon, Thomas: 2309.15913, 2311.09284)

More realistic theories: need "pole @ infinity" or numerators: Tr $\phi^3$ (projective inv.) vs. $\phi^p$ , pions, even gluons?

What are "zeros" of (tree) amplitudes? already highly non-trivial for Tr $\phi^3$: pattern of zeros (some $s_{i,j} = 0$) & surprising factorizations near them; hidden in Feynman diagrams, manifest by geometries!

The same zeros+ factorizations are also present for non-linear sigma model & Yang-Mills:

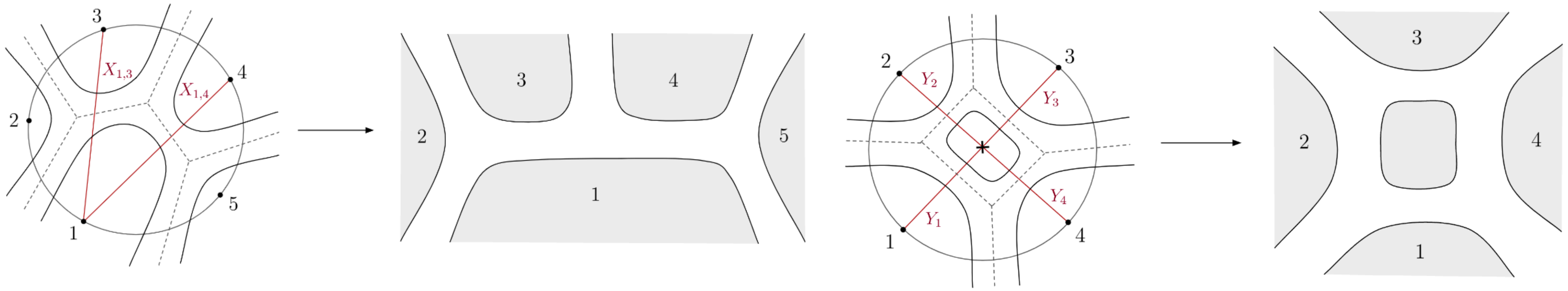
tree amps of Tr $\phi^3$, pions & gluons given by one and same function at different kinematic points !

***—> all-loop NLSM & (conjecturally) YM contained in all-loop stringy Tr $\phi^3$***

# Tr $\phi^3$ amplitudes [Arkani-Hamed, Bai, SH, Yan, '17; Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas, '23,...]

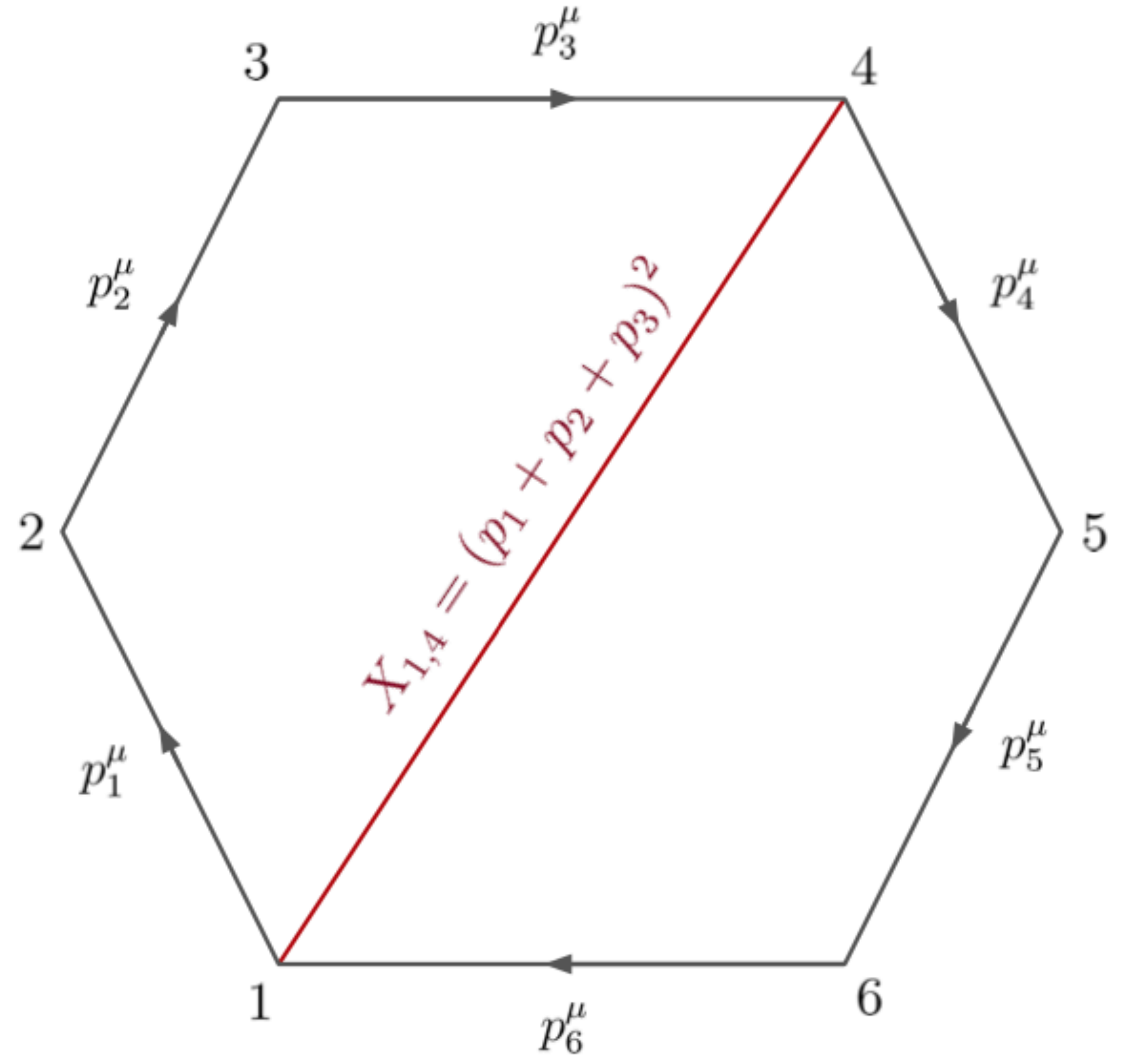
$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

$\phi$ : N by N matrix -> fat graphs, genus expansion (only planar graphs for $N \to \infty$)




planar variables: all poles of tree amps

$$X_{i,j} = (p_i + \cdots + p_{j-1})^2.$$




tree amp= sum over n-gon triangulations, e.g.

$$\mathcal{A}_4 = \frac{1}{X_{13}} + \frac{1}{X_{24}},$$

$$\mathcal{A}_5 = \frac{1}{X_{1,3}X_{1,4}} + \frac{1}{X_{2,4}X_{2,5}} + \frac{1}{X_{1,3}X_{3,5}} + \frac{1}{X_{1,4}X_{2,4}} + \frac{1}{X_{2,5}X_{3,5}}.$$

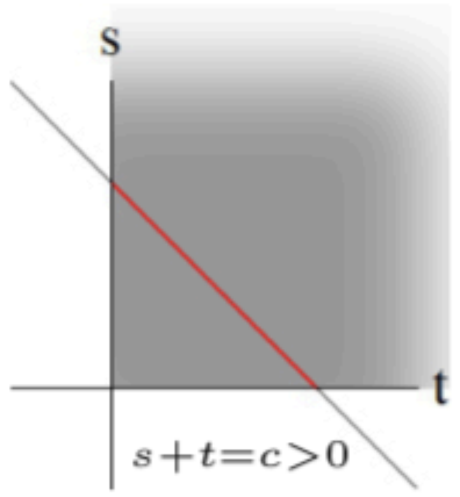
non-planar Mandelstam variables

$$c_{i,j} := -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1},$$

# Tree amplitude from associahedron [ABHY; Arkani-Hamed, SH, Salvatori, Thomas '19]

$\mathscr{A}_{n-3}: \{X_{i,j} \geq 0\} \cap$ (n-3)-dim subspace

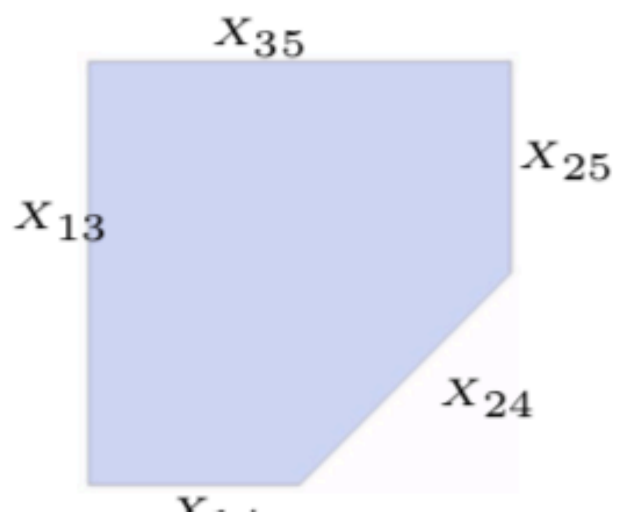
$c_{i,j} = \text{const.} > 0$ e.g. for $1 \leq i < j \leq n-1$

