Hadronic light-by-light contribution to muon $g-2$ from lattice QCD

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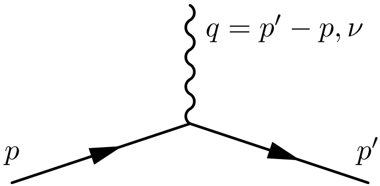
University of Connecticut

Oct 20, 2024

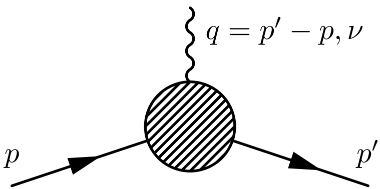
第九届手征有效场论研讨会
长沙潇湘华天大酒店



Dirac equation implies:

$$\bar{u}(p')\gamma_\nu u(p)$$

$$g = 2$$



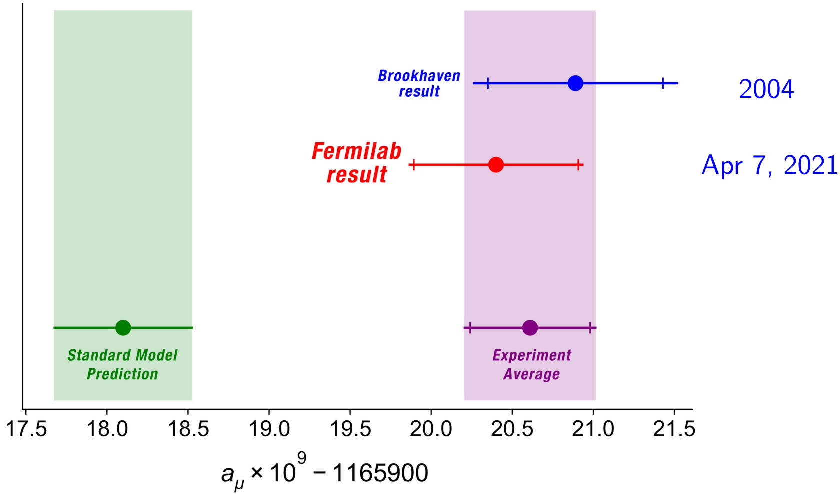
$$\bar{u}(p')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)[\gamma_\nu, \gamma_\rho]q_\rho}{4m}\right)u(p)$$

(Euclidean space time)

$$a = F_2(q^2 = 0) = \frac{g-2}{2}$$

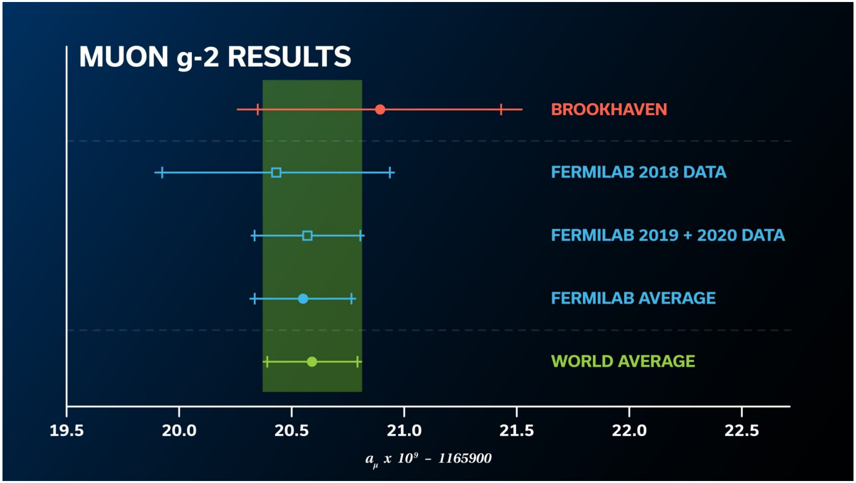
- The quantity a is called the anomalous magnetic moments.
- Its value comes from quantum correction.

Muon anomalous magnetic moment $(g-2)$



- "So far we have analyzed less than 6% of the data that the experiment will eventually collect. Although these first results are telling us that there is an intriguing difference with the Standard Model, we will learn much more in the next couple of years." – Chris Polly, Fermilab scientist, co-spokesperson for the Fermilab muon $g-2$ experiment.

New release of Muon $g-2$ experimental results



- New results (August 10, 2023) reduced the uncertainty by a factor of two.
- All results are consistent.
- J-PARC is working on a separate muon $g-2$ experiment with very different setup.
- Standard model prediction is missing.

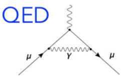
Muon $g-2$ from the Standard Model

Muon $g-2$ Theory Initiative White paper posted 10 June 2020.

