第九届手征有效场论研讨会

Lattice spectra of DDK three-body system with Lorentz covariant kinematic

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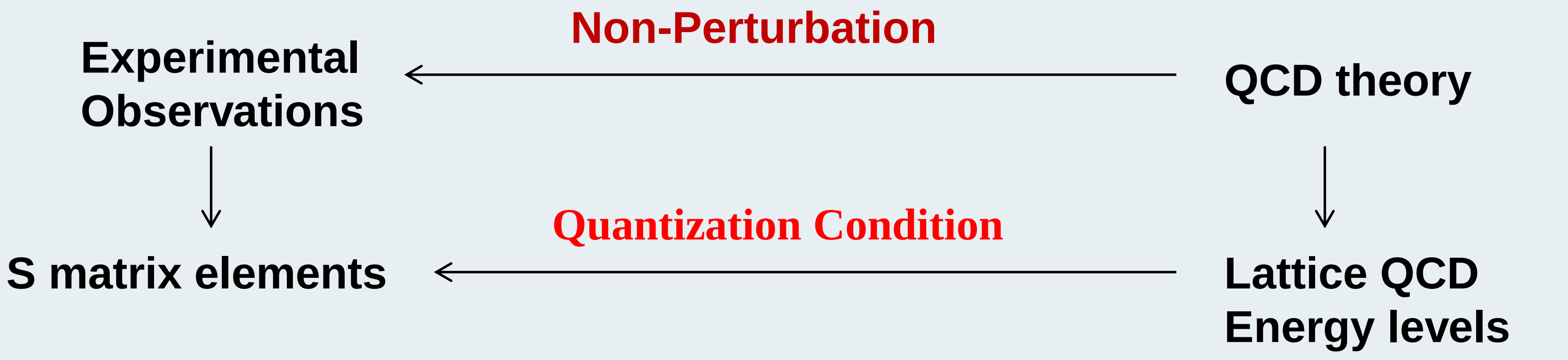
Outline

- The Quantization Condition (QC) in the finite volume
- The covariant QC of three-particle by NREFT
- DDK three-body system
- Summary and Outlook

The quantization condition in the finite volume



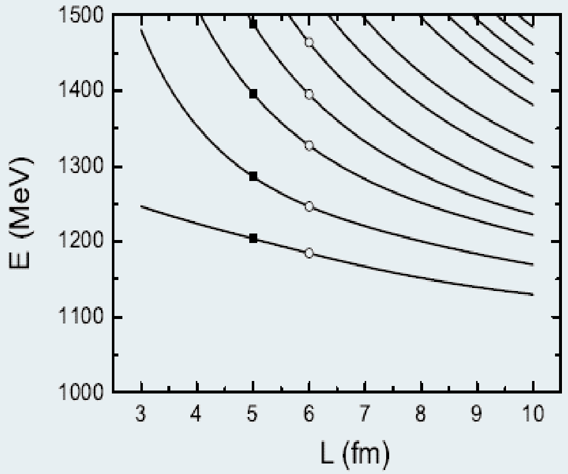
Non-Perturbation

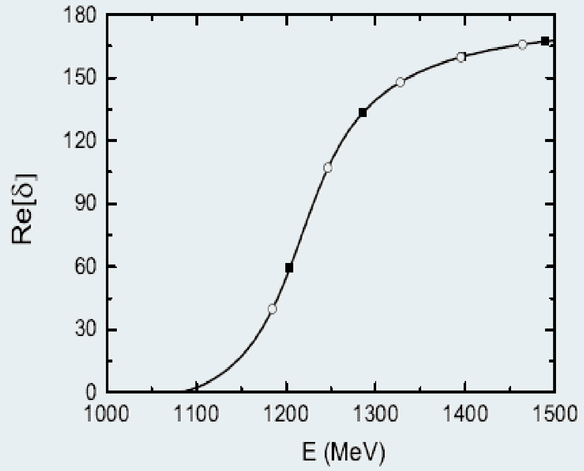
Experimental Observations ← **QCD theory**

Quantization Condition

S matrix elements ← **Lattice QCD Energy levels**

For the 2-body system, Lüscher’s method,
Hamiltonian Effective Field Theory, …

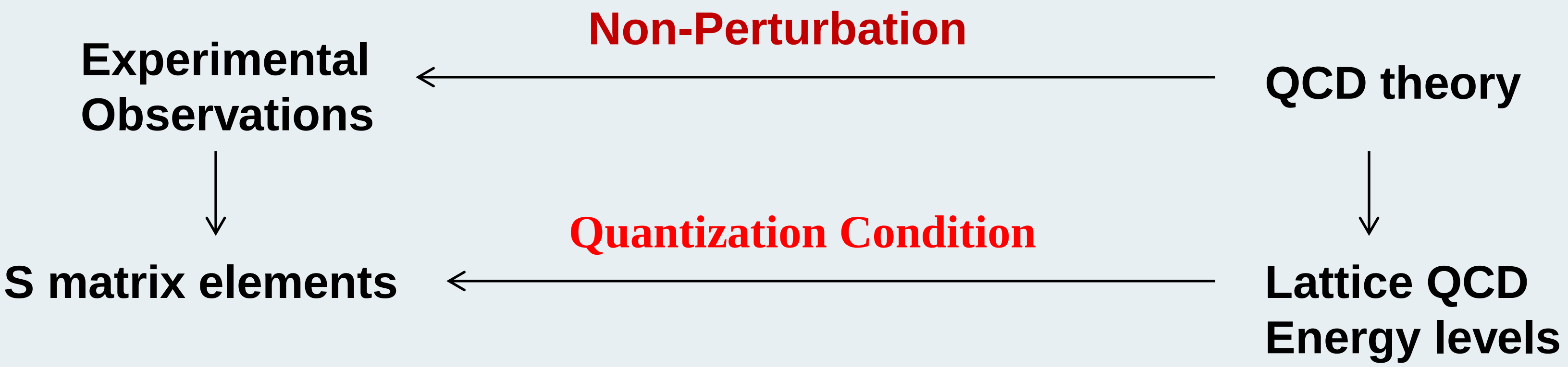




$$N\pi \to N\pi$$

Δ Resonance

The quantization condition in the finite volume



For the 3-body system, it becomes very complicated,

1. There are two free momenta
2. How to define the S matrix of three body
3. How to deal with the divergent of three-body re-scattering

But the 3-body system is extremely important to describe low energy resonances, such as η –> 3π, ω –> 3π, N*(1440) –>Nππ .