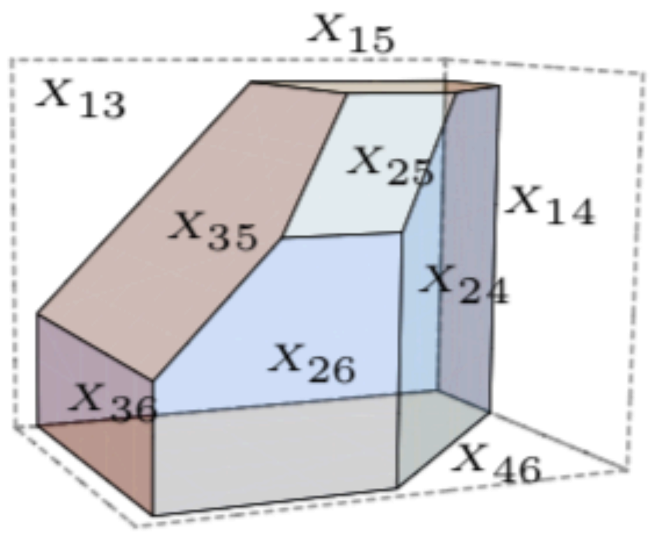



$e.g.\ \mathcal{A}_1 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$

$\mathcal{A}_2 = \{X_{13}, \cdots, X_{25} > 0\} \cap \{-s_{13} = c_{13}, -s_{14} = c_{14}, -s_{24} = c_{24}\}$

$$\mathscr{A}_2 : (n=5) \begin{cases} X_{1,3} > 0 \\ X_{1,4} > 0 \\ X_{2,4} > 0 \Leftrightarrow c_{1,3} - X_{1,3} + X_{1,4} > 0 \\ X_{2,5} > 0 \Leftrightarrow c_{1,3} + c_{1,4} - X_{1,3} > 0 \\ X_{3,5} > 0 \Leftrightarrow c_{1,4} + c_{2,4} - X_{1,4} > 0 \end{cases}$$




$$\Omega(\mathscr{A}_2) = d^2X \left( \frac{1}{X_{13}X_{14}} + \frac{1}{X_{13}X_{35}} + \frac{1}{X_{25}X_{35}} + \frac{1}{X_{25}X_{24}} + \frac{1}{X_{24}X_{14}} \right)$$

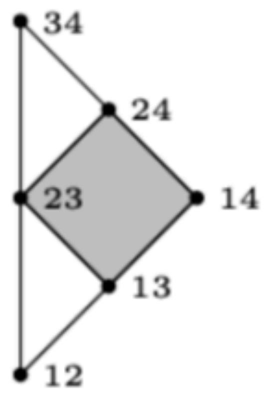
$$\Omega(\mathscr{A}_1) = \left(\frac{dX_{13}}{X_{13}} - \frac{dX_{24}}{X_{24}}\right)\Big|_{X_{13}+X_{24}=c_{13}} = dX_{13}\left(\frac{1}{X_{13}} + \frac{1}{c_{13} - X_{13}}\right)$$

geometric picture: FD expansion = a special triangulation, others → new formula & recursion for $\phi^3$ amps

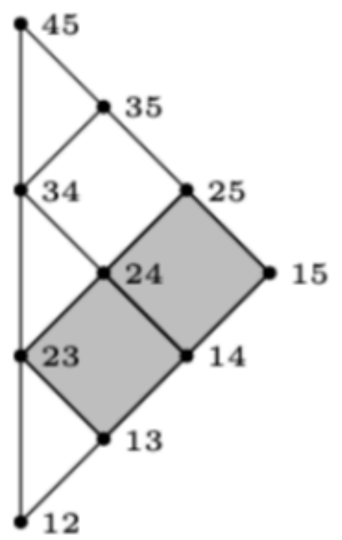
hidden symmetry of $\phi^3$ amps (invisible in FD's), analog of dual conformal symmetry, manifest by geometry!

# A surprise: zeros of Tr $\phi^3$ on the mesh [Arkani-Hamed, Cao, Dong, Figuereido, SH, 2312.16282]

$$n = 4 : c_{13} = 0 \implies \frac{1}{X_{13}} + \frac{1}{X_{24}} = \frac{c_{13}}{X_{13}X_{24}} = 0$$




$$n = 5 : c_{13} = c_{14} = 0\,, \text{ or } c_{14} = c_{24} = 0\,, \text{etc}\,.$$




the big cubic polynomial $N^{(3)} = 0$

Very difficult to see in Feynman diagrams:

$$\frac{1}{X_{13}X_{14}} + \frac{1}{X_{13}X_{35}} + \frac{1}{X_{25}X_{35}} + \frac{1}{X_{25}X_{24}} + \frac{1}{X_{24}X_{14}} = \frac{N^{(3)}(\{X\})}{X_{13}X_{24}X_{35}X_{14}X_{25}}$$

n-pt: highly non-trivial linear subspaces of the big numerator e.g.

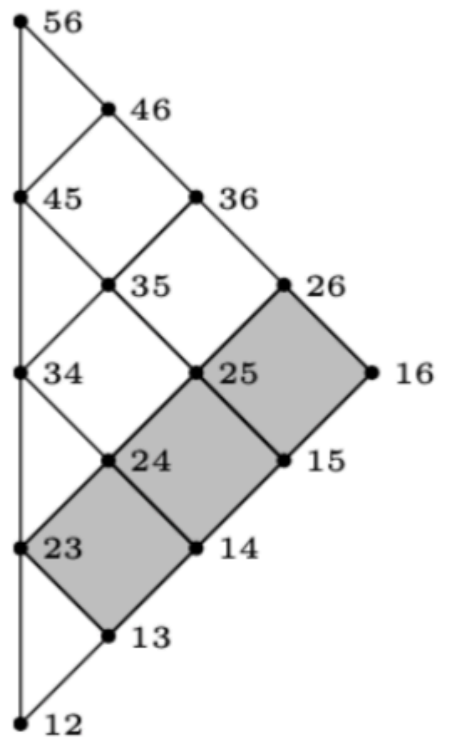
skinny rectangle ("soft limit")

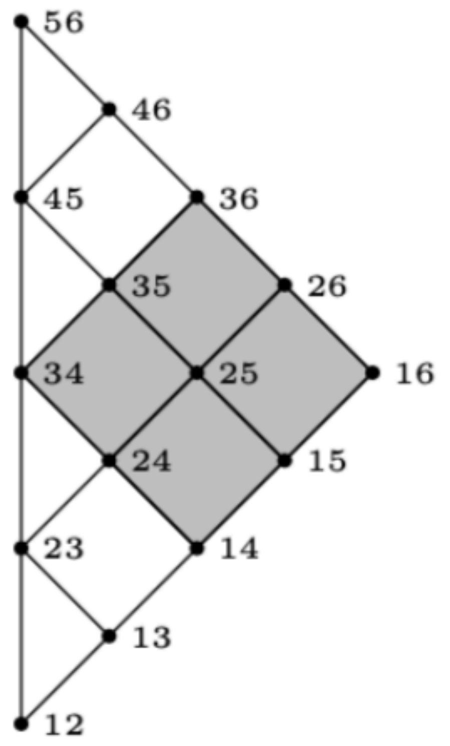
$$c_{13} = c_{14} = \cdots = c_{1,n-1} = 0$$