132 authors from worldwide theory + experiment community. [Phys. Rept. 887 (2020) 1-166]

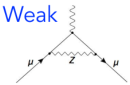
$$\boxed{a_\mu = a_\mu(\mathrm{QED}) + a_\mu(\mathrm{Weak}) + a_\mu(\mathrm{Hadronic})}$$

QED



$+\ldots$ $\quad 116\,584\,718.9\,(1) \times 10^{-11}$ $\quad 0.001$ ppm

Weak

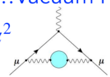


$+\ldots$ $\quad 153.6\,(1.0) \times 10^{-11}$ $\quad 0.01$ ppm

Hadronic…

…Vacuum Polarization (HVP)

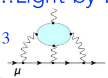
α^2



$+\ldots$ $\quad 6845\,(40) \times 10^{-11}$ [0.6%] $\quad 0.34$ ppm

…Light-by-Light (HLbL)

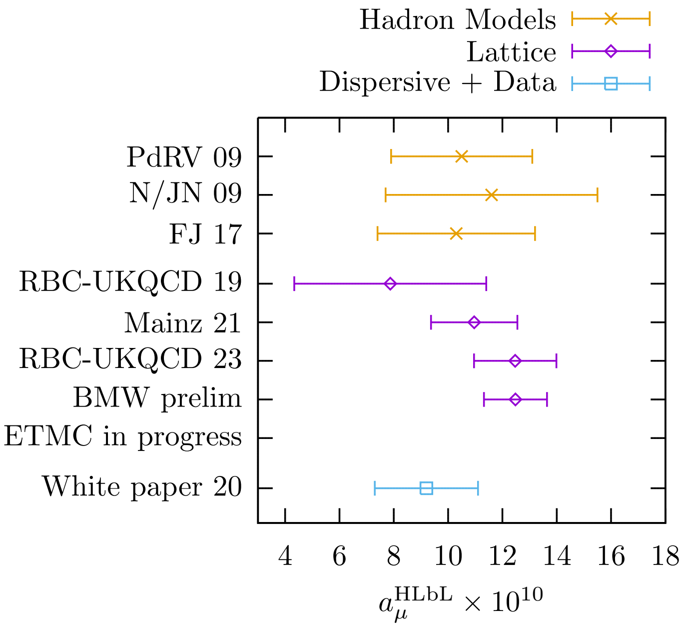
α^3



$+\ldots$ $\quad 92\,(18) \times 10^{-11}$ [20%] $\quad 0.15$ ppm

From Aida El-Khadra's theory talk during the Fermilab $g-2$ result announcement.

- Two methods: dispersive + data $\leftrightarrow$ lattice QCD
- Plan to release new white paper early next year.


Hadron Models
Lattice
Dispersive + Data
PdRV 09
N/JN 09
FJ 17
RBC-UKQCD 19
Mainz 21
RBC-UKQCD 23
BMW prelim
ETMC in progress
White paper 20
4
6
8
10
12
14
16
18
$a_\mu^{\mathrm{HLbL}} \times 10^{10}$

Contributions HLbL White paper

- "Long distance": under good control
  - Dispersive method: Berne group around G. Colangelo
  - π^0 (and η, η') pole: $93.8(4.0) \cdot 10^{-11}$
  - Pion and kaon box (pure): $-16.4(2) \cdot 10^{-11}$
  - $\pi\pi$-rescattering (include scalars below 1 GeV): $-8(1) \cdot 10^{-11}$
- Charm (beauty, top) loop: $3(1) \cdot 10^{-11}$
- "Short and medium distance" Main source of the error
  - Scalars, tensors: $-1(3) \cdot 10^{-11}$
  - Axial vector: $6(6) \cdot 10^{-11}$
  - Short-distance: $15(10) \cdot 10^{-11}$
- $\boxed{a_\mu^{HLbL-Analytic} = 92(19) \cdot 10^{-11}}$

This slide is from Johan Bijnens's talk in The Seventh Plenary Workshop of the Muon $g-2$ Theory Initiative at KEK (2024/09/13). Summary for the theory white paper 2020.

- RBC-UKQCD 2023 T. Blum et al 2023 (arXiv:2304.04423 [hep-lat])

Hadronic light-by-light contribution to the muon anomaly from lattice QCD with infinite volume QED at physical pion mass

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(RBC and UKQCD Collaborations)

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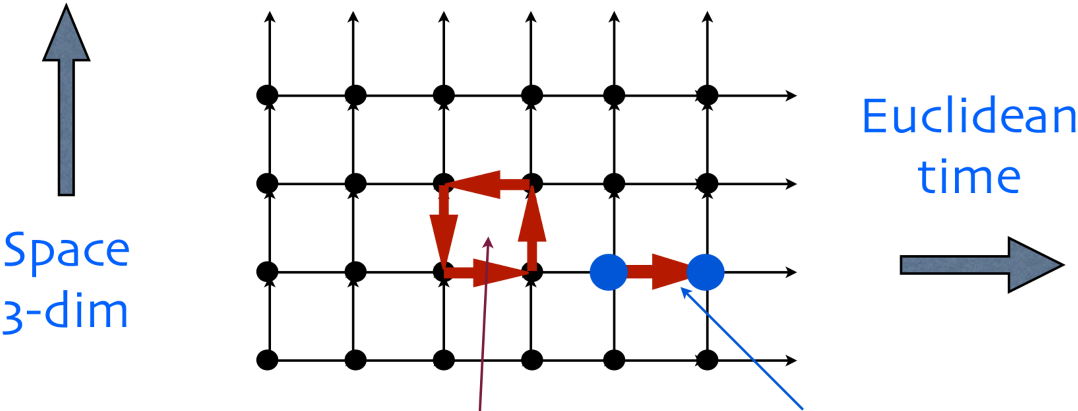
(Dated: April 11, 2023)