- **RFT** Hansen, Sharpe [PRD90,116003,2014], [PRD92,114509,2015]
- **NREFT** Hammer, JYP, Rusetsky [JHEP09, 109,2017], [JHEP10,115,2017]
- **FVU** Mai, Döring [EPJA53, 240, 2017], [PRL, 122, 062503, 2019]

The covariant QC of three-particle by NREFT

Why invariant ?

Typical momenta on the lattice $(2\pi/L)$ is comparable with π mass, then purely kinematic effect of the relativistic invariant treatment, especially for the **moving frames**. The true amplitude is invariant.

Why a manifestly invariant formulation ?

Fingding an explicit parameterization of the three-body force that keep the solution of the Faddeev equation invariant! Too difficult and impossiblea proper short-range three-body force. But it is not only very difficult. Making the tree-level kernel of the Faddeev equation relativistic invariant order by order in the EFT expansion also does not work, since cutoff regularization will break counting rules.

On the contrary, a manifestly invariant formulation will keep the three-body force parameterize in terms of Lorentz-invariant structures, where the coupling constants before these terms are indepenedent and the expansion of the short-range part can be organized in accordance with the will-defined counting rules.

How to obtain a manifestly form ?

$$\frac{1}{2w(\mathbf{l})}\,\frac{1}{w(\mathbf{p})+w(\mathbf{q})+w(\mathbf{l})-K^0}$$
$$\to \frac{1}{2w(\mathbf{l})}\,\frac{1}{w(\mathbf{p})+w(\mathbf{q})+w(\mathbf{l})-K^0}+\frac{1}{2w(\mathbf{l})}\,\frac{1}{w(\mathbf{p})+w(\mathbf{q})-w(\mathbf{l})-K^0}=\frac{1}{b^2-m^2}\,,$$

➢**RFT** Hansen, Sharpe [PRD90,116003,2014], [PRD92,114509,2015]

Weak Point: additional term will bring new pole when the cut is high.

The covariant QC of three-particle by NREFT

How to obtain a manifestly form ?

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$$

$$\mathcal{L}_1 = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi$$

$$\mathcal{L}_2 = -\frac{C_0}{2}\psi^\dagger\psi^\dagger\psi\psi - \frac{C_2}{4}\left(\psi^\dagger\nabla^2\psi^\dagger\psi\psi + h.c.\right) + \cdots$$

$$\mathcal{L}_3 = -\frac{D_0}{6}\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi - \frac{D_2}{12}\left(\psi^\dagger\psi^\dagger\nabla^2\psi^\dagger\psi\psi\psi + h.c.\right) + \cdots$$

$$\mathcal{L}_1 = \psi^\dagger 2w_v(i(v\partial) - w_v)\psi$$

$$\mathcal{L}_2 = -\frac{C_0}{2}\psi^\dagger\psi^\dagger\psi\psi - \frac{C_2}{4}\left\{((w_{v\mu}\psi)^\dagger(w_v^\mu\psi)^\dagger - m^2\psi^\dagger\psi^\dagger)\,\psi\psi \ + h.c.\right\} + \cdots$$

$$\mathcal{L}_3 = -\frac{D_0}{6}\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi - \frac{D_2}{12}\left\{\psi^\dagger\,((w_{v\mu}\psi)^\dagger\,(w_v^\mu\psi)^\dagger - m^2\psi^\dagger\psi^\dagger)\,\psi\psi\psi + h.c.\right\} + \cdots$$

Key point, find a new operator replace ∂_0 and $\vec{\nabla}$

Introduce a four-velocity: $\mathrm{v}^\mu = P^\mu/\sqrt{P^2}$

A new energy operator: $\omega_\mathrm{v} = \sqrt{m^2 + \partial^2 - (\partial \cdot \mathrm{v})^2}$

A new four momentum operator: $\omega_\mathrm{v}^\mu = \mathrm{v}^\mu\,\omega_\mathrm{v} + \mathrm{i}\partial_\perp^\mu$

where $\partial_\perp^\mu = \partial^\mu - \mathrm{v}^\mu\,(\partial \cdot \mathrm{v})$

Just rely on $\vec{\partial}$

When $\mathrm{v}^\mu = \left(1, \vec{0}\right)$, in REST FRAME

$\omega_\mathrm{v} = \sqrt{m^2 - \vec{\partial}^2}$; $\partial_\perp^\mu = \left(0, \vec{\partial}\right)$; $\omega_\mathrm{v}^\mu = (\omega_\mathrm{v}, \mathrm{i}\vec{\partial})$

We construct several Lorentz-invariant operators just by $\vec{\partial}$ and v^μ !!!

The covariant QC of three-particle by NREFT

S-wave

$$\mathcal{L}_1 = \psi^\dagger 2w_v(i(v\partial) - w_v)\psi$$

$$\mathcal{L}_2 = -\frac{C_0}{2}\psi^\dagger\psi^\dagger\psi\psi - \frac{C_2}{4}\left\{\left((w_{v\mu}\psi)^\dagger(w_v^\mu\psi)^\dagger - m^2\psi^\dagger\psi^\dagger\right)\psi\psi \;\; + h.c.\right\} + \cdots$$

$$\mathcal{L}_3 = -\frac{D_0}{6}\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi - \frac{D_2}{12}\left\{\psi^\dagger\left((w_{v\mu}\psi)^\dagger(w_v^\mu\psi)^\dagger - m^2\psi^\dagger\psi^\dagger\right)\psi\psi\psi + h.c.\right\} + \cdots$$