2 by 2 square

$$n = 6 : \text{also } c_{14} = c_{15} = c_{24} = c_{25} = 0\,;$$

generally any rectangle of the mesh

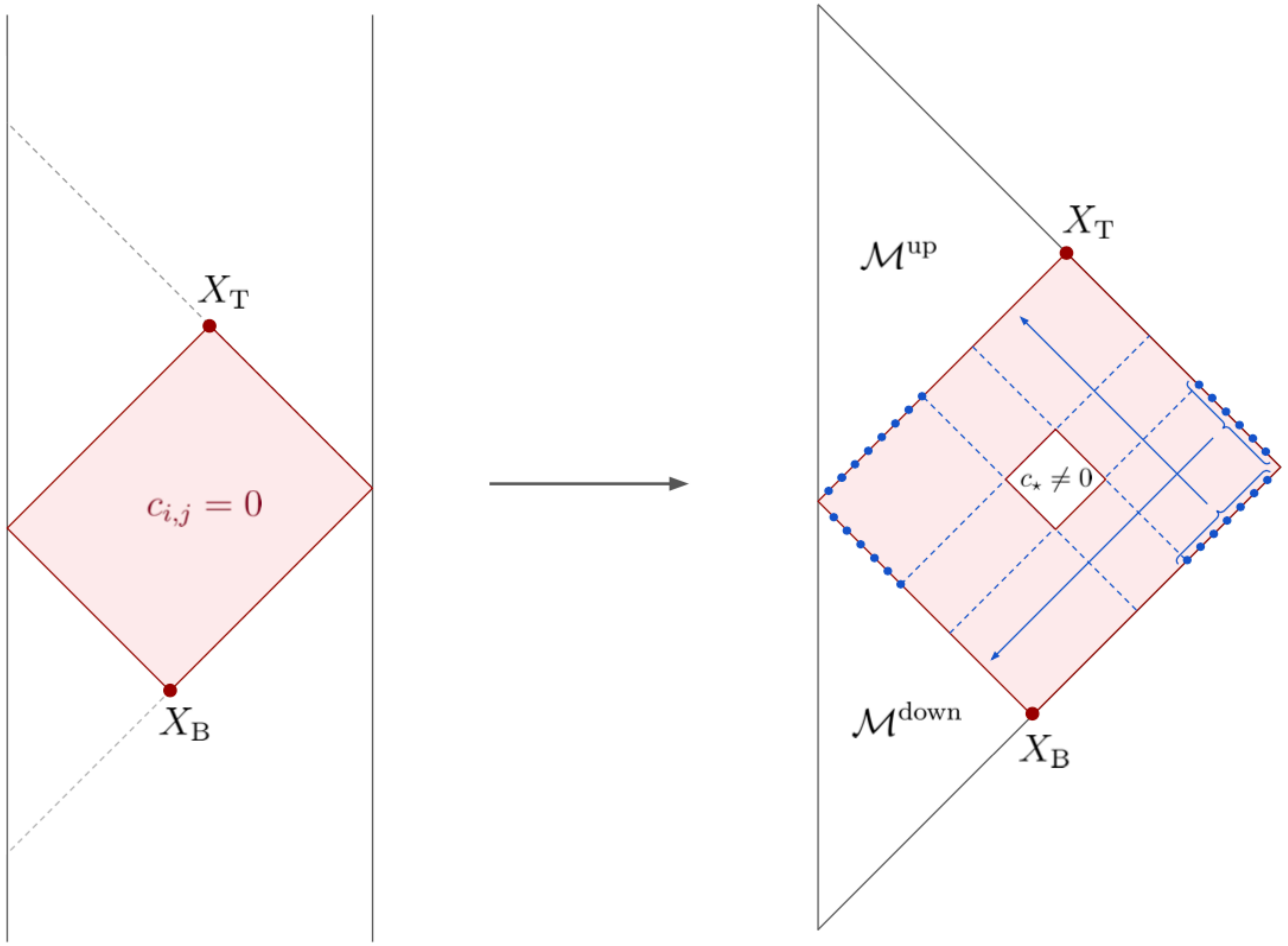


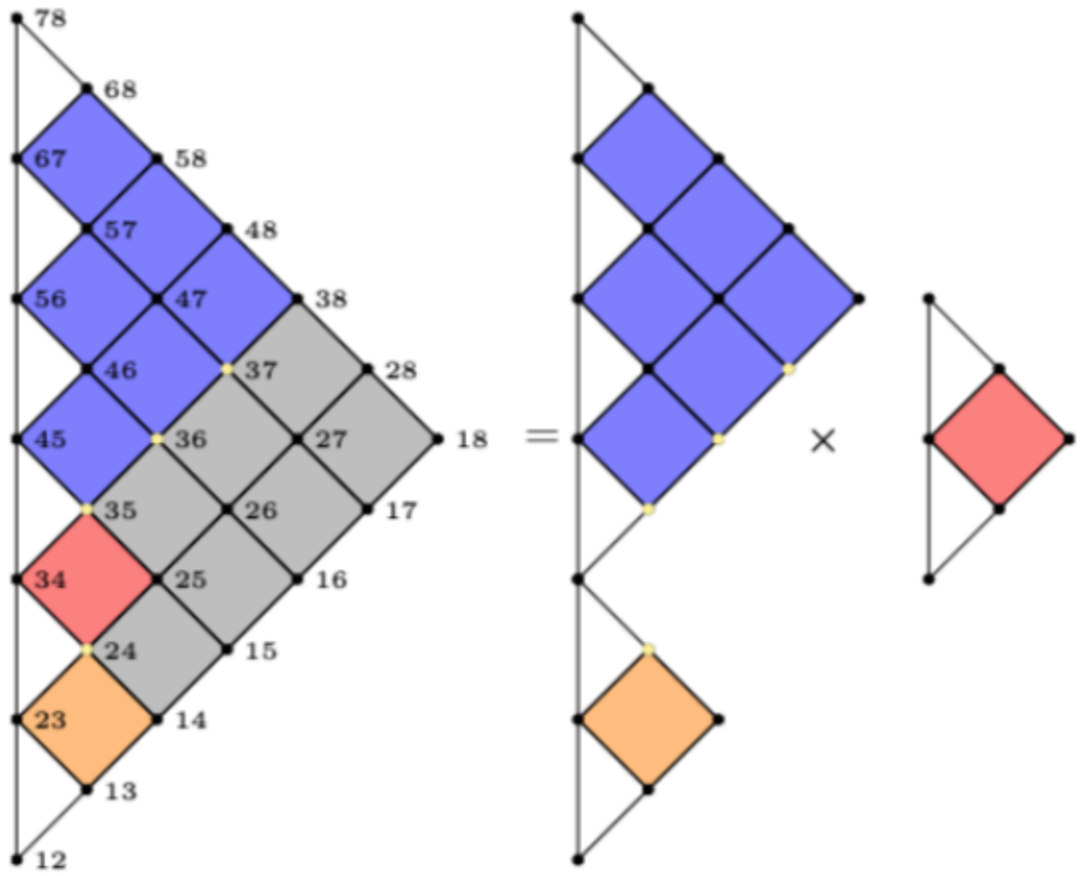
# General zeros & factorizations

$$\mathcal{M}_n(c_\star \neq 0) = \left(\frac{1}{X_{\mathrm{B}}} + \frac{1}{X_{\mathrm{T}}}\right) \times \mathcal{M}^{\mathrm{down}} \times \mathcal{M}^{\mathrm{up}}.$$

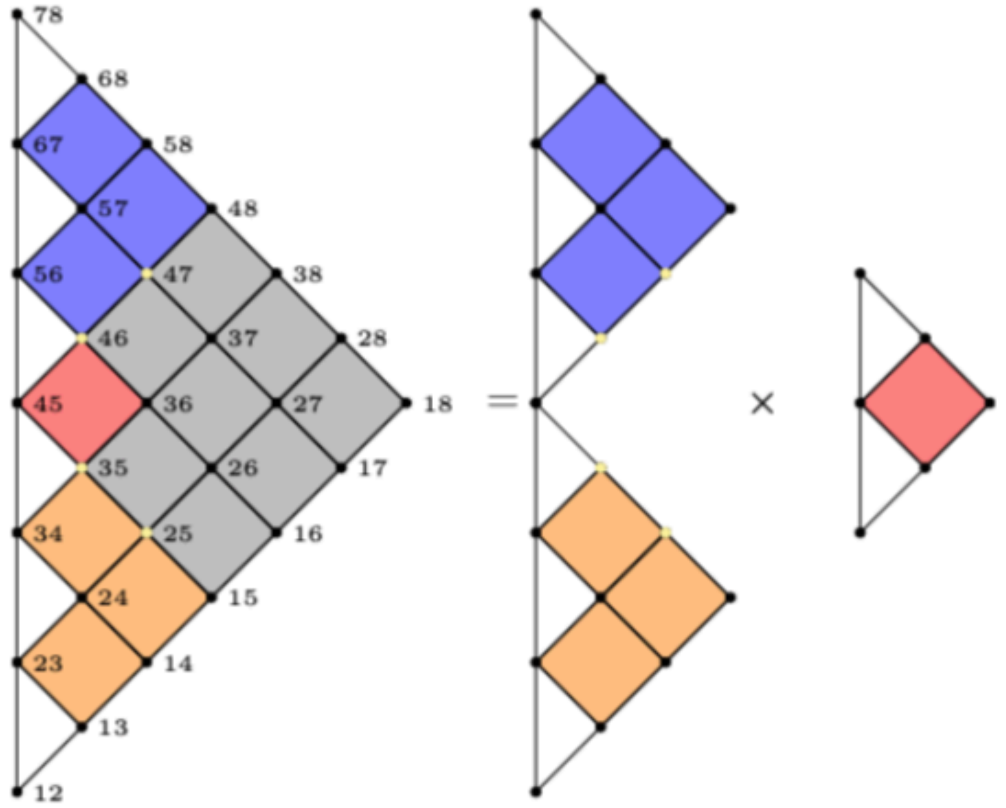
$$\tilde{X} = X_{\mathrm{T}} + X, \quad X = X_B + \tilde{X}.$$




8-> 6*4*4



8-> 5*4*5



shifted kinematics: in terms of momenta, these are currents (with an off-shell leg)

# Zeros of string amplitude [see also D'Adda, Sciuto, D'Auria, Gliozzi, 71']

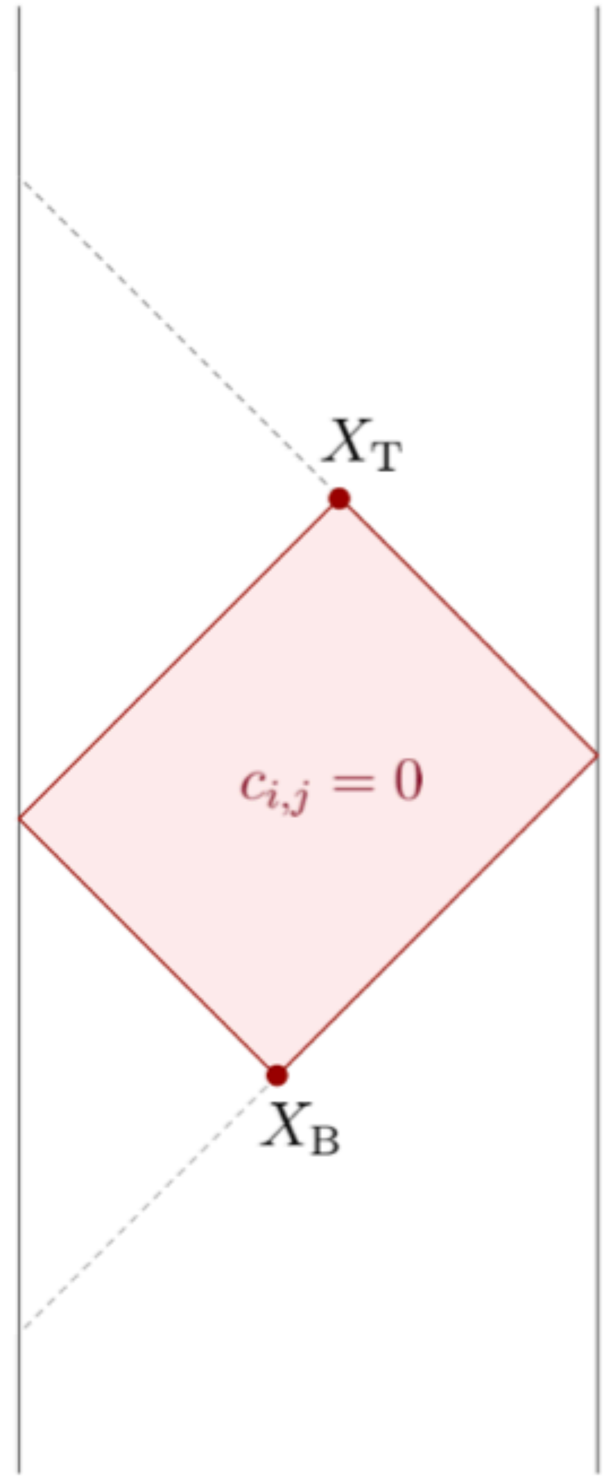
$$\mathcal{I}_4^{\mathrm{Tr}(\phi^3)} = \int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}} (1+y_{1,3})^{-\alpha' c_{13}} = \frac{\Gamma[\alpha' X_{1,3}]\Gamma[\alpha'(c_{1,3} - X_{1,3})]}{\Gamma[\alpha' c_{1,3}]}$$

any non-positive integer works: e.g.

