The m_π dependence of σ and $N^*(920)$ within linear σ model

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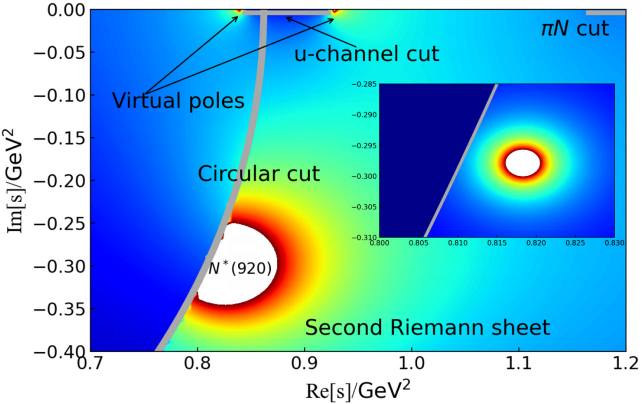
1. A subthreshold pole in πN S_{11} channel:
   - PKU representation [W.F. Wang, D.L. Yao, and H.Q. Zheng, 2018, 2019],
   - K-matrix [Y. Ma, W.Q. Niu, et al, 2020]
   - N/D method [QZL, Y. Ma, W.Q. Niu, et al., 2022]
   - Roy-Steiner equation [X.H. Cao, QZL, and H.Q. Zheng, 2022], [M. Hoferichter, J. R. de Elvira, B. Kubis, and U.-G. MeiSSner, 2024]



2. How to understand this pole?
3. Inspiration from the trajectories of σ the

m_π (MeV)	139	239	283	330	391
$O(N)$ (LO) [Y.L. Lyu, QZL, Z.G. Xiao, and H.Q. Zheng, 2024]	$356 - i148$	$448 - i57$	558(VS I) 438(VS II)	660(VS I) 451(VS II)	780(BS) 489(VS II)
N/D modified $O(N)$	$348 - i180$	469(BS) 426(VS II) 168(VS III)	527(BS) 422(VS II) 264(VS III)	585(BS) $396 - i28$ (Sub. pole)	658(BS) $466 - i77$ (Sub. pole)
lattice + K-matrix [A. Rodas, J. J. Dudek, and R. G. Edwards, 2017,2023]		$(487 \sim 809$ $-i136 \sim 304)$	$(476 \sim 579$ $-i0 \sim 129)$	657^{+3}_{-4}(BS)	758 ± 4(BS)
lattice + Roy Eq.		[X.H. Cao, QZL, Z.H. Guo, and H.Q. Zheng, 2023] $(416 \sim 644$ $-i176 \sim 307)$	$522 \sim 562$ (VS I&II) [A. Rodas, J. J. Dudek, and R. G. Edwards, 2024]		[X.H. Cao, QZL, Z.H. Guo, and H.Q. Zheng, 2023] 759^{+7}_{-16}(BS) $269^{+40}_{-25} - i211^{+26}_{-23}$ (Sub. pole)

4. Linear σ model [M. Gell-Mann and M. Levy, 1960] :
   - Chiral symmetry spontaneous breaking.

The linear σ model Lagrangian with nucleon:

$$\begin{aligned}\mathscr{L} =&\bar{\Psi} i\gamma^\mu \partial_\mu \Psi - g\bar{\Psi}\left(\sigma - i\gamma_5 \vec{\tau}\cdot\boldsymbol{\pi}\right)\Psi \\ &\frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}\right) - \frac{\mu^2}{2}\left(\sigma^2+\boldsymbol{\pi}^2\right) - \frac{\lambda}{4!}\left(\sigma^2+\boldsymbol{\pi}^2\right)^2 \\ &+ C\sigma,\end{aligned} \tag{1}$$

1. Ψ: nucleon field, isospin doublet.
2. $\mu^2 < 0$: $\langle\sigma\rangle = v \neq 0$, spontaneous symmetry breaking .
3. $C\sigma$: explicit symmetry breaking term.
4. $\boldsymbol{\pi}$: π meson triplet.
5. $\sigma \to \sigma + v$ such that $\langle\sigma\rangle = 0$.