Dimer picture

$(C_0,\, C_2,\, D_0,\, D_2) \sim (\sigma,\, f_1,\, h_0,\, h_2)$

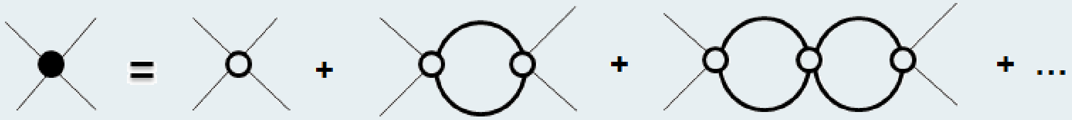
$$\mathcal{L}_1 = \psi^\dagger 2w_v(i(v\partial) - w_v)\psi$$

$$\mathcal{L}_2 = \sigma T^\dagger T + \frac{1}{2}\left[T^\dagger\left(\psi\psi + f_1((w_{v\mu}\psi)(w_v^\mu\psi) - m^2\psi\psi) + \cdots\right) + h.c.\right] + \cdots$$

$$\mathcal{L}_3 = h_0 T^\dagger T\psi^\dagger\psi + h_2 T^\dagger T\left(\psi^\dagger w_v^2\psi + h.c.\right) + \cdots$$

The covariant QC of three-particle by NREFT

$$\mathcal{L}_2 = -\frac{C_0}{2}\psi^\dagger\psi^\dagger\psi\psi - \frac{C_2}{4}\left\{\left((w_{v\mu}\psi)^\dagger(w_v^\mu\psi)^\dagger - m^2\psi^\dagger\psi^\dagger\right)\psi\psi \;+h.c.\right\}+\cdots$$



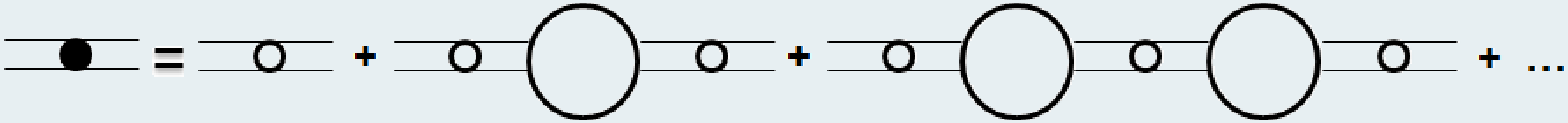
$$T_0 = (4C_0) + (4C_0)^2\frac{1}{2}I + (4C_0)^3\frac{1}{4}I^2 + \cdots = \frac{1}{(4C_0)^{-1} - \frac{1}{2}I}\,.$$

$$I(s) = J(s) + \frac{i\sigma}{16\pi} \qquad \sigma = \left(1 - \frac{4m^2}{s+i\varepsilon}\right)^{1/2}$$

$$I = \int\frac{d^Dk}{(2\pi)^D i}\,\frac{1}{2w_v(k)(w_v(k) - vk - i\varepsilon)}\,\frac{1}{2w_v(P-k)(w_v(P-k) - v(P-k) - i\varepsilon)}$$

Dimer picture

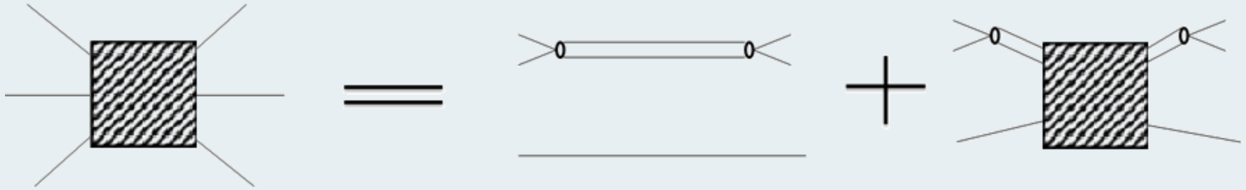
$$\mathcal{L}_2 = \sigma T^\dagger T + \frac{1}{2}\left[T^\dagger\left(\psi\psi + f_1\left((w_{v\mu}\psi)(w_v^\mu\psi) - m^2\psi\psi\right)+\cdots\right) + h.c.\right]+\cdots$$



$$\tau_{\rm L}(s) = \frac{1}{(p_s^{2\rm L}T^{\rm L}_{\rm tree})^{-1} - \frac{1}{2}I(s)} \overset{s>4m^2}{=} \frac{16\pi\sqrt{s}}{16\pi\sqrt{s}\left((p_s^{2\rm L}T^{\rm L}_{\rm tree})^{-1} - \frac{1}{2}J(s)\right) - ip_s} \qquad p_s = \sqrt{\frac{s}{4} - m^2}$$

$$16\pi\sqrt{s}\left((T^{\rm L}_{\rm tree})^{-1} - \frac{1}{2}p_s^{2\rm L}J(s)\right) = p_s^{2\rm L+1}\cot\delta_{\rm L}(s)$$

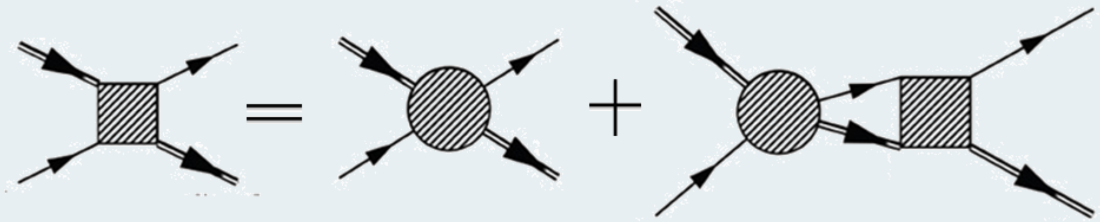
The covariant QC of three-particle by NREFT