by setting $\alpha' c_{1,3} = -n$,

$$\mathcal{I}_4^{\mathrm{Tr}(\phi^3)} \to \sum_{k=0}^{n} \underbrace{\int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}+k}}_{=0} = 0.$$

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \sum_{k_{a_1,b_1},\ldots,k_{a_N,b_N}=0}^{n_{a_1,b_1},\ldots,n_{a_N,b_N}} (\text{remaining integrals}) \times \underbrace{\int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}+k_{a_1,b_1}+\cdots+k_{a_N,b_N}}}_{=0} = 0$$



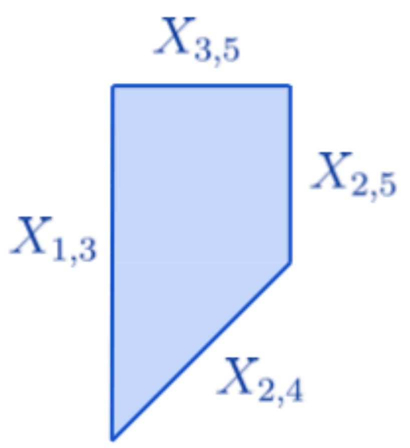

$$c_{i,j} = -n_{ij}\,, \quad 1 \leq i < a-1,\ a \leq j < n$$

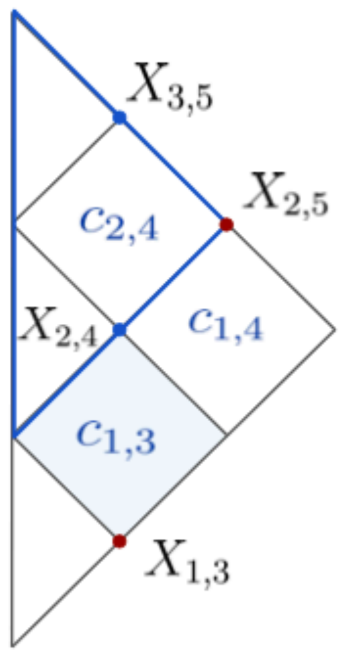
n(n-3)/2 infinite families of zeros

# Factorizations

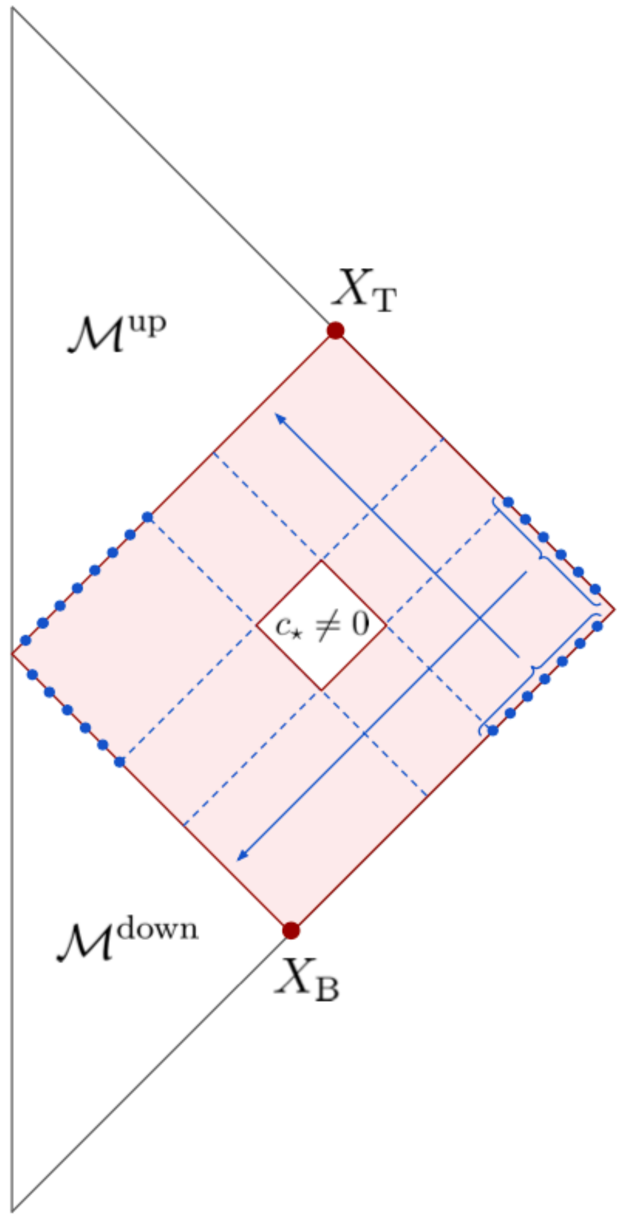
$$c_{1,3}=0\,,\quad \text{but}\quad c_{1,4}\neq 0\,,\qquad \mathcal{I}_5^{\mathrm{Tr}\,\phi^3}\to\int_0^\infty \frac{\mathrm{d}y_{1,3}}{y_{1,3}}y_{1,3}^{\alpha' X_{1,3}}\int_0^\infty \frac{\mathrm{d}y_{1,4}}{y_{1,4}}y_{1,4}^{\alpha' X_{1,4}}\,(1+y_{1,4})^{-\alpha' c_{2,4}}\,(1+y_{1,4}+y_{1,3}y_{1,4})^{-\alpha' c_{1,4}}$$

$$\begin{aligned}\mathcal{I}_5^{\mathrm{Tr}\,\phi^3}&\to\int_0^\infty \frac{\mathrm{d}\tilde{y}_{1,3}}{\tilde{y}_{1,3}}\tilde{y}_{1,3}^{\alpha' X_{1,3}}\,(1+\tilde{y}_{1,3})^{-\alpha' c_{1,4}}\int_0^\infty \frac{\mathrm{d}y_{1,4}}{y_{1,4}}y_{1,4}^{\alpha'(X_{1,4}-X_{1,3})}\,(1+y_{1,4})^{-\alpha'(c_{2,4}+c_{1,4}-X_{1,3})}\\&=\mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,3},\alpha'(c_{1,3}-X_{1,3}))\times\mathcal{I}_4^{\mathrm{up},\mathrm{Tr}\,\phi^3}(\alpha'(X_{1,4}-X_{1,3}),\alpha'(c_{2,4}+c_{1,4}-X_{1,4}))\\&=\mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,3},\alpha' X_{2,5})\times\mathcal{I}_4^{\mathrm{up},\mathrm{Tr}\,\phi^3}(\alpha' X_{2,4},\alpha' X_{3,5}).\end{aligned}$$




$$\left(\frac{1}{X_{1,3}}+\frac{1}{X_{2,5}}\right)\times\mathcal{A}_4(X_{2,4},X_{3,5})$$




$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3}\to\mathcal{I}_i^{\mathrm{down},\mathrm{Tr}\,\phi^3}\times\mathcal{I}_{n-i+2}^{\mathrm{up},\mathrm{Tr}\,\phi^3}\times\mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,i},\alpha'(c_{km}-X_{1,i})).$$

$$X_{l,i}\to X_{l,i}+X_{1,i}=X_{l,n},\text{for}\quad l=2,\dots,k.$$
$$X_{i-1,j}\to X_{i-1,j}-X_{i-1,n}=X_{1,j},\text{for}\quad j=m,\dots,n-1.$$

non-positive integers: sum of such factorizations

# Deformed to the real world [ACDFH 23]

$$\mathcal{I}^{\delta}_{2n} = \int_{\mathbb{R}^{2n-3}_{>0}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left( \frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha'\delta}, \qquad \mathcal{I}^{\delta}_{2n} = \mathcal{I}^{\mathrm{Tr}\,\phi^3}_{2n}[\alpha' X_{e,e} \to \alpha'(X_{e,e}+\delta), \alpha' X_{o,o} \to \alpha'(X_{o,o}-\delta)]\,.$$

key: all $c_{i,j} = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$ are preserved, thus all zero + fact. unchanged!

$\alpha'\delta = 0$

$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

$0 < \alpha'\delta < 1 \quad (\text{or } \mathbb{R}/\mathbb{Z})$

$\alpha' \to 0$

$$\mathcal{L}_{\mathrm{NLSM}} = \frac{1}{8\lambda^2}\mathrm{Tr}\left(\partial_\mu \mathrm{U}^\dagger \partial^\mu \mathrm{U}\right), \quad \text{with} \quad \mathrm{U} = (\mathbb{I} + \lambda\Phi)(\mathbb{I} - \lambda\Phi)^{-1}$$

$\alpha'\delta = \pm 1$

$$\mathcal{L}_{\mathrm{YMS}} = -\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\phi^I D_\mu\phi^I - \frac{g_{\mathrm{YM}}^2}{4}\sum_{I\neq J}[\phi^I,\phi^J]^2\right)$$

2n-pt Tr $\phi^3$ string amps => 2n-pion in NLSM or 2n-scalar (n-gluon) in YMS: same function @ different pts!