At the tree level, particle masses can be read as :

$$\begin{aligned} m_N &= gv, \\ m_\sigma^2 &= \mu^2 + \frac{1}{2}\lambda v^2, \\ m_\pi^2 &= \mu^2 + \frac{1}{6}\lambda v^2. \end{aligned} \tag{2}$$

Tadpole vanishes:

$$(\mu^2 + \frac{1}{6}\lambda v^2)v - C = 0.$$

- Goldstone theorem: $C = 0 \Rightarrow m_\pi^2 v = 0$.
- PCAC: $v = f_\pi$, f_π pion decay constant.

Beyond tree level[B.W. Lee, 1969], [J. A. Mignaco and E. Remiddi, 1971]:

$$\begin{cases} \psi_0 = \sqrt{Z_\psi}\psi, \quad (\sigma_0, \boldsymbol{\pi}_0) = \sqrt{Z_\phi}(\sigma, \boldsymbol{\pi}), \\ \mu_0^2 = \frac{1}{Z_\phi}\left(\mu^2 + \delta\mu^2\right), \quad g_0 = \frac{Z_g}{Z_\psi\sqrt{Z_\phi}}g, \quad \lambda_0 = \frac{Z_\lambda}{Z_\phi^2}\lambda, \end{cases}$$

Renormalization conditions:

1. Z_ϕ, Z_λ and $\delta\mu^2$:
   - Full π propagator $\Delta_\pi(s)$:

$$i\Delta_\pi^{-1}(m_\pi^2) = 0, \quad i\frac{\mathrm{d}\Delta_\pi^{-1}(s)}{\mathrm{d}s}\bigg|_{s=m_\pi^2} = 1 \ , \tag{3}$$

   - Full σ propagator $\Delta_\sigma(s)$:

$$\mathrm{Re}[i\Delta_\sigma^{-1}(m_\sigma^2)] = 0 \ . \tag{4}$$

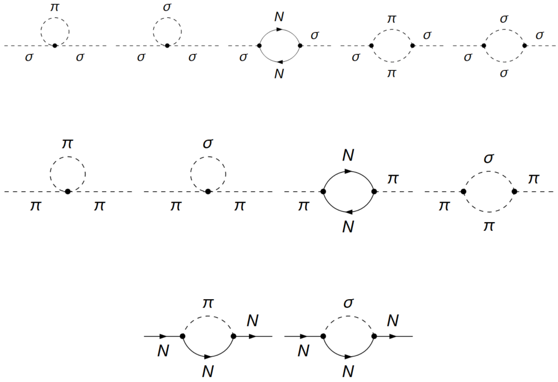
   If $m_\sigma > 2m_\pi$, σ can decay into pions, leading to the propagator become complex at m_σ^2.
   - The relation between m_π and m_σ:

$$m_\sigma^2 = m_\pi^2 + \frac{1}{3}\lambda v^2 \ . \tag{5}$$

2. Z_g, Z_ψ:
   - The full nucleon propagator $\Delta_N(\not{p})$:

$$i\Delta_N^{-1}(m_N) = 0, \quad i\frac{\mathrm{d}\Delta_N^{-1}(\not{p})}{\mathrm{d}\not{p}}\Big|_{\not{p}=m_N} = 1. \tag{6}$$

   - $m_N = gv$.



The full propagator of σ can be expressed:

$$\Delta_\sigma(s) = \frac{i}{s - m_\sigma^2 + \Sigma(s)} \ , \tag{7}$$

Renormalization condition:

$$\mathrm{Re}[i\Delta_\sigma^{-1}(m_\sigma^2)] = 0 \ , \tag{8}$$

or

$$\mathrm{Re}[\Sigma(m_\sigma^2)] = 0 \ . \tag{9}$$

The explicit expression of $\Sigma(s)$ after renormalization:

$$\begin{aligned} \Sigma(s) =& \frac{\lambda^2 v^2}{96\pi^2}\left(-\mathrm{Re}[B_0\left(m_\sigma^2, m_\pi^2, m_\pi^2\right)] + B_0\left(s, m_\pi^2, m_\pi^2\right)\right) \\ &+ \frac{\lambda^2 v^2}{32\pi^2}\left(B_0\left(s, m_\sigma^2, m_\sigma^2\right) - B_0\left(m_\sigma^2, m_\sigma^2, m_\sigma^2\right)\right) \\ &+ \frac{\lambda^2 v^2 B_0'\left(m_\pi^2, m_\pi^2, m_\sigma^2\right)}{144\pi^2}\left(-s + m_\sigma^2\right) . \end{aligned} \tag{10}$$