$$T_3(p_1,p_2,p_3;q_1,q_2,q_3) = T_3^{\mathsf{disc}} + T_3^{\mathsf{conn}}\,,$$

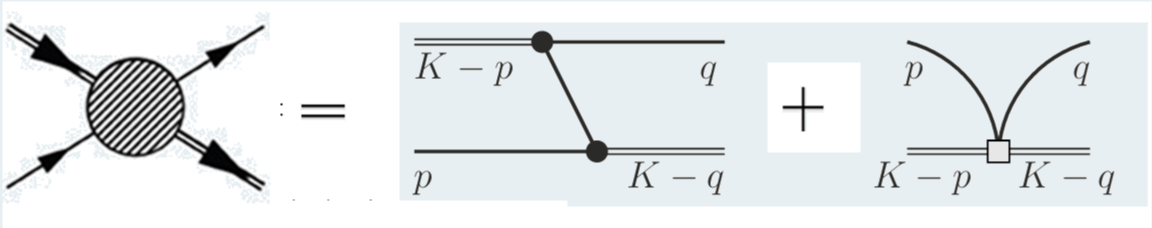
$$T_3^{\mathsf{disc}} = \sum_{i,j=1}^{3}(2\pi)^3\delta^3(p_{i\perp}-q_{j\perp})2w_v(p_i)\tau((K-p_i)^2)\,,$$

$$T_3^{\mathsf{conn}} = \sum_{i,j=1}^{3}\tau((K-p_i)^2)\mathcal{M}(p_i,q_j,K)\tau((K-q_j)^2)\,.$$



$$\mathcal{M}(p,q;P) = Z(p,q;P) + \int\frac{d^4k}{(2\pi)^4}\Theta_v(k)Z(p,k;P)\tau((P-k)^2)\mathcal{M}(k,q;P).$$

$$\Theta_v(k) = 2\pi\delta(k^2-m^2)\theta(\Lambda^2+k^2-(v\cdot k)^2)$$



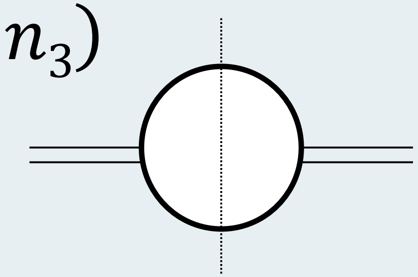
$$\mathcal{Z}(p,q) = \frac{1}{2w_v(K-p-q)(w_v(p)+w_v(q)+w_v(K-p-q)-vK-i\varepsilon)} + \frac{H_0(\Lambda)}{\Lambda^2}$$

$$Z(\vec{p},\vec{q},E) = \frac{1}{\vec{p}^2+\vec{q}^2+\vec{p}\cdot\vec{q}-mE} + h_0$$

$$\tau(s) = \frac{16\pi\sqrt{s}}{-\frac{1}{a}-8\pi\sqrt{s}J(s)-ip(s)} \qquad \tau(\vec{k},E) = \frac{1}{-\frac{1}{a_0}+\sqrt{\frac{3}{4}k^2-mE}}$$

The covariant QC of three-particle by NREFT

- In Box, the “p” become discrete (2π/L) $\vec{n}$, $\vec{n} = (n_1, n_2, n_3)$
- Propagator of dimer, $\tau_{\mathrm{L}}(s) = \dfrac{1}{(p_s^{2\mathrm{L}} T_{\text{tree}}^{\mathrm{L}})^{-1} - \frac{1}{2} I(s)}$



$$I = \int \frac{d^D k}{(2\pi)^D i} \frac{1}{2w_v(k)(w_v(k) - vk - i\varepsilon)} \frac{1}{2w_v(P-k)(w_v(P-k) - v(P-k) - i\varepsilon)}$$

Box

$$I^{FV} = \frac{1}{L^3} \sum_{\mathbf{k}} \int \frac{dk^0}{2\pi i} \frac{1}{2w_v(k)(w_v\ (k) - v \cdot k - i\epsilon)} \frac{1}{2w_v(P-k)(w_v(P-k) - v \cdot (P-k) - i\epsilon)}$$

$$\tau^{FV} = \frac{1}{\left(p_s^{2L} T_{tree}^{L}\right)^{-1} - \frac{1}{2} I^{FV}} = \frac{1}{\left(p_s^{2L} T_{tree}^{L}\right)^{-1} - \frac{1}{2} Re[I^\infty] - \frac{1}{2}(I^{FV} - Re[I^\infty])} = \frac{16\sqrt{s}\,\pi}{p_s cot\delta_L - \frac{1}{32\sqrt{s}\,\pi}(I^{FV} - Re[I^\infty])}$$

$$= \frac{16\sqrt{s}\,\pi}{p_s cot\delta_L - \frac{2}{\sqrt{\pi} L\gamma} Z_{00}^{d}(1; q_0^2)}$$

$$s = P^2,\ \gamma = \left(1 - \frac{\mathbf{P}^2}{P_0^2}\right)^{-1/2},\ \mathbf{d} = \frac{\mathbf{P}L}{2\pi},\ q_0^2 = \frac{L^2}{4\pi^2}\left(\frac{s}{4} - m^2\right), \text{ and}$$

$$Z_{00}^{\mathbf{d}}(1; q_0^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{r} \in P_d} \frac{1}{\mathbf{r}^2 - q_0^2},\ \ P_d = \{\mathbf{r} = \mathbb{R}^3 | r_\parallel = \gamma^{-1}\left(n_\parallel - \frac{1}{2}|\mathbf{d}|\right), \mathbf{r}_\perp = \mathbf{n}_\perp,\ \mathbf{n} \in \mathbb{Z}^3\}$$

Spurious Pole **JHEP 07 (2022) 019**

$$\tau \to \tau - \frac{R_{\mathrm{f}}}{s - s_{\mathrm{f}}} + (\text{polynomial in } \delta s).$$

The covariant QC of three-particle by NREFT

- Scattering equation, $\mathcal{M}(p,q;P)=Z(p,q;P)+\int\frac{d^4k}{(2\pi)^4}\Theta_v(k)Z(p,k;P)\tau((P-k)^2)\mathcal{M}(k,q;P).$

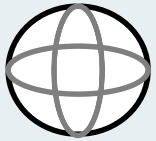
$$\Theta_v(k)=2\pi\delta(k^2-m^2)\theta(\Lambda^2+k^2-(v\cdot k)^2)$$

Box ⇨ $\mathcal{M}^L(p,q;P)=Z(p,q;P)+\frac{1}{L^3}\sum_{\mathbf{k}}\tilde{\Theta}_v(k)Z(p,k;P)\tau^L((P-k)^2)\mathcal{M}^L(k,q;P)$ $\quad\tilde{\Theta}_v(k)=\frac{\theta(\Lambda^2+m^2-(v\cdot k)^2)}{2\omega(\mathbf{k})}$