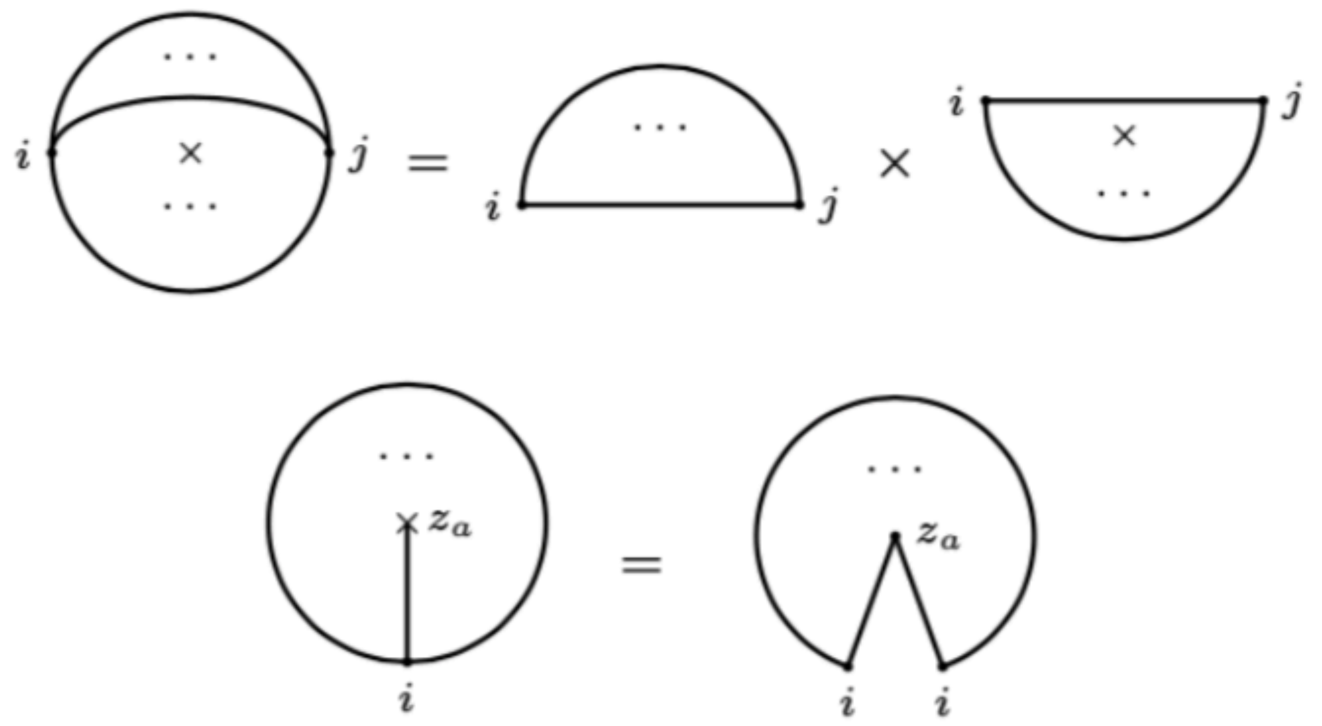
# All-loop NLSM contained in Tr $\phi^3$ [ACDFH]

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(e,e)} u_{e,e}^{\alpha'(X_{e,e}+\delta)} \times \prod_{(o,o)} u_{o,o}^{\alpha'(X_{o,o}-\delta)} \times \prod_{(o,e)} u_{o,e}^{\alpha' X_{o,e}}$$
$$\to \mathcal{A}_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

Field-theory directly take $\delta \to \infty$: $$A_{2n}^{\mathrm{NLSM}} = \lim_{\delta\to\infty} \delta^{n-1} A_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

Same shift works for planar integrand of NLSM: $X_{e,e} \to X_{e,e} + \delta$, $X_{o,o} \to X_{o,o} - \delta$ (inc. loop punctures)

$$\lim_{\delta\to\infty} \sum_{z_{a=1,\cdots,L}\ \text{even/odd}}^{2^L} (\delta)^{n+2L-2} A_{n,L}^{\delta} = A_{n,L}^{\mathrm{NLSM}}.$$



"Adler zero": soft limit -> scaleless integrals! Very practical, e.g. 3-loop n-pt NLSM integrand

# Scaffolded gluons: combinatorial origin of YM [ACDFH, 2024]

$\alpha'\delta = 1$ gives 2n-scalar stringy amplitude = 2n-scalar in bosonic string!

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,2n) = \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \left( \prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j} \right) \exp\left( \sum_{i \neq j} 2 \frac{\epsilon_i \cdot \epsilon_j}{z_{i,j}^2} - \frac{\sqrt{\alpha'} \epsilon_i \cdot p_j}{z_{i,j}} \right) \Bigg|_{\text{multi-linear in } \epsilon_i},$$

$$p_i \cdot \epsilon_j = 0, \quad \forall\ (i,j) \in (1,\ldots,2n),$$

special component $\epsilon_1 \cdot \epsilon_2 \ \ldots \ \epsilon_{2n-1} \cdot \epsilon_{2n}$

$$\epsilon_i \cdot \epsilon_j = \begin{cases} 1 & \text{if } (i,j) \in \{(1,2);(3,4);(5,6);\ldots;(2n-1,2n)\}, \\ 0 & \text{otherwise.} \end{cases}$$

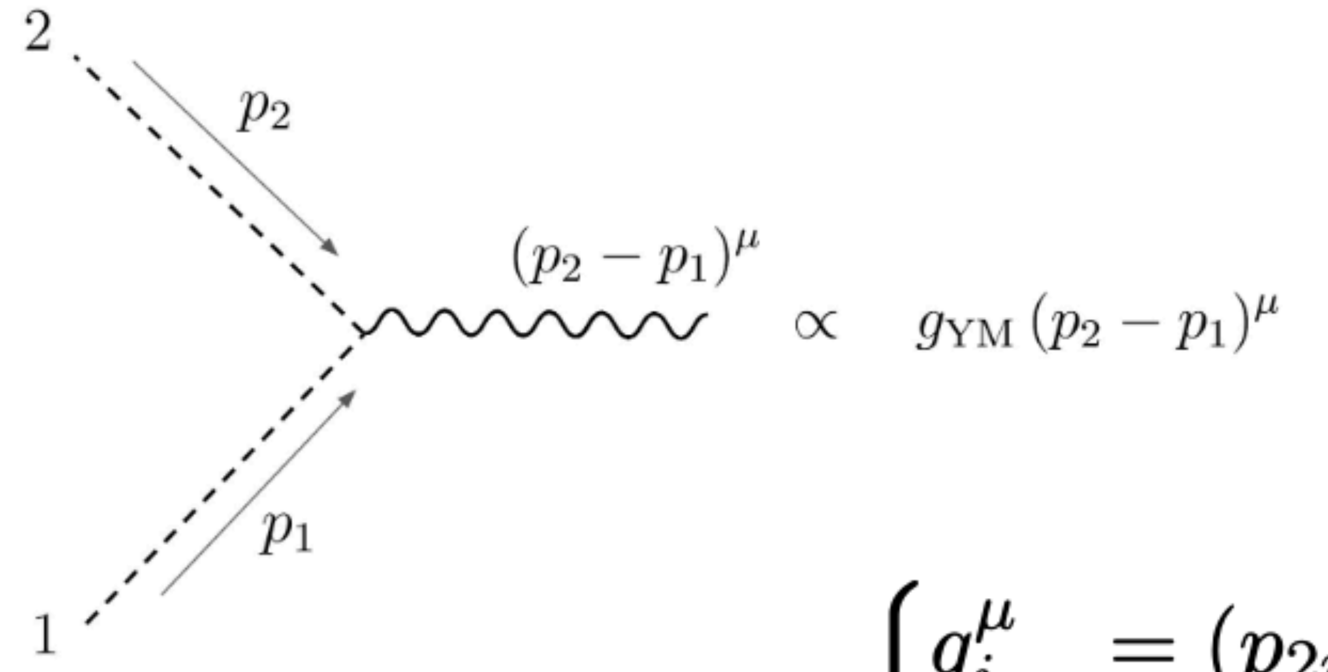
$$\mathcal{A}_{2n}(1,2,\ldots,2n) \xrightarrow{\text{special kinematics}} \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j} \frac{1}{z_{1,2}^2 z_{3,4}^2 z_{5,6}^2 \cdots z_{2n-1,2n}^2}$$

$$= \underbrace{\int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \frac{1}{z_{1,2} z_{2,3} z_{3,4} \cdots z_{2n,1}} \prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j}}_{\text{Stringy Tr}\,\phi^3} \frac{z_{2,3} z_{4,5} z_{6,7} \cdots z_{2n,1}}{z_{1,2} z_{3,4} z_{5,6} \cdots z_{2n-1,2n}} \qquad \left( \prod u_{e,e} / \prod u_{o,o} \right)$$

exactly corresponds to $\alpha'\delta = 1$: n pairs of scalars in bosonic string, $(1,2)(3,4)\cdots(2n-1,2n)$

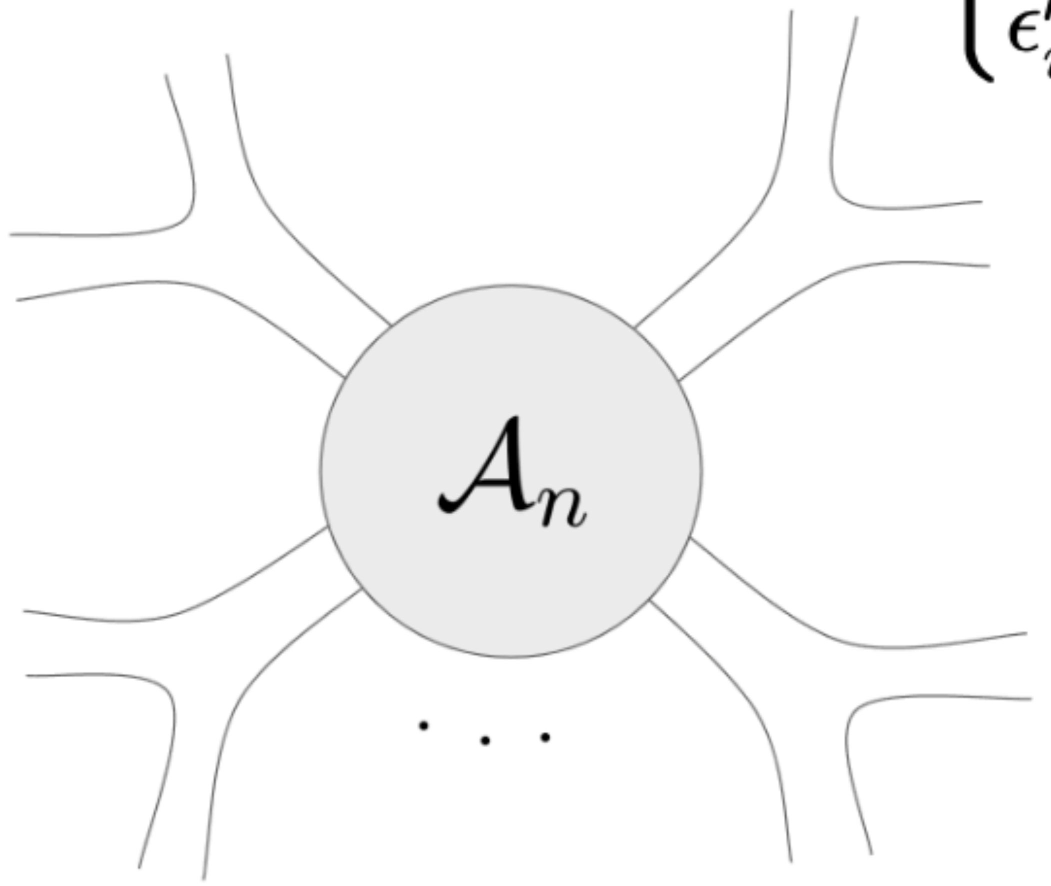
note $\alpha'\delta = -1$ => $(2,3)(4,5)\cdots(2n,1)$

by taking n ``scaffolding residues” -> n-gluon bosonic string amp in X (scalar) language!