The hadronic light-by-light scattering contribution to the muon anomalous magnetic moment, $(g-2)/2$, is computed in the infinite volume QED framework with lattice QCD. We report $a_\mu^{\mathrm{HLbL}} = 12.47(1.15)(0.99)\times 10^{-10}$ where the first error is statistical and the second systematic. The result is mainly based on the 2+1 flavor Möbius domain wall fermion ensemble with inverse lattice spacing $a^{-1} = 1.73$ GeV, lattice size $L = 5.5$ fm, and $m_\pi = 139$ MeV, generated by the RBC-UKQCD collaborations. The leading systematic error of this result comes from the lattice discretization. This result is consistent with previous determinations.

- Earlier work by RBC-UKQCD:

  RBC-UKQCD 2019 T. Blum et al 2020 (PRL 124, 13, 132002)

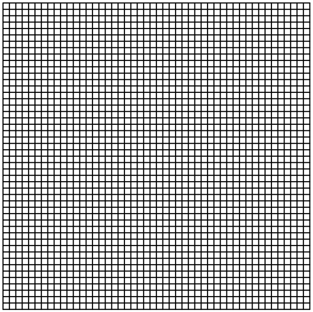


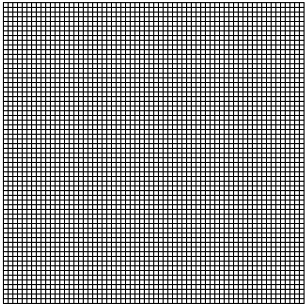
$$S_E^{\text{latt}} = -\sum_{\square} \frac{6}{g^2} \,\text{Re}\, \text{tr}_N (U_{\square,\mu\nu}) - \sum_q \overline{q}(D_\mu^{\text{lat}}\gamma_\mu + am_q)q$$

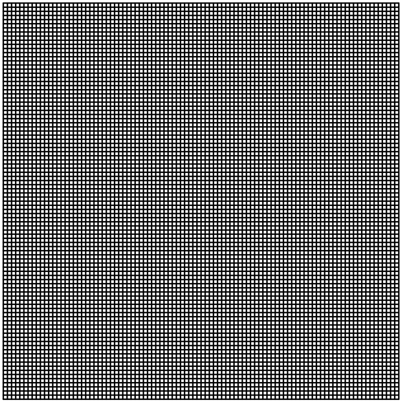
Wilson gauge action Lattice fermion action

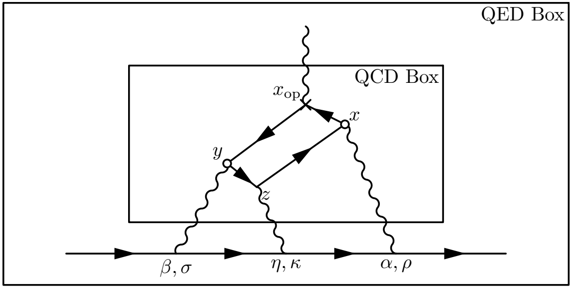
Figure credit: Stephen R. Sharpe.

- RBC-UKQCD Domain wall fermion action and Iwasaki gauge action ensembles.
- At physical pion mass (almost).
- 48I, 64I, 96I with $a^{-1} = 1.73,\ 2.36,\ 2.68$ GeV, $L = 5.47,\ 5.36,\ 7.06$ fm.



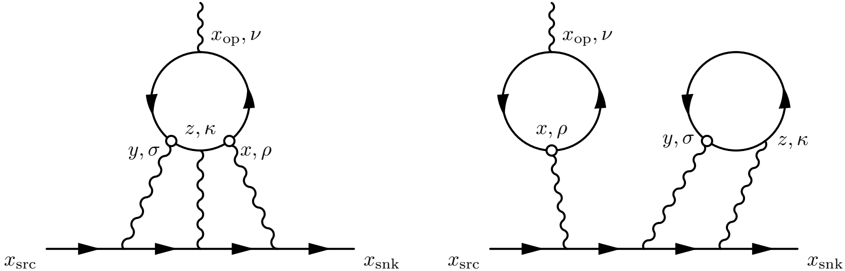






- RBC-UKQCD 2019: $L_{\rm QED} = L_{\rm QCD}$: $4.67 \sim 6.22$ fm, m_π: $135 \sim 144$ MeV, Domain wall fermion. T. Blum et al 2020 (PRL 124, 13, 132002)
- Mainz 2021: $L_{\rm QED} = \infty$, m_π: $200 \sim 422$ MeV, Wilson fermion. E.H. Chao et al. 2021 (EPJC 81, 7, 651), 2022 (EPJC 82, 8, 664)
- RBC-UKQCD 2023: $L_{\rm QED} = \infty$: 5.5 fm, $m_\pi = 139$ MeV, $a^{-1} = 1.73$ GeV, Domain wall fermion. T. Blum et al 2023 (arXiv:2304.04423 [hep-lat])
- RBC-UKQCD on-going: $L_{\rm QED} = \infty$: 5.4 fm, $m_\pi = 139$ MeV, $a^{-1} = 2.36$ GeV, Domain wall fermion.