Two-point loop function B_0:

$$B(p^2, m_1^2, m_2^2) = \frac{-i}{16\pi^2}\int d^4k \frac{1}{k^2 - m_1^2}\frac{1}{(p-k)^2 - m_2^2} .$$

- $m_\sigma > 2m_\pi$: $B_0\left(m_\sigma^2, m_\pi^2, m_\pi^2\right)$ is complex.

$$\mathrm{Re}[\Sigma(m_\sigma^2)] = 0 \nRightarrow i\Delta^{-1}(m_\sigma^2) = 0.$$

  σ propagator is finite when $s = m_\sigma^2$, which means σ is not a bound state.

- $m_\sigma < 2m_\pi$: $B_0\left(m_\sigma^2, m_\pi^2, m_\pi^2\right)$ is real.

$$\mathrm{Re}[\Sigma(m_\sigma^2)] = 0 \Rightarrow \Sigma(m_\sigma^2) = 0 \Rightarrow i\Delta^{-1}(m_\sigma^2) = 0.$$

  m_σ^2 is a pole of σ propagator, resulting σ becomes a bound state.

$IJ(00)$ channel of $\pi\pi$ scattering amplitude

$$\pi^\delta(p_1) + \pi(p_2)^\beta \to \pi^\delta(p_3) + \pi^\gamma(p_4).$$

$\pi\pi$ elastic scattering amplitude is written:

$$T(s,t,u) = A(s,t,u)\delta_{\alpha\beta}\delta_{\gamma\delta} + B(s,t,u)\delta_{\alpha\gamma}\delta_{\beta\delta} + C(s,t,u)\delta_{\alpha\delta}\delta_{\gamma\beta}.$$

- $\alpha, \beta, \gamma, \delta$: isospin indexes.
- s, t, u : Mandelstam variables, $s + u + t = 4m_\pi^2$.
- $A(s,t,u), B(s,t,u), C(s,t,u)$: Lorentz invariant amplitudes

Total isospin $I = 0$ amplitude $T^0(s,t,u)$ can be given:

$$T^0(s,t,u) = 3A(s,t,u) + B(s,t,u) + C(s,t,u)\ .$$

Partial wave amplitude is defined:

$$T_J^I(s) = \frac{1}{32\pi(s-4m_\pi^2)} \int_{4m_\pi^2-s}^{0} P_J\left(1+\frac{2t}{s-4}\right) T^I(s,t,u)\ , \quad (11)$$

- P_J: Legendre polynomial.

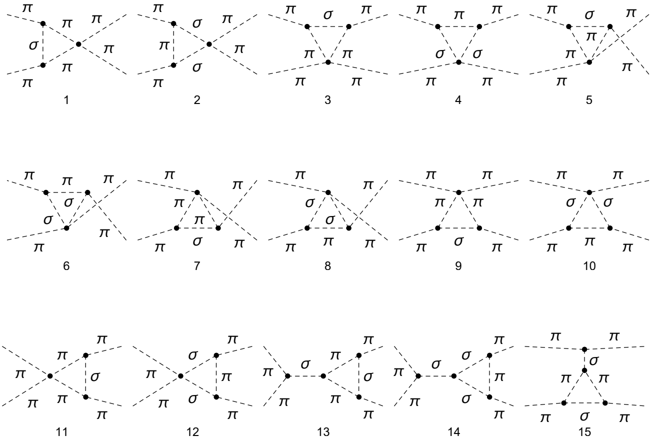
The elastic unitarity reads:

$$\mathrm{Im}\ T_J^I(s) = \rho(s, m_\pi, m_\pi)|T_J^I(s)|^2, \quad s > 4m_\pi^2\ , \qquad (12)$$

with

$$\rho(s, m_1, m_2) = \frac{\sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}{s}\ . \qquad (13)$$

Some one-loop diagrams contribute to $\pi\pi$ elastic scatterings:

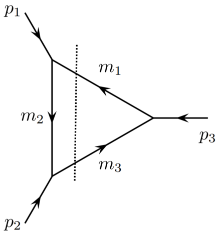


Numerical tips

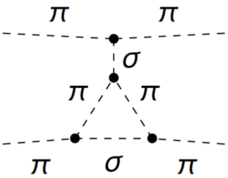
- Loop functions: dispersion relation[Peter Stoffer, 2014].