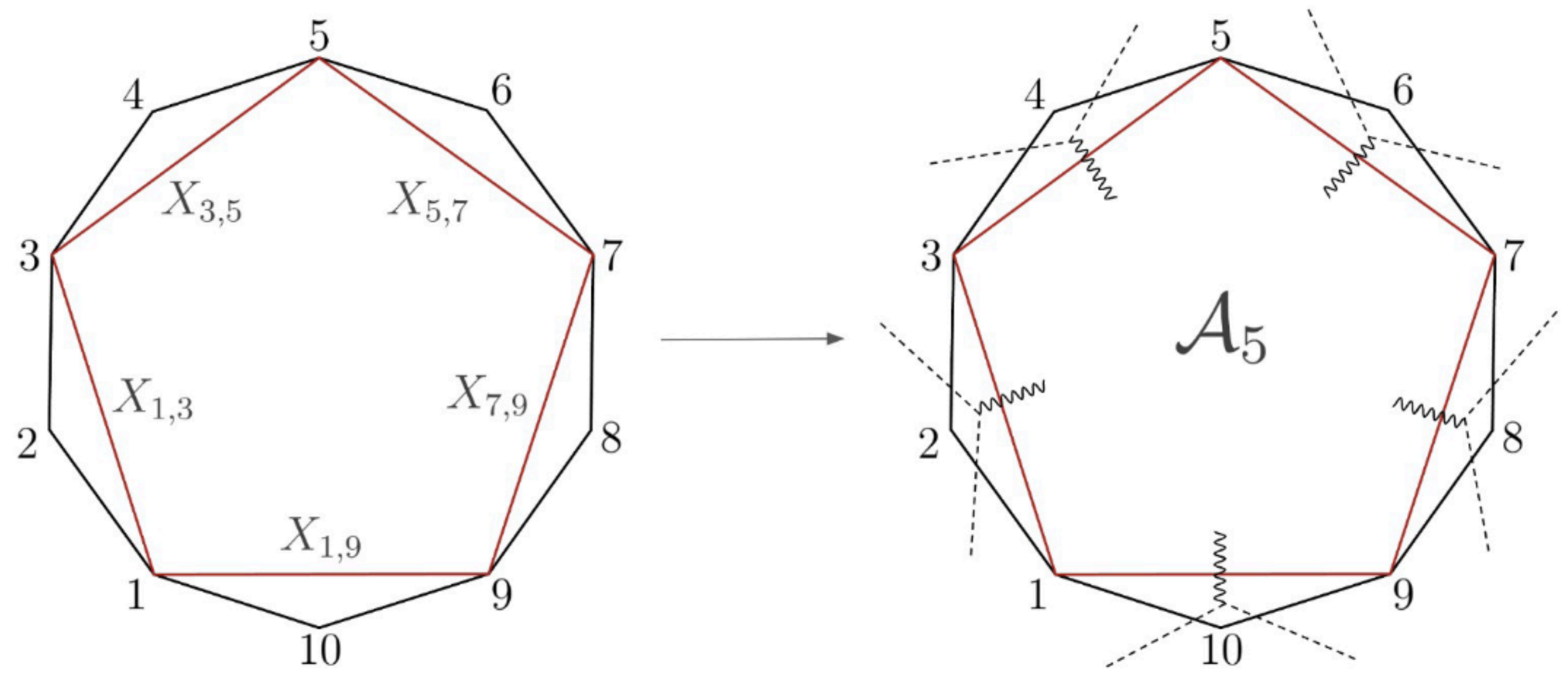



$$\mathcal{I}^{\delta}_{2n} = \int_{\mathbb{R}^{2n-3}_{>0}} \underbrace{\prod_{i=1}^{n} \frac{dy_{2i-1,2i+1}}{y^2_{2i-1,2i+1}} \prod_{I\in\mathcal{T}'} \frac{dy_I}{y_I^2} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}}}_{\Omega_{2n}},$$

$$\begin{cases} q_i^\mu &= (p_{2i}+p_{2i-1})^\mu \\ \epsilon_i^\mu &\propto (p_{2i}-p_{2i-1})^\mu \end{cases},$$




$$X_{1,3} = X_{3,5} = \cdots = X_{1,2n-1} = 0.$$




$$\mathcal{I}^{\rm gluon}_n = \int_{\mathbb{R}^{n-3}_{>0}} {\rm Res}_{y_{1,3}=0}\left({\rm Res}_{y_{3,5}=0}\left(\ldots\left({\rm Res}_{y_{1,2n-1}=0}\left(\Omega_{2n}\right)\right)\ldots\right)\right)\Big|_{X_{2i-1,2i+1}=0}$$

$$A_3^{\rm gluon} = \alpha'^2\,(c_{1,3}c_{1,5}+c_{1,3}c_{2,5}+c_{1,3}c_{3,5}+c_{1,4}c_{3,5}+c_{1,5}c_{3,5}+c_{1,5}c_{3,6}) \\ -\alpha'^3\,(X_{1,4}X_{2,5}X_{3,6})$$

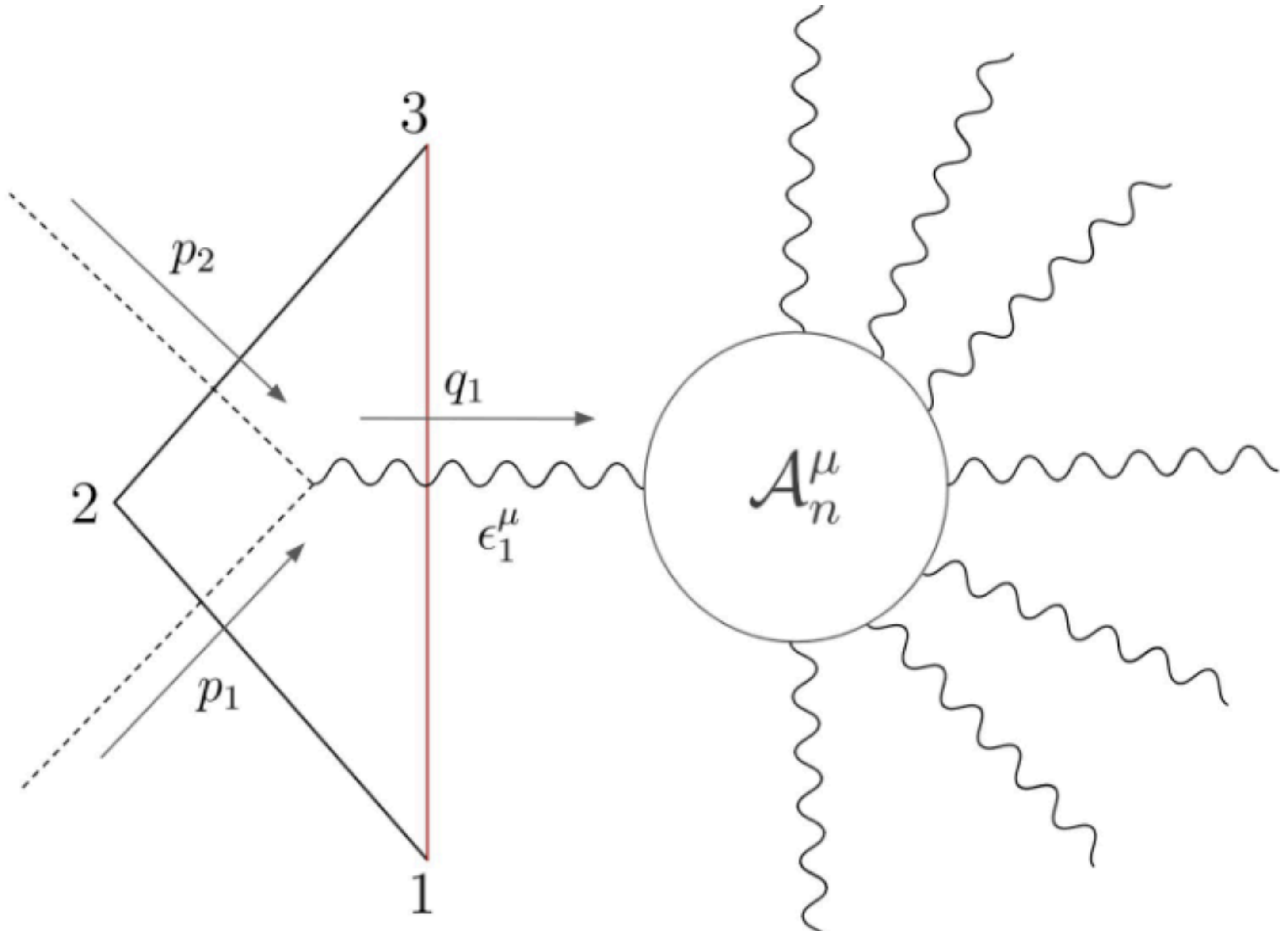
$$A_3^{\rm YM}(1,2,3) = \frac{1}{2}(\epsilon_1\cdot\epsilon_3\;\epsilon_2\cdot q_1 - \epsilon_1\cdot q_2\;\epsilon_2\cdot\epsilon_3 + \epsilon_1\cdot\epsilon_2\;q_2\cdot\epsilon_3).$$

$$A_3^{F^3}(1,2,3) = \epsilon_1\cdot q_3\;\epsilon_2\cdot q_1\;\epsilon_3\cdot q_2,$$