HLbL QED$_\infty$: formulation - $x_{\text{ref}}(x, y, z)$



$$a_\mu^{\text{HLbL}} = \frac{2me^2}{3}\frac{1}{VT}\sum_{x_{\text{op}}}\sum_{x,y,z}\frac{1}{2}\epsilon_{i,j,k}\left(x_{\text{op}} - x_{\text{ref}}(x,y,z)\right)_j (6e^4)\mathcal{H}_{k,\rho,\sigma,\lambda}(x_{\text{op}},x,y,z)\mathcal{M}_{i,\rho,\sigma,\lambda}(x,y,z)$$

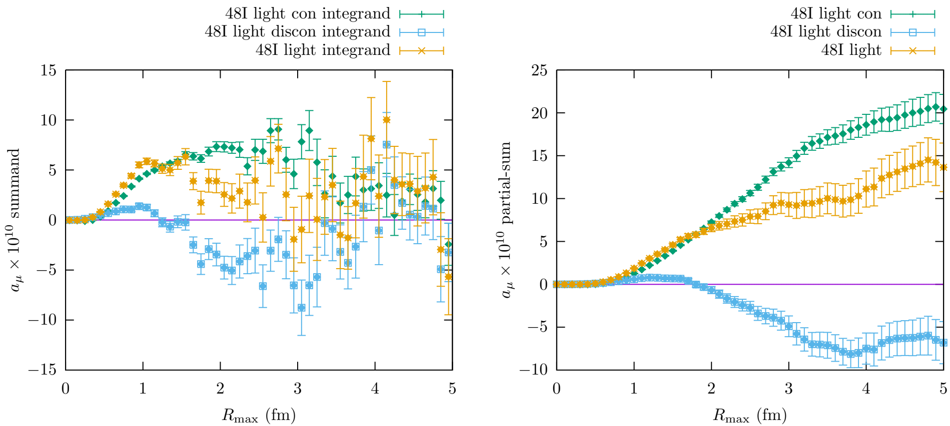
$$x_{\text{ref}}(x,y,z) = x_{\text{ref-far}}(x,y,z) \tag{1}$$

$$= \begin{cases} x & \text{if } |y-z| < \min(|x-y|,|x-z|) \\ y & \text{if } |x-z| < \min(|x-y|,|y-z|) \\ z & \text{if } |x-y| < \min(|x-z|,|y-z|) \\ \frac{1}{3}(x+y+z) & \text{otherwise} \end{cases}$$

$$x_{\text{ref-discon}} = x. \tag{2}$$

T. Blum et al 2023 (arXiv:2304.04423 [hep-lat])

HLbL: light quark results



$$R_{\max} = \max(|x-y|, |y-z|, |x-z|)$$

- Already, we have $a_\mu^{\text{hlbl,light-quark}}(R_{\max} < 4\ \text{fm}) = 11.11(2.11) \times 10^{-10}$.
- Contributions mostly come from $1 \sim 3$ fm distance (depends on the choice of subtraction scheme of the QED weighting function).
- Significant cancellation between the connected and disconnected diagrams (almost statistically independent).

HLbL: long distance - π^0 exchange

- Try to take advantage of the known theoretical ratio of the connected diagram and disconnected diagram contribution at long distance.

$$R_{\text{max}} = \max(|x-y|, |y-z|, |x-z|),$$

- Use $a_\mu(R_{\text{max}} > R_{\text{max}}^{\text{cut}})$ denote contribution in the region where R_{max} larger than $R_{\text{max}}^{\text{cut}}$. We have:

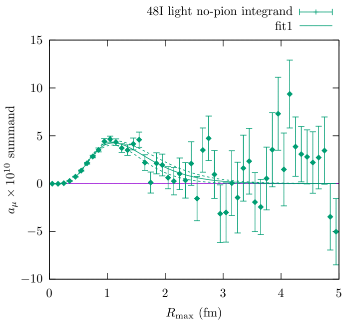
$$\lim_{R_{\text{max}}^{\text{cut}} \to \infty} \frac{a_\mu^{\text{discon}}(R_{\text{max}} > R_{\text{max}}^{\text{cut}})}{a_\mu^{\text{con}}(R_{\text{max}} > R_{\text{max}}^{\text{cut}})} = -\frac{25}{34}.$$

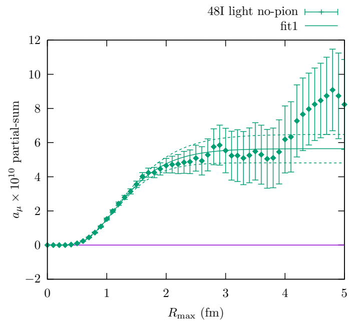
- Define contribution:

$$a_\mu^{\text{no-pion}} = a_\mu^{\text{discon}} + \frac{25}{34} a_\mu^{\text{con}},$$

- We have:

$$a_\mu^{\text{discon}} = a_\mu^{\text{no-pion}} - \frac{25}{34} a_\mu^{\text{con}},$$
$$a_\mu^{\text{total}} = a_\mu^{\text{no-pion}} + \frac{9}{34} a_\mu^{\text{con}}.$$