For example, scalar three-point function:

$$C_0(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2) = \frac{-i}{16\pi^2}\int \frac{d^4k}{D_1 D_2 D_3}$$
$$= \frac{1}{\pi}\int_{(m_1+m_3)^2}^{\infty} ds' \frac{\mathrm{Im}[C_0(s')]}{s' - p_3^2}$$



- $D_1 = k^2 - m_1^2$
- $D_2 = (k + p_1)^2 - m_2^2$
- $D_3 = (k + p_1 + p_2)^2 - m_3^2$
- $\mathrm{Im}[C_0(s')]$: Cutkosky' s rules[R. Cutkosky, 1960].



$$\propto \quad \frac{1}{t - m_\sigma^2} C_0(m_\pi^2, m_\pi^2, t, m_\pi^2, m_\sigma^2, m_\pi^2)$$

Partial-wave projection:

$$\int_{4m_\pi^2 - s}^{0} dt P_J \left(1 + \frac{2t}{s-4}\right) \frac{1}{t - m_\sigma^2} C_0(m_\pi^2, m_\pi^2, t, m_\pi^2, m_\sigma^2, m_\pi^2)$$

$$= \frac{1}{\pi} \int_{4m_\pi^2} dt' \, \mathrm{Im}[C_0(t')] \int_{4m_\pi^2 - s}^{0} dt \frac{1}{(t' - t)(t - m_\sigma^2)}$$

Padé amplitude

Perturbative unitarity:

$$\mathrm{Im}\, T^0_{0l}(s) = \rho(s, m_\pi, m_\pi)|T^0_{0t}(s)|^2 \,, \quad 4m_\pi^2 < s < 4m_\sigma^2$$

- $T^0_{0t}(s)$, $T^0_{0l}(s)$: tree-level and one-loop partial-wave amplitudes.

$[1,1]$ Padé approximants:

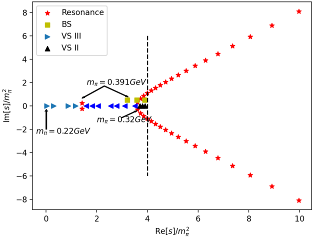
$$T_0^{0[1,1]}(s) = \frac{T^0_{0t}(s)}{1 - T^0_{0l}(s)/T^0_{0t}(s)} \,, \tag{14}$$

S matrix:

$$S(s) = 1 + 2i\rho(s, m_\pi, m_\pi)\, T_0^{0[1,1]}(s)$$

σ resonances: zero points of S matrix on the first Riemann sheet or poles of S matrix on the second Riemann sheet.

Fixing $m_N = 0.938\text{GeV}$, $m_\sigma = 0.7\text{GeV}$ and $v = 0.093\text{GeV}$.



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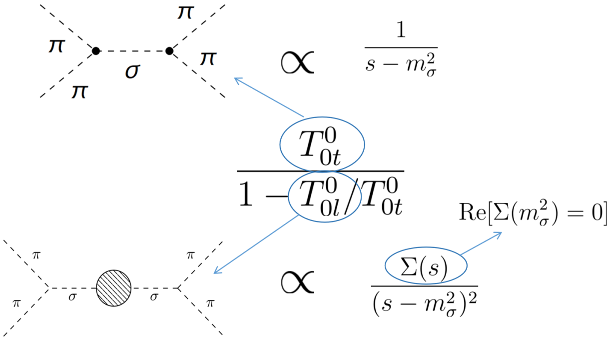
(a) $m_\pi \in (0.139, 0.32)\text{GeV}$: $\text{Re}\, z_0 \downarrow$ and $\text{Im}\, z_0 \downarrow$ until $\text{Re}\, z_0 < 4m_\pi^2$ and $\text{Im}\, z_0 = 0$, generating two virtual states.