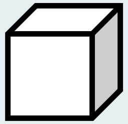
- QC: $\det\left(\delta_{pq}-\frac{1}{L^3}\tilde{\Theta}_v(q)Z(p,q;P)\tau^L((P-q)^2)\right)=0$

Project to the irreducible representation (irrep)

- Symmetry breaking
- Infinite Volume -> SO(3)



- Box Volume -> O$_h$



$$Z^{\Gamma}(r,r^{'})=\sum_{g}\left(\mathscr{T}^{\Gamma}(g)\right)Z(gp_r^{(0)},q_{r'}^{(0)})$$

$$\mathcal{M}^{\Gamma}(r,r^{'};P)=Z^{\Gamma}(r,r^{'};P)+\frac{1}{L^3}\sum_{r^{''}}\frac{\vartheta_{r''}}{|G|}\tilde{\Theta}_v(r^{''})\,Z^{\Gamma}(r,r^{''};P)\tau((P-k_{r''})^2)\mathcal{M}^{\Gamma}(r^{''},r^{'};P).$$

r indicate the momentum shell, $\mathcal{J}$ is the matrix of irrep Γ, g is element of group, |G| and ϑ are the order of group and the shell.

- QC for fixed irrep: $\det\left(\delta_{rr'}-\frac{1}{L^3}\frac{\vartheta_{r'}}{|G|}\tilde{\Theta}_v(r^{'})Z^{\Gamma}(r,r^{'};P)\tau((P-k_{r'})^2)\right)=0$

Example: DDK three-body system

- 3-body bound state of DDK $J^P = 0^-,\ I = \frac{1}{2},\ S = 1,\ C = 2$

  Binding energy 70 MeV   DK attractive interaction

Tian-Wei Wu, Ming-Zhu Liu, Li-Sheng Geng, Emiko Hiyama, Manuel Pavon Valderrama PRD100(2019)3, 034029

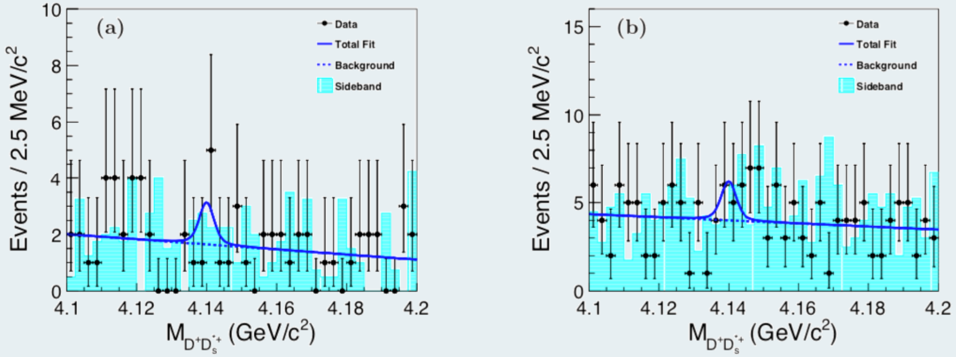


FIG. 4: The invariant-mass spectra of $D^+D_s^{*+}$ in the (a) $\Upsilon(1S)$ and (b) $\Upsilon(2S)$ data samples. The cyan shaded histograms are from the normalized M_{D^+} and $M_{D_s^{*+}}$ sideband events. The blue solid curves show the fitted results with the R^{++} mass fixed at 4.14 GeV/c^2 and width fixed at 2 MeV, and the blue dashed curves are the fitted backgrounds.

Belle collaboration, Phys.Rev.D 102 (2020) 11, 112001

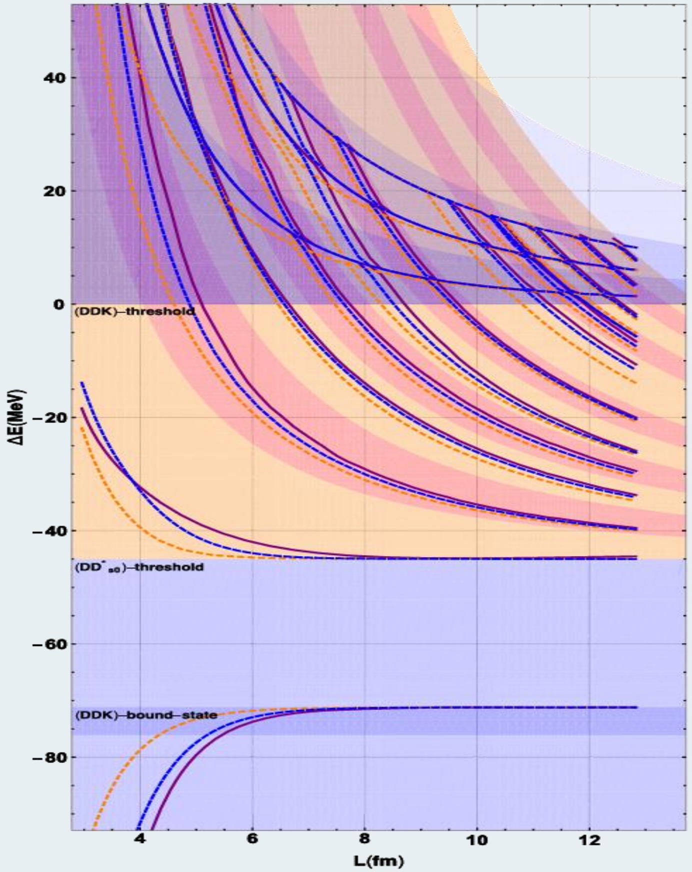


FIG. 7. DDK states in finite volume. The orange curves are calculated at $O(p^0)$, the blue curves at $O(p^2)$, and the purple curves at $O(p^4)$.