- Fit function (fit range 0.5 fm to 4 fm.)

$$f(R_{\text{max}}) = A\frac{R_{\text{max}}^6}{R_{\text{max}}^3 + C^3} e^{-BR_{\text{max}}}$$

Best fit parameters: $A \times 10^{10} = 130.58\ \text{fm}^{-1},\ B = 0.63\ \text{GeV},\ C = 0.66\ \text{fm}.$

- Based on fit (assign 100% systematic error):

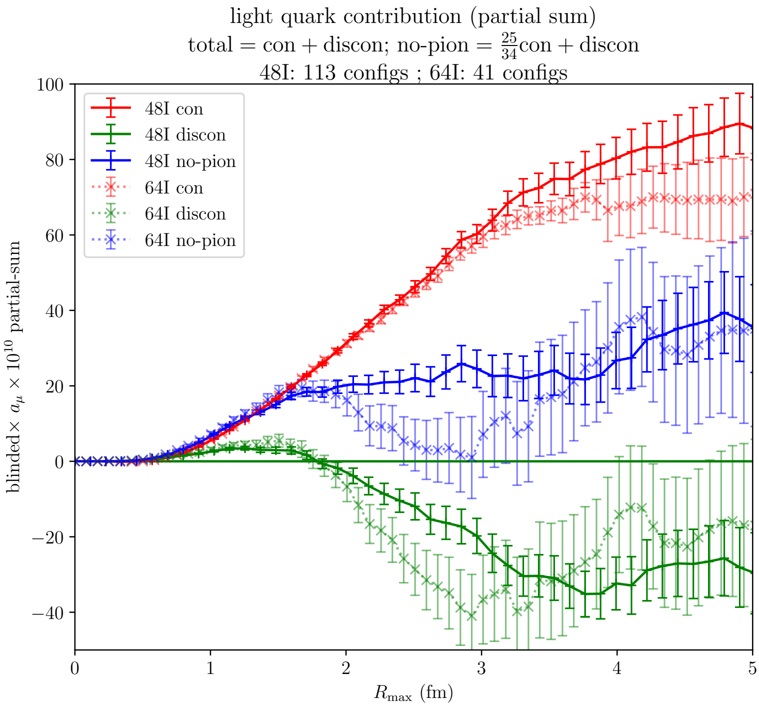
$$a_\mu^{\text{no-pion}}(R_{\text{max}} > 2.5\ \text{fm}) \times 10^{10} = 0.31(0.22)_{\text{stat}}(0.31)_{\text{syst}}\ [0.38]$$

HLbL: compilation - light quark (RBC-UKQCD 2023)

Contribution name	a_μ	$\times 10^{10}$
48I light con $R_{\max} < 4\text{fm}$	18.61	$(1.22)_{\text{stat}}$
48I light no-pion $R_{\max} > 2.5\text{fm}$	0.31	$(0.22)_{\text{stat}}(0.31)_{\text{syst}}$ [0.38]
48I light discon $R_{\max} < 4\text{fm}$	-7.49	$(1.82)_{\text{stat}}$
48I light discon $R_{\max} < 4\text{fm}$ hybrid-2.5fm	-8.28	$(1.31)_{\text{stat}}(0.31)_{\text{syst}}$ [1.35]
48I light $R_{\max} < 4\text{fm}$	11.11	$(2.11)_{\text{stat}}$
48I light $R_{\max} < 4\text{fm}$ hybrid-2.5fm	10.32	$(0.99)_{\text{stat}}(0.31)_{\text{syst}}$ [1.04]
48I light $R_{\max} > 4\text{fm}$	2.00	$(0.11)_{\text{stat}}(0.28)_{\text{syst}}$ [0.30]
long distance π^0 exchange		Norman Christ & Cheng Tu
48I light FV-corr (LMD model)	-0.47	$(0.11)_{\text{syst}}$
48I light m_π-corr (340 MeV ensemble)	0.35	$(0.07)_{\text{stat}}(0.17)_{\text{syst}}$ [0.19]
48I light a^2-corr (48I/64I strange)	0.00	$(0.83)_{\text{syst}}$
light total	12.99	$(2.11)_{\text{stat}}(0.90)_{\text{syst}}$ [2.29]
light total hybrid-2.5fm	12.20	$(1.01)_{\text{stat}}(0.95)_{\text{syst}}$ [1.38]

HLbL: comments

- The hybrid "no-pion" trick reduces the statistical error from 2.11 to 0.99.
- Current largest systematic error is from the estimated a^2 effects (8%).
  - The light quark result is entirely from 48I – NO continuum extrapolation is performed.
  - The estimated error is based on the strange connected contribution, which we did on both the 48I and 64I ensemble, and lead to 8% correction on the 48I result.
  - Calculate the light quark contribution on the 64I ensemble is needed to reduce and more reliably estimate this effect.