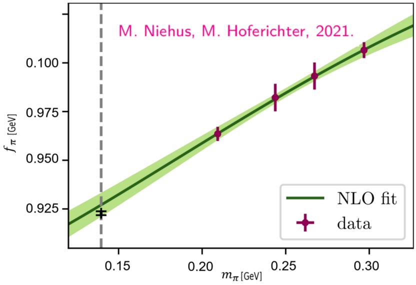
(b) $m_\pi \in (0.32\text{GeV}, m_c)$: $z_0^{II} \to 4m_\pi^2$ and finally crosses threshold to the first Riemann sheet, turning out a bound state.

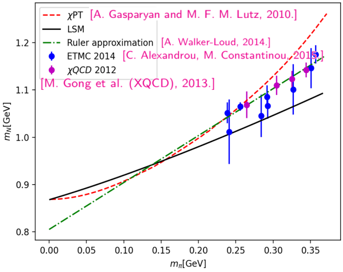
(c) $m_\pi \in (0.32, 0.391)\text{GeV}$: $z_0^{I} \downarrow$, $z_0^{III} \uparrow$ and colliding with each other, resulting in a pair of resonances.

- $m_c = \frac{m_\sigma}{2}$;
- $m_\pi < m_c$: $\Sigma(m_\sigma^2) \neq 0 \Rightarrow T_0^{0[1,1]}(m_\sigma^2) \neq \infty$;
- $m_\pi > m_c$: $\Sigma(m_\sigma^2) = 0 \Rightarrow T_0^{0[1,1]}(m_\sigma^2) = \infty$;



1. PCAC: $v = f_\pi$;
2. Basing on lattice calculation, pion decay constant and pion mass : $f_\pi = km_\pi + b$, $k \simeq 0.046, b \simeq 0.087\text{GeV}$;.
3. $m_N = gv = gf_\pi$;
4. $m_\pi = 0.139\text{GeV}, m_N = 0.938\text{GeV}, v = 0.093\text{GeV}$, $g \simeq 10$.





S_{11} channel of πN scattering amplitude

$$\pi^a(p) + N_i(q) \to \pi^{a'}(p') + N_f(q')$$

Isospin amplitude decomposed as:

$$T = \chi_f^\dagger \left(\delta^{aa'} T^+ + \frac{1}{2} \left[\tau^{a'}, \tau^a \right] T^- \right) \chi_i$$

1. τ^a: Pauli matrices;
2. χ_i, χ_f: isospin wave functions;

Total isospins $I = 1/2, 3/2$:

$$T^{I=1/2} = T^+ + 2T^- \ ,$$
$$T^{I=3/2} = T^+ - T^- \ .$$

$I = 1/2, 3/2$ Lorentz structure:

$$T^I = \bar{u}^{(s')}(q') \left[A^I(s,t) + \frac{1}{2} \left(\not{p} + \not{p}' \right) B^I(s,t) \right] u^{(s)}(q)$$

$$T_{\pm}^{I,J} = T_{++}^{I,J}(s) \pm T_{+-}^{I,J}(s), \quad L = J \mp \frac{1}{2},$$

1. I, J and L: total isospin, total angle momentum and orbit angle momentum;
2. partial-wave helicity amplitudes $T_{+\pm}^{I,J}$:

$$T_{++}^{I,J} = 2m_N A_C^{I,J}(s) + \left(s - m_\pi^2 - m_N^2\right) B_C^{I,J}(s)$$
$$T_{+-}^{I,J} = \frac{-1}{\sqrt{s}}\left[\left(s - m_\pi^2 + m_N^2\right) A_S^{I,J}(s) + m_N\left(s + m_\pi^2 - m_N^2\right) B_S^{I,J}(s)\right]$$

$$F_{C/S}^{I,J}(s) = \int_{-1}^{1} \mathrm{d}z_s F^I(s,t)\left[P_{J+1/2}(z_s) \pm P_{J-1/2}(z_s)\right], \quad F = A, B$$

$z_s = \cos\theta$ with θ scattering angle

3. elastic unitarity condition:
$\mathrm{Im}\, T_{\pm}^{I,J}(s) = \rho(s, m_\pi, m_N)|T_{\pm}^{I,J}(s)|^2, s > s_R = (m_\pi + m_N)^2.$

N/D method corresponding to solve an integral equation about $N(s)$ function:

$$N(s) = N(s_0) + U(s) - U(s_0) + \frac{(s-s_0)}{\pi}\int_{s_R}^{\infty}\frac{(U(s)-U(s'))\rho(s',m_\pi,m_N)N(s')}{(s'-s_0)(s'-s)}ds'$$

1. s_0: subtraction point;
2. $N(s_0)$: subtraction constant;
3. $U(s)$ function: analytical when $s > s_R$;
4. $N(s)$ function: only containing left hand cuts.