# Conjecture: all-loop YM in stringy Tr $\phi^3$ [ACDFH, 2024]

Generalize stringy tree amp (disk) to loops (higher-genus surfaces):

$$A_n^{\text{gluon}} = \int_0^\infty \prod_i \frac{\mathrm{d}y_i}{y_i^2} \,\text{Res}_{y_{s_1}=0}\left(\text{Res}_{y_{s_2}=0}\left(\dots\left(\text{Res}_{y_{s_n}=0}\Omega_{2n}\right)\dots\right)\right).$$

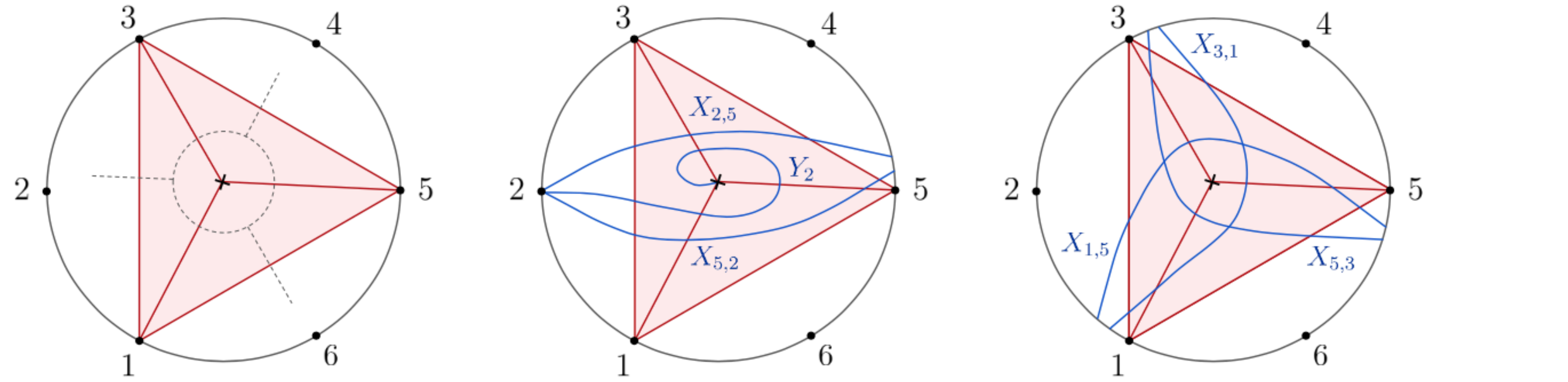



e.g. 1-loop w. self-intersecting curves & closed curve $\Delta$ (absent for scalars)

$$\mathcal{I}_{2n}^{\text{1-loop}}(1,2,\dots,2n) = \int_0^\infty \prod_i \frac{\mathrm{d}y_i}{y_i^2} \prod_C u_C^{\alpha' X_C} \times \prod_{C' \in \text{ s.i.}} u_{C'}^{\alpha' X_{C'}} \times u_\Delta^\Delta$$

$$\epsilon_1^\mu \propto (X_3 - X_2)^\mu - (X_2 - X_1)^\mu \propto X_2^\mu - \frac{(X_3+X_1)^\mu}{2}$$

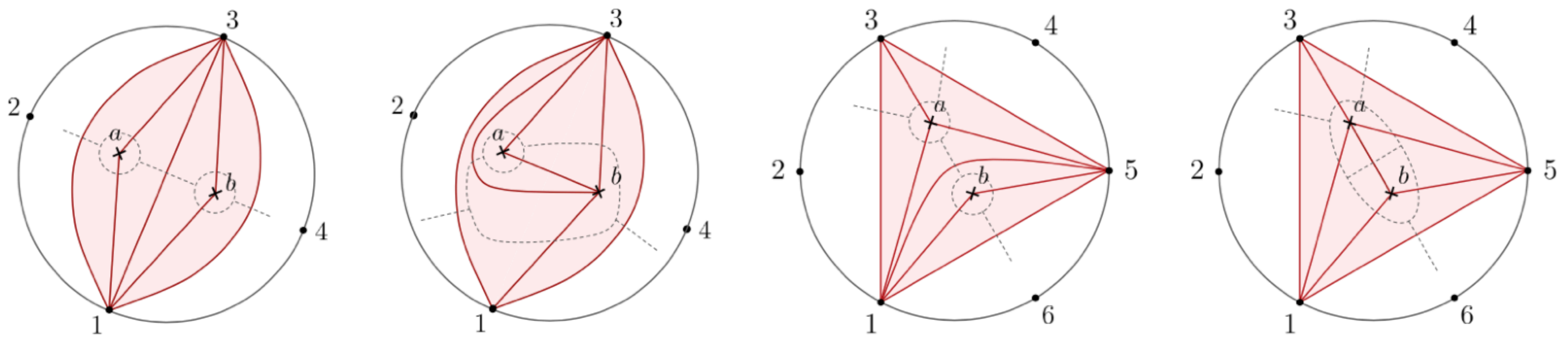
not bosonic string beyond tree, but conjecturally gives all-loop integrands!



extend the notion of (loop) gauge invariance + factorization from surfaceology!

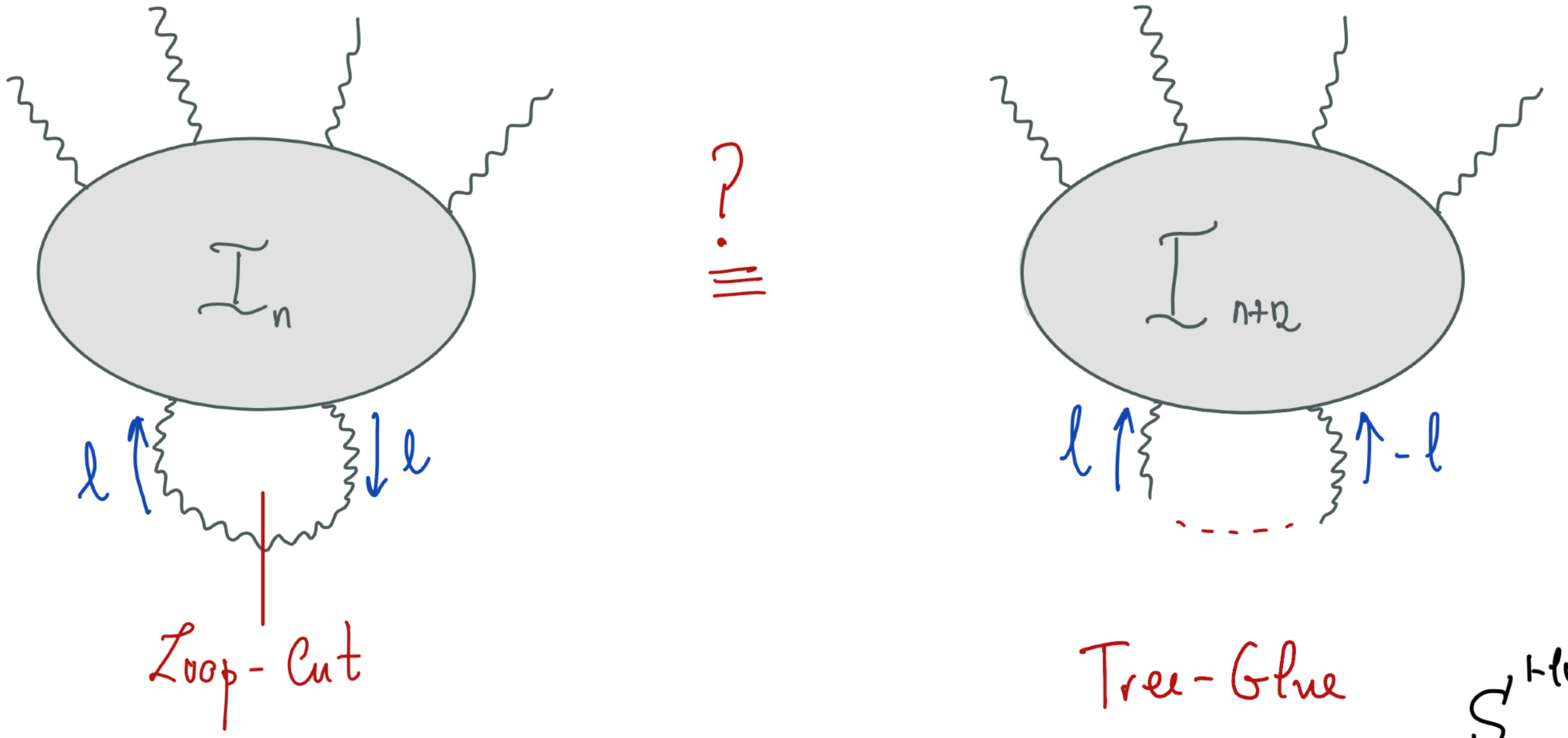
strong evidence from leading singularities (residues only): checked up to 2 loops;

LS = residue of $\int \prod \frac{dy}{y^2} \prod u^X$ = gluing of 3pt (in X space) iff $\Delta = 1 - D$.