$$R_{\max} = \max(|x-y|, |y-z|, |x-z|)$$

Contribution	a_μ	$\times 10^{10}$
light con	25.70	$(1.33)_{\text{stat}}(1.99)_{\text{syst}}$ [2.39]
light discon	-12.71	$(1.87)_{\text{stat}}(1.17)_{\text{syst}}$ [2.20]
light discon hybrid-2.5fm	-13.50	$(1.36)_{\text{stat}}(1.21)_{\text{syst}}$ [1.82]
light total	12.99	$(2.11)_{\text{stat}}(0.90)_{\text{syst}}$ [2.29]
light total hybrid-2.5fm	12.20	$(1.01)_{\text{stat}}(0.95)_{\text{syst}}$ [1.38]
strange con	0.35	$(0.01)_{\text{stat}}$
strange discon hybrid-2.5fm	-0.36	$(0.22)_{\text{stat}}(0.03)_{\text{syst}}$ [0.22]
strange total hybrid-2.5fm	-0.00	$(0.22)_{\text{stat}}(0.03)_{\text{syst}}$ [0.23]
sub-leading discon	0.00	$(0.07)_{\text{syst}}$ [Mainz 2021]
charm total	0.28	$(0.05)_{\text{syst}}$ [Mainz 2022]
con	26.36	$(1.33)_{\text{stat}}(1.99)_{\text{syst}}$ [2.39]
discon	-13.12	$(2.30)_{\text{stat}}(1.18)_{\text{syst}}$ [2.59]
discon hybrid-2.5fm	-13.89	$(1.47)_{\text{stat}}(1.22)_{\text{syst}}$ [1.91]
total	13.24	$(2.53)_{\text{stat}}(0.90)_{\text{syst}}$ [2.68]
total hybrid-2.5fm	12.47	$(1.15)_{\text{stat}}(0.95)_{\text{syst}}$ [1.49]

- Domain wall fermion, Iwasaki gauge ensemble by RBC-UKQCD collaborations.
- $L_{\text{QED}} = \infty$, $L_{\text{QCD}} = 5.5$ fm, $m_\pi = 139$ MeV, $a^{-1} = 1.73$ GeV.
- The **subtracted** infinite volume QED weighting function.
- Rearrange the connected and disconnected diagrams to form $a_\mu^{\text{no-pion}}$.
- Separate lattice calculation of the long distance π^0 exchange contribution ($R_{\max} > 4$ fm).
- Finite volume correction for $R_{\max} < 4$ fm use π^0-pole amplitude via LMD model.
- Correction $m_\pi = 139$ MeV $\to$ 135 MeV from the 24DH (341 MeV) and 32D (142 MeV) ensembles ($a^{-1} \approx 1$ GeV).
- Subleading disconnected diagrams and charm quark contribution from Mainz 21.

$$a_\mu^{\text{HLbL}} \times 10^{10} = 12.47(1.15)_{\text{stat}}(0.95)_{\text{syst}}\ [1.49],$$

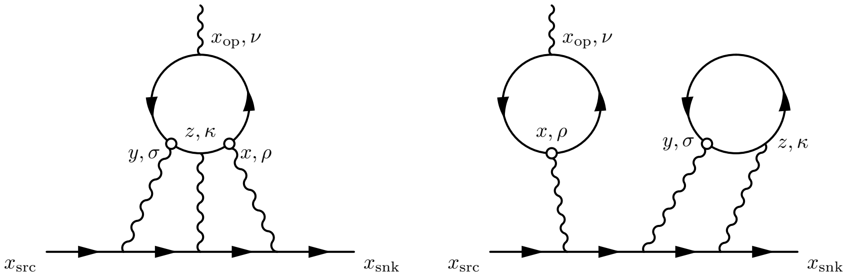
HLbL: ALCC @ SUMMIT & Frontier

Thank You!

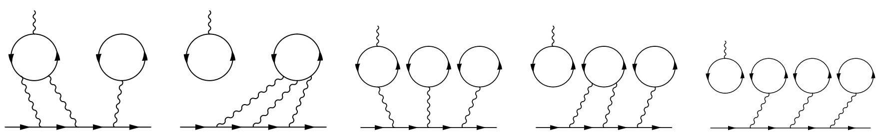
HLbL: comparison with QED_L

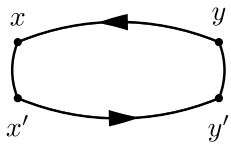
	a_μ	$\times 10^{10}$
all con QED_L	24.46	$(2.35)_{\mathrm{stat}}(5.11)_{\mathrm{syst}}$ [5.62]
all con diff	1.90	$(2.76)_{\mathrm{stat}}(5.48)_{\mathrm{syst}}$ [6.14]
all discon QED_L	-16.45	$(2.09)_{\mathrm{stat}}(3.99)_{\mathrm{syst}}$ [4.50]
all discon diff	2.56	$(2.57)_{\mathrm{stat}}(4.17)_{\mathrm{syst}}$ [4.90]
total QED_L	8.17	$(3.03)_{\mathrm{stat}}(1.77)_{\mathrm{syst}}$ [3.51]
total diff	4.30	$(3.25)_{\mathrm{stat}}(2.01)_{\mathrm{syst}}$ [3.82]

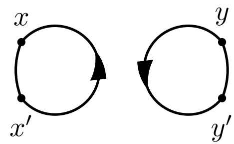
- Note: diff $=$ QED_∞ results $-$ QED_L results.
- Charm quark contribution is added to the previous QED_L results.