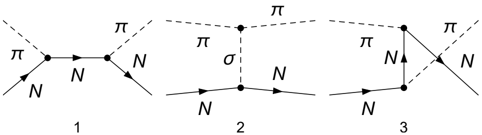
The partial-wave amplitude $M(s)$ satisfying the unitarity condition:

$$M(s) = \frac{N(s)}{D(s)} \,,$$

$$D(s) = 1 - \frac{s-s_0}{\pi}\int_{s_R}^{\infty}\frac{\rho(s')N(s')}{(s'-s)(s'-s_0)}ds'.$$

$$N(s) = N(s_0) + U(s) - U(s_0) + \frac{(s-s_0)}{\pi}\int_{s_R}^{\infty}\frac{(U(s)-U(s'))\rho(s',m_\pi,m_N)N(s')}{(s'-s_0)(s'-s)}ds'$$

1. Introducing a cutoff Λ, $(s_R,\infty)\to(s_R,\Lambda)$, matrix inverse method;
2. s_0 and Λ fixed at s_R and $(m_N+m_\sigma)^2$;
3. Tree level: $U(s)$ equal to tree-level partial-wave amplitude $M_t(s)$ and $N(s_0)=U(s_0)$;

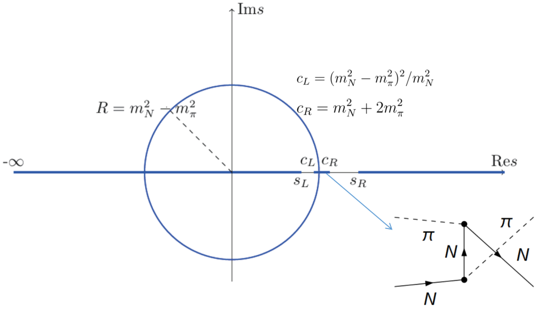


4. Up to one-loop level:

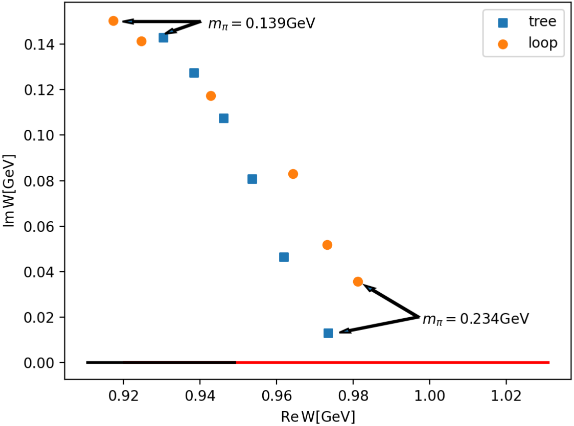
$$U(s) = M_t(s) + M_l(s) - \frac{s}{\pi}\int_{s_R} \frac{\rho(s', m_\pi, m_N) M_t^2(s')}{s'(s'-s)} ,$$

for the one-loop partial-wave amplitude $M_l(s)$, due to perturbative unitarity:

$$\mathrm{Im}\, M_l(s) = \rho(s, m_\pi, m_N)|M_t(s)|^2, \quad s > s_R .$$



- S matirx: $S(s) = 1 + 2i\rho(s, m_\pi, m_N)M(s)$;
- Fixing $m_\sigma = 0.55\text{GeV}$;
- $N^*(920)$: zero points of $S(s)$:



Summary

1. The trajectories of σ resonances with m_π variations are consistent with lattice results and $\mathcal{O}(N)$ model.

2. Due to the complexity of the analytic structure of the πN scattering amplitude, $N^*(920)$ will tend to u-cuts with m_π increasing.

3. The properties of $N^*(920)$ remain to be further studied: lattice calculations, finite temperature and density.

Thank you!