Jin-Yi Pang, Li-Sheng Geng, Jia-Jun Wu Phys.Rev.D 102 (2020) 11, 114515

Non-covariant form Finite volume spectra of DDK system

Example: DDK three-body system

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3.$$

$$\mathcal{L}_1 = D^\dagger 2w_v\big(v \cdot i\partial - w_v\big)D + K^\dagger 2w_v\big(v \cdot i\partial - w_v\big)K$$
$$+ \sigma_{DK} T^\dagger_{DK} T_{DK} + \sigma_{DD} T^\dagger_{DD} T_{DD}.$$

$$\mathcal{L}_2 = T^\dagger_{DK}(D\mathscr{F}_{DK}K) + \frac{1}{2}T^\dagger_{DD}(D\mathscr{F}_{DD}D) + \text{h.c.}$$

$$D\mathscr{F}_{DD}D = DD + \frac{1}{8}f_{DD}\big(D\bar{w}^\mu_\perp \bar{w}_{\perp\mu}D - (\bar{w}^\mu_\perp D)(\bar{w}_{\perp\mu}D)\big).$$
$$D\mathscr{F}_{DK}K = DK + \frac{1}{8}f_{DK}\big(u^2_D D\bar{w}^\mu_\perp \bar{w}_{\perp\mu}K + u^2_K K\bar{w}^\mu_\perp \bar{w}_{\perp\mu}D$$
$$- 2u_D u_K(\bar{w}^\mu_\perp D)(\bar{w}_{\perp\mu}K)\big).$$

$$\mathcal{L}_3 = T^\dagger_{DK} D^\dagger \mathscr{H} T_{DK} D,$$

$$\mathscr{H} = h_0 + h_2\Delta + h_2^{\prime}(\overleftarrow{\Delta}_T + \Delta_T) + \cdots \qquad \Delta \to (s - s_{th}),\ \Delta_T \to \sigma^2 - s^{(DK)}_{th}$$

Example: DDK three-body system

Two dimer fields for DK and DD

$$\tau_{DK}(s) = \frac{1}{b_0^{(DK)} + b_1^{(DK)}(s - s_{\text{th}}) - \Sigma_{DK}(s)}, \quad \tau_{DD}(s) = \frac{2}{b_0^{(DD)} + b_1^{(DD)}(s - s_{\text{th}}) - \Sigma_{DD}(s)}.$$

$$\Sigma_{DK}(s) = \frac{1}{16\pi^2}\left(\frac{2p^*_{DK}}{\sqrt{s}} \log \frac{m_D^2 + m_K^2 - s + 2p^*_{DK}\sqrt{s}}{2m_D m_K} - (m_D^2 - m_K^2)\left(\frac{1}{s} - \frac{1}{(m_D + m_K)^2}\right) \log \frac{m_D}{m_K}\right)$$

$$\Sigma_{DD}(s) = \frac{1}{16\pi^2} \frac{2p^*_{DD}}{\sqrt{s}} \log \frac{2m_D^2 - s + 2p^*_{DK}\sqrt{s}}{2m_D^2}$$

$$\tau^L_{DK}(P) = \frac{1}{J_{DK}(s) - \text{Re}\Sigma_{DK}(s) - S^L_{DK}(P)},$$

$$\tau^L_{DD}(P) = \frac{2}{J_{DD}(s) - \text{Re}\Sigma_{DD}(s) - S^L_{DD}(P)}.$$

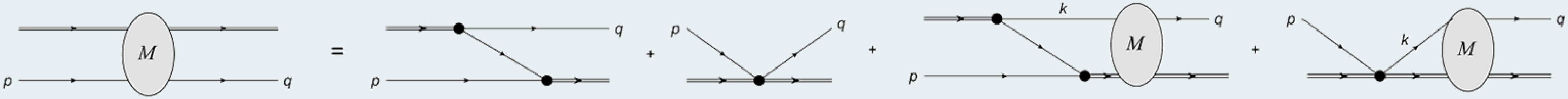
$$J(s) = \left[\frac{1}{b_0 + b_1(s - s_{\text{th}}) - \Sigma(s)} - \frac{R_{\text{f}}}{s - s_{\text{f}}} + c_0 + c_1(s - s_{\text{th}})\right]^{-1} + \Sigma(s).$$

Spurious Pole

$$S^L_{DK} = \frac{1}{4\pi^{3/2} L\gamma\sqrt{s}} Z^d_{00}(1; \eta^2_{DK}) \quad S^L_{DD} = \frac{1}{4\pi^{3/2} L\gamma\sqrt{s}} Z^d_{00}(1; \eta^2_{DD})$$

	b_0	b_1	s_{f}	s^*	R_{f}	c_1	c_2
DK	-0.0019	0.0152	1.4283	1.5364	276.0684	1612.274	-9415.88
DD	0.0053	0.0723	3.8834	$-$	33.1097	284.077	-2437.35

Example: DDK three-body system



$$\mathcal{M}(p,q;P) = Z(p,q;P) + \int \frac{d^4k}{(2\pi)^4}\Theta_v(k)Z(p,k;P)\tau((P-k)^2)\mathcal{M}(k,q;P)$$

$$\mathcal{M}^L(p,q;P) = Z(p,q;P) + \frac{1}{L^3}\sum_{\mathbf{k}}\tilde{\Theta}_v(k)Z(p,k;P)\tau^L((P-k)^2)\mathcal{M}^L(k,q;P).$$

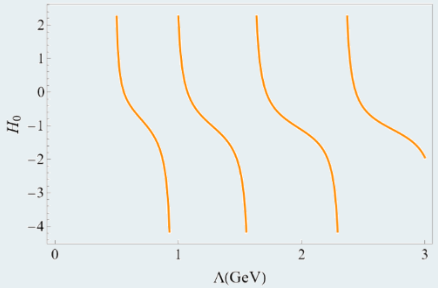
$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}, \quad Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \quad \tau = \begin{pmatrix} \tau_{DK} & \\ & \tau_{DD} \end{pmatrix}$$