- We use two point sources quark propagators to calculate the hadronic part of the diagram. (Two point sources locations denoted as small circle.)
- The following sub-leading disconnected diagrams are suppressed by flavor SU(3). Mainz 2021 [E.H. Chao et al. 2021 (EPJC 81, 7, 651)]: explicitly calculated these diagrams and obtained $\pm 0.07 \times 10^{-10}$.







For the four-point-function, when its two ends, x and y, are far separated, but x' is close to x and y' is close to y, the four-point-function is dominated by π^0 exchange.

Both the connected and the disconnected diagram will contribute in these region. We can find a connection between the connnected diagram and the disconnected diagram by first investigating the η correlation function.

$$\langle \bar{u}\gamma_5 u(x)(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d)(y)\rangle \sim e^{-m_\eta|x-y|} \tag{24}$$

$$\langle \bar{u}\gamma_5 u(x)(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)(y)\rangle + 2\langle \bar{u}\gamma_5 u(x)\bar{d}\gamma_5 d(y)\rangle \sim e^{-m_\eta|x-y|} \tag{25}$$

That is

$$\langle \bar{u}\gamma_5 u(x)\bar{d}\gamma_5 d(y)\rangle = -\frac{1}{2}\langle \bar{u}\gamma_5 u(x)(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)(y)\rangle + \mathcal{O}\big(e^{-m_\eta|x-y|}\big) \tag{26}$$

Above is a relation between disconnected diagram π^0 exchange (left hand side) and connected diagram π^0 exchange (right hand side).

HLbL: long distance - π^0 exchange

The nearby two current operater can be viewed as an interpolating operator for π^0, just like $\bar{u}\gamma_5 u$ or $\bar{d}\gamma_5 d$ with appropriate charge factors.

Multiplied by appropriate charge factors:

$$\text{Connected contribution} \quad \left[\left(\frac{2}{3}\right)^4 + \left(-\frac{1}{3}\right)^4\right] = \frac{17}{81} \tag{27}$$

$$\text{Disconnected contribution} \quad \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right]^2 \left(-\frac{1}{2}\right) = \frac{25}{81}\left(-\frac{1}{2}\right) \tag{28}$$

$$\text{Connected : Disconnected} \;=\; 34 : -25 \tag{29}$$

Different approach by J. Bijnens and J. Relefors: **JHEP 1609 (2016) 113**.

HLbL: some lattice calculation details

- RBC-UKQCD 48I ensemble ($48^3 \times 96$). 5.5 fm, $m_\pi = 139$ MeV, $a^{-1} = 1.73$ GeV.
- Uniformly random sample 2048 point locations per config. Calculate point source light quark propagators for each point. Overall, we calculated 113 configs.
- **Same set of propagators** used in the calculation of the **connected** and **disconnected** diagrams.
- Computational techniques:
  - locally-coherent Lanczos approach (arXiv:1710.06884 [hep-lat])
  - ZMobius (arXiv:1701.07792 [hep-lat])
  - AMA (arXiv:1208.4349 [hep-lat])

- RBC-UKQCD 48I ensemble ($48^3 \times 96$). 5.5 fm, $m_\pi = 139$ MeV, $a^{-1} = 1.73$ GeV.
- Uniformly random sample 2048 point locations per config. Calculate point source light quark propagators for each point. Overall, we calculated 113 configs.
- **Connected diagrams**: sample two-point-pairs (x, y) formed using these 2048 points based on the empirical probability:

$$p(r) = \begin{cases} 1 & \text{if } 8 \geq r > 0 \\ \frac{1}{(r/8)^3} & \text{if } L \geq r > 8 \\ 0 & \text{if } r > L \end{cases},$$

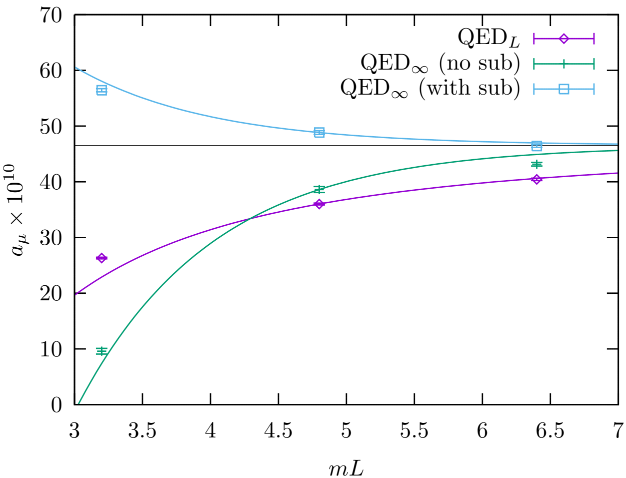
Compute 57,000 pairs per config on average. We also do stochastic sparsening for the other two points x_{op}, z with ratio $1/16$, which saves both computational time and storage (more important).