# How to determine “perfect” YM loop integrands?

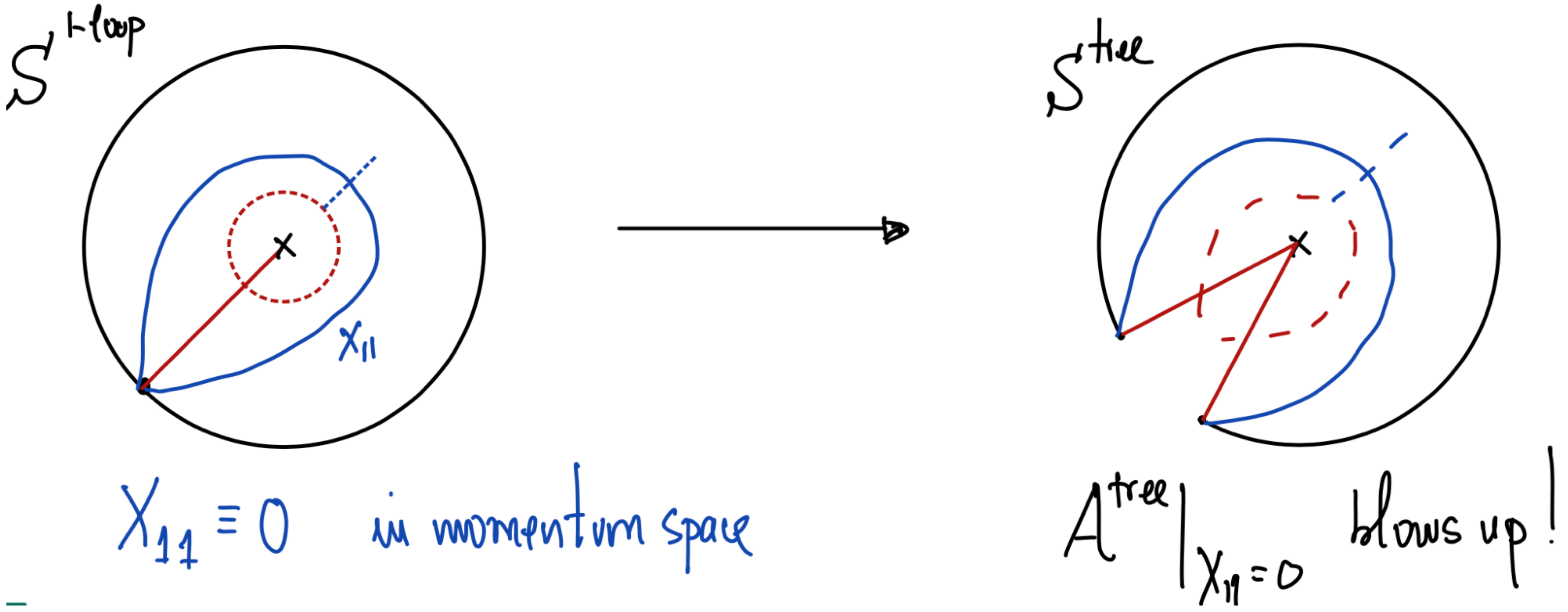
Similar to tree factorization on poles, just need loop cuts: e.g. 1-loop single-cut = forward limit (gluing tree)




naively divergent => “the” integrand
(e.g. Adler zero, gauge inv.) ill defined!

no issues for scalars, but for gluons 1/0 !
(cancels in super-Yang-Mills)

pic from Carolina’s talk @ Strings



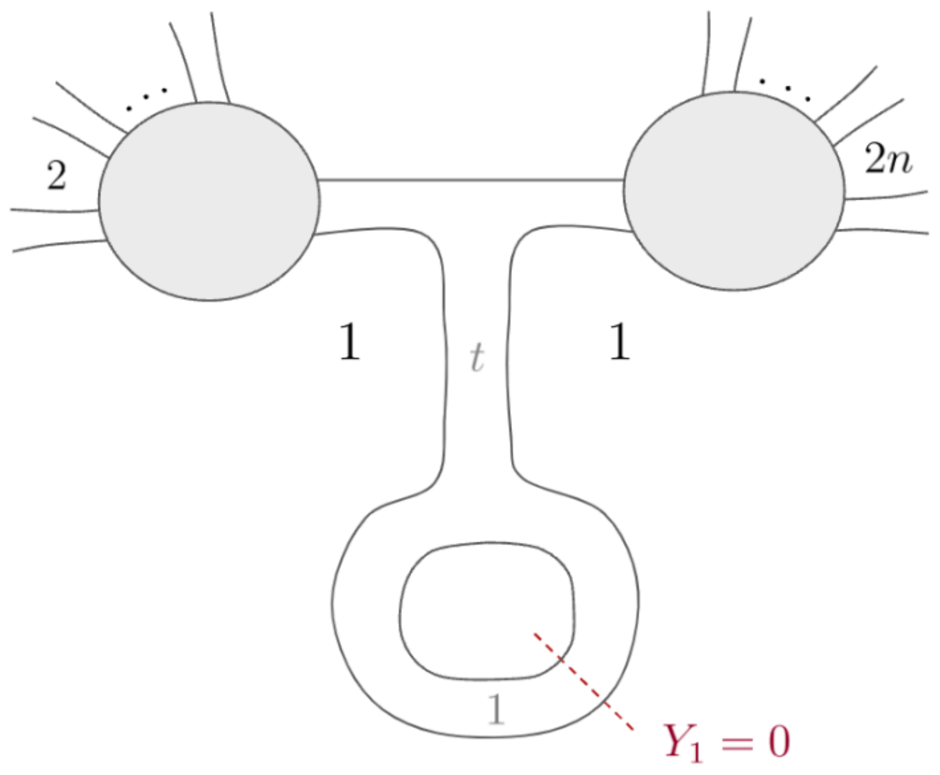

surface provides a natural way out: curves without standard momentum (e.g. tadpoles) => “perfect” integrand

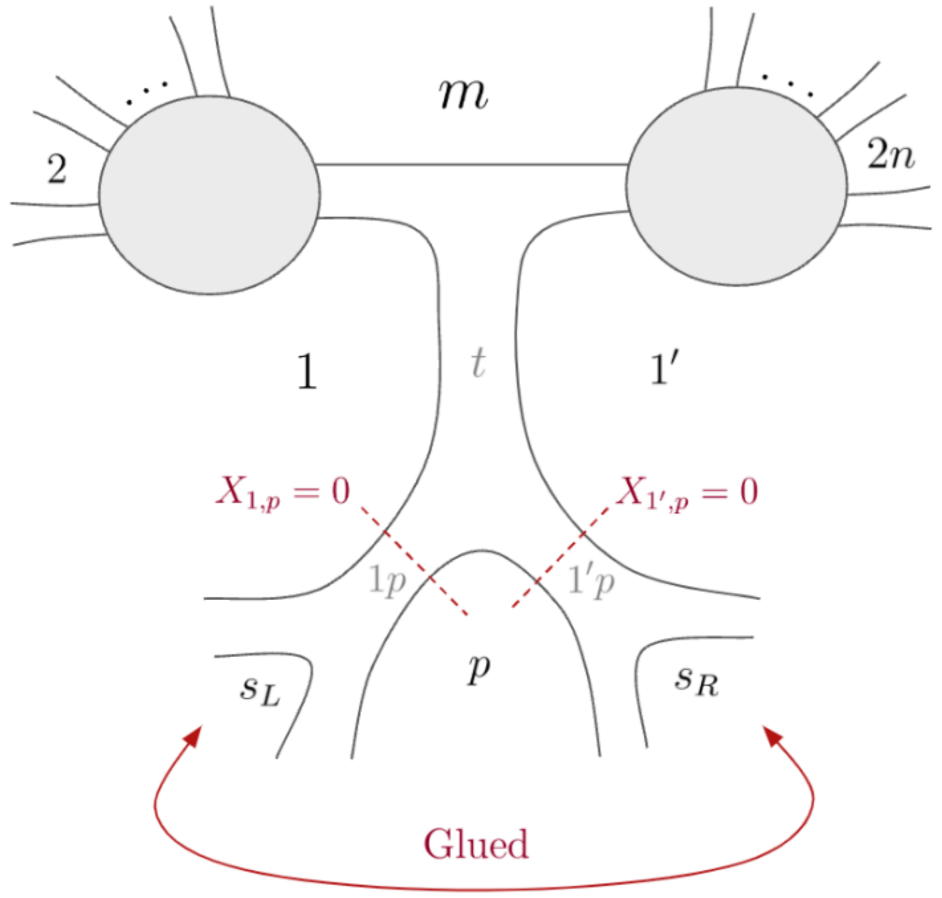
“doubling” variables: similar to Lorentzian -> complex in 4d tree kinematics

# Recursion relations for YM loop integrands [ACDFH, to appear]

Surface makes it clear that all-loop “perfect” integrands exist (also beyond planar limit); can be reconstructed from these “residues” => recursions for perfect 1-loop integrand & all-loop integrand up to scaleless terms!

$$\tilde{\mathcal{A}}^{\rm YM}_{n,L} = \int_0^1 \frac{dt}{t} \sum_{i=1}^{n} \sum_{a=1}^{L} \tilde{X}_{2i-1,z_a} \left( \sum_{j,k} (X_{z_a,j} + X_{z_a,k} - X_{j,k}) \frac{\partial^2 \tilde{\mathcal{A}}^{\rm YM}_{n+2,L-1}(1,\ldots,i',z_a,i,\ldots,n)}{\partial X_{2i',j}\partial X_{2i'+2,k}} - D \frac{\partial \tilde{\mathcal{A}}^{\rm YM}_{n+2,L-1}(1,\ldots,i',z_a,i,\ldots,n)}{\partial X_{2i',2i'+2}} \right) \Bigg|_{i=i',\ \tilde{X}_{2i-1,z_a} \to t\tilde{X}_{2i-1,z_a}} .$$




“perfect”: a notion of surface gauge invariance+ factorization

in practice, e.g. explicit results up to 1-loop 6pt, 2-loop 4pt (D-dim) integrands -> reproduce correct amps after loop integrations!

huge simplifications when going back to 4d spinor-helicity!

# Summary

Scattering Amplitudes: exciting frontier of hep-th (intersections of QFT, Strings & math)
wide applications in particle physics, precision + integrability, gravity + cosmology, string theory

New formulations of QFT: twistor-strings, Grassmannian, CHY formulation, Riemann surfaces etc.

New relations: Goldstone particles, gluons, gravitons, (strings)… *double copy* for quantum & classical gravity

A new theme: combinatorial geometries encoding QM + spacetime e.g. amplituhedron (SYM/ABJM), correlahedron -> energy correlator, AdS/dS + cosmological amplitudes

Surfacehedra (Tr $\phi^3$), binary geometries ("strings") => Real world: all-loop amps of pions, gluons …

*Thank you!*