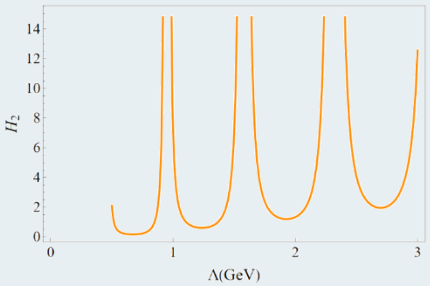
$$Z_{11}(p,q;P) = \frac{1}{2w_v^K(P-p-q)\left(w_v^K(P-p-q) - v\cdot(P-p-q)\right)} + \frac{H_0(\Lambda)}{\Lambda^2} + \frac{H_2(\Lambda)}{\Lambda^4}(s - s_{\rm th})$$

$$Z_{12}(p,q;P) = Z_{21}(q,p;P) = \frac{1}{2w_v^D(P-p-q)(w_v^D(P-p-q) - v\cdot(P-p-q))} \qquad Z_{22} = 0$$

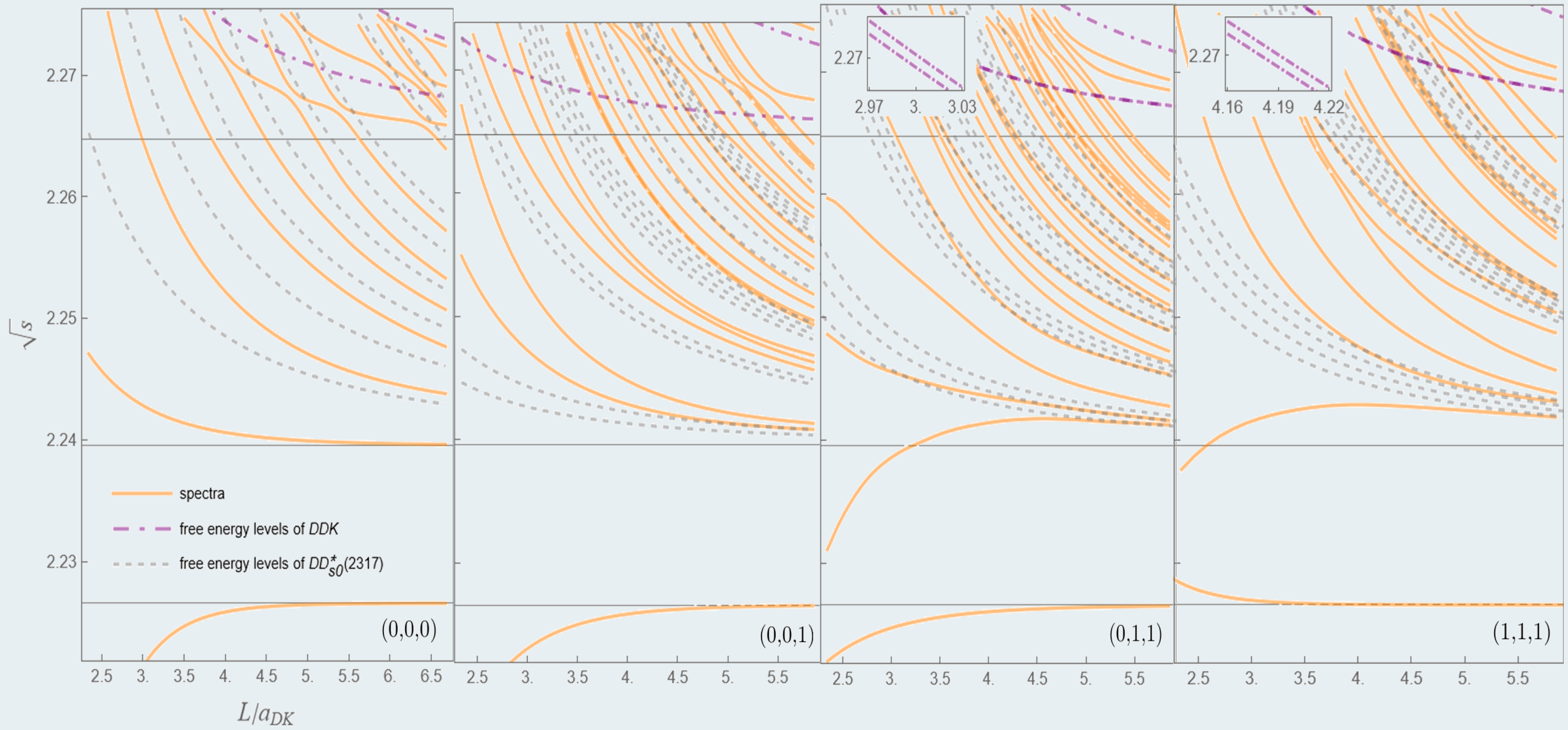
To fix $H_0(\Lambda)$ and $H_2(\Lambda)$

- $DD_{s0}^*(2317)$ scattering length $a = a(H_0)$
- DDK 3-body bound state $B_3 = B_3(H_0, H_2)$





Example: DDK three-body system



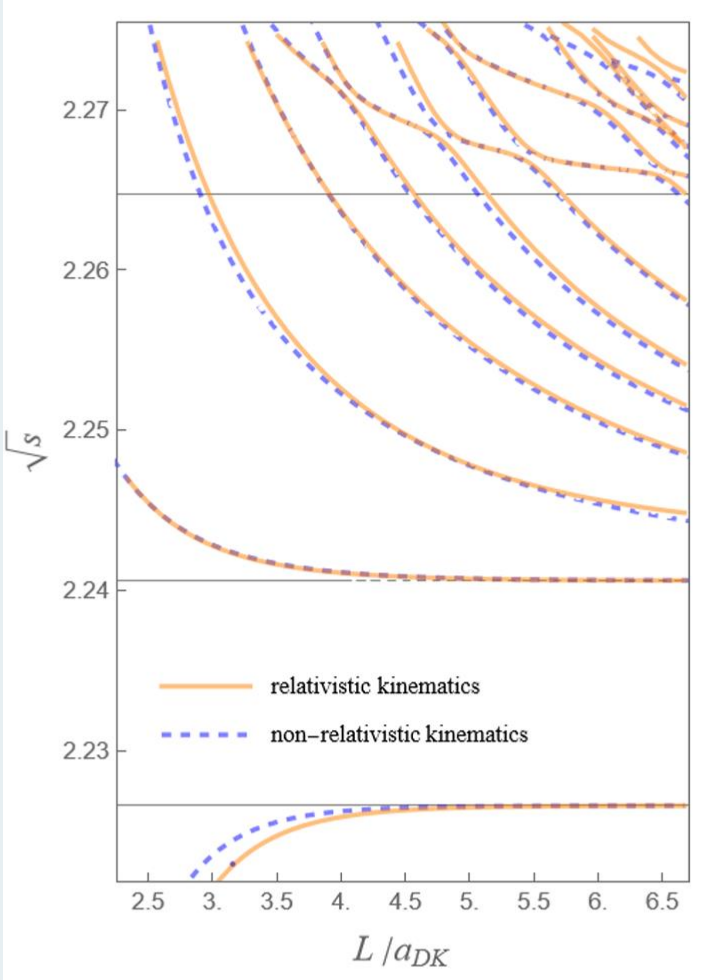
Threshold of DDK

Threshold of DD(2317)