- RBC-UKQCD 48I ensemble ($48^3 \times 96$). 5.5 fm, $m_\pi = 139$ MeV, $a^{-1} = 1.73$ GeV.
- Uniformly random sample 2048 point locations per config. Calculate point source light quark propagators for each point. Overall, we calculated 113 configs.
- Disconnected diagrams: calculate all possible two-point-pairs formed with these 2048 points. To make it affordable, we aggressively sparsen when summing over z with "adaptive sampling". Note that the procedure is NOT biased!

$$n(z,y) = \sum_{\kappa,\sigma} \Big| \mathrm{Tr}\Big(\gamma_\kappa S_q(z,y) \gamma_\sigma S_q(y,z) - \langle \gamma_\kappa S_q(z,y) \gamma_\sigma S_q(y,z) \rangle_{\mathrm{QCD}} \Big) \Big|^2$$

$$p_y(z) = \begin{cases} 1 & \text{if } n(z,y) \geq t_0^2 \text{ and } |z-y| \leq L \\ \sqrt{n(z,y)}/t_0 & \text{if } n(z,y) < t_0^2 \text{ and } |z-y| \leq L \\ 0 & \text{if } |z-y| > L \end{cases},$$

where $t_0 = 5 \times 10^{-5}$. In short, we sample z with probability determined based on the magnitude of the value quark loop evaluated for this config and point source location y.



- QED$_L$: $\mathcal{O}(1/L^2)$ finite volume effects
- QED$_\infty$ (no sub) $\mathfrak{G}^{(1)}$: $\mathcal{O}(e^{-mL})$ finite volume effects
- QED$_\infty$ (with sub) $\mathfrak{G}^{(2)}$: smaller $\mathcal{O}(e^{-mL})$ finite volume effects

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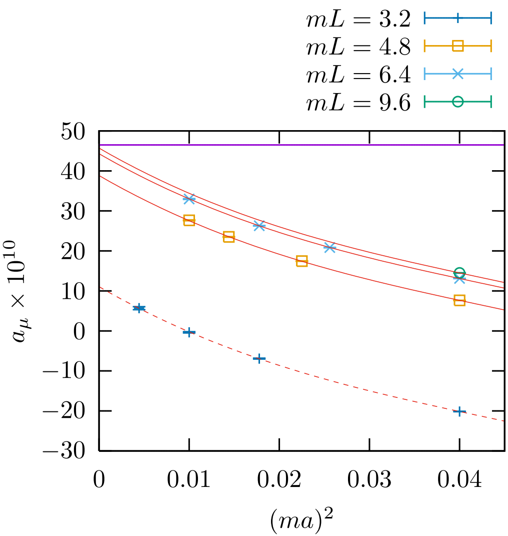
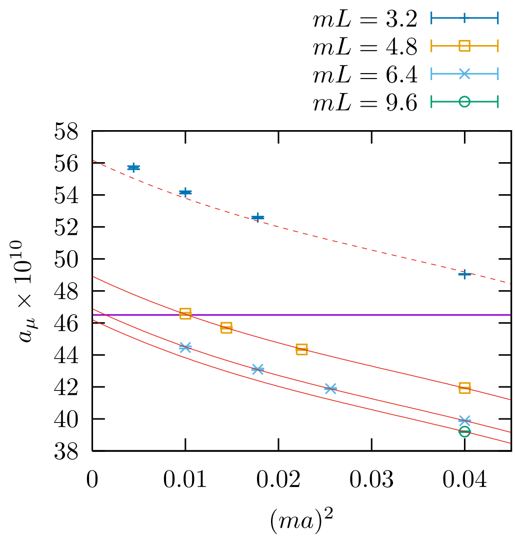
- Compare the two $\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)$ in **pure QED computation**.



Original QED kernel



Subtracted QED kernel

- Notice the vertical scales in the two plots are different.

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$$\langle \mathcal{O}(U,q,\bar{q})\rangle = \frac{\int [\mathcal{D}U]\prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}} \mathcal{O}(U,q,\bar{q})}{\int [\mathcal{D}U]\prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}}}$$
$$= \frac{\int [\mathcal{D}U] e^{-S_{\text{gauge}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}}\gamma_\mu + am_q\right) \tilde{\mathcal{O}}(U)}{\int [\mathcal{D}U] e^{-S_{\text{gauge}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}}\gamma_\mu + am_q\right)}$$

Monte Carlo:

- The integration is performed for all the link variables: U. Dimension is $L^3 \times T \times 4 \times 8$.
- Sample points the following distribution:

$$e^{-S_{\text{gauge}}^{\text{latt}}(U)} \prod_q \det\left(D_\mu^{\text{latt}}(U)\gamma_\mu + am_q\right)$$

- Therefore:

$$\langle \mathcal{O}(U,q,\bar{q})\rangle = \frac{1}{N_{\text{conf}}} \sum_{k=1}^{N_{\text{conf}}} \tilde{\mathcal{O}}(U^{(k)})$$

- Parameters in lattice QCD calculations (e.g. isospin symmetric ($m_u = m_d = m_l$) and three flavor u, d, s theory):

$$g \qquad am_l \qquad am_s$$

  Note that lattice spacing a is determined by g via the renormalization group equation.

- The experimental inputs needed to determine these parameters can be: m_π/m_Ω, m_K/m_Ω.