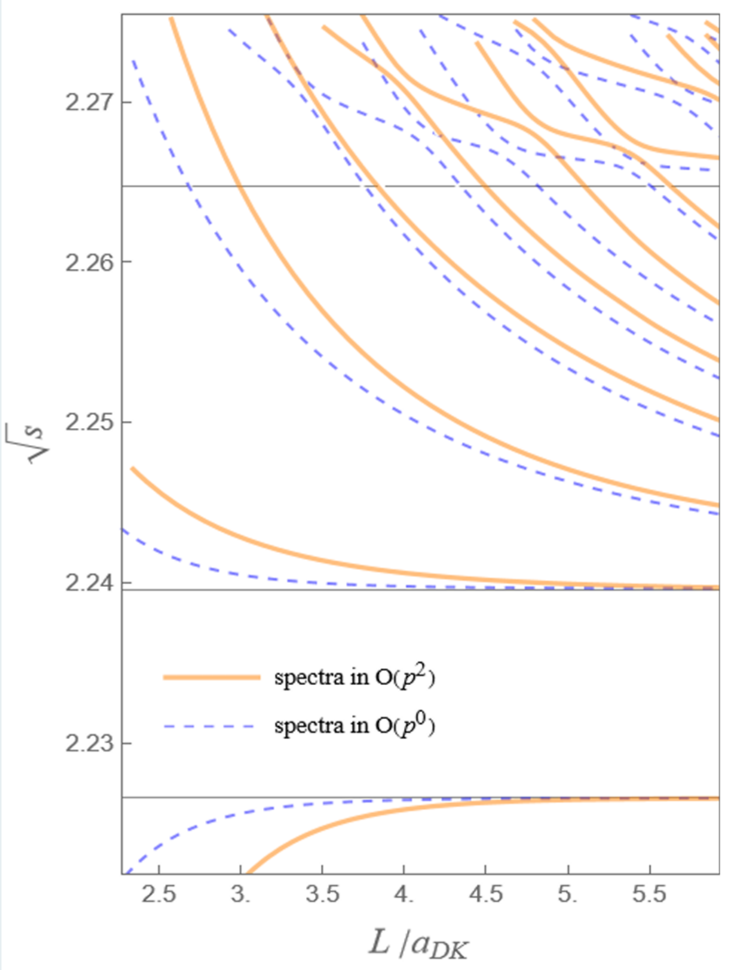
Bound state of DDK

The finite volume spectra for different total momentum ($\sqrt{s}$ is scaled by m_D)

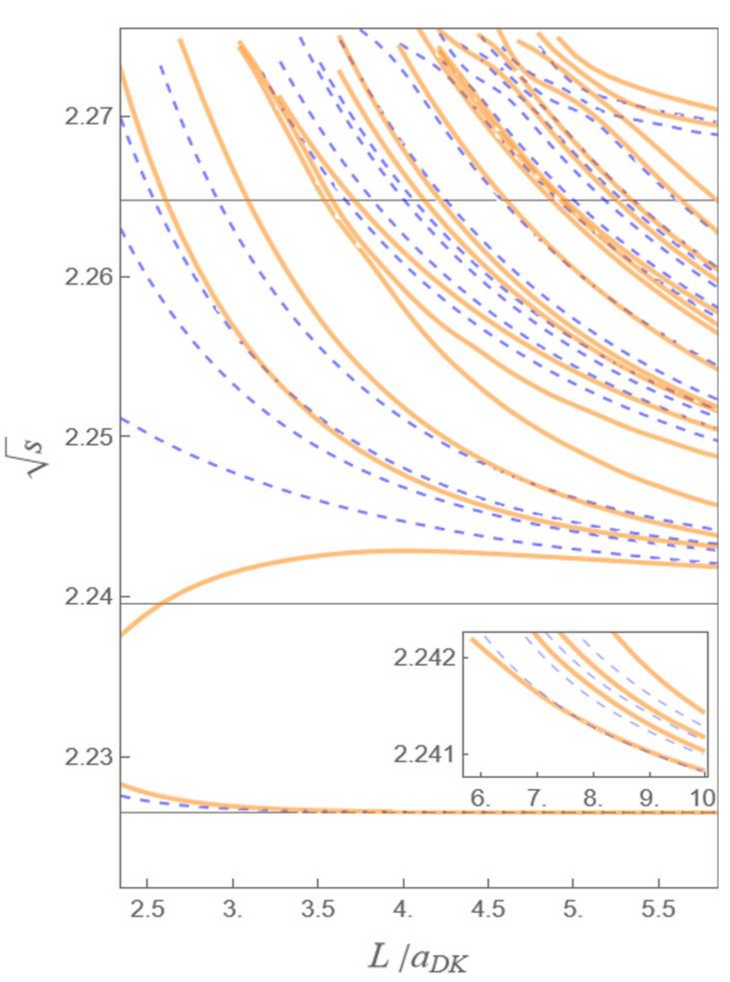
Example: DDK three-body system



Compare with non-relativistic



O(p^0) vs O(p^2) in (0,0,0)



O(p^0) vs O(p^2) in (1,1,1)

Summary and Outlook

We present a covariant form for three-body system by using NREFT for the QC in finite volume and scattering equation in infinite volume.

We present the finite spectra of DDK system.

A lot of things can be done !
For example : to $DD\pi$ system to study T_{cc}

Thanks very much !